Algèbres de Nakayama dont le cube du radical de la catégorie des modules est nul
Nakayama algebras whose module category is radical cubed zero

par

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SOMMAIRE

Dans ce mémoire, nous entendons par algèbre toute K-algèbre de dimension finie sur un corps algébriquement clos K. Si A est une K-algèbre, on entend par A-module tout A-module à gauche de type fini, sauf indication contraire.

Tout d'abord, nous introduisons les notions nécessaires et les définitions des catégories des modules. Ensuite, nous démontrons certains concepts de la théorie des représentations des algèbres et des théorèmes de structure sur les algèbres de Nakayama, sur les modules projectifs et injectifs, afin de prouver le théorème qui stipule que le cube radical de la catégorie de module avec certaines conditions est nul pour les algèbres de Nakayama.

ABSTRACT

In this thesis, we mean by algebra any K-algebra of finite dimension over an algebraically closed field K. If A is a K-algebra, we mean by A-module any left A-module of finite type, unless otherwise specified.

First, we introduce the necessary notions and the definitions of the category of modules. Then, we prove certain concepts of the theory of representations of algebras and the structure theorems on Nakayama algebras and projective and injective modules, in order to prove the theorem which states that the radical cube of the category of module with some conditions is zero for Nakayama algebras.

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INTRODUCTION

Let A be a finite dimensional K-algebra over an algebraically closed field K; and mod A the category of finitely generated left A-modules. It is well known that A is of finite representation type if and only if the radical of mod A is nilpotent. Also, by the Harada-Sai lemma, the radical of mod$A$ is nilpotent with a nilpotency index bounded by $2^b - 1$, where b is the maximal dimension of indecomposable A-modules; see [4].

In this thesis, we would like to prove that, when $A = KQ/I$ is a Nakayama algebra, where Q is a finite connected quiver and I is an admissible ideal of kQ. Then $\text{rad}^3(\text{mod}A) = 0$ if and only if one of the following cases occurs :

(1) $Q = \vec{\mathbb{A}}_n$ with $1 \le n \le 3$ and $I = 0$.

(2) $Q = \tilde{\mathbb{A}}_n$ with $n \ge 1$ and I is generated by all paths of length two in Q.

(3) $Q = \vec{\mathbb{A}}_n$ with $n \ge 3$ and I is generated by all paths of length two in Q.

To reach this goal, we need to understand some important definitions and lemmas about the radical of the module categories, representations of algebras, theory of Auslander-Reiten and Nakayama algebras.

So, we will start the first chapter by introducing some notions on algebras and modules. The second chapter will be devoted to quivers and algebras. Then, chapter three consists of Auslander-Reiten theory and all important lemmas about almost split morphisms and irreducible morphisms.

In the last chapter, we need to understand the Nakayama algebras and the lemmas which we need to solve our main result. Finally, we will see the main result and proof.

CHAPTER 1

Algebras and Modules

The content presented in this chapter is taken from [3]. Throughout this thesis, K denotes an algebraically closed field. We shall consider only finite dimensional K-algebras.

1.1 Algebras

Throughout this thesis, we need to study categories of modules over algebras. It is natural to start the first section with algebras.

Definition 1.1.1. *Let A be a K-algebra. We define the* opposite algebra A^{op} *to have the same elements as A, but the multiplication $*$ in A^{op} is defined as follows : $a * b = ba$, for all $a, b \in A$.*

Definition 1.1.2. *The algebra A is said to be* finite dimensional *if its dimension as a K-vector space is finite.*

Definition 1.1.3. *Let A be a finite dimensional K-algebra. A K-vector subspace I of A is a* right ideal *(or* left ideal*) of A if $ia \in I$ (or $ai \in I$, respectively) for all $i \in I$ and $a \in A$.*

A two-sided ideal *of A is a K-vector subspace I of A which is a right ideal and left ideal of A.*

Definition 1.1.4. *Let A be a finite dimensional K-algebra. The intersection of all maximal left ideals of A is called the (Jacobson)* radical *of A, written as* rad A.

Lemma 1.1.5. ([8], I.3.1) *Let A be a finite dimensional K-algebra. Then* rad A *is the intersection of all maximal right ideals of A, which is also a two-sided ideal of A. Moreover,*

$$\begin{aligned} \mathrm{rad} A &= \{a \in A \mid 1 - xa \text{ is left invertible, for all } x \in A\} \\ &= \{a \in A \mid 1 - ax \text{ is right invertible, for all } x \in A\}. \end{aligned}$$

Definition 1.1.6. *A finite dimensional K-algebra A is said to be* local *if A has a unique maximal left ideal.*

Lemma 1.1.7. ([3], I.4.6) *Let A be a finite dimensional K-algebra. The following statements are equivalent.*

(1) *A is a local algebra.*

(2) *A has a unique maximal right ideal.*

(3) rad A *consists of all noninvertible elements of A.*

(4) *The set of all invertible elements of A is a two-sided ideal of A.*

(5) *For any $a \in A$, one of the elements a or $1 - a$ is invertible.*

(6) $A/\mathrm{rad}\ A$ *is a division algebra.*

Definition 1.1.8. *Let A be a finite dimensional K-algebra. An element $x \in A$ is said to be* nilpotent *if there exists $m \in \mathbb{N}$ such that $x^m = 0$.*

Lemma 1.1.9. *Let A be a finite dimensional K-algebra. If $a \in A$ is nilpotent, then $1 - a$ is invertible.*

Proof. If $a^n = 0$, then $(1-a)(1+a+\cdots+a^{n-1}) = 1 - a^n = 1$. Thus, $1 - a$ is invertible. □

Definition 1.1.10. *Let A be a finite dimensional K-algebra. An element $e \in A$ is called* idempotent *if $e^2 = e$. Two idempotents $e, e' \in A$ are called* orthogonal *if $ee' = e'e = 0$. The idempotent e is said to be* primitive *if e cannot be written as a sum $e = e' + e''$, where e', e'' are nonzero orthogonal idempotents of A.*

Definition 1.1.11. *Let A be a finite dimensional K-algebra. A set $\{e_1, \ldots, e_n\}$ of idempotents of A is called a* complete set of orthogonal primitive idempotents *if the e_i are pairwise orthogonal primitives idempotents of A such that $1 = e_1 + \cdots + e_n$.*

Proposition 1.1.12. ([6], 2.27) *Let A be a finite dimensional K-algebra with an idempotent e. The following statements hold.*

(1) *$eAe = \{eae \mid a \in A\}$ is an algebra with e being the identity.*

(2) *We have an anti-isomorphism of algebras as follows:*

$$\phi : \mathrm{End}_A(Ae) \to eAe : f \mapsto f(e).$$

Corollary 1.1.13. ([6], 2.3) *Let A be a finite dimensional K-algebra. Then,*

$$\phi_A : \mathrm{Hom}_A({}_AA, {}_AA) \to A : f \to f(1)$$

is an anti-isomorphism of algebras.

1.2 Linear Categories

In this section, we present an introduction to general categories.

Definition 1.2.1. *A* category *is a triple* $\mathcal{C} = (\mathrm{Ob}\,\mathcal{C}, \mathrm{Hom}\,\mathcal{C}, \circ)$*, where* $\mathrm{Ob}\,\mathcal{C}$ *is called the class of* objects*,* $\mathrm{Hom}\,\mathcal{C}$ *is called the class of* morphisms*, and* $\circ$ *is called the* composition of morphisms*, satisfying the following conditions.*

(1) *To each pair of objects* X, Y*, we associate a set* $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ *of* morphisms *from* X *to* Y *with the following properties*:

  (a) *for each object* X*, there exists an element* $id_X \in \mathrm{Hom}_{\mathcal{C}}(X,X)$*, called the* identity morphism *on* X;

  (b) *the intersection of* $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ *and* $\mathrm{Hom}_{\mathcal{C}}(Z,U)$ *is empty*; *in case* $(X,Y) \neq (Z,U)$.

(2) *The composition* $\circ$ *is only partially defined for objects* X, Y, Z *as follows*:

$$\circ : \mathrm{Hom}_{\mathcal{C}}(Y,Z) \times \mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{C}}(X,Z) : (g,f) \mapsto g \circ f$$

with the following two properties :

(a) $h \circ (g \circ f) = ((h \circ g) \circ f$*, for all morphisms* $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$*,* $g \in \mathrm{Hom}_{\mathcal{C}}(Y,Z)$*,* $h \in \mathrm{Hom}_{\mathcal{C}}(Z,U)$; *and*

(b) $f \circ 1_X = f$ *and* $1_X \circ g = g$*, for all objects* X, Y, Z *and morphisms* $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$ *and* $g \in \mathrm{Hom}_{\mathcal{C}}(Z,X)$.

Definition 1.2.2. *Let* $\mathcal{C}$ *be a category. A morphism* $f : M \to N$ *in* $\mathcal{C}$ *is called*

(1) *a* section *if there exists a morphism* $g : N \to M$ *such that* $gf = id_M$.

(2) *a* retraction *if there exists a morphism* $g : N \to M$ *such that* $fg = id_N$.

Lemma 1.2.3. *Let $\mathcal{C}$ be a category with morphisms $f : M \to N$ and $g : N \to L$.*

(1) *If f, g are sections, then gf is a section.*

(2) *If f, g are retractions, then gf is a retraction.*

Proof. (1) Suppose that there exist $f' : N \to M$ and $g' : L \to N$ such that $f'f = id_M$ and $g'g = id_N$. Then, $(f'g')(gf) = f'f = id_M$. So gf is a section.

(2) Suppose that there exist $f' : N \to M$ and $g' : L \to N$ such that $ff' = id_N$ and $gg' = id_L$. Then, $(gf)(f'g') = gg'f = id_L$. So gf is a retraction. □

Definition 1.2.4. *A category $\mathcal{C}$ is called K-*linear *if*

(1) *for all objects X, Y, the set* $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ *is a K-vector space;*

(2) *the composition of morphisms is K-bilinear, that is, for all morphisms* $f_1, f_2 \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, $g_1, g_2 \in \mathrm{Hom}_{\mathcal{C}}(Y, Z)$ *and all scalars* $\lambda_1, \lambda_2, \mu_1, \mu_2 \in K$, *we have*

$$g \circ (\lambda_1 f_1 + \lambda_2 f_2) = \lambda_1 (g \circ f_1) + \lambda_2 (g \circ f_2)$$

$$(\mu_1 g_1 + \mu_2 g_2) \circ f = \mu_1 (g_1 \circ f) + \mu_2 (g_2 \circ f)$$

(3) *every finite family of objects of $\mathcal{C}$ admits a product and co-product in $\mathcal{C}$.*

Definition 1.2.5. *Let $\mathcal{C}$ be a K-linear category and $f : M \to N$ a morphism in $\mathcal{C}$.*

(1) *A* kernel *of f, denoted by* $\mathrm{Ker}(f)$, *is a pair (U, u), where U is an object of $\mathcal{C}$ and $u : U \to M$ is a morphism such that :*

  (a) $fu = 0$.

  (b) *If $u' : U' \to M$ is a morphism such that $fu' = 0$, then there exists a unique morphism $g : U' \to U$ such that $u' = ug$.*

(2) *A* cokernel *of f, denoted by* $\mathrm{Coker}(f)$, *is a pair (V, v), where V is an object of $\mathcal{C}$ and $v : N \to V$ is a morphism such that :*

(a) $vf = 0$.

(b) *If* $v' : N \to V'$ *is a morphism such that* $v'f = 0$*, then there exists a unique morphism* $h : V \to V'$ *such that* $v' = hv$*.*

Definition 1.2.6. *Let* $\mathcal{C}$ *be a* K*-linear category. A class* $\mathcal{I}$ *of morphisms of* $\mathcal{C}$ *is called a* two-sided ideal *in* $\mathcal{C}$ *if* $\mathcal{I}$ *has the following properties :*

(1) *for any* $X, Y \in \operatorname{Ob}\mathcal{C}$*, the set* $\mathcal{I}(X,Y)$ *of morphisms* $f : X \to Y$ *in* $\mathcal{I}$ *is a* K*-vector subspace of* $\operatorname{Hom}_{\mathcal{C}}(X,Y)$*;*

(2) *if* $f \in \mathcal{I}$ *and* g *is a morphism in* $\mathcal{C}$ *that is left-composable with* f*, then* $g \circ f \in \mathcal{I}$ *;*

(3) *if* $f \in \mathcal{I}$ *and* h *is a morphism in* $\mathcal{C}$ *that is right-composable with* f*,then* $f \circ h \in \mathcal{I}$*.*

Definition 1.2.7. *Let* $\mathcal{C}$ *be a linear category. The* radical *of* $\mathcal{C}$ *is a two-sided ideal* $\operatorname{rad}_{\mathcal{C}}$ *in* $\mathcal{C}$*, defined by the following formula*

$$\operatorname{rad}_{\mathcal{C}}(X,Y) = \{h \in \operatorname{Hom}_{\mathcal{C}}(X,Y) : id_X - g \circ h \text{ is invertible for any } g \in \operatorname{Hom}_{\mathcal{C}}(Y,X)\}$$

for any objects X, Y *of* $\mathcal{C}$*.*

Lemma 1.2.8. *Let* $\mathcal{C}$ *be a linear category with* $f \in \operatorname{rad}_{\mathcal{C}}(M,N)$*.*

(1) *If* M *is not zero, then* f *is not a section.*

(2) *If* N *is not zero, then* f *is not retraction.*

Proof. (1) Suppose that M is not zero. If f is a section, then there exists $f' : N \to M$ such that $f'f = id_M$. Then, $0 = id_M - f'f$, which is invertible. This is impossible as M is not zero. So f is not a section.

(2) Suppose that N is not zero. If f is a retraction, then there exists $f' : N \to M$ such that $ff' = id_N$. Then, $0 = id_N - ff'$, which is invertible. This is impossible as N is not zero. So f is not a retraction. $\square$

1.3 Category of Modules

To understand the radical of a module category better, we start this section with module categories.

Definition 1.3.1. *Let A be a finite dimensional K-algebra. The* category of *left A-*modules *is a category whose objects are left modules and whose morphisms are all module homomorphisms between left A-modules. So, we can say that the category* mod A *of all finitely generated left A-modules is a K-linear category.*

Lemma 1.3.2. *Let A be a finite dimensional K-algebra with $f : M \to N$ a morphism in* mod A*. If* $\dim(M) = \dim(N)$*, then f is an isomorphism if and only if f is a monomorphism if and only if f is an epimorphism.*

Proof. It is well known that $\dim(M) = \dim(\mathrm{Ker} f) + \dim(\mathrm{Im} f)$. Assume that

$\dim(M) = \dim(N)$.

If f is an epimorphism, then $\dim(M) = \dim(N) + \dim(\mathrm{Ker} f)$, and so, $\dim(\mathrm{Ker} f) = 0$. Then f is a monomorphism, and hence, an isomorphism.

If f is a monomorphism, then $\dim(\mathrm{Ker} f) = 0$. Therefore, $\dim(N) = \dim(M) = \dim(\mathrm{Im} f)$. Then, f is an epimorphism, and hence, an isomorphism. □

Definition 1.3.3. *Let A be a finite dimensional K-algebra. A non-zero module M in* mod A *is said to be* indecomposable *if $M = M_1 \oplus M_2$ implies that $M_1 = 0$ or $M_2 = 0$.*

Definition 1.3.4. *An algebra A is called* representation-finite *if its module category admits only finitely many isoclasses of indecomposable objects. It is called* representation-infinite *if it is not representation-finite.*

Lemma 1.3.5. ([8], I. 8.7)

Let A be a K-algebra, $e \in A$ an idempotent, and M be a module in mod A. *Then the K-linear map $\theta : \mathrm{Hom}_A(Ae, M) \to eM$, defined by $\theta(\phi) = \phi(e) = e\phi(e)$ for $\phi \in \mathrm{Hom}_A(Ae, M)$, is an isomorphism of left eAe-modules.*

Lemma 1.3.6. ([3], I. 4.7) *Let A be a finite dimensional K-algebra. A module M in* $\mathrm{mod}\, A$ *is indecomposable if and only if* $\mathrm{End}(M)$ *is local.*

Proposition 1.3.7. ([1], VII. 6.13) *Let A be a finite dimensional K-algebra with M a non-zero module in* mod A. *Then, $M \cong M_1 \oplus \ldots \oplus M_n$, where each M_i is indecomposable. Moreover, if $M \cong N_1 \oplus \ldots \oplus N_m$ with each N_j indecomposable, then $m = n$ and there exists a permutation σ on $\{1, \ldots, n\}$ such that $M_i \cong N_{\sigma(i)}$, for any $1 \le i \le n$.*

Definition 1.3.8. *Let A be a finite dimensional K-algebra. A non-zero module S in* mod A *is called* simple *if any submodule of S is either zero or S.*

Proposition 1.3.9. *Let A be a finite dimensional K-algebra and M a nonzero module in* mod A. *The following conditions are equivalent.*

(1) *M is simple.*

(2) *Every morphism $f : M \to L$ in* $\mathrm{mod}\, A$ *is zero or a monomorphism.*

(3) *Every morphism $g : L \to M$ in* $\mathrm{mod}\, A$ *is zero or an epimorphism.*

Proof. (1) implies (2). Suppose that M is simple. Let $f : M \to L$ be a morphism. Since M is simple and $\mathrm{Ker}(f) \subseteq M$, then $\mathrm{Ker}(f) = 0$ or $\mathrm{Ker}(f) = M$.

If $\mathrm{Ker}(f) = 0$, then f is a monomorphism.

If $\mathrm{Ker}(f) = M$, then $f(x) = 0_L$ for any $x \in M$. So f is zero.

(2) implies (3). Suppose that every A-linear map $f : M \to L$ is zero or a monomorphism. Let $g : L \to M$ be an A-linear map. Consider the canonical projection $p : M \to M/\mathrm{Im}(g)$. If p is zero, then $M/\mathrm{Im}(g) = 0$. So $M = \mathrm{Im}(g)$. Then g is an epimorphism. Otherwise, p is a monomorphism. So $\mathrm{Im}(g) = \mathrm{Ker}(p) = 0$. So g is zero.

(3) implies (1). Suppose that every A-linear map $g : L \to M$ is zero or an epimorphism. Let N be a submodule of M. Consider the inclusion map $j : N \to M$. If j is zero, then $N = 0$. Otherwise, j is surjective. So $M = \mathrm{Im}(j) = N$. Then M is simple. □

Corollary 1.3.10. ([8], I. 5.1) *Let A be a finite dimensional K-algebra. If S is a simple module in* $\operatorname{mod} A$*, then* $\mathrm{End}_A(S)$ *is a division algebra.*

Definition 1.3.11. *Let A be a finite dimensional K-algebra with M a nonzero module in* mod A*. A chain*

$$0 = M_0 \subset M_1 \subset \cdots \subset M_m = M$$

of submodules of M is called composition series *of M if M_{i+1}/M_i is simple, for*

$i = \{0, 1, \ldots, m-1\}$.

Theorem 1.3.12. ([3], I.3) *If A is a finite dimensional K-algebra. Let*

$$0 = M_0 \subset M_1 \subset \cdots \subset M_m = M$$

and

$$0 = N_0 \subset N_1 \subset \cdots \subset N_n = M$$

be two composition series of a module M in mod A*. Then $m = n$, and there exists a permutation σ of* $\{1, \ldots, m\}$ *such that $M_{j+1}/M_j \cong N_{\sigma(j)+1}/N_{\sigma(j)}$, for any*

$j \in \{0, 1, \ldots, m-1\}$.

Note that, if a module M in mod A has a composition series, then any finite increasing series of submodules of M can be refined to a composition series.

1.4 Semi-simple Modules and Semi-simple Algebras

The main objective of this section is to introduce the notions of semi-simple modules and semi-simple algebras.

Definition 1.4.1. *Let A be a finite dimensional K-algebra. A non-zero module M in* mod A *is called* semi-simple *if it is a sum of simple submodules.*

Lemma 1.4.2. ([6], 2.32) *Let A be a K-algebra with M a module in* mod A. *If M is semi-simple, then $M = S_1 \oplus \cdots \oplus S_n$ where S_i is simple.*

Definition 1.4.3. *Let A be a finite dimensional K-algebra. We say that A is* semi-simple *if the left A-module ${}_A A$ is semi-simple.*

Proposition 1.4.4. ([3], I.3) *Let A be a finite dimensional K-algebra. The following conditions are equivalent.*

(1) *A is semi-simple.*

(2) *Every right A-module is semisimple.*

(3) *Every left A-module is semisimple.*

(4) $\mathrm{rad} A = 0$.

Proposition 1.4.5. ([1], VII.4.4) *Let A be a finite dimensional K-algebra. Then,*

$\bar{A} = A/\mathrm{rad}(A)$ is a semi-simple K-algebra.

1.5 Radical of Modules

In this section, we see the definitions of radical of module, top and socle. Hence, we state some important lemmas about them.

Definition 1.5.1. *Let A be a finite dimensional K-algebra and M a nonzero module in* mod A*. A submodule L of M is* maximal *if $L \neq M$ and if L' is submodule of M such that $L \subseteq L' \subseteq M$, then $L' = L$ or $L' = M$.*

Definition 1.5.2. *Let A be a finite dimensional K-algebra with M a module in* mod A*. We define the* radical of M*, denoted by* rad M*, to be zero if $M = 0$, and otherwise, to be the intersection of all maximal submodules of M.*

Proposition 1.5.3. ([1], VII. 1.4) *Let A be a K-algebra with M a module in* mod A*. If $M = M_1 \oplus \cdots \oplus M_s$, then* $\text{rad } M = \text{rad } M_1 \oplus \cdots \oplus \text{rad } M_s$*.*

Lemma 1.5.4. *Let A be a finite dimensional K-algebra and M a module in* mod A*. If M is semi-simple, then* $\text{rad } M = 0$*.*

Proof. If M is semi-simple, then by Lemma 1.4.2, $M = M_1 \oplus \cdots \oplus M_t$, where M_i are simple modules. Then, by Proposition 1.5.3, $\text{rad } M = \text{rad } M_1 \oplus \cdots \oplus \text{rad } M_t$. Since 0 is the only maximal submodule of M_i, $\text{rad } M_i = 0$. So $\text{rad } M = 0$. □

Lemma 1.5.5. ([6], 2.6.7) *Let A be a K-algebra. If $f : M \to N$ is a morphism in* mod A*, then* $f(\text{rad}(M)) \subseteq \text{rad}(N)$*.*

Proposition 1.5.6. *Let A be a finite dimensional K-algebra and M a module in* mod A*. Then,* $\text{rad } M = (\text{rad}A)M$*.*

Proof. Let $x \in M$. Then, $f_x : A \to M : a \mapsto ax$ is an A-linear map. Then, by Proposition 1.5.5, $(\text{rad}(A))x = f_x(\text{rad}(A)) \subseteq \text{rad}M$. So $\text{rad}(A)M \subseteq \text{rad}M$. Then, $\bar{M} = M/\text{rad}(A)M$ is a module over $\bar{A} = A/\text{rad}(A)$. By Proposition 1.4.5, $\bar{A}$ is semi-simple.

Let $x \in M$ with $x \notin \text{rad}(A)M$. Then $\bar{x}$ is a non-zero element in $\bar{M}$. By Proposition 1.6.9, $\bar{M}$ is a semi-simple $\bar{A}$-module. Also, $\bar{M}$ is a semi-simple A-module. Thus, there exists map $f : \bar{M} \to S$, where S is simple, such that $f(\bar{x}) \neq 0$. Let $p : M \to \bar{M}$ be the canonical projection. Then, $(fp)(x) = f(\bar{x}) \neq 0$. That is, $x \notin \mathrm{Ker}(fp)$.

Then, $x \notin \mathrm{rad}\ M$. So, $\mathrm{rad}\ M \subseteq (\mathrm{rad}A)M$. □

Lemma 1.5.7. ([3], I.3.8) *Let M be a module in* $\mathrm{mod}\ A$. *If L is a maximal submodule of M such that M/L is semisimple, then* $\mathrm{rad}\ M \subseteq L$.

Proposition 1.5.8. ([8], I.5.16) *Let A be a finite dimensional K-algebra, $B = A/\mathrm{rad}A$, e a nonzero idempotent of A and $\bar{e} = e + \mathrm{rad}\ A$ the associated idempotent of B. The following conditions are equivalent.*

(1) *Ae is an indecomposable left A-module.*

(2) $\mathrm{rad}Ae$ *is a unique maximal left A-submodule of Ae.*

(3) *$B\bar{e}$ is a simple left B-module.*

Definition 1.5.9. *Let A be a finite dimensional K-algebra and M a nonzero module in* $\mathrm{mod}\ A$. *We assign to M two semisimple A-modules*

$$\mathrm{top}(M) = M/\mathrm{rad}M$$

called the top *of M, and*

$$\mathrm{soc}(M) = \textstyle\sum_{S \in \Lambda} S$$

where Λ is the set of all simple submodules of M, called the socle *of M.*

Definition 1.5.10. *A left A-submodule X of a module M in* $\mathrm{mod}\ A$ *is said to be* essential *if $X \cap Y \neq 0$ for any non-zero left A-submodule Y of M.*

Lemma 1.5.11. *Let A be a finite dimensional K-algebra and M a nonzero module in* $\mathrm{mod}\ A$. *Then,* $\mathrm{soc}(M) = \{x \in M \mid (\mathrm{rad}A) \cdot x = 0\}$.

Proof. Write $T = \{x \in M \mid (\text{rad}A) \cdot x = 0\}$, which is clearly a submodule of M. We know that $\text{soc}(M)$ is semi-simple and by Proposition 1.9.7, $\text{rad}(\text{soc}M) = 0$. Also by Proposition 1.5.6, we have $(\text{rad}A)(\text{soc}M) = 0$. So it means that $\text{soc}(M) \subseteq T$.

Let $x \in T$ be nonzero. We consider $S = Ax$. Then $\text{rad}(S) = (\text{rad}A)(Ax) = ((\text{rad}A)A) \cdot x = \text{rad}(A) \cdot x = 0$. So S is semi-simple by Proposition 1.4.4. This means that $S \subseteq \text{soc}M$. Therefore, $T \subseteq \text{soc}(M)$. Hence, $\text{soc}(M) = T$. □

Lemma 1.5.12. *Let A be a finite dimensional K-algebra and M a nonzero module in* mod A*.*

(1) *If* $\text{top}M$ *is simple, then M is indecomposable.*

(2) *If* $\text{soc}M$ *is simple, then M is indecomposable.*

Proof. (1) Assume that $M = M_1 \oplus M_2$ such that, $M_i \neq 0$ for any $i \in \{1, 2\}$. Then, we have $\text{top}M \cong \text{top}M_1 \oplus \text{top}M_2$. Since M_i is finite dimensional, $\text{rad}M_i \neq M_i$. Then, $M_i/\text{rad}M_i \neq 0$. So $\text{top}M$ is not simple, a contradiction.

(2) Assume that $M = M_1 \oplus M_2$ such that, $M_i \neq 0$ for any $i \in \{1, 2\}$. Since, $\text{soc}M$ is the sum of simple submodule of M, $\text{soc}M \cap M_i \neq 0$ is essential for any $i \in \{1, 2\}$. Then, we have $0 \neq (\text{soc}M \cap M_i) \subseteq \text{soc}M$ and $\text{soc}M \subseteq M_i$ for any $i \in \{1, 2\}$. Hence, $\text{soc}M \subseteq M_1 \cap M_2$, a contradiction. □

1.6 Projective and Injective Modules

Now, we introduce the notions of projective and injective modules.

Definition 1.6.1. *Let A be a finite dimensional K-algebra. An A-module P in* mod A *is said to be* projective *if for any epimorphism $u : M \to N$ the induced map* $\text{Hom}_A(P, u)$ *is* *surjective, or for any epimorphism* $u : M \to N$ *and any morphism* $f : P \to N$*, there exists a morphism* $f' : P \to M$ *such that* $f = uf'$.

Lemma 1.6.2. *Let* A *be a* K*-algebra. If* P *is a projective module in* mod A*, then every epimorphism* $f : X \to P$ *is a retraction.*

Proof. Suppose that P is projective module. Let $f : X \to P$ be an epimorphism. Consider $id_P : P \to P$. So, by the definition of projective modules, there exists $v : P \to X$ such that $fv = id_P$. So f is a retraction. □

Proposition 1.6.3. ([8], I. 8.2)

Let A *be a finite dimensional* K*-algebra with* $M = M_1 \oplus \cdots \oplus M_n$ *, where* M_i *are modules in* mod A*. Then* M *is projective if and only if* M_i *is projective, for* $i \in \{1, \ldots, n\}$.

Proposition 1.6.4. *Let* A *be a* K*-algebra. Then* ${}_AA$ *is projective.*

Proof. Let $f : M \to N$ be an epimorphism. If $g : A \to N$ is A-linear, then $x = g(1) \in N$. As f is epimorphism, $x = f(y)$ for $y \in M$. We claim that the following map h is A-linear.

$$h : A \to M : a \mapsto ay$$

Indeed, for any $\alpha, \beta \in A$, and $a_1, a_2 \in A$, we have

$$h(\alpha a_1 + \beta a_2) = (\alpha a_1 + \beta a_2)y = \alpha a_1 y + \beta a_2 y = \alpha h(a_1) + \beta h(a_2).$$

Moreover, for any $a \in A$, $g(a) - fh(a) = g(a) - f(ay) = ag(1) - af(y) = ax - ax = 0$. Then, $g = fh$. So A is projective. □

Corollary 1.6.5. ([6], 2.22) *Let* A *be a finite dimensional* K*-algebra. If* e *is an idempotent in* A*, then* $A = Ae \oplus A(1 - e)$*. In particular,* Ae *is projective.*

Proposition 1.6.6. ([8], I.8.2) *Let A be a finite dimensional K-algebra and $e_1, \dots, e_n$ a set of primitive orthogonal idempotents of A with $1_A = e_1 + \cdots + e_n$. Then,*

(1) *$A = Ae_1 \oplus \cdots \oplus Ae_n$ is a decomposition of A into a direct sum of indecomposable projective left A-modules.*

(2) *Every nonzero projective module P in $\operatorname{mod} A$ is a direct sum $P = P_1 \oplus \cdots \oplus P_m$, where each module P_j, $j \in \{1, \dots, m\}$, is isomorphic to a e_iA with $i \in \{1, \dots, n\}$.*

Definition 1.6.7. *An A-module I is said to be* injective *if, for any monomorphism*

$v : M \to N$, the induced map $\operatorname{Hom}_A(v, I) : \operatorname{Hom}_A(N, I) \to \operatorname{Hom}_A(M, I)$ is injective, or equivalently, for any monomorphism $v : M \to N$ and any morphism $g : M \to I$, there exists a morphism $g' : N \to I$ such that $g = g'v$.

Lemma 1.6.8. *Let A be a K-algebra. If I is an injective A-module, then any monomorphism $g : I \to X$ is a section.*

Proof. Suppose that I is an injective module. Let $g : I \to X$ be a monomorphism. Consider $id_I : I \to I$, by the definition of injective, there exists $v : X \to I$, such that $vg = id_I$. So g is a section. □

Proposition 1.6.9. ([1], V.7.1) *Let A be a finite dimensional K-algebra. The following conditions are equivalent.*

(1) *A is semi-simple.*

(2) *Any module in $\operatorname{mod} A$ is semi-simple.*

(3) *Any module in $\operatorname{mod} A$ is projective.*

(4) *Any module in $\operatorname{mod} A$ is injective.*

Corollary 1.6.10. *Let A be a finite dimensional K-algebra. If A is semi-simple, then $A = P_1 \oplus \cdots \oplus P_n$, where P_i is simple and projective.*

Proof. If ${}_AA$ is semi-simple, by Lemma 1.4.2, ${}_AA = P_1 \oplus \ldots \oplus P_n$, where P_i is simple. Since ${}_AA$ is projective by Proposition 1.6.4, each P_i is projective by Lemma 1.6.3. □

1.7 Projective Cover and Injective Envelope

In this section, we introduce the concepts of projective covers and injective envelopes.

Definition 1.7.1. *Let A be a finite dimensional K-algebra. A submodule L of a module M in* $\operatorname{mod} A$ *is called* superfluous *if for every submodule X of M the equality $L + X = M$ implies $X = M$.*

Definition 1.7.2. *Let A be a finite dimensional K-algebra. An epimorphism $f : M \to N$ in* mod A *is said to be* minimal *if* $\operatorname{Ker} f$ *is superfluous in M. A morphism $f : P \to M$ in* mod A *is called a* projective cover *of M if P is a projective module and f is a minimal epimorphism.*

Theorem 1.7.3. ([8], I.8.4) *Let A be a finite dimensional K-algebra. For any nonzero module M in* mod A*, there exists a projective cover*

$$h : P(M) \to M.$$

Moreover, the induced homomorphism $\operatorname{top}(h) : \operatorname{top}(P(M)) \to \operatorname{top}(M)$ *of semisimple modules in* mod A *is an isomorphism.*

Definition 1.7.4. *A monomorphism $g : L \to M$ in* mod A *is called* minimal *if every nonzero submodule X of M has a nonzero intersection with* $\operatorname{Im} g$*. A monomorphism*

$g : L \to I$ *in* mod A *is called an* injective envelope *of* L *if* I *is an injective module and* g *is a minimal monomorphism.*

Theorem 1.7.5. ([8], I.8.18) *Let* A *be a finite dimensional* K*-algebra. For any nonzero module* M *in* mod A*, there exists an injective envelope*

$$u : M \to E(M)$$

such that the induced homomorphism $\operatorname{soc}(u) : \operatorname{soc}(M) \to \operatorname{soc}(E(M))$ *is an isomorphism.*

Proposition 1.7.6. ([1], VIII.2.1, 2.4) *Let* A *be a finite dimensional* K*-algebra with* M *a module in* $\operatorname{mod} A$*.*

(1) *An epimorphism* $f : P \to M$ *in* $\operatorname{mod} A$ *is a projective cover of* M *if and only if* P *is projective with* $\operatorname{Ker}(f) \subseteq \operatorname{rad} P$*.*

(2) *A monomorphism* $g : M \to I$ *in* $\operatorname{mod} A$ *is an injective enveloppe of* M *if and only if* I *is injective with* $\operatorname{Soc} I \subseteq \operatorname{Im}(g)$*.*

1.8 Exact Sequences of Modules

Before studying almost split sequences in chapter 3, we need a few notions in this section.

Definition 1.8.1. *Let* A *be a finite dimensional algebra. A sequence*

$$\cdots \to X_{n-1} \xrightarrow{h_{n-1}} X_n \xrightarrow{h_n} X_{n+1} \to \cdots$$

of morphisms in mod A *is called* exact *at* X_n *if* $\operatorname{Ker} h_n = \operatorname{Im} h_{n-1}$ *for any* n*. In particular, an exact sequence*

$$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$$

is called a short exact sequence*.*

Lemma 1.8.2. *Let A be a finite dimensional algebra. A sequence $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ in* mod A *is a short exact sequence if f is a monomorphism, g is an epimorphism and* $\mathrm{Im} f = \mathrm{Ker} g$.

Lemma 1.8.3. ([1], II.3.6) *Consider a commutative diagram with exact rows of A-modules and linear maps.*

$$\begin{array}{ccccccc} & & L & \xrightarrow{u} & M & \xrightarrow{v} & N & \longrightarrow & 0 \\ & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & L' & \xrightarrow{u'} & M' & \xrightarrow{v'} & N' & & \end{array}$$

There is an exact sequence $\mathrm{Ker} f \xrightarrow{u_1} \mathrm{Ker} g \xrightarrow{v_1} \mathrm{Ker} h \xrightarrow{\sigma} \mathrm{Coker} f \xrightarrow{u_2} \mathrm{Coker} g \xrightarrow{v_2} \mathrm{Coker} h$ *where u_1, v_1 are deduced by* Ker *and u_2, v_2 are deduced by* Coker. *In addition,*

(1) *If u is a monomorphism, then u_1 is a monomorphism.*

(2) *If v' is an epimorphism, then v_2 is an epimorphism.*

Definition 1.8.4. *Let A be a finite dimensional K-algebra. Two short exact sequences in* mod A, $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ *and* $0 \to L' \xrightarrow{f'} M' \xrightarrow{g'} N' \to 0$ *are said to be* isomorphic *if there is a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \xrightarrow{f} & M & \xrightarrow{g} & N & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} & & \\ 0 & \longrightarrow & L' & \xrightarrow{f'} & M' & \xrightarrow{g'} & N' & \longrightarrow & 0 \end{array}$$

in mod A, *where u, v, w are isomorphisms. We note that v is an isomorphism if u and w are isomorphisms.*

Definition 1.8.5. *A short exact sequence $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ in* mod A *is said to* split, *where f' is the canonical injection, g' is the canonical projection and if there exists a morphism $h : Y \to X \oplus Z$ such that the diagram commutes.*

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle id_X} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle id_Z} & & \\
0 & \longrightarrow & X & \xrightarrow{f'} & X \oplus Z & \xrightarrow{g'} & Z & \longrightarrow & 0
\end{array}
$$

Proposition 1.8.6. ([6], 2.1.7)

Let A be a finite dimensional algebra with $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ a short exact sequence in mod A. *The following statement are equivalent.*

(1) *The short exact sequence splits.*

(2) *f is a section.*

(3) *g is a retracction.*

1.9 Radical of the Module Category

In this section, we have some results about the radical rad(mod A) of mod A. A map in rad(mod A) will be called a radical map. The following result gives us a link between the indecomposable A-modules and rad(mod A).

Proposition 1.9.1. ([6], 2.7) *Let A be a finite dimensional K-algebra, and let $f : M \to N$ be a morphism in* mod A.

(1) *If M is indecomposable, then $f \in \operatorname{rad}(M, N)$ if and only if f is not a section.*

(2) *If N is indecomposable, then $f \in \operatorname{rad}(M, N)$ if and only if f is not a retraction.*

(3) *If M and N are indecomposable, then $f \in \operatorname{rad}(M, N)$ if and only if f is not an isomorphism.*

Corollary 1.9.2. *Let A be a K-algebra with S a simple module in* mod A.

(1) *If S is projective, then* $\mathrm{rad}(M,S) = 0$*, for any A-module M.*

(2) *If S is injective, then* $\mathrm{rad}(S,N) = 0$*, for any A-module N.*

Proof. (1) Let $f \in \mathrm{rad}(M,S)$. Suppose that f is not zero. Since S is simple, by Proposition 1.3.9(3), f is an epimorphism. Since S is projective, f is a retraction, which contradicts Proposition 1.9.1 (2). So $\mathrm{rad}(M,S) = 0$.

(2) Let $g \in \mathrm{rad}(S,M)$. Suppose that g is not zero. Since S is simple, by Proposition 1.3.9(2), g is a monomorphism. Since S is injective, g is a section, which contradicts Proposition 1.9.1(1). So $\mathrm{rad}(S,M) = 0$. □

Lemma 1.9.3. ([8], III.1.4) *Let X and Y be indecomposable modules in* $\mathrm{mod}\,A$*. Then, the following statements hold.*

(1) $\mathrm{rad}_A(X,Y)$ *is the subspace of* $\mathrm{Hom}_A(X,Y)$ *formed by all nonisomorphisms.*

(2) $\mathrm{rad}_A(X,Y) = \mathrm{Hom}_A(X,Y)$ *if $X \ncong Y$.*

Proposition 1.9.4. *Let A be a K-algebra and $f \in \mathrm{Hom}_A(A,A)$. Then f is in* $\mathrm{rad}(A,A)$ *if and only if* $f(1) \in \mathrm{rad}(A)$.

Proof. By Corollary 1.1.13, an endomorphism $f : A \to A$ of K-algebras is right invertible if and only if $f(1)$ is left invertible.

Necessity. Suppose that $f \in \mathrm{rad}({}_AA, {}_AA)$. Consider $f(1) \in A$. For any $x \in A$, by Corollary 1.1.13, there exists $g \in \mathrm{Hom}_A(A,A)$ such that $g(1) = x$. Since $f \in \mathrm{rad}({}_AA, {}_AA)$, $\mathrm{id}_A - fg$ is right invertible. Thus, $(\mathrm{id}_A - fg)(1)$ is left invertible.
Now, $1 - xf(1) = 1 - g(1)f(1) = 1 - f(g(1)\cdot 1) = \mathrm{id}_A(1) - (fg)(1)$, which is left invertible. So $f(1) \in \mathrm{rad}(A)$.

Sufficiency. Suppose that $f(1) \in \mathrm{rad}(A)$. For any $g \in \mathrm{Hom}_A(A,A)$, we have $g(1) \in A$. Then $(\mathrm{id}_A - fg)(1) = \mathrm{id}_A(1) - f(g(1)\cdot 1) = 1 - g(1)f(1)$, which is left invertible. Then, $\mathrm{id}_A - fg$ is right invertible. So, $f \in \mathrm{rad}({}_AA, {}_AA)$. □

Proposition 1.9.5. ([5], 1.15)

Let A be a finite dimensional K-algebra. Then rad A *is the unique nilpotent ideal of A such that $A/\text{rad}A$ is semi-simple.*

Proposition 1.9.6. ([6], 2.8) *Let A be a finite dimensional K-algebra. If* $\text{rad}^n(\text{mod } A) = 0$ *for some $n \geq 1$, then* $\text{rad}^n(A) = 0$.

Proposition 1.9.7. *Let A be a finite dimensional K-algebra. Then,* $\text{rad}(\text{mod } A) = 0$ *if and only if A is semi-simple.*

Proof. Necessity. If $\text{rad}(\text{mod } A) = 0$, by Proposition 1.9.6, $\text{rad}(A) = 0$. Then A is semi-simple.

Sufficiency. Suppose that A is semi-simple. Let M, N be two non-zero A-modules. Then, by Proposition 1.6.9, $M = S_1 \oplus \ldots \oplus S_t$, where S_i are simple and injective modules. Then by Corollary 1.9.2, $\text{rad}(M, N) = \bigoplus_{i=1}^{t} \text{rad}(S_i, N) = 0$. □

1.10 Standard Duality

Now, we introduce the principle of duality, which will be a powerful tool in the proofs.

Let A be a finite dimensional K-algebra. Recall that mod A stands for the category of finite dimensional left A-modules. We denote by mod A^{op} the category of finite dimensional right A-modules. We define the functor

$$D : \text{mod}A \to \text{mod}A^{\text{op}}$$

by assigning to each module M in mod A the dual K-vector space

$$M^* = \text{Hom}_K(M, K)$$

endowed with the right A-module structure given by the formula $(\varphi a)(m) = \varphi(am)$ for $\varphi \in \mathrm{Hom}_K(M,K)$, $a \in A$ and $m \in M$, and to each A-linear morphism $h : M \to N$ the dual K-linear morphism

$$D(h) = \mathrm{Hom}_K(h,K) : D(N) \to D(M) : \varphi \to \varphi h,$$

which is a morphim of right A-modules. It is well known that D is a duality of categories, called the *standard K-duality*. The quasi-inverse of the duality D is also denoted by

$$D : \mathrm{mod} A^{\mathrm{op}} \to \mathrm{mod} A$$

and is defined by assigning to each right A-module Y the dual K-vector space $D(Y) = Y^* = \mathrm{Hom}_K(Y,K)$ endowed with the left A-module structure given by the formula $(a\varphi)(y) = \varphi(ya)$ for $\varphi \in \mathrm{Hom}_K(Y,K)$, $a \in A$ and $y \in Y$. It is easy to verify that the evaluation map $ev : M \to M^{**}$, given by the formula $ev(m)(\varphi) = \varphi(m)$ for $m \in M$ and $\varphi \in D(M)$, defines natural equivalences of functors $1_{\mathrm{mod}\, A} \cong D \circ D$ and $1_{\mathrm{mod}\, A^{\mathrm{op}}} \cong D \circ D$.

Proposition 1.10.1. ([8], I.8.16) *Let A be a finite dimensional K-algebra and D the standard duality between* mod A *and* modA^{op}.

1. *A module E in* mod A *is injective if and only if the module $D(E)$ in* modA^{op} *is projective.*
2. *A module P in* mod A *is projective if and only if the module $D(P)$ in* modA^{op} *is injective.*
3. *A module M in* mod A *is indecomposable if and only if $D(M)$ is indecomposable.*
4. *A module S in* mod A *is simple if and only if the module $D(S)$ in* modA^{op} *is simple.*
5. *A module M in* mod A *is semisimple if and only if the module $D(M)$ in* modA^{op} *is semisimple.*

(6) *For every nonzero module* M *in* $\operatorname{mod} A$*, we have* $D(\operatorname{top} M) \cong \operatorname{soc}(D(M))$ *and* $D(\operatorname{soc} M) \cong \operatorname{top}(D(M))$.

Proposition 1.10.2. ([6], 2.9) *Let* A *be a finite dimensional* K*-algebra. For any* $n \geq 1$, $\operatorname{rad}^n(\operatorname{mod} A) = 0$ *if and only if* $\operatorname{rad}^n(\operatorname{mod} A^{\mathrm{op}}) = 0$.

Proposition 1.10.3. *Let* A *be a finite dimensional of* K*-algebra. Then* A *is semi-simple if and only if* A^{op} *is semi-simple.*

Proof. By Proposition 1.9.7, A is semi-simple if and only if $\operatorname{rad}(\operatorname{mod} A) = 0$ if and only if $\operatorname{rad}(\operatorname{mod} A^{\mathrm{op}}) = 0$ if and only if A^{op} is semi-simple. □

Definition 1.10.4. *Let* A *be a finite dimensional* K*-algebra with a complete set of orthogonal primitive idempotents* $\{e_1, \cdots, e_n\}$*. We say that* A *is* basic *if* $Ae_i \ncong Ae_j$*, for all* $i \neq j$.

Proposition 1.10.5. ([8], I. 8.2, 5.17, 8.19) *Let* A *be a basic finite dimensional* K*-algebra with a complete set of orthogonal primitive idempotents* $\{e_1, \cdots, e_n\}$*. Put* $P_i = Ae_i$, $S_i = P_i/\operatorname{rad} P_i$ *and* $I_i = D(e_i A)$*, for* $i = 1, \ldots, n$.

(1) $\{P_1, \ldots, P_n\}$ *is a complete set of representatives of the isomorphism classes of indecomposable projective modules in* $\operatorname{mod} A$.

(2) $\{S_1, \ldots, S_n\}$ *is a complete set of representatives of the isomorphism classes of simple modules in* $\operatorname{mod} A$.

(3) $\{I_1, \ldots, I_n\}$ *is a complete set of representatives of the isomorphism classes of indecomposable injective modules in* $\operatorname{mod} A$.

CHAPTER 2

Quivers and Algebras

We introduce the basic concept of quivers and path algebras which we need to prove the main theorem. The definitions and results are taken from [3, 6, 2, 7].

2.1 Quivers

A quiver is a graphical object in which one can encode much of the structural information of an algebra. So, in this section, we see the definition of quiver and path.

Definition 2.1.1. *A* quiver *is a quadruple* $Q = (Q_0, Q_1, s, t)$*, where* Q_0 *is the set of* vertices*,* Q_1 *is the set of* arrows*, and* $s, t : Q_1 \to Q_0$ *are maps, which associate to each arrow* $\alpha \in Q_1$ *its source* $s(\alpha)$ *and target* $t(\alpha)$*, respectively. Moreover, an arrow* α *is written as* $\alpha : s(\alpha) \to t(\alpha)$*.*

A quiver Q is said to be *finite* if Q_0 and Q_1 are finite sets. Throughout this thesis, all quivers are finite.

Example 2.1.2. *The following graph is an example of a quiver :*

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \circlearrowright \gamma$$

where $Q_0 = \{1,2,3\}$ *and* $Q_1 = \{\alpha, \beta, \gamma\}$. *Then, we have* $s(\alpha) = 1, t(\alpha) = s(\beta) = 2, s(\gamma) = t(\beta) = t(\gamma) = 3$.

Definition 2.1.3. *Let* $Q = (Q_0, Q_1, s, t)$ *be a quiver and* $a, b \in Q_0$. *A* path *of* length $l \geqslant 1$ *with source* a *and target* b *is a sequence*

$$(b|\alpha_l, \cdots, \alpha_2, \alpha_1|a)$$

where $\alpha_k \in Q_1$ *such that* $s(\alpha_1) = a$, $s(\alpha_{k+1}) = t(\alpha_k)$ *for* $1 \leq k < l$, *and* $t(\alpha_l) = b$. *Such a path will be written as* $\alpha_l \cdots \alpha_2 \alpha_1$ *and may be visualized as follows*

$$a = a_0 \xrightarrow{\alpha_1} a_1 \to \cdots \xrightarrow{\alpha_l} a_l = b,$$

where α_1 *is called the* initial *arrow, and* α_l *the* terminal *arrow. Moreover, with each vertex* $a \in Q_0$, *one associates a path* $= (a||a)$ *of length* 0, *called the* trivial path *at* a *and denoted by* ε_a. *Given any* $l \geq 0$, *the set of all paths of length* l *in* Q *is denoted by* Q_l.

Definition 2.1.4. *A path of length* $l \geqslant 1$ *is called an* oriented cycle *when its source and target coincide. A quiver is called* acyclic *if it contains no oriented cycles.*

2.2 Algebras Given by a Quiver

The usual composition of paths in a quiver can be used to define an algebraic structure. In this section, we see the definition of path algebra and propositions that we need.

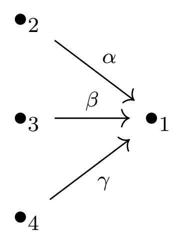
Definition 2.2.1. *Let* Q *be a quiver. The* path algebra KQ *of* Q *over* K *is the* K*-algebra whose underlying* K*-vector space has as basis the set of all paths* $(b|\alpha_l, \cdots, \alpha_1|a)$ *of length*

$l \geqslant 0$ in Q and such that the product of two basis vectors $(b|\alpha_l, \cdots, \alpha_1|a)$ and $(d|\alpha_k, \cdots, \beta_1|c)$ of KQ is defined by

$$(b|\alpha_l, \cdots, \alpha_1|a)(d|\alpha_k, \cdots, \beta_1|c) = \delta_{bc}(d|\alpha_k, \cdots, \beta_1, \alpha_l, \cdots, \alpha_1|a)$$

where δ_{bc} denotes the Kronecker delta. In other words, the product of two paths $\alpha_1 \cdots \alpha_l$ and $\beta_1 \cdots \beta_k$ is equal to zero if $t(\alpha_l) \neq s(\beta_1)$ and is equal to the concatenation $\alpha_1 \cdots \alpha_l \beta_1 \cdots \beta_k$ if $t(\alpha_l) = s(\beta_1)$.

Example 2.2.2. *Consider the following quiver :*



Then, a K-basis of KQ is $\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \alpha, \beta, \gamma\}$. Also, there is a K-algebra isomorphism

$$KQ \cong \begin{bmatrix} k & 0 & 0 & 0 \\ k & k & 0 & 0 \\ k & 0 & k & 0 \\ k & 0 & 0 & k \end{bmatrix}$$

Lemma 2.2.3. ([3], II.1.5) *Let Q be a finite quiver. The element $1 = \sum \varepsilon_a$ for any $a \in Q_0$ is the identity of KQ and the set $\{\varepsilon_a \mid a \in Q_0\}$ of all the trivial paths $\varepsilon_a = (a||a)$ is a complete set of primitive orthogonal idempotents for KQ.*

Definition 2.2.4. *Let Q be a finite quiver. The two-sided ideal of the path algebra KQ generated by the arrows of Q is called the* arrow ideal *of KQ and is denoted by R_Q. Note that R_Q, as a K-vector space, has a direct sum decomposition*

$$R_Q = KQ_1 \oplus KQ_2 \oplus \cdots \oplus KQ_l \oplus \cdots,$$

where kQ_l denotes the K-vector space with Q_l as a basis. Moreover, one denotes by R_Q^m the two-sided ideal of KQ generated by Q_m. The underlying K-vector space of R_Q^m is a direct sum

$$R_Q^m = \bigoplus_{l \geq m} KQ_l.$$

So R_Q^m, as a K-vector space, is generated by the paths of length $\geq m$ in Q.

Definition 2.2.5. *Let Q be a finite quiver and R_Q be the arrow ideal of the path algebra KQ. A two-sided ideal I of KQ is said to be* admissible *if there exists $m \geq 2$ such that*

$$R_Q^m \subseteq I \subseteq R_Q^2.$$

In this case, the pair (Q, I) is said to be a bound quiver *and the quotient algebra KQ/I is said to be the algebra of the bound quiver (Q, I), or simply, a* bound quiver algebra.

Proposition 2.2.6. *Let Q be a finite quiver.*

(1) *If I is an admissible ideal of kQ, then $A = KQ/I$ is finite dimensional.*

(2) *The zero ideal of KQ is admissible if and only if Q is acyclic.*

Proof. (1) Let I be an admissible ideal of KQ, then there exists $m \geq 2$ such that $R^m \subseteq I$, where R is the arrow ideal R_Q of KQ.

Then, there exists an epimorphism $KQ/R^m \to KQ/I$. Thus, it is enough to prove that KQ/R^m is finite dimensional. The residual classes of the paths of length less than m form a basis of KQ/R^m as a K-vector space. Because there are only finitely many such paths, our statement follows.

(2) The zero ideal is admissible if and only if there exists $m \geq 2$ such that $R_Q^m = 0$, so any product of m arrows in KQ is zero. This is the case if and only if Q is acyclic. □

Proposition 2.2.7. ([6], 3.2.7) *Let Q be a quiver of n vertices and R_Q the arrow ideal of KQ. Then, $KQ/R_Q \cong K^n$, the product of n copies of K, which is a semi-simple algebra.*

Let $A = KQ/I$, where Q is a quiver and I is an admissible ideal of KQ. For any $\rho \in kQ$, we write $\bar{\rho} = \rho + I$. In particular, we write $e_a = \epsilon_a + I$, where ϵ_a is the trivial path at a.

Proposition 2.2.8. *Let $A = KQ/I$, where Q is a finite quiver and I is an admissible ideal of KQ.*

(1) $\mathrm{rad}(A) = R_Q/I = \{\sum_i \lambda_i \bar{p}_i \mid \lambda_i \in K; p_i$ *are non-trivial paths in* $Q\}$.

(2) $\{e_a \mid a \in Q_0\}$ *is a complete set of orthogonal primitive idempotents of A.*

(3) $P_a = Ae_a$ *is an indecomposable projective module in* mod A.

(4) $I_a = D(e_a A)$ *is an indecomposable injective module in* mod A*, where D is the standard duality from* mod A^{op} *onto* mod A.

Proof. (1) Since I is admissible ideal of KQ, there exists $m \geq 2$ such that $R^m \subseteq I$. Consequently, $(R/I)^m = 0$. So R/I is a nilpotent ideal of KQ/I. By Proposition 2.2.7, $(KQ/I)(R/I) \cong KQ/R$ is isomorphic to a direct product of K. Then, by Proposition 1.9.5, $\mathrm{rad}(A) = R/I$.

(2) We have the canonical homomorphism $KQ \to KQ/I$, where e_a is the image of ε_a. By Lemma 2.2.3, the given set is a complete set of orthogonal idempotents. We need to show that e_a is primitive, that is, the only idempotents of $e_a(KQ/I)e_a$ are 0 and e_a. We can write any idempotent e of $e_a(KQ/I)e_a$ in this form $e = \lambda\varepsilon_a + w + I$, where $\lambda \in K$ and w is a linear combination of cycles through a of length ≥ 1. $e^2 = e$ gives

$$(\lambda^2 - \lambda)\varepsilon_a + (2\lambda - 1)w + w^2 \in I.$$

Since R_Q is the arrow ideal of KQ and $I \subseteq R_Q^2$, we must have $\lambda^2 - \lambda = 0$. So $\lambda = 0$ or $\lambda = 1$. If $\lambda = 0$, then $e = w + I$, where w is idempotent modulo I. Also, $R_Q^m \subseteq I$ for some $m \geq 2$, so we have $w^m \in I$. Thus, w is also nilpotent modulo I. Consequently, $w \in I$ and e are zero. If $\lambda = 1$, then $e_a - e = -w + I$ is an idempotent in $e_a(KQ/I)e_a$. So, w is again idempotent modulo I. Like before, it is also nilpotent modulo I. So, it must belong to I and consequently, $e_a = e$.

(3) Let $u = \lambda e_a + \sum_{i=1}^{n} \gamma_i(\alpha_i + I)$ in $e_a A e_a$ where α_i are oriented cycles from a to a, λ and γ_i are in K. Then, $(\sum_{i=1}^{n} \gamma_i(\alpha_i + I))^m = \sum_{i=1}^{l} \mu_i \beta_i + I$ where β_i are oriented cycles from a to a of length at least m. Thus, $\beta_i \in R^m \subseteq I$. Thus, $(\sum_{i=1}^{l} \gamma_i(\alpha_i + I))^m = 0 + I$. If $\lambda = 0$, then u is nilpotent. So by the lemma 1.1.9, $e_a - u$ is invertible. If $\lambda \neq 0$, let $u' = \frac{u}{\lambda} = e_a + \sum_{i=1}^{n} \frac{\gamma_i}{\lambda}(\alpha_i + I)$. So, $\sum_{i=1}^{n} \frac{\gamma_i}{\lambda}(\alpha_i + I)$ is nilpotent, and by lemma 1.1.9, u' is invertible. This implies that $u = \lambda u'$ is invertible. Therefore, by proposition 1.1.7, the algebra $e_a A e_a$ is local. Also by proposition 1.1.12, $\mathrm{End}(Ae_a) \cong (e_a A e_a)^{\mathrm{op}}$. Since $e_a A e_a$ is local, $\mathrm{End}(Ae_a)$ is also local and then, by Proposition 1.3.6, Ae_a is indecomposable. Moreover, as e_a is an idempotent, by lemma 1.6.5, $P_a = Ae_a$ is projective.

(4) By Propositions 1.10.5(3) and 2.2.8(3), I_a is an indecomposable injective module.

□

Lemma 2.2.9. *Let $A = kQ/I$, where Q is a finite quiver and I is an admissible ideal of KQ. If $\alpha : a \to b$ is an arrow in Q, then we have a radical A-linear map*

$$P[\bar{\alpha}] : P_b \to P_a : v \mapsto v\bar{\alpha}.$$

Proof. First, $\bar{\alpha} = \bar{\alpha} e_a = e_b \bar{\alpha}$. Given $v \in P_b$, we have $v\bar{\alpha} = (v\bar{\alpha})e_a \in \mathrm{rad} P_a$. Thus, $P[\bar{\alpha}]$ defines a map from P_b to P_a, whose image is contained in $\mathrm{rad} P_a$. In particular, $P[\bar{\alpha}]$ is not an epimorphism. Moreover, for any $u \in A$, we have $P[\bar{\alpha}](uv) = (uv)\bar{\alpha} = u(v\bar{\alpha}) = uP[\alpha](v)$. So $P[\bar{\alpha}]$ is a morphism in $\mathrm{mod}\, A$. Since P_a and P_b are indecomposable, by Proposition 1.8.1, $P[\bar{\alpha}] \in \mathrm{rad}(P_b, P_a)$. □

2.3 Monomial Algebras

In this section, we consider monomial algebras to calculate the indecomposable projective and injective modules with a point $a \in Q_0$. Also, we will see the properties of radical and socle with an admissible monomial ideal.

Definition 2.3.1. *A two-sided ideal I in KQ is called* m*onomial if it is generated as a two-sided ideal by a set $\{\rho_1, \ldots, \rho_r\}$ of paths of length at least two in Q. In this case, A/I is called a monomial algebra. Moreover, a path in Q is called non-zero if $p \notin I$, or equivalently, $\bar{p}$ is non-zero in A/I.*

Lemma 2.3.2. *Let Q be a finite quiver and I be a monomial ideal of kQ. If $u = \sum_i \lambda_i \zeta_i \in I$, where $\lambda_1, \ldots, \lambda_r \in K^*$ and $\zeta_1, \ldots, \zeta_r$ are pairwise different paths in Q, then $\zeta_1, \ldots, \zeta_r \in I$.*

Proof. By the hypothesis, $I =< \rho_1, \ldots, \rho_r >$, where $\rho_1, \ldots, \rho_r$ are paths of length at least two in Q. Let $u = \sum_i \lambda_i \zeta_i \in I$, where $\lambda_1, \ldots, \lambda_r \in K^*$ and $\zeta_1, \ldots, \zeta_r$ are pairwise different paths in Q. Then, $\sum_i \lambda_i \zeta_i = \sum_j \mu_j p_j \xi_j q_j$, where $\mu_j \in K$ and p_j, q_j are paths in Q and $\xi_j \in \{\rho_1, \ldots, \rho_r\}$ such that the $p_j \xi_j q_j$ are pairwise different. Recall that the paths are linearly independent in kQ. So, for each i, we have $\zeta_i = p_j \xi_j q_j$ for some j. Since $\xi_j \in I$, we have $\zeta_i \in I$ for any i. □

Lemma 2.3.3. *Let $A = kQ/I$, where Q is a finite quiver and I is an admissible monomial ideal. If ρ, ζ are non-zero paths in Q, then $\bar{\rho} = \bar{\zeta}$ if and only if $\rho = \zeta$.*

Proof. The sufficiency is evident. Let ρ, ζ be non-zero paths in Q such that $\bar{\rho} = \bar{\zeta}$. Then, $\rho - \zeta \in I$. If $\rho \neq \zeta$, then $\rho, \zeta \in I$ by Lemma 2.3.2. This contradicts our assumption. So, $\rho = \zeta$. □

Proposition 2.3.4. *Let $A = KQ/I$ where Q is a quiver and I is admissible monomial ideal of KQ. For any point a in Q_0, put $P_a = Ae_a$. Then,*

(1) P_a *has as a* K*-basis the set of classes modulo* I *of non-zero paths starting with* a.

(2) $\text{rad}P_a$ *has as a* K*-basis the set of classes modulo* I *of non-zero non-trivial paths starting with* a.

(3) $S_a = P_a/radP_a$ *is simple.*

Proof. (1) Since I is admissible and monomial, $I =< \rho_1, \ldots, \rho_s >$, where $\rho_1, \ldots, \rho_s$ are paths of length at least two in Q. Moreover, there exists an integer $m \geq 2$ such that $R^m \subseteq I$. In particular, every path of length at least m lies in I. So the non-zero paths are of length $< m$. Since Q is finite, the number of non-zero paths is finite. Let $p_1, p_2, \ldots, p_t$ be the pairwise different non-zero paths starting with a, where $p_1 = \epsilon_a$. By Lemma 2.3.2, $p_1 + I, p_2 + I, \ldots, p_t + I$ are linearly independent.

Let $u = ve_a \in P_a$, where $v \in A$. Write $v = \sum_{i=1}^{t} \lambda_i(p_i + I) + \sum_j \mu_j(q_j + I)$, where $\lambda_i, \mu_j \in K$, and q_j are non-zero paths not starting with a. Then

$$u = ve_a = \sum_{i=1}^{t} \lambda_i(p_i + I)e_a + \sum_i \mu_j(q_j + I)e_a = \sum_i \lambda_i(p_i + I) + (0 + I) = \sum_{i=1}^{t} \lambda_i(p_i + I).$$

So P_a has a K-basis $\{p_1 + I, \ldots, p_t + I\}$.

(2) Let $\{p_2, \ldots, p_t\}$ be the set of non-trivial paths starting with a.
As seen above, $p_2 + I, \ldots, p_t + I$ are linearly independent.
On the other hand, $\text{rad}P_a = (\text{rad}A)(Ae_a) = (\text{rad}A)e_a$. Let $u \in \text{rad}P_a$. Then $u = ve_a$ with $v \in \text{rad}A$. By Lemma 2.2.8(1), $\text{rad}(A) = R_Q/I$. Thus, $v = \sum_{i=2}^{t} \lambda_i(p_i + I) + \sum_j \mu_j(q_j + I)$, where $\lambda_i, \mu_j \in K$, and the q_j are non-trivial non-zero paths not stating with a. Then

$$u = ve_a = \sum_{i=2}^{t} \lambda_i(p_i + I)e_a + \sum_i \mu_j(q_j + I)e_a = \sum_{i=2}^{t} \lambda_i(p_i + I) + (0 + I) = \sum_{i=2}^{t} \lambda_i(p_i + I).$$

So $\{p_2 + I, \cdots, p_t + I\}$ is a K-basis of $\text{rad}P_a$.

(3) By Statement (2), $e_a \notin \text{rad}P_a$. Thus, $e_a \neq 0$. Let $u + \text{rad}P_a \in P_a/\text{rad}P_a$, where $u \in P_a$. By Statement (2), $u = \mu e_a + \sum_i \lambda_i(p_i + I)$, where $\lambda_i, \mu \in K$ and the p_i are non-trivial path

starting with a. By Statement (2), $u - \mu e_a \in \mathrm{rad}P_a$. Hence, $\{e_a + \mathrm{rad}P_a\}$ is a K-basis of S_a. In particular, the dimension of S_a is 1. So S_a is a simple module. □

Fix a point a in Q_0. Let $Q_I(-,a)$ denote the set of non-zero paths ending at a in Q. Put $P_a^{o} = e_aA \in \mathrm{mod}A^{\mathrm{op}}$ and $I_a = \mathrm{Hom}_K(e_aA, K) = D(P_a^{o}) \in \mathrm{mod}A$. By the dual of Proposition 2.3.4, P_a^{o} is an indecomposable projective module in $\mathrm{mod}A^{\mathrm{op}}$, which has as a K-basis $\{\bar\rho \mid \rho \in Q_I(-,a)\}$. For each $\rho \in Q_I(-,a)$, we denote by $\bar\rho^*$ the K-linear function from e_aA to K such, for any $\eta \in Q_I(-,a)$, that $\bar\rho^*(\bar\eta) = 1$ if $\eta = \rho$ and $\bar\rho^*(\bar\eta) = 0$ if $\rho \neq \eta$.

Proposition 2.3.5. *Let $A = KQ/I$, where Q is a finite quiver and I is an admissible monomial ideal of KQ. For each vertex $a \in Q$, the indecomposable injective module I_a has as a K-basis $\{\bar\rho^* \mid \rho \in Q_I(-,a)\}$, called the dual basis of $\{\bar\rho \mid \rho \in Q_I(-,a)\}$.*

Proof. Write $Q_I(-,a) = \{\rho_1, \ldots, \rho_n\}$. Let $f : e_aA \to K$ and put $\lambda_i = f(\bar\rho_i) \in K$. We claim that $f = \sum_{i=1}^n \lambda_i\bar\rho_i^*$. For any $u = \sum_{j=1}^n \mu_j\bar\rho_j \in e_aA$, where $\mu_j \in K$, then we have $f(u) = \sum_{j=1}^n \mu_j f(\bar\rho_j) = \sum_{j=1}^n \mu_j\lambda_j$. On the other hand, we have

$$(\sum_{i=1}^n \lambda_i\rho_i^*)(u) = \sum_{i,j=1}^n \lambda_i\bar\rho_i^*(\mu_j\bar\rho_j) = \sum_{i,j=1}^n \lambda_i\mu_j\bar\rho_i^*(\bar\rho_j) = \sum_{j=1}^n \lambda_j\mu_j\bar\rho_j^*(\bar\rho_j) = \sum_{j=1}^n \lambda_j\mu_j.$$

So $f = \sum_{i=1}^n \lambda_i\bar\rho_i^*$. It remains to show that $\bar\rho_1^*, \ldots, \bar\rho_n^*$ are linearly independent. Assume that $\sum\limits_{i=1}^n \lambda_i\bar\rho_i^* = 0$, where $\lambda_i \in K$. Then, $(\sum\limits_{i=1}^n \lambda_i\bar\rho_i^*)(u) = \sum\limits_{i=1}^n \lambda_i\bar\rho_i^*(u) = 0$, for any $u \in e_aA$. Taking $u = \bar\rho_j$, we have

$$0 = \sum_{i=1}^n \lambda_i\bar\rho_i^*(\bar\rho_j) = \lambda_j\bar\rho_j^*(\bar\rho_j) + \sum_{i\neq j} \lambda_i\bar\rho_i^*(\bar\rho_j) = \lambda_j \cdot 1 + \sum_{i\neq j} \lambda_i \cdot 0 = \lambda_j, \text{ for } j = 1, \ldots, n.$$

So $\bar\rho_1^*, \ldots, \bar\rho_i^*$ are linearly independent. Hence, $\{\bar\rho_1^*, \ldots, \bar\rho_n^*\}$ is a K-basis of I_a. □

Lemma 2.3.6. *Let $A = KQ$, where Q is a finite quiver and I is an admissible monomial ideal of KQ. Consider the indecomposable injective module I_a for some $a \in Q_0$.*

(1) *If $\eta \in Q_I(-,a)$, then $\bar\eta \cdot \bar\eta^* = e_a^*$.*

(2) *If η is a non-trivial path in Q, then $\bar{\eta} \cdot e_a^* = 0$.*

Proof. (1) Let $\eta \in Q_I(-,a)$. Then, $(\bar{\eta} \cdot \bar{\eta}^*)(e_a) = \bar{\eta}^*(e_a \cdot \bar{\eta}) = \bar{\eta}^*(\bar{\eta}) = 1$. Consider a non-trivial path $\zeta \in Q_I(-,a)$. Then, $(\bar{\eta} \cdot \bar{\eta}^*)(\bar{\zeta}) = \bar{\eta}^*(\bar{\zeta} \cdot \bar{\eta})$. If ζ, η are not composable, then $\bar{\zeta} \cdot \bar{\eta} = 0$, and hence, $(\bar{\eta} \cdot \bar{\eta}^*)(\bar{\zeta}) = 0$. Otherwise, $\zeta\eta$ is a path whose length is greater than the length of η. In particular, $\zeta\eta \neq \eta$. By Lemma 2.3.3, $\overline{\zeta\eta} \neq \bar{\eta}$. Therefore, we have $(\bar{\eta} \cdot \bar{\eta}^*)(\bar{\zeta}) = \bar{\eta}^*(\bar{\zeta} \cdot \bar{\eta}) = \bar{\eta}^*(\overline{\zeta\eta}) = 0$. So $\bar{\eta} \cdot \bar{\eta}^* = e_a^*$.

(2) Let η be a non-trivial path in Q. Given any $\rho \in Q_I(-,a)$, by definition, we have $(\bar{\eta} \cdot e_a^*)(\bar{\rho}) = e_a^*(\bar{\rho} \cdot \bar{\eta})$. Now, either $\rho\eta = 0$ or $\rho\eta$ is a non-trivial path in $Q_I(-,a)$. In either case, $e_a^*(\bar{\rho} \cdot \bar{\eta}) = 0$. Therefore, $\bar{\eta} \cdot e_a^* = 0$. □

Lemma 2.3.7. *Let $A = KQ$, where Q is a finite quiver and I an admissible monomial ideal of KQ. For any $a \in Q_0$, $\mathrm{soc} I_a = K < e_a^* > \cong S_a$.*

Proof. We write $Q_I(-,a) = \{\varepsilon_a, \rho_1, \ldots, \rho_t\}$, where $\rho_1, \ldots, \rho_t$ are non-trivial paths. By Proposition 2.3.5, I_a has a K-basis $\{e_a^*, \bar{\rho}_1^*, \ldots, \bar{\rho}_t^*\}$. For any $f \in I_a$, by Lemma 1.5.11, $f \in \mathrm{soc} I_a$ if and only if $(\mathrm{rad} A) \cdot f = 0$. By Lemma 2.3.6(2), $\bar{\eta} \cdot e_a^* = 0$, for any non-trivial η in Q. That is, $(\mathrm{rad} A) \cdot e_a^* = 0$. Hence, $K < e_a^* > \subseteq \mathrm{soc} I_a$. For the other inclusion, assume that $f \in \mathrm{soc} I_a$. We can write $f = \lambda_a e_a^* + \sum_{i=1}^t \lambda_i \bar{\rho}_i{}^*$, where $\lambda_a, \lambda_i \in K$. For any j such that $1 \leq j \leq t$, by Lemma 2.3.6(2), we have

$$
\begin{aligned}
0 &= (\bar{\rho}_j \cdot f)(e_a) \\
&= \lambda_a (\bar{\rho}_j \cdot e_a^*)(e_a) + \textstyle\sum_{i=1}^t \lambda_i (\bar{\rho}_j \bar{\rho}_i{}^*)(e_a) \\
&= \lambda_a e_a^*(e_a \cdot \rho_j) + \textstyle\sum_{i=1}^t \lambda_i (\bar{\rho}_i{}^*)(\rho_j) \\
&= \lambda_a e_a^*(\rho_j) + \lambda_j \bar{\rho}_j^*(\rho_j) \\
&= \lambda_j.
\end{aligned}
$$

So it means that $f = \lambda_a e_a^* \in K < e_a^* >$. Then $\mathrm{soc} I_a \subseteq K < e_a^* >$. Therefore, $\mathrm{soc} I_a = K < e_a^* >$, which is a simple module. Finally, by Lemma 2.3.6(1), $e_a \cdot e_a^* = e_a^* \neq 0$. Thus, $e_a \cdot \mathrm{soc} I_a \neq 0$. So $\mathrm{soc} I_a \cong S_a$. □

Let $\alpha : a \to b$ be an arrow in Q. Then we have a right A-linear map

$$P^{\mathrm{o}}[\alpha] : e_a A \to e_b A : v \mapsto \bar{\alpha} v.$$

Applying $D = \mathrm{Hom}_K(-, k)$, we obtain a left A-linear map $I[\alpha] = D(P^{\mathrm{o}}[\alpha]) : I_b \to I_a$. If $f \in I_b = \mathrm{Hom}_K(e_b A, K)$, then $I[\alpha](f) = D(P^{\mathrm{o}}[\alpha])(f) = f \circ P^{\mathrm{o}}[\alpha]$. Therefore, for $v \in e_a A$, we obtain

$$I[\alpha](f)(v) = (f \circ P^{\mathrm{o}}[\alpha])(v) = f(P^{\mathrm{o}}[\alpha])(v)) = f(\bar{\alpha} v).$$

This A-linear map is explicitly described in the following statement.

Lemma 2.3.8. *Let $A = KQ/I$, where Q is a finite quiver and I is an admissible monomial ideal of KQ. Given an arrow $\alpha : a \to b$ in Q, the A-linear map $I[\bar{\alpha}] : I_b \to I_a$ is a radical map. Moreover, for any $\rho \in Q_I(-, b)$, we have*

$$I[\bar{\alpha}](\bar{\rho}^*) = \begin{cases} \bar{\eta}^*, & \text{if } \rho = \alpha\eta, \text{ for some } \eta \in Q_I(-, a); \\ 0, & \text{if } \alpha \text{ is not the terminal arrow of } \rho. \end{cases}$$

Proof. Let $\rho \in Q_I(-, b)$. Assume first that $\rho = \alpha\eta$, for some $\eta \in Q_I(-, a)$. We claim that $I[\bar{\alpha}](\bar{\rho}^*) = \bar{\eta}^*$. By using the multiplication mentioned above, we have

$$I[\bar{\alpha}](\bar{\rho}^*)(\bar{\eta}) = \bar{\rho}^*(\bar{\alpha} \cdot \bar{\eta}) = \bar{\rho}^*(\bar{\rho}) = 1.$$

If $\zeta \in Q_I(-, a)$ with $\zeta \neq \eta$, then $\alpha \cdot \zeta \neq \alpha \cdot \eta = \rho$. Since I is monomial, $\overline{\alpha\zeta} \neq \bar{\rho}$. Therefore,

$$I[\bar{\alpha}](\bar{\rho}^*)(\bar{\zeta}) = \bar{\rho}^*(\bar{\alpha} \cdot \bar{\zeta}) = \bar{\rho}^*(\overline{\alpha\zeta}) = 0.$$

This shows that $I[\bar{\alpha}](\bar{\rho}^*) = \bar{\eta}^*$.

Assume now that α is not a terminal arrow of ρ. For any $\zeta \in Q_I(-, a)$, we have $\alpha \cdot \zeta \neq \rho$. Since I is monomial, $\overline{\alpha\zeta} \neq \bar{\rho}$. So

$$I[\bar{\alpha}](\bar{\rho}^*)(\bar{\zeta}) = \bar{\rho}^*(\bar{\alpha} \cdot \bar{\zeta}) = \bar{\rho}^*(\overline{\alpha\zeta}) = 0$$

So $I[\bar{\alpha}](\bar{\rho}^*) = 0$. □

2.4 Representations of a Quiver

As we have seen in previous sections, quivers provide a convenient way to visualise finite dimensional algebras. In this section, we explain how to use quivers to visualise modules.

Definition 2.4.1. *Let Q be a finite quiver. A K-*linear representation*, or simply, a* representation*, M of Q is defined by the following data :*

(1) *With each vertex a in Q_0 is associated a K-vector space $M(a)$.*

(2) *With each arrow $\alpha : a \to b$ in Q_1 is associated a K-linear map*
$M(\alpha) : M(a) \to M(b)$.

Let M, N be representations of Q. A representation morphism *$f : M \to N$ consists of a family of K-linear maps $f_a : M(a) \to N(a)$, with $a \in Q_0$, such that for each arrow $\alpha : a \to b$, the following diagram is commutative :*

$$\begin{array}{ccc} M(a) & \xrightarrow{M(\alpha)} & M(b) \\ {\scriptstyle f_a}\downarrow & & \downarrow{\scriptstyle f_b} \\ N(a) & \xrightarrow{N(\alpha)} & N(b). \end{array}$$

A representation M of Q is called finite dimensional *if $M(a)$ is finite dimensional, for every $a \in Q_0$. The category of finite dimensional k-linear representations of Q will denoted by* $\mathrm{rep}(Q)$.

Example 2.4.2. *Let Q be the quiver*

$$\begin{array}{ccccc} 1 & & & & \\ & \overset{\alpha}{\searrow} & & & \\ & & 3 & \xleftarrow{\gamma} & 4. \\ & \underset{\beta}{\nearrow} & & & \\ 2 & & & & \end{array}$$

We have the following representation

$$\begin{array}{ccccc} K & & & & \\ & \searrow^{\begin{bmatrix}1\\0\end{bmatrix}} & & & \\ & & K^2 & \xleftarrow{\begin{bmatrix}1\\1\end{bmatrix}} & K. \\ & \nearrow_{\begin{bmatrix}0\\1\end{bmatrix}} & & & \\ K & & & & \end{array}$$

Let $A = KQ$, where Q is a finite acyclic quiver. Given a module M in mod A, we associate a representation $\hat{M}$ in $\text{rep}(Q)$ as follows. For $a \in Q_0$ we put $\hat{M}(a) = e_a M$; and for $\alpha : a \to b$ in Q_1, we define a K-linear map

$$\hat{M}(\alpha) : \hat{M}(a) \to \hat{M}(b) : u \mapsto \alpha \cdot u.$$

In this way, we obtain the following result.

Proposition 2.4.3. ([3], III.1.6) *Let $A = KQ$, where Q is a finite acyclic quiver. There exists an equivalence of K-linear categories*

$$F : \text{mod}A \xrightarrow{\cong} \text{rep}(Q) : M \mapsto \hat{M}.$$

In the sequel, for the simplicity of the notation, we shall identify a module M in mod A with the representation $\hat{M}$ in $\text{rep}(Q)$.

Lemma 2.4.4. ([3], III.2.4) *Let $A = KQ$, where Q is a finite acyclic quiver. For $a \in Q_0$, the indecomposable projective module P_a can be identified with the representation $P_a = \{P_a(x), P_a(\alpha)\}_{x \in Q_0; \alpha : x \to y \in Q_1}$, where $P_a(x)$ is the K-vector space having $Q(a, x)$ as a basis, and $P_a(\alpha) : P_a(x) \to P_a(y)$ is the left multiplication by α.*

Lemma 2.4.5. ([3], III.2.6) *Let $A = KQ$, where Q is a finite acyclic quiver. For $a \in Q_0$, the indecomposable injective module I_a can be identified with the representation*

$I_a = \{I_a(x), I_a(\alpha)\}_{x\in Q_0;\alpha\in Q_1}$, *where* $I_a(x) = D(e_a A e_x)$ *is the* K*-vector space having* $Q(x,a)$ *as a basis, and* $I_a(\alpha) = D(P_a^{o}(\alpha)) : D(e_a A e_x) \to D(e_a A e_y)$ *for each arrow* $\alpha : x \to y$.

Proposition 2.4.6. ([3], III.2) *Let* $A = KQ$, *where* Q *is a finite acyclic quiver. For each vertex* $a \in Q_0$, *the simple module* S_a *at* a *is identified with the representation* $S_a = (S_a(b), S_a(\alpha))_{b\in Q_0;\,\alpha\in Q_1}$, *where*

$$S_a(b) = \begin{cases} K, & \text{if } b = a; \\ 0, & \text{if } b \neq a, \end{cases}$$

and $S_a(\alpha) = 0$, *for any* $\alpha \in Q_1$.

Lemma 2.4.7. ([3], III.2.2) *Let* Q *be a finite acyclic quiver. If* M *is a finite dimensional representation of* Q, *then* $\mathrm{rad}M$ *is the representation with* $(\mathrm{rad}M)(a) = \sum_{\beta\in Q_1(-,a)} \mathrm{Im}(M(\beta))$ *for every* $a \in Q_0$, *and* $(\mathrm{rad}M)(\alpha) : (\mathrm{rad}M)(a) \to (\mathrm{rad}M)(b)$ *is obtained by restricting the* K*-linear map* $M(\alpha) : M(a) \to M(b)$ *for every arrow* $\alpha : a \to b$.

Example 2.4.8. *Let* A *be the path algebra of the following linearly oriented quiver*

$$a \xrightarrow{\alpha} b \xrightarrow{\beta} c\ .$$

Then

(1) $\mathrm{rad}P_a \cong P_b$ *and* $\mathrm{rad}P_b \cong P_c \cong S_c$.

(2) $I_c/\mathrm{soc}I_c \cong I_b$ *and* $I_b/\mathrm{soc}I_b \cong I_a \cong S_a$.

(3) $P_a \cong I_c$.

Proof. We know that $A = K < \varepsilon_a, \varepsilon_b, \varepsilon_c, \alpha, \beta, \beta\alpha >$. By using the Propositions 2.4.4 and 2.4.3, we have the indecomposable projective modules as follows :

$$P_a = A\varepsilon_a = K < \varepsilon_a, \alpha, \beta\alpha >$$

By Proposition 2.4.3, P_a is represented by the following K-linear representation :

$$P_a : \qquad K < \varepsilon_a > \xrightarrow{\alpha\cdot} K < \alpha > \xrightarrow{\beta\cdot} K < \beta\alpha >,$$

where $\alpha\cdot$ and $\beta\cdot$ denote the left multiplications by α and β respectively. Then, we have the following isomorphism of representations :

$$\begin{array}{ccccc} K < \varepsilon_a > & \xrightarrow{\alpha\cdot} & K < \alpha > & \xrightarrow{\beta\cdot} & K < \beta\alpha > \\ {\scriptstyle f_1}\downarrow & & {\scriptstyle f_2}\downarrow & & {\scriptstyle f_3}\downarrow \\ K & \xrightarrow{\text{id}} & K & \xrightarrow{\text{id}} & K, \end{array}$$

where f_1, f_2 and f_3 are the K-linear isomorphisms such that $f_1(\varepsilon_a) = f_2(\alpha) = f_3(\beta\alpha) = 1$.

By Proposition 2.3.4, $\text{rad}P_a = K < \alpha, \beta\alpha >$, which is represented by the following K-linear representation :

$$\text{rad}P_a : \qquad 0 \longrightarrow K < \alpha > \xrightarrow{\beta\cdot} K < \beta\alpha > .$$

Using proposition 2.4.3, we obtain an isomorphism of representations :

$$\text{rad}P_a : \qquad \begin{array}{ccccc} 0 & \longrightarrow & K < \alpha > & \xrightarrow{\beta\cdot} & K < \beta\alpha > \\ {\scriptstyle 0}\downarrow & & {\scriptstyle f_2}\downarrow & & {\scriptstyle f_3}\downarrow \\ 0 & \longrightarrow & K & \xrightarrow{\text{id}} & K \end{array}$$

Similarly, $P_b = A\varepsilon_b = K < \varepsilon_b, \beta >$, which is represented as follows :

$$P_b : \qquad 0 \longrightarrow K < \varepsilon_b > \xrightarrow{\beta\cdot} K < \beta > .$$

Moreover, we have an isomorphism of representations :

$$\begin{array}{ccccc} 0 & \longrightarrow & K < \varepsilon_b > & \xrightarrow{\beta\cdot} & K < \beta > \\ {\scriptstyle f_1}\downarrow & & {\scriptstyle f_2}\downarrow & & {\scriptstyle f_3}\downarrow \\ 0 & \longrightarrow & K & \xrightarrow{\text{id}} & K \end{array}$$

where f_1, f_2 and f_3 are the K-linear isomorphisms such that $f_1 = 0$ and $f_2(\varepsilon_b) = f_3(\beta) = 1$. So, we see that $\text{rad}P_a \cong P_b$.

By Proposition 2.3.4, $\text{rad}P_b = K < \beta >$ and is represented by the following K-linear representation :

$$\text{rad}P_b : \qquad 0 \longrightarrow 0 \longrightarrow K < \beta > .$$

By Proposition 2.4.3, we obtain an isomorphism of representations :

$$\text{rad}P_b : \qquad \begin{array}{ccccc} 0 & \longrightarrow & 0 & \longrightarrow & K < \beta > \\ {\scriptstyle 0}\downarrow & & {\scriptstyle f_2}\downarrow & & {\scriptstyle f_3}\downarrow \\ 0 & \longrightarrow & 0 & \longrightarrow & K \end{array}$$

Also, $P_c = A\varepsilon_c = K < \varepsilon_c >$ and is represented by the following K-linear representation :

$$P_c : \qquad 0 \longrightarrow 0 \longrightarrow K < \varepsilon_c > .$$

Then, we have an isomorphism of representations :

$$\begin{array}{ccccc} 0 & \longrightarrow & 0 & \xrightarrow{0} & K < \varepsilon_c > \\ {\scriptstyle f_1}\downarrow & & {\scriptstyle f_2}\downarrow & & {\scriptstyle f_3}\downarrow \\ 0 & \longrightarrow & 0 & \longrightarrow & K \end{array}$$

where f_1, f_2 and f_3 are the K-linear isomorphisms such that $f_1 = f_2 = 0$ and $f_3(\varepsilon_c) = 1$. We have $\text{rad}P_c = 0$.

Indeed, by Proposition 2.4.6, the simple module is $S_a = P_a/\text{rad}P_a$. Since we show the calculation for P_a and $\text{rad}P_a$, we have $S_a = K < \varepsilon_a > \quad S_c = K < \varepsilon_c >$.

By Proposition 2.4.3, S_a is represented by the following K-linear representation :

$$S_a : \qquad K < \varepsilon_a > \longrightarrow 0 \longrightarrow 0.$$

Then, we have the following isomorphism of representations :

$$S_a: \qquad \begin{array}{ccccc} K < \varepsilon_a > & \longrightarrow & 0 & \longrightarrow & 0 \\ \cong\downarrow & & \downarrow & & \downarrow \\ K & \longrightarrow & 0 & \longrightarrow & 0. \end{array}$$

Similarly, $S_b = K < \varepsilon_b >$, which is represented by the following K-linear representation :

$$S_b: \qquad 0 \longrightarrow K < \varepsilon_b > \longrightarrow 0.$$

Thus, we have the following isomorphism of representations :

$$S_b: \qquad \begin{array}{ccccc} 0 & \longrightarrow & K < \varepsilon_b > & \longrightarrow & 0 \\ \downarrow & & \cong\downarrow & & \downarrow \\ 0 & \longrightarrow & K & \longrightarrow & 0. \end{array}$$

Also, $S_c = K < \varepsilon_c >$ and is represented by the following K-linear representation :

$$S_c: \qquad 0 \longrightarrow 0 \longrightarrow K < \varepsilon_c > .$$

So, we have the following isomorphism of representations :

$$S_c: \qquad \begin{array}{ccccc} 0 & \longrightarrow & 0 & \longrightarrow & K < \varepsilon_c > \\ \downarrow & & \downarrow & & \cong\downarrow \\ 0 & \longrightarrow & 0 & \longrightarrow & K. \end{array}$$

In conclusion, we have $\text{rad}P_b \cong P_c \cong S_c$.

To obtain the indecomposable injective left modules, we first have the indecomposable projective right A-modules as follows.

$$P_a^{\text{o}} = \varepsilon_a A = K < \varepsilon_a >; P_b^{\text{o}} = \varepsilon_b A = K < \varepsilon_b, \alpha >; P_c^{\text{o}} = \varepsilon_c A = K < \varepsilon_c, \beta, \beta\alpha > .$$

So by using the Lemma 2.4.5, $I_a = D(\varepsilon_a A) = K < \varepsilon_a^* >$, which is identified with the following K-linear representation :

$$I_a: \qquad K < \varepsilon_a^* > \longrightarrow 0 \longrightarrow 0.$$

Then, we have an isomorphism of representations as follows :

$$I_a : \qquad \begin{array}{ccccc} K < \varepsilon_a^* > & \longrightarrow & 0 & \longrightarrow & 0 \\ \cong\big\downarrow & & \big\downarrow & & \big\downarrow \\ K & \longrightarrow & 0 & \longrightarrow & 0, \end{array}$$

where the first vertical isomorphism sends $\lambda\varepsilon_a^*$ to λ, for all $\lambda \in k$. Also, $I_a \cong S_a$ and by using the lemma 2.3.7, we have $\mathrm{soc}I_a \cong S_a$ and $I_a/\mathrm{soc}I_a \cong 0$.

Similarly, $I_b = D(\varepsilon_b A) = K < \varepsilon_b^*, \alpha^* >$, which is represented by the following K-linear representation :

$$I_b : \qquad K < \alpha > \longrightarrow K < \varepsilon_b^* > \longrightarrow 0.$$

Then, we have an isomorphism of representations as follows :

$$\begin{array}{ccccc} K < \alpha^* > & \longrightarrow & K < \varepsilon_b^* > & \longrightarrow & 0 \\ \cong\big\downarrow & & \cong\big\downarrow & & \big\downarrow \\ K & \longrightarrow & K & \longrightarrow & 0 \end{array}$$

By lemma 1.5.11, we have $\mathrm{soc}I_b \cong S_b$ and $I_b/\mathrm{soc}I_b \cong I_a$.

Also, $I_c = D(\varepsilon_c A) = K < \varepsilon_c^*, \beta^*, (\beta\alpha)^* >$, which is represented by the following K-linear representation :

$$I_c : \qquad K < (\beta\alpha)^* > \longrightarrow K < \beta^* > \longrightarrow K < \varepsilon_c^* > .$$

Then, we have an isomorphism of representations as follows :

$$\begin{array}{ccccc} K < (\beta\alpha)^* > & \longrightarrow & K < \beta^* > & \longrightarrow & K < \varepsilon_c^* > \\ \cong\big\downarrow & & \cong\big\downarrow & & \cong\big\downarrow \\ K & \longrightarrow & K & \longrightarrow & K \end{array}$$

Now, we see $I_c \cong P_a$ and by lemma 1.5.11, we have $\mathrm{soc}I_c \cong S_c$ and $I_c/\mathrm{soc}I_c \cong I_b$.

CHAPTER 3

Auslander-Reiten Theory

In the previous chapter, we saw some quiver theory techniques for visualizing finite dimensional algebras and their modules. However, to actually compute the indecomposable modules and the morphisms between them, we will present in this chapter other useful tools of the Auslander-Reiten Theory. In particular, the existence of almost split sequences and irreducible morphisms in mod A. Throughout this chapter, A is a finite dimensional K-algebra, and the A-modules and the morphisms are in mod A.

3.1 Almost Split Morphisms

The concept of almost split morphisms is playing an important part to understand the result of our thesis.

Definition 3.1.1. *Let A be a finite dimensional K-algebra. A morphism $f : M \to N$ in* mod A *is said to be*

(1) left split *if any morphism $g : M \to X$ factors through f.*

(2) right split *if any morphism* $h : Y \to N$ *factors through* f.

Lemma 3.1.2. *Let A be a finite dimensional K-algebra. A morphism $f : M \to N$ in* mod *A is left split or right split if and only it is a section or a retraction, respectively.*

Proof. Suppose that f is left split. Consider $id_M : M \to M$, there exists $h : N \to M$ such that $id_M = hf$. Then f is section. Now, suppose that f is section.
Then, there exists $h : N \to M$ such that $hf = id_M$. Let $g : M \to X$ be a morphism. We have $g = g \circ id_M = (gh)f$, with $gh : N \to X$. Hence, f is left split.

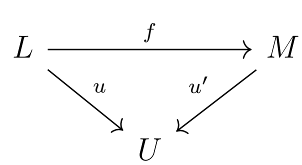
Now suppose that f is right split. Consider $id_N : N \to N$, there exists $h : N \to M$ such that $id_N = fh$. Then f is retraction. Suppose that f is retraction. Then, there exists $h : N \to M$ such that $fh = id_N$. Let $g : Y \to N$ be a morphism. We have $g = id_N \circ g = f(hg)$, with $hg : Y \to M$. Hence, f is right split. □

Definition 3.1.3. *Let A be a finite dimensional K-algebra.*

(1) *A morphism $f : L \to M \in$ mod$A$ is said to be* left almost split *if the following conditions are satisfied.*

  (a) *f is not section.*

  (b) *For every non-section morphism $u : L \to U$ in* mod *A, there exists a morphism $u' : M \to U$ such that $u = u'f$, that is, the diagram*

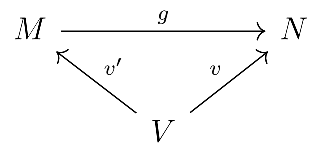


  commutes.

(2) *A morphism $g : M \to N \in$ mod$A$ is said to be* right almost split *if the following conditions are satisfied.*

(a) *g is not retraction.*

(b) *For every non-retraction morphism $v : V \to N$, there exists a morphism $v' : V \to M$ such that $v = gv'$, that is, the diagram*



commutes.

Lemma 3.1.4. *Let A be a finite dimensional K-algebra.*

(1) *If $f : L \to M$ is a left almost split morphism in* mod A*, then L is indecomposable.*

(2) *If $g : M \to N$ is a right almost split morphism in* mod A*, then N is indecomposable.*

Proof. (1) Assume that $L = L_1 \oplus L_2$, with both L_1 and L_2 non-zero and let $p_1 : L \to L_1$ and $p_2 : L \to L_2$ be the canonical projections. Then $\mathrm{Ker}p_1 = L_2 \neq 0$ and $\mathrm{Ker}p_2 = L_1 \neq 0$ imply that p_1 and p_2 are not sections. Since f is a left almost split morphism in mod A, there exists homomorphisms $u_1 : M \to L_1$ and $u_2 : M \to L_2$ in mod A such that $p_1 = u_1 f$ and $p_2 = u_2 f$. Consider the homomorphism

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} : M \to L_1 \oplus L_2 = L.$$

Then, we have $(uf)(x) = u(f(x)) = (u_1(f(x))), u_2(f(x))) = (p_1(x), p_2(x)) = x = id_L(x)$, for each $x \in L$. Hence, $uf = id_L$ and f is a section in mod A, which contradicts the fact that f is a left almost split morphism in mod A.

The proof of (2) is dual. □

Definition 3.1.5. *Let A be a finite dimensional K-algebra.*

(1) *A morphism $f : L \to M$ in* mod A *is said to be* left minimal *if every morphism $h \in \mathrm{End}_A(M)$ with $hf = f$ is an isomorphism.*

(2) *A morphism* $g : M \to N$ *in* mod A *is said to be* right minimal *if every morphism* $h \in \mathrm{End}A(M)$ *with* $gh = g$ *is an isomorphism.*

Definition 3.1.6. *Let* A *be a finite dimensional* K*-algebra.*

(1) *A morphism* $f : L \to M$ *in* mod A *is called* minimal left almost split *if it is both left minimal and left almost split.*

(2) *A morphism* $g : M \to N$ *in* mod A *is called* minimal right almost split *if it is both right minimal and right almost split.*

Lemma 3.1.7. *Let* A *be a finite dimensional* K*-algebra with* X, Y *two modules in* mod A*. The following statements hold.*

(1) *The zero morphism* $0 : X \to Y$ *is minimal left almost split if and only if* X *is a simple injective module and* $Y = 0$*.*

(2) *The zero morphism* $0 : X \to Y$ *is minimal right almost split if and only if* $X = 0$ *and* Y *is a simple projective module.*

Proof. (1) Assume that $0_{X,Y} : X \to Y$ is a minimal left almost split morphism in mod A. Since $0_Y : Y \to Y$ is such that $0_{X,Y} = 0_Y \circ 0_{X,Y}$, by the left minimality of $0_{X,Y}$, the zero morphism 0_Y is an isomorphism, or equivalently, $Y = 0$. Furthermore, since $0_{X,Y}$ is not a section, X is non-zero. If X is not a simple module, then it has a proper submodule L. Then the canonical projection $v : X \to X/L$ is non-zero. Not being a monomorphism, v is not a section. Thus, $v = v' \circ 0_{X,Y}$ for some morphism $v' : Y \to V$, so $v = 0$, a contradiction. So, X is simple. The injective envelope $u : X \to E(X)$ of X is a non-zero monomorphism. So u cannot factor through $0_{X,Y}$. Hence, u is a section. By Proposition 1.10.5, $E(X)$ is an indecomposable injective A-module. Hence, u is an isomorphism, and consequently, X is a simple injective module.

Conversely, assume that X is a simple injective module. In particular, $0 : X \to 0$ is not a section. If $g : X \to N$ is a non-section morphism in mod A then, by Lemma 1.9.1(1)

$g \in \text{rad}(X, N)$. Then, by Corollary 1.9.2(2), $g = 0$. In particular, g factors through 0. That is, $0 : X \to Y$ is minimal left almost split.

(2) Assume that $0_{X,Y} : X \to Y$ is a minimal right almost split morphism in mod A. Since $0_X : X \to X$ is such that $0_{X,Y} = 0_{X,Y}0_X$, by the right minimality of $0_{X,Y}$, the zero morphism 0_X is an isomorphism, or equivalently, $X = 0$. Furthermore, since $0_{X,Y}$ is not a retraction, Y is non-zero. If Y is not a simple module, then it has a proper submodule L. Then the canonical injection $u : Y/L \to Y$ is non-zero. Not being an epimorphism, u is not a retraction. Thus, $u = 0_{X,Y}u'$ for some morphism $u' : V \to X$, so $u = 0$, a contradiction. So, Y is simple. The projective cover $h : P(Y) \to Y$ of Y is a non-zero epimorphism. So h cannot factor through $0_{X,Y}$. Hence, h is a retraction. By Proposition 1.10.5, $P(Y)$ is an indecomposable projective A-module. Hence, h is an isomorphism, and consequently, Y is a simple projective module.

Conversely, assume that Y is a simple projective module. In particular, $0 : 0 \to Y$ is not a retraction. If $g' : M \to Y$ is a non-retraction morphism in mod A, then, by Lemma 1.9.1(2), $g \in \text{rad}(M, Y)$. Then, by corollary, 1.9.2(1) $g' = 0$. In particular, g' factors through 0. That is, $0 : X \to Y$ is minimal right almost split. □

Lemma 3.1.8. *Let A be a finite dimensional K-algebra with P an indecomposable projective module in* mod A. *Then the inclusion map $i : \text{rad}P \to P$ is minimal right almost split.*

Proof. Since P is indecomposable projective, by Propositions 1.5.8 and 1.6.6, rad$P$ is the unique maximal submodule of P. Since $i : \text{rad}P \to P$ is not an epimorphism, it is not a retraction. Let $v : V \to P$ be a non-retraction morphism in mod A. By Lemma 1.6.2, v is not an epimorphism. Hence $\text{Im}(v)$ is a proper submodule of P, and so $\text{Im}(v) \subseteq \text{rad}P$. Therefore, we obtain $v = iv'$, where $v' : V \to \text{rad}P$ is the co-restriction of v to rad$P$. This shows that i is a right almost split morphism in mod A. If $ih = i$, for some $h \in \text{End}_A(\text{rad}P)$, then $h = \mathrm{id}_{\mathrm{rad}P}$. Hence $i : \mathrm{rad}P \to P$ is also right minimal. Therefore, i is a minimal right almost split morphism in mod A. □

Lemma 3.1.9. *Let A be a finite dimensional K-algebra with I an indecomposable injective module in* mod A*. Then the canonical projection $p : I \to I/\mathrm{soc}I$ is minimal left almost split.*

Proof. Suppose that I is an indecomposable injective module in mod A. Let $h : I \to L$ be a morphism, which is not section. Since h is not injective, Ker$h$ is not zero. Since soc$I$ is essential in I, Ker$h \bigcap$ soc$I$ is not zero. But soc$I$ is simple and then, soc$I \subseteq$ Ker$h$. Then, $h(\mathrm{soc}I) = 0$ and h is factorized by $I/\mathrm{soc}I$. Hence, there exists $u : I/\mathrm{soc}I \to L$ such that $h = up$. □

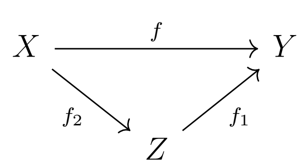
3.2 Irreducible Morphisms

In this section, we see the relation between radical square and irreducible morphisms, and some useful lemmas which bring us closer to prove the main result.

Definition 3.2.1. *A homomorphism $f : X \to Y$ in* mod A *is said to be* irreducible *if*

(1) *f is neither a section nor a retraction;*

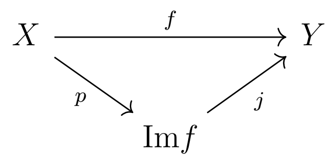
(2) *if $f = f_1 f_2$, either f_1 is a retraction or f_2 is a section*



Also, this notion is self-dual, that is $f : X \to Y$ is irreducible in mod A if and only if $Df : DY \to DX$ is irreducible in modA^{op}.

Lemma 3.2.2. *Let $f : X \to Y$ be an irreducible morphism . Then, f is a proper monomorphism or f is a proper epimorphism.*

Proof. Let $f = jp$ be canonical factorization of f



where p is the co-restriction and j is the inclusion map. Since f is irreducible, we conclude that either p is a section or j is a retraction. Moreover, f is not an isomorphism in mod A. Assume that f is not a proper epimorphism. Then $j : \mathrm{Im} f \to Y$ is not a retraction in mod A, because otherwise $\mathrm{Im} f = Y$ and f is a proper epimorphism. Therefore, $p : X \to \mathrm{Im} f$ is a section in mod A, hence an isomorphism. This implies that $f : X \to Y$ is a proper monomorphism. □

Lemma 3.2.3. *Let $f : X \to Y$ be an irreducible morphism in* mod A*.*

(1) *If X and Y are projective modules, then f is monomorphism.*

(2) *If X and Y are injective modules, then f is epimorphism.*

Proof. (1) Assume that f is not monomorphism. By Lemma 3.2.2, f is an epimorphism, and hence, by lemma 1.6.2 a retraction. This contradicts the definition of an irreducible morphism.

(2) Assume that f is not an epimorphism. By Lemma 3.2.2, f is monomorphism, and hence, by lemma 1.6.8 a section. This contradicts the definition of an irreducible morphism. □

Lemma 3.2.4. *Let $f : X \to Y$ be a morphism in* mod A*. If X or Y is indecomposable, and f is irreducible, then f is radical.*

Proof. Assume that X is indecomposable, and $f : X \to Y$ is irreducible. Then f is not section. Because of lemma 1.9.1, it is radical. The proof is similar if Y is indecomposable. □

Lemma 3.2.5. *Let X, Y be indecomposable modules in* mod A. *A morphism $f : X \to Y$ is irreducible if and only if $f \in \operatorname{rad}_A(X,Y) \backslash \operatorname{rad}_A^2(X,Y)$.*

Proof. Assume that f is irreducible. Then, f is not an isomorphism, because f is neither a section nor a retraction. Applying lemma 1.9.1(3), $f \in \operatorname{rad}_A(X,Y)$. If $f \in \operatorname{rad}_A^2(X,Y)$, then f can be written as $f = gh$, where $h \in \operatorname{rad}_A(X,Z)$ and $g \in \operatorname{rad}_A(Z,Y)$ for some $Z \in \operatorname{mod} A$. Since X, Y are indecomposable, h is not a section and g is not a retraction. Thus, f is not irreducible. a contradiction. Therefore, $f \notin \operatorname{rad}_A^2(X,Y)$.

Conversely, assume that $f \in \operatorname{rad}_A(X,Y) \backslash \operatorname{rad}_A^2(X,Y)$. Since X, Y are indecomposable, f is neither a section nor a retraction and not an isomorphism by lemma 1.2.8(3).

Suppose that $f = gh$, for some morphisms $h : X \to Z$ and $g : Z \to Y$. Then Z is non-zero. Decomposing Z into indecomposable summands as $Z = \bigoplus_{i=1}^{t} Z_i$, also we can write $h = \begin{bmatrix} h_1 \\ \cdot \\ \cdot \\ h_t \end{bmatrix} : X \to \bigoplus_{i=1}^{t} Z_i$ and $g = \begin{bmatrix} g_1 & \dots & g_t \end{bmatrix} : \bigoplus_{i=1}^{t} Z_i \to Y$ so that $f = \sum_{i=1}^{t} g_i h_i$. Because $f \notin \operatorname{rad}_A^2(X,Y)$, there exists $i \in \{1, \dots, t\}$ such that $h_i \notin \operatorname{rad}_A(X,Z)$ or there exists $j \in \{1, \dots, t\}$ such that $g_j \notin \operatorname{rad}_A(Z,Y)$. By lemma 1.9.3, we obtain that either $h_i : X \to Z_i$ for some $i \in \{1, \dots, t\}$, or $g_j : Z_j \to Y$ for some $j \in \{1, \dots, t\}$ is an isomorphism. Hence either h is a section or g is a retraction in $\operatorname{mod} A$. Therefore, $f : X \to Y$ is irreducible. □

Lemma 3.2.6. *Let $f : X \to Y$ be a non-zero morphism in* mod A.

(1) *If f is minimal right almost split, then f is irreducible.*

(2) *If f is minimal left almost split, then f is irreducible.*

Proof. (1) Suppose that f is minimal right almost split. Then, f is not a retraction and by Lemma 3.1.4(2), Y is indecomposable. Then by Proposition 1.9.1(2), we have $f \in \operatorname{rad}(X,Y)$. In particular, by lemma 1.2.8(1), f is not a section. Suppose that $f = f_1 f_2$ with $f_1 : Z \to Y$ and $f_2 : X \to Z$ are morphisms in mod A. If f_1 is not a retraction, since f is right almost split, there exists $f' : Z \to X$ such that $f_1 = f f_1'$. So $f = f_1 f_2 = f f_1' f_2$. Since f is right minimal, $f_1' f_2$ is an automorphism. Hence, f_2 is a section. Consequently, f is irreducible.

(2) Suppose that f is left minimal almost split. Then, f is not a section and by Lemma 3.1.4(1), X is indecomposable. Then by Proposition 1.9.1(1), we have $f \in \operatorname{rad}(X,Y)$.
In particular, by lemma 1.2.8(2), f is not a retraction. Suppose that $f = f_1 f_2$ with $f_1 : Z \to Y$, $f_2 : X \to Z$ for $Z \in \operatorname{mod} A$. If f_2 is not a section, since f is a left almost split, there exists $f_2' : Y \to Z$ such that $f_2 = f_2' f$. So $f = f_1 f_2 = f_1 f_2' f$. Since f is left minimal, then $f_1 f_2'$ is an automorphism. Hence, f_1 is a retraction. Consequently, f is irreducible. □

Proposition 3.2.7. ([2], IV.1.10)

(1) *Let $f : M \to N$ be a minimal left almost split morphism in* mod A*. A non-zero morphism $g : M \to X$ is irreducible if and only if $g = hf$ where $h : N \to X$ is a retraction.*

(2) *Let $f : M \to N$ be a minimal right almost split morphism in* mod A*. A non-zero morphism $g : X \to N$ is irreducible if and only if $g = fh$ where $h : X \to M$ is a section.*

Corollary 3.2.8. ([6], 4.13)

(1) *Let $f : M \to N$ be a minimal left almost split morphism. Then, there exists an irreducible morphism $g : M \to L$ if and only if L is a non-zero direct summand of N.*

(2) *Let $f : M \to N$ be a minimal right almost split morphism. Then, there exists an irreducible morphism $g : L \to N$ if and only if L is a non-zero direct summand of M.*

Proposition 3.2.9. ([3], IV.5.6)

Let A be a finite dimensional K-algebra. If A is of finite representation type, then any nonzero nonisomorphism between indecomposable modules in mod A *is a sum of composites of irreducible morphisms.*

As a consequence of the above result, we have the following statement, which is important for the proof of our main result.

Proposition 3.2.10. *Let A be a finite dimensional K-algebra of finite representation type and n a positive integer. If the composite of every chain of n irreducible morphisms between indecomposable modules is zero, then* $\operatorname{rad}^n(\operatorname{mod} A) = 0$.

Proof. Suppose that every chain of n irreducible morphisms between indecomposable modules has a zero composition. Assume that $\operatorname{rad}^n(\operatorname{mod} A) \neq 0$. Then there exists a non-zero morphism $f \in \operatorname{rad}^n(M, N)$, where M, N are indecomposable modules in mod A. By definition, $f = \sum_{i=1}^{s} f_{in} \circ \cdots \circ f_{i1}$, where the f_{ij} are radical maps between indecomposable modules. Since $f \neq 0$, one of the $f_{in} \circ \cdots \circ f_{i1}$ is non-zero. We can assume that $f_{1n} \circ \cdots \circ f_{11} \neq 0$. Put $g = g_n \circ \cdots \circ g_1$, where $g_j = f_{1j}$, for $j = 1, \ldots, n$. Then $g \neq 0$.

By Proposition 3.2.9, $g_j = \sum_{i_j=1}^{s_j} g_{i_j,j}$, where $s_j \geq 1$ and $g_{i_j,j}$ is a composite of irreducible morphisms between indecomposable modules, for $j = 1, \ldots, n$. This yields

$$g = g_n \circ \cdots \circ g_1 = \sum_{i_n=1}^{s_n} \cdots \sum_{i_1=1}^{s_1} g_{i_n,n} \circ \cdots \circ g_{i_1,1} \neq 0.$$

Thus, we may assume that $g_{1,n} \circ \cdots \circ g_{1,1} \neq 0$. Now, g_{1j} is the composite of t_j (≥ 1) irreducible morphisms between indecomposable modules, for $j = 1, \ldots, n$. Thus, $g_{1,n} \circ \cdots \circ g_{1,1}$ is a non-zero composite of $t_1 + \cdots t_n (\geq n)$ irreducible morphisms between indecomposable modules. Since $t_1 + \cdots t_n \geq n$, we obtain a desired contradiction. $\square$

3.3 Almost Split Sequences

The main aim of this section is to prove the propositions of the Auslander-Reiten theorem and almost split sequences in the module category of finite dimensional algebras, and explain their characterizations.

Lemma 3.3.1. ([3], IV.1.7) *Let* $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ *be a non-split short exact sequence in* $\operatorname{mod} A$.

(1) *The homomorphism* $f : L \to M$ *is irreducible if and only if, for every homomorphism* $v : V \to N$, *there exists* $v_1 : V \to M$ *such that* $v = gv_1$ *or* $v_2 : M \to V$ *such that* $g = vv_2$.

(2) *The homomorphism* $g : M \to N$ *is irreducible if and only if, for every homomorphism* $u : L \to U$, *there exists* $u_1 : M \to U$ *such that* $u = u_1 f$ *or* $u_2 : U \to M$ *such that* $f = u_2 u$.

Corollary 3.3.2. (1) *If* $f : L \to M$ *is an irreducible monomorphism, then* $N = \operatorname{Coker} f$ *is indecomposable.*

(2) *If* $g : M \to N$ *is an irreducible epimorphism, then* $L = \operatorname{Ker} g$ *is indecomposable.*

Proof. (1) Let $g : M \to N$ be the cokernel of f. Then, we have a short exact sequence

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0.$$

Assume that $N = N_1 \oplus N_2$ with N_1 and N_2 nonzero. Let $q_i : N_i \to N$ be the canonical injections and $N \to N_i$ the canonical projections, for $i = 1, 2$. Since N_1, N_2 are non-zero, q_1 and q_2 are not isomorphism. If there exists a morphism $u_i : M \to N_i$ such that $g = q_i u_i$

for some $1 \leq i \leq 2$, because g is an epimorphism, q_i is also an epimorphism, and hence, q_i an isomorphism, a contradiction. Then, by Lemma 3.3.1, there exists a homomorphism $v_i : N_i \to M$ such that $gv_i = q_i$, for $i = 1, 2$. Then $v = v_i p_i : M \to N$ is such that $gv = 1_N$. So g is a retraction. By Proposition 1.8.6, f is a section. This contradicts the fact that f is irreducible. One can prove (2) in a dual fashion. □

Definition 3.3.3. *A short exact sequence in* mod A

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

is called an almost split sequence *(or an* Auslander-Reiten sequence*) if f is a minimal left almost split morphism and g is a minimal right almost split morphism.*

Proposition 3.3.4. ([3], IV.1.13) *Consider a short exact sequence*

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

in mod A*. The following conditions are equivalent.*

(1) *The given sequence is an almost split sequence.*

(2) *L is indecomposable, and g is right almost split.*

(3) *N is indecomposable, and f is left almost split.*

(4) *The map f is minimal left almost split.*

(5) *The map g is minimal right almost split.*

(6) *The maps f and g are irreducible.*

Definition 3.3.5. *Let A be a finite dimensional K-algebra with M an indecomposable module in* mod A*. Let*

$$P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} M \longrightarrow 0$$

be an exact sequence in $\mathrm{mod}\, A$ *such that* $p_0 : P_0 \to M$ *and* $p_1 : P_1 \to \mathrm{Ker} p_0$ *are projective covers. Applying the contravariant functor* $(-)^t = \mathrm{Hom}_A(-, A)$*, we obtain an exact sequence*

$$0 \longrightarrow M^t \xrightarrow{p_0^t} P_0^t \xrightarrow{p_1^t} \mathrm{Coker}(p_1^t) \longrightarrow 0$$

in $\mathrm{mod}\, A^{\mathrm{op}}$*. We denote* $\mathrm{Coker}(p_1^t)$ *by* $\mathrm{Tr}M$ *and call it the* transpose *of* M*.*

Definition 3.3.6. *The* Auslander-Reiten translation *is defined to be the compositions of* D *with* Tr*. We set* $\tau = D\mathrm{Tr}$ *and* $\tau^{-1} = \mathrm{Tr}D$.

Proposition 3.3.7. ([3]. IV.2.10) *Let* M *and* N *be indecomposable modules in* $\mathrm{mod}\ A$*.*

(1) *The module* τM *is zero if and only if* M *is projective.*

(2) *The module* $\tau^{-1}N$ *is zero if and only if* N *is injective.*

(3) *If* M *is a non-projective module, then* τM *is indecomposable non-injective and* $\tau^{-1}\tau M \cong M$*.*

(4) *If* N *is a non-injective module, then* $\tau^{-1}N$ *is indecomposable non-projective and* $\tau\tau^{-1}N \cong N$*.*

(5) *If* M, N *are non-projective, then* $M \cong N$ *if and only if* $\tau M \cong \tau N$*.*

(6) *If* M, N *are non-injective, then* $M \cong N$ *if and only if* $\tau^{-1}M \cong \tau^{-1}N$*.*

Proposition 3.3.8. ([3]. IV.3.1)

(1) *For any indecomposable non-projective module* $M \in \mathrm{mod}\, A$*, there exists an almost split sequence* $0 \to \tau M \to E \to M \to 0$ *in* $\mathrm{mod}\ A$*.*

(2) *For any indecomposable non-injective module* $N \in \mathrm{mod}\, A$*, there exists an almost split sequence* $0 \to N \to F \to \tau^{-1}N \to 0$ *in* $\mathrm{mod}\ A$*.*

Corollary 3.3.9. *Let* M *be an indecomposable module in* $\mathrm{mod}\ A$*.*

(1) *There exists a minimal right almost split morphism* $g : N \to M$.

(2) *There exists a minimal left almost split morphism* $f : M \to N$.

Proof. (1) If M is not projective then, by Proposition 3.3.8(1), there exists an almost split sequence $0 \to \tau M \to N \to M \to 0$. In particular, there exists a minimal right almost split morphism $g : N \to M$. If M is projective then, by Lemma 3.1.8(1), the inclusion map $i : \mathrm{rad} M \to M$ is a minimal right almost split morphism.

(2) If M is not injective then, by Proposition 3.3.8(2), there exists an almost split sequence $0 \to M \to N \to \tau^{-1} M \to 0$. In particular, there exists a minimal left almost split morphism $f : M \to N$. If M is injective then, the canonical projection $p : M \to M/\mathrm{soc} M$ is a minimal left almost split morphism, by Lemma 3.1.9(1). □

Lemma 3.3.10. ([2], II.2.24) *Let A be a finite dimensional K-algebra with M an indecomposable module in* mod A.

(1) *There exists an irreducible morphism $f : X \to M$ if and only if there exists a morphism $h : Y \to M$ such that $\begin{bmatrix} f & h \end{bmatrix} : X \oplus Y \to M$ is minimal right almost split.*

(2) *There exists an irreducible morphism $g : M \to X$ if and only if there exists a morphism $h : M \to Y$ such that $\begin{bmatrix} g \\ h \end{bmatrix} : M \to X \oplus Y$ is minimal left almost split.*

Corollary 3.3.11. *Let A be a finite dimensional K-algebra with $f : M \to N$ an irreducible morphism in* mod A.

(1) *If M is indecomposable and f is not a minimal left almost morphism, then there exists an irreducible morphism $g : M \to L$, where L is indecomposable such that $\begin{bmatrix} f \\ g \end{bmatrix} : M \to N \oplus L$ is irreducible.*

(2) *If N is indecomposable and f is not a minimal right almost morphism, then there exists an irreducible morphism $g : L \to N$, where L is indecomposable, such that $\begin{bmatrix} f & g \end{bmatrix} : M \oplus L \to N$ is irreducible.*

Proof. (1) Suppose that M is indecomposable and f is not a minimal left almost morphism. Then, by Lemma 3.3.10(2), there exists a nonzero morphism $h : M \to Y$ such that $\begin{bmatrix} f \\ h \end{bmatrix} : M \to N \oplus Y$ is minimal left almost split. Since $Y \neq 0$, we can decompose h as $h = \begin{bmatrix} g \\ h' \end{bmatrix} : M \to L \oplus Y'$, where L is indecomposable. So, we have a minimal left almost split map $\begin{bmatrix} f \\ g \\ h' \end{bmatrix} : M \to N \oplus L \oplus Y'$. So, by lemma 3.3.10(2), both $g : M \to L$ and $\begin{bmatrix} f \\ g \end{bmatrix} : M \to N \oplus L$ are irreducible. The proof for (2) is dual. □

The following two results say that the components of an irreducible morphism are irreducible.

Lemma 3.3.12. *Let* $f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} : X \to Y_1 \oplus Y_2$ *be an irreducible morphism in* mod A, *with* X *indecomposable and* Y_1, Y_2 *nonzero. Then,* f_1, f_2 *are irreducible.*

Proof. By Corollary 3.3.9(2), there exists a minimal left almost split $g : X \to Y$. By Proposition 3.2.7(1), $f = hg$, where $h : Y \to Y_1 \oplus Y_2$ is a retraction. Let $p_i : Y_1 \oplus Y_2 \to Y_i$ be the canonical projection, for $i = 1, 2$. Then $f_i = p_i f = (p_i h) g$. Since p_i is a retraction, by Lemma 1.2.3(2), $p_i h$ is retraction. So, by Proposition 3.2.7(1), f_i is irreducible. □

Lemma 3.3.13. *Let* $f = \begin{bmatrix} f_1 & f_2 \end{bmatrix} : X_1 \oplus X_2 \to Y$ *be an irreducible morphism in* mod A, *with* Y *indecomposable and* X_1, X_2 *nonzero. Then,* f_1, f_2 *are irreducible.*

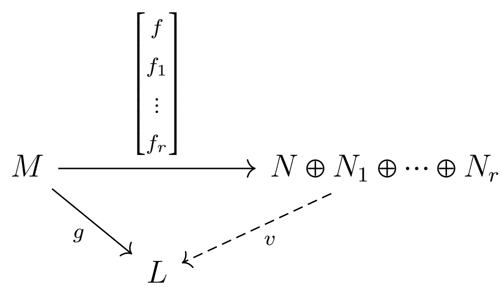
Proof. By Corollary 3.3.9(1), there exists a minimal right almost split $g : Y \to X$. By Proposition 3.2.7(2), $f = gh$, where $h : X_1 \oplus X_2 \to X_i$ is a section. Let $u_i : X_i \to X_1 \oplus X_2$ be the inclusion map, for $i = 1, 2$. Then $f_i = f u_i = g(h u_i)$. Since u_i is a section, by Lemma 1.2.3(1), $h u_i$ is a section. So, by Proposition 3.2.7(2), f_i is irreducible. □

Lemma 3.3.14. *Let* A *be a finite dimensional* K*-algebra, and let* M, N, L *be indecomposable modules in* mod A *with* $N \ncong L$.

(1) *If* $f : M \to N$ *and* $g : M \to L$ *are irreducible, then so is* $\begin{bmatrix} f \\ g \end{bmatrix} : M \to N \oplus L$.

(2) *If* $f : N \to M$ *and* $g : L \to M$ *are irreduicble morphisms, then so is* $\begin{bmatrix} f & g \end{bmatrix} : N \oplus L \to M$.

Proof. Let $f : M \to N$ and $g : M \to L$ be irreducible morphisms. If f is left minimal almost split, then $g = hf$, where $h : N \to L$. Since f is irreducible, h is a retraction. Since N and L are indecomposable, h is an isomorphism, a contradiction. Thus, f is not minimal left almost split. So, we have a commutative diagram



where the upper morphism is minimal left almost split and

$$v = \begin{bmatrix} h & h_1 & \cdots & h_r \end{bmatrix} : N \oplus N_1 \oplus \cdots \oplus N_r \to L$$

with $r \geq 1$ and the N_i are indecomposable. Since $N \ncong L$, we see that h is not an isomorphism, that is, $h \in \operatorname{rad}(N, L)$. If none of the h_i with $1 \leq i \leq r$ is an isomorphism, then

$$g = \begin{bmatrix} h & h_1 & \cdots & h_r \end{bmatrix} \begin{bmatrix} f \\ f_1 \\ \vdots \\ f_r \end{bmatrix} = hf + h_1 f_1 + \cdots + h_r f_r \notin \operatorname{rad}^2(M, L),$$

a contradiction to Lemma 3.2.5. Thus, we may assume that h_1 is an isomorphism.

Consider another commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\begin{bmatrix} f \\ f_1 \\ \vdots \\ f_r \end{bmatrix}} & N \oplus N_1 \oplus \cdots \oplus N_r \\ {\scriptstyle 1_M}\downarrow & & \downarrow{\scriptstyle w} \\ M & \xrightarrow[\begin{bmatrix} f \\ g \end{bmatrix}]{} & N \oplus L \end{array}$$

where

$$w = \begin{bmatrix} 1_N & 0 & \cdots & 0 \\ h & h_1 & \cdots & h_r \end{bmatrix} : N \oplus N_1 \oplus \cdots \oplus N_r \to N \oplus L$$

Since $h = h_1 h_1^{-1} h$, we see that

$$w \begin{bmatrix} 1_N & 0 \\ -h_1^{-1}h & h_1^{-1} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1_N & 0 & \cdots & 0 \\ h & h_1 & \cdots & h_r \end{bmatrix} \begin{bmatrix} 1_N & 0 \\ -h_1^{-1}h & h_1^{-1} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1_N & 0 \\ 0 & 1_L \end{bmatrix}.$$

That is, w is a retraction. By Proposition 3.2.7, $\begin{bmatrix} f \\ g \end{bmatrix} : M \to N \oplus L$ is irreducible. The proof for (2) is dual. □

Proposition 3.3.15. *Let A be a finite dimensional K-algebra.*

(1) *Let M be an indecomposable non-projective module in* mod A. *There exists an irreducible morphism $f : X \to M$ if and only if there exists an irreducible morphism $f' : \tau M \to X$.*

(2) *Let N be an indecomposable non-injective module in* mod A. *There exists an irreducible morphism $g : N \to Y$ if and only if there exists an irreducible morphism $g' : Y \to \tau^{-1} N$.*

Proof. (1) Assume that $f : X \to M$ is irreducible. By Lemma 3.3.10(1), there exists a morphism $h : Y \to M$ such that $\begin{bmatrix} f & h \end{bmatrix} : X \oplus Y \to M$ is right minimal almost split. Note that $\begin{bmatrix} f & h \end{bmatrix}$ is an epimorphism, because M is not projective. Put $L = \operatorname{Ker}\begin{bmatrix} f & h \end{bmatrix}$. By Proposition 3.3.4, the short exact sequence

$$0 \to L \xrightarrow{\begin{bmatrix} f' \\ h' \end{bmatrix}} X \oplus Y \xrightarrow{\begin{bmatrix} f & h \end{bmatrix}} M \to 0$$

is almost split. So, $f' : L \to X$ is irreducible and there exists an isomorphism $g : \tau M \to L$. Consequently, $f'g : \tau M \to X$ is an irreducible morphism. We can prove the necessity of (1) by a dual fashion. Moreover, since $\tau^{-1}\tau N \cong N$ and $\tau^{-1}\tau M \cong M$, the statement (2) follows from the statement (1). □

Corollary 3.3.16. *Let A be a finite dimensional K-algebra.*

(1) *Let S be a simple projective module in* mod A. *If $f : S \to M$ is irreducible, then M is projective and S is not injective.*

(2) *Let S be a simple injective module in* mod A. *If $g : M \to S$ is irreducible, then M is injective and S is not projective.*

Proof. (1) Assume that $f : S \to M$ is irreducible. By Corollary 1.9.2(2), S is not injective. Let N be an indecomposable summand of M. By lemma 3.3.13, there exist an irreduicble morphism $f' : S \to N$. If N is not projective then, by Proposition 3.3.15, there exists an irreducible morphism $\tau M \to S$, and this contradicts Corollary 1.9.2(1). Thus, N is projective. And consequently, M is projective. We can prove (2) in a dual fashion. □

Lemma 3.3.17. *Let $f : X \to Y$ be an irreducible morphism. If Y is indecomposable and non-projective, then there exists an almost split sequence*

$$0 \to \tau Y \xrightarrow{\begin{pmatrix} g \\ g' \end{pmatrix}} X \oplus X' \xrightarrow{(f,f')} Y \to 0.$$

Proof. Suppose that $f : X \to Y$ is irreducible. By Lemma 3.3.10, there exists a morphism $f' : X' \to Y$ such that $(f, f') : X \oplus X' \to Y$ is minimal right almost split, which is an epimorphism because Y is not projective. By Proposition 3.3.4, there exists an almost split sequence

$$0 \to N \xrightarrow{\begin{pmatrix} g \\ g' \end{pmatrix}} X \oplus X' \xrightarrow{(f,f')} Y \to 0.$$

So, there exist an isomorphism $h : \tau Y \to N$. □

3.4 Auslander-Reiten Quiver

If M, N are indecomposable modules in mod A, a morphism $f : M \to N$ is irreducible if and only if it belongs to $\mathrm{rad}_A(M,N) \backslash \mathrm{rad}_A^2(M,N)$ by lemma 3.2.5. Thus, the quotient vector space $\mathrm{rad}_A(M,N)/\mathrm{rad}_A^2(M,N)$ can be considered as a measure for the set of irreducible morphisms from M to N.

Definition 3.4.1. *Let M and N be indecomposable in* mod A. *The* space of irreducible morphisms *is the K-vector space*

$$\mathrm{Irr}_A(M,N) = \frac{\mathrm{rad}_A(M,N)}{\mathrm{rad}_A^2(M,N).}$$

Definition 3.4.2. *Let A be a finite dimensional K-algebra, where K is an algebraically closed field. The* Auslander-Reiten quiver $\Gamma(\text{mod } A)$ *of* mod A *is a translation quiver defined as follows.*

(1) *The vertices of* $\Gamma(\text{mod } A)$ *are the isomorphism classes $[M]$, where M ranges over the indecomposable modules in* mod A;

(2) *Given two vertices $[M]$ and $[N]$, the number of arrows from $[M]$ to $[N]$ is equal to the K-dimension of* $\mathrm{Irr}(M,N)$.

(3) *The translation* τ *is defined so that* $\tau[M] = [\mathrm{DTr}M]$, *for any indecomposable non-projective module* M *in* mod A.

In the sequel, for the sake of simplicity, we shall identify an indecomposable module M in mod A with the corresponding vertex $[M]$ in $\Gamma(\text{mod } A)$.

Lemma 3.4.3. ([2], IV.1.3) *Let* A *be a finite dimensional* K*-algebra with* $M = \oplus_{i=1}^{t} M_i^{m_i}$ *a module in* mod A, *where* $m_i > 0$ *and the* M_i *are indecomposable and pairwise non-isomorphic.*

(1) *If* $f : L \to M$ *is a minimal left almost morphism, then* $\dim_K \mathrm{Irr}_A(L, M_i) = m_i$, *for* $i = 1, \dots, t$

(2) *If* $g : M \to N$ *is a minimal right almost morphism, then* $\dim_K \mathrm{Irr}_A(M_i, N) = m_i$, *for* $i = 1, \dots, t$

Lemma 3.4.4. ([2], IV.1.5) *Let* A *be a finite dimensional* K*-algebra with* M, N *indecomposable modules in* mod A. *If* $\dim_K \mathrm{Hom}_A(M, N) \leq 1$, *then there exists no irreducible morphism* $f : M \to N \oplus N$ *or* $g : M \oplus M \to N$.

Lemma 3.4.5. *Let* A *be a finite dimensional* K*-algebra with* $M \to N$ *an arrow in* $\Gamma(\text{mod } A)$. *The following statements hold.*

(1) $M \ncong N$.

(2) *If* N *is not projective, then there exists an arrow* $\tau N \to M$.

(3) *If* M *is not injective, then there exists an arrow* $N \to \tau^{-1} M$.

Proof. By definition of Auslander Reiten quiver, there exists an irreducible morphism $f : M \to N$. In particular, f is not an isomorphism.

(1) By Lemma 3.2.2, f is an epimorphism or a monomorphism. Suppose that $M \cong N$. In particular, $\dim(M) = \dim(N)$. By Lemma 1.3.2, f is an isomorphism, a contradiction.

(2) Assume that N is not projective. By Lemma 3.3.15(1), there exists an irreduicble morphism $g : \tau N \to M$. Hence, we have an arrow $\tau N \to M$ in $\Gamma(\text{mod } A)$.

(3) Assume that M is not injective. By Lemma 3.3.15(2), there exists an irreduicble morphism $g : N \to \tau^{-1} M$. Hence, we have an arrow $N \to \tau^{-1} M$ in $\Gamma(\text{mod } A)$. □

CHAPTER 4

Radical Nilpotence of the Module Category over a Nakayama Algebra

Now that we have almost all prerequisites, we demonstrate in this chapter the main result of our research. We start by describing Nakayama algebras and their properties which are important for our final result.

4.1 Nakayama Algebras

We let A denote a finite dimensional K-algebra and all A-modules are, unless otherwise specified, finite dimensional left A-modules.

Definition 4.1.1. *For a module M in* mod A*, since* rad A *is nilpotent, there exists a minimal positive integer m such that* $\operatorname{rad}^m M = 0$*. The integer m is called the Loewy length of M and is denoted by $\ell\ell(M)$. In this case, we obtain a decreasing chain*

$$M \supset \operatorname{rad} M \supset \cdots \supset \operatorname{rad}^{m-1} M \supset \operatorname{rad}^m M = 0$$

of submodules of M, called the r*adical series of M, where* $\text{rad}^i M = (\text{rad}^i A)M$*, for all* $i = 1, \ldots, m$*.*

It is clear that $\ell\ell(M) \leq \dim_K M$, for every module M in mod A.

Definition 4.1.2. *A non-zero module M in* mod A *is called uniserial if it admits a unique composition series.*

Clearly, every simple module S in mod A is uniserial with a unique composition series $0 \subset S$. Note, however, that there exist uniserial modules which are not simple. Also, if M is uniserial, every submodule and quotient of M is uniserial.

Lemma 4.1.3. ([3], V. 2.2) *A module M in* mod A *is uniserial if and only if its radical series is a composition series.*

Lemma 4.1.4. *Let A be a finite dimensional K-algebra with M a uniserial module in* mod A*. The top and the socle of M are simple. In particular, M is indecomposable.*

Proof. Since M is uniserial, let $0 \subset A_1 \cdots \subset A_n = M$ be the Jordan Hölder filtration of M. Now, let N be a simple submodule of M. Then, consider the Jordan Hölder filtration of M/N, say $0 \subset M_1/N \cdots \subset M/N$. This would give us a Jordan Hölder filtration of M as $0 \subset N \subset M_1 \cdots \subset M$. Since M has a unique decomposition, $N = A_1$. Therefore, M has a unique simple module and hence, soc$M$ is simple.

Moreover, let $N = \text{rad}M$ be a unique maximal submodule of M. Then we consider the Jordan Hölder filtration of M/N, say $0 \subset M_1/N \cdots \subset M/N$. Since M is uniserial, the radical series is a composition series. Then, each composition factor is simple. Hence, top$M$ is simple. Also, by Lemma 1.5.12, M is indecomposable.

□

Definition 4.1.5. *A finite dimensional K-algebra A is called a* Nakayama algebra *if all indecomposable projective modules and all indecomposable injective modules in* mod A *are uniserial.*

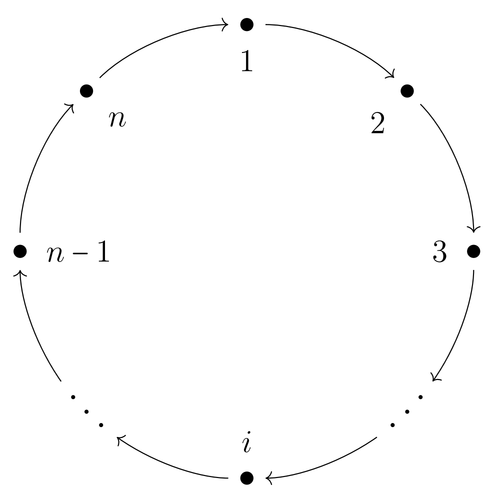
Proposition 4.1.6. ([8], I. 10.6) *Let A be a finite dimensional K-algebra. If A is a Nakayama algebra, then every indecomposable module in* mod A *is uniserial.*

4.2 Nakayama Algebras Given By a Bound Quiver

In this section, we study Nakayama algebras given by a bound quiver and their indecomposable projective modules and indecomposable injective modules. For this purpose, for any integer $n \geq 1$, we consider a quiver

$$\vec{\mathbb{A}}_n : 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow \cdots \longrightarrow n-1 \longrightarrow n$$

and a quiver $\tilde{\mathbb{A}}_n$ as follows :



Proposition 4.2.1. ([8], I. 10.3) *Let $A = kQ/I$, where Q is a finite connected quiver and*

I is an admissible ideal of kQ. Then A is a Nakayama algebra if and only if $Q = \vec{\mathbb{A}}_n$ or $\tilde{\mathbb{A}}_n$, for some $n \geq 1$. In this case, I is a monomial ideal of kQ.

Proposition 4.2.2. *Let $A = KQ/I$ be a connected Nakayama algebra, and let p be a maximal path in $Q_I(a,-)$, for some $a \in Q_0$.*

(1) *If $p = \varepsilon_a$, then $P_a = S_a$.*

(2) *If p is of the form $a = a_0 \xrightarrow{\alpha_1} a_1 \to \cdots \to a_{r-1} \xrightarrow{\alpha_r} a_r$, where $r \geq 1$ and $\alpha_1, \ldots, \alpha_r$ are arrows, then P_a has a K-basis $\{e_a, \bar{\alpha}_1, \overline{\alpha_2\alpha_1}, \cdots, \overline{\alpha_r \cdots \alpha_1}\}$.*

Proof. By Proposition 2.3.4, P_a has as a K-basis the set of classes modulo I of paths in $Q_I(a,-)$. Since $Q = \vec{\mathbb{A}}_n$ or $\tilde{\mathbb{A}}_n$ by Proposition 4.2.1, there exists at most one arrow in Q starting in any given vertex.

(1) By Proposition 2.3.4 (3), $S_a = P_a/\text{rad}P_a$. If $p = \varepsilon_a$, then $Q_I(a,-) = \{\varepsilon_a\}$. So by proposition 2.3.4 (2), $\text{rad}P_a = 0$, and hence, $P_a = S_a$.

(2) Let p be of the form $a = a_0 \xrightarrow{\alpha_1} a_1 \to \cdots \xrightarrow{\alpha_r} a_r$, where $r \geq 1$. It amounts to prove that $Q_I(a,-) = \{\varepsilon_a, \alpha_1, \ldots, \alpha_r \cdots \alpha_1\} := \mathscr{P}$.
Indeed, since $\alpha_r \cdots \alpha_1$ is a non-zero path, ε_a, α_1, $\ldots$, $\alpha_r \cdots \alpha_1$ are all non-zero paths. Thus, $\mathscr{P} \subseteq Q_I(a,-)$.

Conversely, let q be a path of length s in $Q_I(a,-)$. We want to use induction on s to show that $q \in \mathscr{P}$. If $s = 0$, then $q = \varepsilon_a \in \mathscr{P}$. Assume that $s = 1$. That is, $q = \beta_1$, an arrow in $Q_I(a,-)$. Since α_1 is an arrow starting in a, by the above stated property, $\beta_1 = \alpha_1$. In particular, $q \in \{\varepsilon_a, \alpha_1, \alpha_2\alpha_1, \ldots, \alpha_r \cdots \alpha_1\}$. Suppose that $s > 1$ and any path of length $s-1$ in $Q_I(a,-)$ lies in $\mathscr{P}$. Write $q = \beta_s\beta_{s-1}\ldots\beta_1$, where the β_i are arrows. Then, $\beta_{s-1}\ldots\beta_1$ is a path of $s-1$ in $Q_I(a,-)$. Since $\alpha_{s-1}\ldots\alpha_1$ is the only path of $s-1$ in $\mathscr{P}$, by the induction hypothesis, $\beta_{s-1}\ldots\beta_1 = \alpha_{s-1}\ldots\alpha_1$. In particular, $s-1 \leq r$ and β_s is an arrow starting in a_{s-1}. Again by the above stated property, $\beta_s = \alpha_s$. That is, $q = \alpha_s \cdots \alpha_1 \in \mathscr{P}$. By induction, $Q_I(a,-) \subseteq \mathscr{P}$. Thus, $Q_I(a,-) = \{\varepsilon_a, \alpha_1, \ldots, \alpha_r \cdots \alpha_1\}$. □

Proposition 4.2.3. *Let $A = KQ/I$ be a connected Nakayama algebra. Let q be a maximal path in $Q_I(-,a)$, for some $a \in Q_0$.*

(1) *If $q = \varepsilon_a$, then $I_a = K < e_a^* > \cong S_a$.*

(2) *If q is of the form $b_s \xrightarrow{\beta_s} b_{s-1} \to \cdots \to b_1 \xrightarrow{\beta_1} b_0 = a$, where $s \geq 1$ and $\beta_1, \ldots, \beta_s$ are arrows, then I_a has as a K-basis $\{e_a^*, \bar{\beta}_1^*, \ldots, \overline{\beta_1 \cdots \beta_s}^*\}$.*

Proof. By Proposition 2.3.5, I_a has as a K-basis $\{\bar{\rho}^* \mid \rho \in Q_I(-,a)\}$, the dual basis of the K-basis $\{\bar{\rho} \mid \rho \in Q_I(-,a)\}$ of $e_a A$. Since $Q = \vec{\mathbb{A}}_n$ or $\tilde{\mathbb{A}}_n$ by Proposition 4.2.1, there exists at most one arrow in Q ending in any given vertex.

(1) If $q = \varepsilon_a$, then $Q_I(-,a) = \{\varepsilon_a\}$. Thus, $I_a = K < e_a^* >$. By Lemma 2.3.7, $I_a \cong S_a$.

(2) Let q be of the form $b_s \xrightarrow{\beta_s} b_{s-1} \to \cdots \to b_1 \xrightarrow{\beta_1} b_0 = a$, where $s \geq 1$. It amounts to show that $Q_I(-,a) = \{\varepsilon_a, \beta_1, \beta_1\beta_2, \ldots, \beta_1 \cdots \beta_1\} := \mathscr{P}$. Since $\beta_1 \cdots \beta_s$ is a non-zero path, ε_a, β_1, $\ldots$, $\beta_1 \cdots \beta_s$ are all non-zero paths. Thus, $\mathscr{P} \subseteq Q_I(-,a)$.

Conversely, let p be a path of length r in $Q_I(-,a)$. We want to use induction on r to show that $p \in \mathscr{P}$. If $r = 0$, then $p = \varepsilon_a \in \mathscr{P}$. If $r = 1$, then $p = \eta_1$, an arrow in $Q_I(-,a)$. Since β_1 is an arrow ending in a, by the above stated property, $\eta_1 = \beta_1 \in \mathscr{P}$. Suppose that $r > 1$ and any path of length $r-1$ in $Q_I(-,a)$ lies in $\mathscr{P}$. Write $p = \eta_1\eta_2 \ldots \eta_r$, where the η_i are arrows. Since $\eta_1 \ldots \eta_{r-1}$ is a path of $r-1$ in $Q_I(-,a)$ and $\beta_1 \ldots \beta_{r-1}$ is the only path of $r-1$ in $\mathscr{P}$, by the induction hypothesis, $\eta_1 \ldots \eta_{r-1} = \beta_1 \ldots \beta_{r-1}$. In particular, $r-1 \leq s$ and η_r is an arrow starting in b_{r-1}. By the above stated property, $\eta_r = \beta_r$. Thus, $p = \eta_1 \ldots \eta_{r-1}\eta_r \in \mathscr{P}$. By induction, $Q_I(-,a) \subseteq \mathscr{P}$. Thus, $Q_I(-,a) = \{\varepsilon_a, \beta_1, \ldots, \beta_1 \cdots \beta_s\}$. □

4.3 Main Statement

Let A be a connected Nakayama algebra given by a bound quiver. Our main result is to state the necessary and sufficient conditions for $\text{rad}^3(\text{mod}A)$ to vanish. We start with the necessary conditions.

Lemma 4.3.1. *Let* $A = KQ/I$ *be a connected Nakayama algebra. If* $\mathrm{rad}^3(\mathrm{mod}A) = 0$*, then one of the following cases occurs*:

(1) $Q = \vec{\mathbb{A}}_n$ *and* $I = 0$*, where* $1 \le n \le 3$.

(2) $Q = \vec{\mathbb{A}}_n$ *and* I *is generated by all the paths of length two in* Q*, where* $n \ge 3$.

(3) $Q = \tilde{\mathbb{A}}_n$ *and* I *is generated by all the paths of length two in* Q*, where* $n \ge 1$.

Proof. By Proposition 4.2.1, $Q = \tilde{\mathbb{A}}_n$ or $\vec{\mathbb{A}}_n$, for some $n \ge 1$. Suppose that $\mathrm{rad}^3(\mathrm{mod}A) = 0$.

If $\mathrm{rad}(\mathrm{mod}A) = 0$, then A is simple, that is, $Q = \vec{\mathbb{A}}_1$ with $I = 0$. Thus, the case (1) occurs.

If $\mathrm{rad}(\mathrm{mod}A) \ne 0$ but $\mathrm{rad}^2(\mathrm{mod}A) = 0$, then $Q = \vec{\mathbb{A}}_2$ and $I = 0$; see ([6], 5.1.7). Thus, the case (1) occurs.

Suppose that $\mathrm{rad}^2(\mathrm{mod}A) \ne 0$. Then $Q = \vec{\mathbb{A}}_n$ with $n \ge 3$; see ([6], 5.1.7) or $Q = \tilde{\mathbb{A}}_n$ with $n \ge 1$. By Proposition 1.9.6, $\mathrm{rad}^3(A) = 0$.

Assume that $\mathrm{rad}^2(A) = 0$. By Proposition 2.2.8, any path of length two lies in I. So, I is generated by all the paths of length two. Thus, the case (2) or (3) occurs.

Assume now that $\mathrm{rad}^2(A) \ne 0$. Then, Q contains a path $a \xrightarrow{\alpha} b \xrightarrow{\beta} c$ such that $\beta\alpha \notin I$. By lemma 2.2.9, we have a sequence of radical morphisms

$$P_c \xrightarrow{P[\bar{\beta}]} P_b \xrightarrow{P[\bar{\alpha}]} P_a$$

between indecomposable projective modules in mod A. Since

$$(P[\bar{\alpha}] \circ P[\bar{\beta}])(e_c) = P[\bar{\alpha}](P[\bar{\beta}](e_c)) = e_c\bar{\beta}\bar{\alpha} = \bar{\beta}\bar{\alpha} \ne 0,$$

we see that $P[\bar{\alpha}] \circ P[\bar{\beta}] \ne 0$. Since $\mathrm{rad}^3(\mathrm{mod}A) = 0$, neither $P[\bar{\beta}]$ nor $P[\bar{\alpha}]$ lies in $\mathrm{rad}^2(\mathrm{mod}A)$. By Lemma 3.2.5, $P[\bar{\alpha}]$ and $P[\bar{\beta}]$ are irreducible. By Lemma 3.2.2, $P[\bar{\alpha}]$ and $P[\bar{\beta}]$ are monomorphisms, and so is $P[\bar{\alpha}] \circ P[\bar{\beta}]$. In particular, by Lemma 1.3.2, the vertices a, b, c are pairwise distinct.

Assume that Q contains an arrow $\gamma : c \to d$. Then, $\bar{\gamma} \in P_c$. As $P[\bar{\alpha}] \circ P[\bar{\beta}]$ is a monomorphism, $(P[\bar{\alpha}] \circ P[\bar{\beta}])(\bar{\gamma}) \neq 0$, that is, $\bar{\gamma}\bar{\beta}\bar{\alpha} \neq 0$. By Proposition 2.2.8, $\text{rad}^3(A) \neq 0$, a contradiction. Thus, Q contains no arrow starting in c.

Dually, by Lemma 2.3.8, we have a sequence of radical morphisms

$$I_c \xrightarrow{I[\bar{\beta}]} I_b \xrightarrow{I[\bar{\alpha}]} I_a$$

between indecomposable injective modules in mod$A$. Since $\beta\alpha \notin I$, by Proposition 4.2.3, $e_c^*, \bar{\beta}^*, \overline{\beta\alpha}^*$ lie in a K-basis of I_c. Moreover, by Lemma 2.3.8, we obtain

$$(I[\bar{\alpha}] \circ I[\bar{\beta}])(e_c^*) = I[\bar{\alpha}](I[\bar{\beta}](e_c^*)) = I[\bar{\alpha}](0) = 0;$$

$$(I[\bar{\alpha}] \circ I[\bar{\beta}])(\bar{\beta}^*) = I[\bar{\alpha}](I[\bar{\beta}](\bar{\beta}^*)) = I[\bar{\alpha}](e_b^*) = 0;$$

$$(I[\bar{\alpha}] \circ I[\bar{\beta}])(\overline{\beta\alpha}^*) = I[\bar{\alpha}](I[\bar{\beta}](\overline{\beta\alpha}^*) = I[\bar{\alpha}](\bar{\alpha}^*) = e_a^*.$$

In particular, $I[\bar{\alpha}] \circ I[\bar{\beta}] \neq 0$. Since $\text{rad}^3(\text{mod}A) = 0$, neither $I[\bar{\beta}]$ nor $I[\bar{\alpha}]$ lies in $\text{rad}^2(\text{mod}A)$. By Lemma 3.2.5, $I[\bar{\beta}]$ and $I[\bar{\alpha}]$ are irreducible. Since I_a, I_b are injective, by Lemma 3.2.3(2), $I[\bar{\beta}]$ and $I[\bar{\alpha}]$ are epimorphisms, and so is $I[\bar{\alpha}] \circ I[\bar{\beta}]$.

Assume that Q contains an arrow $\delta : d \to a$ in Q. Then, $\bar{\delta}^* \in I_a$. Since $\text{rad}^3(A) = 0$, we have $\overline{\beta\alpha\delta} = 0$. Therefore, $\beta\alpha$ is a maximal path in $Q_I(-, c)$. By Lemma 4.2.3, I_c has a K-basis $\{e_c^*, \bar{\beta}^*, \overline{\beta\alpha}^*\}$. Since $I[\bar{\alpha}] \circ I[\bar{\beta}]$ is an epimorphism,

$$\bar{\delta}^* = (I[\bar{\alpha}] \circ I[\bar{\beta}])(\lambda_0 \cdot e_c^* + \lambda_1 \cdot \bar{\beta}^* + \lambda_2 \cdot \overline{\beta\alpha}^*),$$

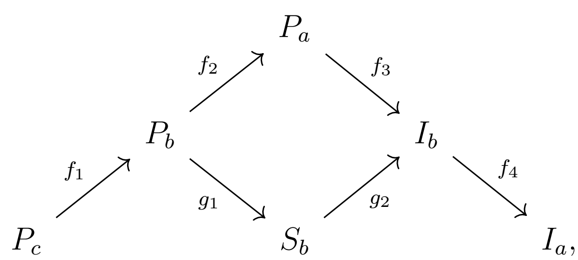
where $\lambda_0, \lambda_1, \lambda_2 \in K$. In view of the calculation stated above, we see that $\lambda_2 e_a^* = \bar{\delta}^*$, a contradiction. Thus, Q contains no arrow ending in a. Being $\tilde{\mathbb{A}}_n$ or $\vec{\mathbb{A}}_n$, the quiver Q consists of the path $a \xrightarrow{\alpha} b \xrightarrow{\beta} c$. Since $\beta\alpha \notin I$, we see that $I = 0$. That is, Case (1) occurs. □

Next, we shall show that the conditions stated in Lemma 4.3.1 are also sufficient for $\text{rad}^3(\text{mod } A)$ to vanish.

Lemma 4.3.2. *Let* $A = K\vec{\mathbb{A}}_3$. *Then* $\mathrm{rad}^3(\mathrm{mod}A) = 0$.

Proof. We may assume that $\vec{\mathbb{A}}_3$ is the quiver $a \xrightarrow{\alpha} b \xrightarrow{\beta} c$. As seen in Example 2.4.8, we obtain $P_a \cong I_c$; $P_b \cong \mathrm{rad}P_a$; $P_c \cong \mathrm{rad}P_b \cong S_c$; $I_a \cong S_a$, $I_b/S_b \cong I_a$ and $I_c/S_c \cong I_b$.

First, we claim that the Auslander-Reiten quiver $\Gamma(\mathrm{mod}\ A)$ of mod A is as follows:



where f_1, f_2 are minimal right almost split monomorphisms, and f_3, f_4 are minimal left almost split epimorphisms.

Since P_c is simple projective, by Proposition 3.1.8, the minimal right almost split morphism for P_c is the zero morphism $0 \to P_c$. Because of Proposition 3.3.10(2), there exists no irreducible map $f : N \to P_c$. So, there exists no arrow in $\Gamma(\mathrm{mod}A)$ ending with P_c.

Now because $P_c \cong \mathrm{rad}P_b$, by Lemma 3.1.8, there exists a minimal right almost split map $f_1 : P_c \to P_b$. Since P_c is indecomposable, $\Gamma(\mathrm{mod}\,A)$ has only one arrow $P_c \to P_b$ ending with P_b.

Suppose that f_1 is not a minimal left almost split morphism. By Corollary 3.3.11(1), there exists an irreducible map $(f_1, f)^T : P_c \to P_b \oplus M$, where M is indecomposable. In particular, $f : P_c \to M$ is irreducible. By Corollary 3.3.16, M is projective. By Lemma 3.3.10, P_c is isomorphic to a direct summand of $\mathrm{rad}M$. Since $\mathrm{rad}P_a \cong P_b$ and $\mathrm{rad}P_c = 0$, we see that $M = P_b$. On the other hand, by Lemma 1.3.5, $\mathrm{Hom}_A(P_c, P_b) \cong e_cAe_b = k < \bar{\beta} >$, which is of dimension one, a contradiction to Lemma 3.4.4. So f_1 is the minimal left almost split morphism for P_c. On the other hand, since $\mathrm{top}P_b \cong S_b$ and $P_c \cong \mathrm{rad}P_b$, we have a short exact sequence

$$(1) \qquad 0 \longrightarrow P_c \xrightarrow{f_1} P_b \xrightarrow{g_1} S_b \longrightarrow 0.$$

Since f_1 is a minimal left almost split morphism, by proposition 3.3.4, this is an almost split sequence. In particular, $g_1 : P_b \to S_b$ is a minimal right almost split map. So, Γ(mod A) has only one arrow $P_c \to P_b$ starting with P_c and only one arrow $P_b \to S_b$ ending with S_b.

Furthermore, since rad$P_a \cong P_b$, there exists a minimal right almost split monomorphism $f_2 : P_b \to P_a$. Since P_b is indecomposable, Γ(mod A) has only one arrow $P_b \to P_a$ ending with P_a.

Since $P_a \ncong S_a$, by lemma 3.3.14(1), the map

$$\begin{pmatrix} f_2 \\ g_1 \end{pmatrix} : P_b \to P_a \oplus S_a$$

is irreducible. Suppose that it is not minimal left almost split. By Corollary 3.3.11, there is an irreducible morphism $g : P_b \to X$ is irreducible with X indecomposable such that

$$\begin{pmatrix} f_2 \\ g_1 \\ g \end{pmatrix} : P_b \to P_a \oplus S_b \oplus X$$

is irreducible. If X is projective, then P_b is a direct summand of rad$X$. In particular, $X \ncong P_b$. And since P_c is simple projective, by Proposition 3.1.8, $X \ncong P_c$. Therefore, $X \cong P_a$. This yields an irreducible morphism

$$\begin{pmatrix} f_2 \\ g \end{pmatrix} : P_b \to P_a \oplus P_a.$$

However, by Lemma 1.3.5, $\mathrm{Hom}_A(P_b, P_a) \cong e_b A e_a = k < \bar{\alpha} >$, which is of dimension one, a contradiction to Lemma 3.4.4. So, X is not projective. By Proposition 3.4.5(2), there is an arrow $\tau X \to P_b$. Since $P_c \to P_b$ is the only arrow ending with P_b, we have $\tau X \cong P_c$, and consequence, $X \cong \tau^{-} P_c = S_b$. So, we have an irreducible morphism

$$\begin{pmatrix} g_1 \\ g \end{pmatrix} : P_b \to S_b \oplus S_b.$$

However, by Lemma 1.3.5, $\mathrm{Hom}_A(P_b, S_b) \cong e_b S_b = S_b$, which is of dimension one, a contradiction to Lemma 3.4.4. So

$$\begin{pmatrix} f_2 \\ g_1 \end{pmatrix} : P_b \to P_a \oplus S_b$$

is a minimal left almost split monomorphism. By proposition 3.3.4, there is an almost split sequence

$$(2) \qquad 0 \longrightarrow P_b \xrightarrow{\begin{pmatrix} f_2 \\ g_1 \end{pmatrix}} P_a \oplus S_b \xrightarrow{(f_3, g_2)} Y \longrightarrow 0.$$

In particular, by lemma 3.3.13, $f_3 : P_a \to Y$ is an irreducible morphism. Now $P_a \cong I_c$ and $I_c / S_c \cong I_b$. So, by Lemma 3.1.9, there is a minimal left almost split epimorphism $h : P_a \to I_b$. Since I_b is indecomposable, $Y \cong I_b$. And we may assume that $Y = I_b$ and $f_3 = h$. In particular, So, $P_a \to I_b$ is the only arrow in $\Gamma(\mathrm{mod}\ A)$ starting with P_a.

Now, consider the irreducible map $g_2 : S_b \to I_b$. Since $S_b = \mathrm{soc} I_b$ with $I_b / S_b \cong I_a$, we have a short exact sequence

$$(3) \qquad 0 \longrightarrow S_b \xrightarrow{g_2} I_b \xrightarrow{f_4} I_a \longrightarrow 0,$$

where $f_4 : I_b \to I_a$ is the canonical projection. By lemma 3.1.9, f_4 is a minimal left almost epimorphism. In particular, f_4 is an irreducible morphism. Since g_2 is also irreducible, by Lemma 3.3.4, this is an almost split sequence. Finally, since I_a is a simple injective module, by lemma 3.1.7, there is no arrow starting with I_a. This establishes our claim.

Now we show that any path of three irreducible morphisms between indecomposable modules stated the above Auslander-Reiten quiver has a zero composite. Indeed, in view of the almost split sequence (2), we have $f_3 f_2 = -g_2 g_1 = 0$. Then, $f_3 f_2 f_1 = -g_2 g_1 f_1 = 0$.

Since $f_3 f_2 = -g_2 g_1 = 0$, and in the view of the almost split sequence (3), we have $f_4 f_3 f_2 = -f_4 g_2 g_1 = 0$. By proposition 3.2.10, we have $\mathrm{rad}^3(\mathrm{mod} A) = 0$. □

In the following lemmas, we check the relations between dimension of any indecomposable modules with respect to the nilpotency of radical square in Nakayama algebra.

Lemma 4.3.3. *Let $A = KQ/I$ be a Nakayama algebra with* $\mathrm{rad}^2 A = 0$. *Consider an indecomposable module M in* mod A.

(1) *The dimension of M is equal to one or two.*

(2) *If* $\dim_K(M) = 2$, *then M is projective and injective.*

Proof. By Proposition 4.1.6, M is uniserial. Since $\mathrm{rad}^2 M = 0$, we see that $\ell\ell(M) \leq 2$.

If $\ell\ell(M) = 1$, by Lemma 4.1.3, we have a radical series $M \supset 0$, which is the composition series for M. Therefore, M is simple, and $\dim_K(M) = 1$.

If $\ell\ell(M) = 2$, then we have a radical series $M \supset \mathrm{rad} M \supset 0$, which is a composition series of M. In particular, $M/\mathrm{rad} M$ and $\mathrm{rad} M$ are simple.
Therefore, $\dim_K(M) = \dim_K(\mathrm{rad}(M)) + \dim_K(M/\mathrm{rad}(M)) = 2$. This proves Statement (1).

Suppose that $\dim_K(M) = 2$. Let $f : P \to M$ be a projective cover of M. Then, by Theorem 1.7.3, $P/\mathrm{rad} P \cong M/\mathrm{rad} M$. Being uniserial, by Lemma 4.1.3, M has a simple top, so does P. By Lemma 1.5.12(1), P is indecomposable. By Statement (1), $\dim_K(P) \leq 2$. Since f is an epimorphism, $\dim(P) \geq \dim(M) = 2$. Therefore, $\dim_K(P) = 2 = \dim(M)$. By Lemma 1.3.2, f is an isomorphism. So M is projective. Now, let $g : M \to I$ be an injective envelope of M.
Then, by theorem 1.7.5, $\mathrm{soc}(M) \cong \mathrm{soc}(I)$. Since M is uniserial, by Lemma 4.1.3, M has a simple socle, and so does I. By Lemma 1.5.12(2), I is indecomposable. By Statement (1), $\dim(I) \leq 2$. Since g is a monomorphism, $\dim(I) \geq \dim(M) = 2$. So, $\dim(I) = 2 = \dim(M)$.

By Lemma 1.3.2, g is isomorphism. Hence, M is injective. Therefore, M is projective and injective. □

Lemma 4.3.4. *Let* $A = KQ/I$ *be a Nakayama algebra with* $\operatorname{rad}^2 A = 0$*. Consider a simple module* S *in* $\operatorname{mod} A$*.*

(1) *If* S *is not projective with a projective cover* P*, then* $\dim(P) = 2$*.*

(2) *If* S *is not injective with an injective envelope* I*, then* $\dim(I) = 2$*.*

Proof. By Lemma 4.3.3(1), $\dim(S) = 1$.

(1) Let S be non-projective with a projective cover $g : P \to S$. In particular, S is the simple top of P. By Lemma 1.5.12(1), P is indecomposable. By Lemma 4.3.3, $\dim(P) \leq 2$. If $\dim(P) = 1 = \dim(S)$ then, by Lemma 1.3.2, g is an isomorphism, and then, S is projective, a contradiction. So, $\dim(P) = 2$.

(2) Let S be non-injective with an injective envelope $f : S \to I$. In particular, S is the simple socle of I. By Lemma 1.5.12(2), I is indecomposable. By Lemma 4.3.3, $\dim(I) \leq 2$. Since f is monomorphism, $\dim(I) \geq \dim(S) = 1$. If $\dim(I) = 1$ then, by Lemma 1.3.2, f is an isomorphism. Then, S is injective, a contradiction. So, $\dim(I) = 2$. □

In the following, there are a few lemmas that help us to prove the sufficiency part of our main result.

Lemma 4.3.5. *Let* $A = KQ/I$ *be a Nakayama algebra with* $\operatorname{rad}^2 A = 0$*. Consider a simple module* S *in* $\operatorname{mod} A$*.*

(1) *If* S *is not projective with a projective cover* $g : P \to S$*, then there exists an almost split sequence*

$$0 \longrightarrow \tau S \xrightarrow{f} P \xrightarrow{g} S \longrightarrow 0,$$

where f *is minimal right almost split and* g *is minimal left almost split.*

(2) *If S is not injective with an injective envelope $f : S \to I$, then there exists an almost split sequence*

$$0 \longrightarrow S \xrightarrow{f} I \xrightarrow{g} \tau^{-}S \longrightarrow 0,$$

where f is minimal right almost split and g is minimal left almost split.

Proof. (1) Assume that S is not projective with a projective cover $g : P \to S$. Then, we have a non-split short exact sequence

$$0 \longrightarrow \mathrm{rad}(P) \xrightarrow{f} P \xrightarrow{g} S \longrightarrow 0.$$

Since P is indecomposable projective. By Lemma 3.1.8, f is minimal right almost split. In particular, f is irreducible. Moreover, by Lemma 4.3.4, $\dim(P) = 2$. So, $\dim(\mathrm{rad}(P)) = 1$. Hence, $\mathrm{rad}(P)$ is a simple submodule of P. Being uniserial, by lemma 4.1.4, P has a simple socle. Thus, $\mathrm{rad}(P) = \mathrm{soc}(P)$, and hence, S is isomorphic to the socle-factor of P. Moreover, by Lemma 4.3.3, P is injective. Therefore, by 3.1.9, g is minimal left almost split. In particular, g is also irreducible. By Lemma 3.3.4, the above sequence is an almost split sequence.

(2) Assume that S is non-injective with an injective envelope $f : S \to I$. By lemma 2.3.7, $S \cong \mathrm{soc}(I)$. Thus, we have a non-split short exact sequence

$$0 \longrightarrow S \xrightarrow{f} I \xrightarrow{g} I/S \longrightarrow 0.$$

Since I is indecomposable injective, by Lemma 3.1.9, g is minimal left almost split. In particular, g is irreducible. Moreover, by Lemma 4.3.4, $\dim(I) = 2$. So, $\dim(I/S) = 1$, and thus, I/S is simple. As we know, I/S is simple if and only if $S \subseteq I$ is a maximal submodule. So, S is a maximal submodule of I. Hence, $\mathrm{rad}(I) \subseteq S$. On the other hand, being uniserial, I has a simple top. By lemma 1.5.7, $\mathrm{rad}(I)$ is also a maximal submodule of I. This gives rise to $S = \mathrm{rad}(I)$. Moreover, by Lemma 4.3.3, I is projective. By Lemma 3.1.8, f is minimal right almost split. In particular, f is also irreducible. By Lemma 3.3.4, the above sequence is an almost split sequence. □

Lemma 4.3.6. *Let* $A = KQ/I$ *be a Nakayama algebra with* $\operatorname{rad}^2 A = 0$*. Consider a sequence* $M \xrightarrow{f} N \xrightarrow{g} L$ *of two irreducible maps between indecomposable modules in* mod A*. If* f *is a monomorphism or* g *is an epimorphism, then,* $gf = 0$*.*

Proof. Assume that g is an epimorphism. Then, $1 \leq \dim(L) < \dim(N)$. By lemma 4.3.3, $\dim(N) \leq 2$. Therefore, $\dim(N) = 2$ and L is simple. By lemma 3.3.2(1), there exists a non-split short exact sequence

$$0 \longrightarrow U \xrightarrow{h} N \xrightarrow{g} L \longrightarrow 0,$$

where $U = \operatorname{Ker}(g)$. By corollary 3.3.16(2), L is not projective, and by lemma 4.3.5(1), there exists an almost split sequence

$$0 \longrightarrow \tau L \xrightarrow{u} P \xrightarrow{v} L \longrightarrow 0,$$

where u is minimal right almost split, v is minimal left almost split, and $\dim(P) = 2$. Since g is not a retraction and v is right almost split, $g = vv'$ for some map $v' : N \to P$. Then, $vv'h = gh = 0$. Since $u = \operatorname{Ker}(v)$, there exists a map $u' : U \to \tau L$ such that $v'h = uu'$. Therefore, we have a commutative diagram as follows :

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U & \xrightarrow{h} & N & \xrightarrow{g} & L & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle u'} & & \downarrow{\scriptstyle v'} & & \| & & \\ 0 & \longrightarrow & \tau L & \xrightarrow{u} & P & \xrightarrow{v} & L & \longrightarrow & 0. \end{array}$$

Since g is irreducible, v' is a section. In particular, v' is a monomorphism. Since $\dim(P) = 2 = \dim(N)$, by Lemma 1.3.2, v' is an isomorphism. By lemma 1.8.3, we have two exact sequences $0 \to \operatorname{Ker}(u') \to \operatorname{Ker}(v')$ and $\operatorname{Ker}(\mathrm{id}_L) \to \operatorname{Coker}(u') \to \operatorname{Coker}(v')$. Since v' is an isomorphism, $\operatorname{Ker}(v') = 0$ and $\operatorname{Coker}(v') = 0$. Similarly, $\operatorname{Ker}(\mathrm{id}_L) = 0$ and $\operatorname{Coker}(\mathrm{id}_L) = 0$. This yields two exact sequences $0 \to \operatorname{Ker}(u') \to 0$ and $0 \to \operatorname{Coker}(u') \to 0$. So, $\operatorname{Ker}(u') = 0$ and $\operatorname{Coker}(u') = 0$. That is, u' is an isomorphism. As a consequence, we have an almost split sequence

$$0 \longrightarrow U \xrightarrow{h} N \xrightarrow{g} L \longrightarrow 0,$$

where h is minimal right almost split. Being irreducible, $f : M \to N$ is not a retraction. Thus, there exists a map $q : M \to U$ such that $f = h \circ q$. Therefore, $gf = ghq = 0$.

Suppose now that f is a monomorphism. Then, $1 \leq \dim(M) < \dim(N)$. By Lemma 4.3.3, $\dim(N) \leq 2$. Therefore, $\dim(N) = 2$ and M is simple. Then, by lemma 3.3.2(1), there exists a non-split short exact sequence

$$0 \longrightarrow M \xrightarrow{f} N \xrightarrow{h} V \longrightarrow 0,$$

where $V = \mathrm{Coker}(f)$. By corollary 3.3.16(1), M is not injective, and by lemma 4.3.5(2), there exists an almost split sequence

$$0 \longrightarrow M \xrightarrow{u} I \xrightarrow{v} \tau^{-1}M \longrightarrow 0,$$

where u is minimal right almost split, v is minimal left almost split, and $\dim(I) = 2$. Since f is not a section and u is left almost split, $f = u'u$ for some map $u' : I \to N$. Then, $hu'u = hf = 0$. Since $v = \mathrm{Coker}(u)$, there exists a map $v' : \tau^{-}M \to V$ such that $hu' = v'v$. Therefore, we have a commutative diagram as follows :

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \xrightarrow{u} & I & \xrightarrow{v} & \tau^{-}M & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle u'} & & \downarrow{\scriptstyle v'} & & \\
0 & \longrightarrow & M & \xrightarrow{f} & N & \xrightarrow{h} & V & \longrightarrow & 0.
\end{array}$$

Since f is irreducible, u' is a retraction. In particular, u' is an epimorphism.
Since $\dim(I) = 2 = \dim(N)$, by Lemma 1.3.2, u' is an isomorphism.
By the Snake Lemma, we have two exact sequences

$$\mathrm{Ker}(u') \to \mathrm{Ker}(v') \to \mathrm{Coker}(\mathrm{id}_M)$$

and

$$\mathrm{Coker}(u') \to \mathrm{Coker}(v') \to \mathrm{Coker}(\mathrm{id}_M).$$

Since u' is an isomorphism, $\mathrm{Ker}(u') = 0$ and $\mathrm{Coker}(u') = 0$. Similarly, $\mathrm{Ker}(\mathrm{id}_M) = 0$ and $\mathrm{Coker}(\mathrm{id}_M) = 0$. This yields two exact sequences $0 \to \mathrm{Ker}(v') \to 0$ and $0 \to \mathrm{Coker}(v') \to 0$.

So, $\text{Ker}(v') = 0$ and $\text{Coker}(v') = 0$. That is, v' is an isomorphism. As a consequence, we have an almost split sequence

$$0 \longrightarrow M \xrightarrow{f} N \xrightarrow{h} V \longrightarrow 0,$$

where h is minimal left almost split. Since g is irreducible, there exists a map $t : V \to L$ such that $g = t \circ h$. Therefore, $gf = thf = 0$. □

Proposition 4.3.7. *Let $A = KQ/I$ be a Nakayama algebra with $\text{rad}^2 A = 0$. Then $\text{rad}^3(\text{mod}A) = 0$.*

Proof. Let $X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} X_4$ be a sequence of three irreducible morphisms between indecomposable modules. Then each f_i is either a monomorphism or an epimorphism. Moreover, by Lemma 4.3.3(1), $\dim(X_i) \le 2$ for any $1 \le i \le 4$.

If $\dim(X_1) = 1$, then X_1 is simple. So, f_1 is a monomorphism. So, by Lemma 4.3.6, $f_2 f_1 = 0$. Then, $f_3 f_2 f_1 = 0$.

Assume $\dim(X_1) = 2$. If f_1 is a monomorphism, then $\dim(X_2) > \dim(X_1) = 2$, a contradiction. So, f_1 is an epimorphism. Then $\dim(X_2) < \dim(X_1) = 2$. So, X_2 is simple. Thus, f_2 is a monomorphism. So, by Lemma 4.3.6, $f_3 f_2 = 0$, and hence, $f_3 f_2 f_1 = 0$. By Proposition 3.2.10, $\text{rad}^3(\text{mod}A) = 0$. □

Now, we are ready to state the main result of this thesis.

Theorem 4.3.8. *Let $A = KQ/I$ be a Nakayama algebra, where Q is a finite connected quiver and I is an admissible ideal of kQ. Then $\text{rad}^3(\text{mod}A) = 0$ if and only if one of the following cases occurs.*

(1) *$Q = \vec{\mathbb{A}}_n$ with $1 \le n \le 3$ and $I = 0$.*

(2) *$Q = \tilde{\mathbb{A}}_n$ with $n \ge 1$ and I is generated by all paths of length two in Q.*

(3) *$Q = \vec{\mathbb{A}}_n$ with $n \ge 3$ and I is generated by all paths of length two in Q.*

Proof. The necessity of the theorem holds by Lemma 4.3.1. It remains to show the sufficiency. Suppose that the case (2) or (3) occurs. By proposition 4.2.1, A is a Nakayama algebra, and by proposition 1.9.6, $\text{rad}^2(A) = 0$. Thus, by Proposition 4.3.7, $\text{rad}^3(\text{mod}A) = 0$.

Now, we suppose that $Q = \vec{\mathbb{A}}_n$ with $1 \leq n \leq 3$ and $I = 0$. If $n = 1$, then A is simple. By Proposition 1.9.7, $\text{rad}(\text{mod}A) = 0$. In particular, $\text{rad}^3(\text{mod}A) = 0$. If $n = 2$, then $\text{rad}^2(\text{mod})\text{A} = 0$; see ([6], 5.2.4). So, $\text{rad}^3(\text{mod}A) = 0$. If $n = 3$ then, by Lemma 4.3.2, $\text{rad}^3(\text{mod}A) = 0$. □

CONCLUSION

In this thesis, after introducing the concept of radical in a K-linear category, we saw in chapter 1, for a finite dimensional K-algebra A, rad(mod A) is zero if and only if A is semi-simple.

Also, in the remaining chapters, we presented the concepts of the theory of representation of algebras and the structure of Nakayama algebras on projective and injective modules.

Then, all concepts of four chapters helped us to show that in Nakayama algebras with a module category, radical cube is zero if and only if Q is type of $Q = \vec{\mathbb{A}}_n$ with $1 \leq n \leq 3$ and $I = 0$, $Q = \tilde{\mathbb{A}}_n$ with $n \geq 1$ and I is generated by all paths of length two in Q and $Q = \vec{\mathbb{A}}_n$ with $n \geq 3$ and I is generated by all paths of length two in Q.

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