Construction des sous-categories inclinantes amassées des categories amassées des types $\mathbb{A}_\infty$ et $\mathbb{A}_\infty^\infty$

Constructing Cluster-tilting Subcategories of Cluster Categories of Types $\mathbb{A}_\infty$ and $\mathbb{A}_\infty^\infty$

par

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Abstract

The main objective of this thesis is to study the cluster-tilting subcategories in a cluster category $\mathscr{C}(Q)$, where Q is a quiver with no infinite path of type $\mathbb{A}_\infty$ or $\mathbb{A}_\infty^\infty$.

We start this work with the τ-rigidity theory in an Auslander-Reiten k-category $\mathcal{A}$, where k is an algebraically closed field and τ is the Auslander-Reiten translation of $\mathcal{A}$. Given a standard Auslander-Reiten component of $\mathcal{A}$ which is a finite wing or, of shape $\mathbb{Z}\mathbb{A}_\infty$ or $\mathbb{Z}\mathbb{A}_\infty^\infty$, we first characterize its maximal τ-rigid sets and then produce a method to construct all of them. This technique also allows us to obtain all the tilting modules over the path algebra of a linearly oriented quiver $\vec{\mathbb{A}}_n$.

We then apply the above mentioned results to our main objective. Indeed, the rigid subcategories of $\mathscr{C}(Q)$ are determined by the τ-rigid sets in its fundamental domain $\mathscr{F}(Q)$, which consists of some standard Auslander-Reiten components of shape $\mathbb{Z}\mathbb{A}_\infty$ or $\mathbb{Z}\mathbb{A}_\infty^\infty$ of the derived category of finite dimensional representations of Q. The above results enable us to characterize and construct the maximal rigid subcategories of $\mathscr{C}(Q)$.

Finally, combining these with the criteria by Holm-Jørgensen and Liu-Paquette for maximal rigid subcategories to be cluster-tilting, we shall be able to obtain a complete characterization of the cluster-tilting subcategories of $\mathscr{C}(Q)$ and provide an explicit method to construct them all.

Sommaire

L'objectif principal de cette thèse est d'étudier les sous-catégories inclinantes amassées d'une catégorie amassée $\mathscr{C}(Q)$, où Q est un carquois sans chemins infinis de type $\mathbb{A}_\infty$ or $\mathbb{A}_\infty^\infty$.

Nous commençons par la théorie de τ-rigidité dans une catégorie d'Auslander-Reiten $\mathcal{A}$, où τ est la translation d'Auslander-Reiten de $\mathcal{A}$. Étant donnée une composante standard d'Auslander-Reiten de $\mathcal{A}$ qui est une aile ou de la forme $\mathbb{Z}\mathbb{A}_\infty$ ou $\mathbb{Z}\mathbb{A}_\infty^\infty$, nous caractérisons d'abord ses ensembles τ-rigides maximaux et produisons ensuite une méthode pour les construire tous. Cette technique nous permet également d'obtenir tous les modules inclinants sur une algèbre héréditaire d'un carquois orienté linéairement de type $\mathbb{A}_n$.

Nous appliquons ensuite les résultats mentionnés ci-dessus à notre objectif principal. Effectivement, les sous-catégories rigides de $\mathscr{C}(Q)$ sont déterminées par les ensembles τ-rigides dans son domaine fondamental $\mathscr{F}(Q)$, qui se compose de certaines composantes d'Auslander-Reiten de la catégorie dérivée des représentations de dimension finie de Q, qui sont toutes standards de type $\mathbb{Z}\mathbb{A}_\infty$ ou $\mathbb{Z}\mathbb{A}_\infty^\infty$. Les résultats ci-dessus nous permettent de caractériser et de construire les sous-catégories maximales rigides de $\mathscr{C}(Q)$.

Enfin combinant nos résultats avec les critères d'Holm-Jørgensen and de Liu-Paquette pour qu'une sous-catégorie maximale rigide soit inclinante amassée, nous pourrons obtenir une caractérisation complète des sous-catégories inclinantes amassées de $\mathscr{C}(Q)$ et fournir une méthode explicite pour les construire.

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Contents

Introduction

Cluster algebras were introduced by Fomin and Zelevinsky in [24, 25], whose initial aim was to study the dual canonical basis and total positivity in Lie theory. It has been revealed that the theory of cluster algebras is connected to various subjects of mathematics, such as representation theory, algebraic geometry and combinatorics.

Let Q be a finite quiver without loops or 2-cycles. Consider the rational function field $\mathbb{Q}(x_1, \ldots, x_n)$, where $x_1, \ldots, x_n$ are labeled by the vertices $1, 2, \ldots, n$ of Q. Using the local information of Q at some vertex k, one mutates $\{x_1, \ldots, x_n\}$ at x_k and obtains a new set $\{x'_1, \ldots, x'_n\}$ of elements of $\mathbb{Q}(x_1, \ldots, x_k)$, which corresponds to a new quiver Q' of vertices $1, 2, \ldots, n$. Using information of Q', one mutates again $\{x'_1, \ldots, x'_n\}$. The sets obtained by all possible mutations are called *clusters*. An element of $\mathbb{Q}(x_1, \ldots, x_n)$ is called a *cluster variable* if it belongs to a cluster. The $\mathbb{Z}$-subalgebra $\mathcal{A}(Q)$ of $\mathbb{Q}(x_1, \ldots, x_n)$ generated by all cluster variables is called the *cluster algebra* associated with Q.

A remarkable connection between representation theory and cluster algebras was discovered by Buan, Marsh, Reineke, Reiten and Todorov in [20]. Indeed, given a finite acyclic quiver Q, they constructed the so-called *cluster category* $\mathscr{C}(Q)$ associated with Q, which is the orbit category of the derived category of finite dimensional representations of Q by the composite of the Auslander-Reiten translation and the shift functor. One calls $\mathscr{C}(Q)$ an *additive categorification* of the cluster algebra $\mathcal{A}(Q)$ associated with Q, in the sense that cluster-tilting objects correspond to clusters; direct summands of cluster-tilting objets correspond to cluster variables; and replacing an indecomposable direct summand of a cluster-tilting object corresponds to mutating the corresponding cluster at the corresponding cluster variable.

In a general 2-Calabi-Yau triangulated category, replacing cluster-tilting objects by cluster-tilting subcategories, Buan, Iyama, Reiten and Scott introduced the notion of a general cluster structure; see [21]. This inspired a great interest in generalizing the construction given in [20] to the infinite case. Indeed, let Q be a locally finite quiver without infinite paths. As shown by Bautista, Liu and Paquette; see [15], the category $\mathrm{rep}(Q)$ of finite dimensional representations of Q over a field is a hereditary abelian category having Auslander-Reiten sequences, and consequently, the derived category $D^b(\mathrm{rep}(Q))$ has Auslander-Reiten triangles; see [53]. By Keller's result in [39], the orbit category $\mathscr{C}(Q)$ of $D^b(\mathrm{rep}(Q))$, constructed in the same fashion as in [20], is a 2-Calabi-Yau triangulated category. Remarkably, it has been shown that the cluster-tilting subcategories in $\mathscr{C}(Q)$ form a cluster structure; see [48, 62, 59]. For this reason, one calls $\mathscr{C}(Q)$ the *cluster category* associated with Q.

As we can see, in order to study the cluster structure in $\mathscr{C}(Q)$, we shall need to study its cluster-tilting subcategories. In case Q is finite of n vertices, a cluster-tilting object in $\mathscr{C}(Q)$ is simply a rigid object of n non-isomorphic indecomposable direct summands; see [20]. In case Q is infinite, it is more difficult to recognize cluster tilting subcategories, not to mention to construct all of them. The objective of this thesis is to deal with these two problems in the $\mathbb{A}_\infty$-case and the $\mathbb{A}_\infty^\infty$-case.

Let Q be a quiver of type $\mathbb{A}_\infty$ or $\mathbb{A}_\infty^\infty$ without infinite paths. Holm-Jørgensen and Liu-Paquette have shown in [34, 48] that the cluster-tilting subcategories in $\mathscr{C}(Q)$ are the maximal rigid ones which are functorially finite in $\mathscr{C}(Q)$. Moreover, a geometric model (that is, an infinite polygon with marked points in the $\mathbb{A}_\infty$-case and an infinite strip with marked points in the $\mathbb{A}_\infty^\infty$-case) has been constructed in such a way that the maximal rigid subcategories correspond to triangulations of the geometric model. This enabled them to characterize the functorial finiteness of a maximal rigid subcategory in terms of the geometric model, that is, they gave a criterion for a maximal rigid subcategory to be cluster-tilting. Note, however, these geometric descriptions do not provide any method to obtain cluster-tilting subcategories. In this thesis, we shall study the cluster-tilting subcategories of $\mathscr{C}(Q)$ from a purely categorical point of view. Recall that cluster-tilting subcategories are *strictly additive*, that is, closed under isomorphisms, finite direct sums and taking direct summands; see [48, Section 1]. Since $\mathscr{C}(Q)$ is Krull-Schmidt, every strictly additive subcategory $\mathscr{T}$ of $\mathscr{C}(Q)$ is determined by a set $\operatorname{ind}\mathscr{T}$ of objects in its Auslander-Reiten quiver $\Gamma_{\mathscr{C}(Q)}$. Thus, we shall work in the framework of $\Gamma_{\mathscr{C}(Q)}$ in order to provide

(1) a characterization of maximal rigid subcategories, and

(2) a method to construct all the maximal rigid subcategories, as well as all the cluster-tilting subcategories.

We should point out that our construction of maximal rigid subcategories is based on an effective construction of all tilting modules over the path algebra of a linearly oriented quiver $\vec{\mathbb{A}}_n$. We shall mention that Assem and Happel have given an implicit method to construct the tilting modules of an algebra of type $\mathbb{A}_n$ in [4].

For the rest of this introduction, we shall present more details. The cluster category $\mathscr{C}(Q)$ admits a *fundamental domain* $\mathscr{F}(Q)$, which is a translation subquiver of the Auslander-Reiten quiver $\Gamma_{D^b(\operatorname{rep}(Q))}$ of $D^b(\operatorname{rep}(Q))$. The canonical projection $\pi : D^b(\operatorname{rep}(Q)) \to \mathscr{C}(Q)$ induces a translation quiver morphism $\pi : \mathscr{F}(Q) \to \Gamma_{\mathscr{C}(Q)}$, acting identically on the underlying quiver. In case Q is of infinite Dynkin type, $\pi : \mathscr{F}(Q) \to \Gamma_{\mathscr{C}(Q)}$ is an isomorphism, and a set of objects in $\Gamma_{\mathscr{C}(Q)}$ is maximal rigid if and only if it is maximal τ_D-rigid in $\mathscr{F}(Q)$, where τ_D is the Auslander-Reiten translation of $D^b(\operatorname{rep}(Q))$; see (5.1.7). Therefore, it amounts to study the maximal τ_D-rigid sets in $\mathscr{F}(Q)$. This is particularly advantageous since in this case every connected component of $\Gamma_{D^b(\operatorname{rep}(Q))}$ is standard, that is, the morphisms between two objects can be described by the paths between them. Moreover, this approach allows us to work under a more general setting, that is, to study maximal τ-rigid sets in a standard component of an Auslander-Reiten category.

For this purpose, we shall need some combinatorial considerations. Let (Γ, τ) be a translation quiver, which is a finite *wing*; see (1.4.4), or is of shape $\mathbb{Z}\mathbb{A}_\infty$ or $\mathbb{Z}\mathbb{A}_\infty^\infty$. A *section-generator* of Γ is a set of vertices of Γ whose convex hull is a section, or equivalently, a set of vertices of a section containing its source vertices and its sink vertices; see (4.1.15), (4.2.24) and (4.3.22). Since it is easy to find all the sections in Γ, we shall be able to find all section-generators of Γ.

To ease our work, we shall introduce a coordinate system for $\mathit{\Gamma}$ so that we may identify a vertex $X \in \mathit{\Gamma}$ with a unique pair (i_X, j_X) of integers; see Chapter 4. In particular, the coordinate system yields an order $\preceq$ over the vertices of $\mathit{\Gamma}$; see (4.1.6), (4.2.7) and (4.3.7). A set $\mathcal{S}$ of vertices of $\mathit{\Gamma}$ is called a *sectional chain* if it is a chain such that, for every minimal cover $X \prec Y$ in $\mathcal{S}$, there is a path in $\mathit{\Gamma}$ connecting X and Y. Indeed, every section-generator of $\mathit{\Gamma}$ is a sectional chain; see (4.1.15), (4.2.24) and (4.3.23).

Assume that $\mathit{\Gamma}$ is a finite wing or of shape $\mathbb{Z}\mathbb{A}_\infty$. Then, every vertex $X \in \mathit{\Gamma}$ is the wing vertex of a unique wing $\mathcal{W}_X$ in $\mathit{\Gamma}$; see (1.4.6). Fix two vertices $X, Y \in \mathit{\Gamma}$. We have $X \preceq Y$ if and if only $\mathcal{W}_X \subseteq \mathcal{W}_Y$. We say that $\mathcal{W}_X, \mathcal{W}_Y$ are *comparable* if X, Y are comparable. Moreover, we shall say that $\mathcal{W}_X, \mathcal{W}_Y$ are *separable* if there is a quasi-simple vertex between them but contained in neither of them; see (1.4.8). If $X \prec Y$ and X, Y are connected by a path, then we define $\mathcal{W}_Y^X$ to be the maximal wing contained in $\mathcal{W}_Y$ such that $\mathcal{W}_X, \mathcal{W}_Y^X$ are separable. In case $\mathit{\Gamma}$ is of shape $\mathbb{Z}\mathbb{A}_\infty$, the quasi-simple vertices of $\mathit{\Gamma}$ can be written as S_i with $i \in \mathbb{Z}$ such that $S_{i+1} = \tau S_i$. We denote by R_i^+ (respectively, R_i^-) the infinite sectional path starting (respectively, ending) with S_i. A set of vertices of $\mathit{\Gamma}$ is called *locally finite* if it contains at most finitely many vertices of each of the R_i^+ and the R_j^-. Given any $n \in \mathbb{Z}$, we define $\mathit{\Gamma}_{<n}^+ = \bigcup_{i<n} R_i^+$, the full subquiver of $\mathit{\Gamma}$ generated by the successors of S_{n-1}, and $\mathit{\Gamma}_{>n}^- = \bigcup_{j>n} R_j^-$, the full subquiver of $\mathit{\Gamma}$ generated by the predecessors of S_{n+1}.

Now, let $\mathcal{A}$ be an *Auslander-Reiten category*; see (2.1.2), with Auslander-Reiten quiver $\mathit{\Gamma}_{\mathcal{A}}$ and Auslander-Reiten translation τ. We are ready to describe our construction of the maximal τ-rigid sets in a standard component $\mathit{\Gamma}$ of $\mathit{\Gamma}_{\mathcal{A}}$, which is a finite wing or is of shape $\mathbb{A}_\infty$ or $\mathbb{A}_\infty^\infty$. Roughly speaking, our construction consists of a choice of one or two section-generators of $\mathit{\Gamma}$ and an addition of some extra objects in some finite wings.

First, let $\mathcal{W}$ be a standard component of $\mathit{\Gamma}_{\mathcal{A}}$ which is a wing of *rank* n; see (1.4.4). In this case, a pair (X, Y) in $\mathcal{W}$ is τ-rigid if and only if $\mathcal{W}_X, \mathcal{W}_Y$ are comparable or separable; see (5.2.1). Since add$\mathcal{W} \cong$ mod$k\vec{\mathbb{A}}_n$; see (2.2.10), the maximal τ-rigid sets in $\mathcal{W}$ correspond to the tilting modules in mod$H$, and consequently, a maximal τ-rigid set in $\mathcal{W}$ is a τ-rigid set of n objects. This allows us to construct inductively the maximal τ-rigid sets in $\mathcal{W}$. Indeed, this is trivial in case $\mathcal{W}$ of rank 1 or 2. In case $n > 2$, we shall take a section-generator $\mathcal{S}$ of $\mathcal{W}$. Being a sectional chain, $\mathcal{S}$ is of the form $X_1 \prec X_2 \prec \cdots \prec X_m$, where X_t, X_{t+1} with $1 \le t < m$ are connected by a path in $\mathcal{W}$. As mentioned above, we obtain wings $\mathcal{W}^{X_t}_{X_{t+1}}$ with $1 \le t < m$, which are pairwise separable of rank less than n; see (5.2.8). Choosing a maximal τ-rigid set Θ_t in $\mathcal{W}^{X_t}_{X_{t+1}}$, which is doable by the induction hypothesis, we obtain a maximal τ-rigid set $\mathcal{S} \cup \Theta_1 \cup \cdots \cup \Theta_{m-1}$ in $\mathcal{W}$. Indeed, every maximal τ-rigid set in $\mathcal{W}$ can be obtained by this way; see (5.2.9). Since we are able to obtain all the section generators of $\mathcal{W}$, this provides a method to obtain all the maximal τ-rigid sets in $\mathcal{W}$.

Secondly, let Γ be a standard component of $\Gamma_{\mathcal{A}}$ of shape $\mathbb{Z}\mathbb{A}_\infty$. Similarly, a pair (X, Y) in Γ is τ-rigid if and only if $\mathcal{W}_X, \mathcal{W}_Y$ are comparable or separable; see (5.3.1). The maximal τ-rigid sets in Γ are characterized in Theorem 5.3.17, in particular, they always contain a section-generator. To construct a maximal τ-rigid set, we start with a section-generator $\mathcal{S}$ of Γ, which is necessarily of the form

$$X_1 \prec X_2 \prec \cdots \prec X_t \prec \cdots ,$$

where X_t, X_{t+1} are connected by a path for all $t \ge 1$. As mentioned in the finite wing case, we obtain finite wings $\mathcal{W}^{X_t}_{X_{t+1}}$ in Γ. Choosing a maximal τ-rigid set Θ_t in $\mathcal{W}^{X_t}_{X_{t+1}}$ for all t, we obtain a set $\Theta = \bigcup_{t=1}^{\infty} \Theta_t$, called an *addend* to $\mathcal{S}$, such that $\mathcal{S} \cup \Theta$ is τ-rigid; see (5.3.24). If $\mathcal{S}$ is *locally finite*; see (4.2.18), then $\mathcal{S} \cup \Theta$ is maximal τ-rigid in Γ. Otherwise, say, $\mathcal{S}$ is *almost contained* in some R^-_m; see (5.3.22). Then, we choose another section-generator $\mathcal{S}'$ of Γ almost contained in R^+_n with $m \ge n+2$, and an addend $\Theta_{\mathcal{S}'}$ to $\mathcal{S}'$. Moreover, we choose a maximal τ-rigid set Θ in the (possibly empty) finite wing $\Gamma^+_{<m} \cap \Gamma^-_{>n}$. Then

$$\mathcal{S} \cup \Theta_{\mathcal{S}} \cup \mathcal{S}' \cup \Theta_{\mathcal{S}'} \cup \Theta$$

is a maximal τ-rigid set in Γ. More importantly, all maximal τ-rigid sets in Γ are constructed by one of these two constructions; see (5.3.24).

For our later purpose, we should point out that every maximal τ-rigid set in $\Gamma^-_{>n}$ (respectively, $\Gamma^+_{<n}$) is of the form $\mathcal{S} \cup \Theta_{\mathcal{S}} \cup \Theta$, where $\mathcal{S}$ is a section-generator almost contained in R^-_j with $j > n$ (respectively, R^+_i with $i < n$), $\Theta_{\mathcal{S}}$ is an addend to $\mathcal{S}$, and Θ is a maximal τ-rigid set in the (possibly empty) finite wing $\Gamma^+_{<j} \cap \Gamma^-_{>n}$ (respectively, $\Gamma^+_{<n} \cap \Gamma^-_{>i}$); see (5.3.28) and (5.3.27).

Thirdly, let Γ be a standard component of $\Gamma_{\mathcal{A}}$ of shape $\mathbb{Z}\mathbb{A}^\infty_\infty$. In this case, a pair (X, Y) in Γ is τ-rigid if and only if X, Y are comparable; see (5.4.1). Thus, the maximal τ-rigid sets in Γ are simply the sections in Γ; see (5.4.3) and (4.3.19).

Finally, we go back to the cluster category $\mathscr{C}(Q)$ of type $\mathbb{A}_\infty$ or $\mathbb{A}^\infty_\infty$. In case Q is of type $\mathbb{A}_\infty$, the fundamental domain $\mathscr{F}(Q)$ for $\mathscr{C}(Q)$ is the connecting component of $\Gamma_{D^b(\text{rep}(Q))}$, which is of shape $\mathbb{Z}\mathbb{A}_\infty$. As explained above, we shall be able to characterize the maximal τ_D-rigid sets in $\mathscr{F}(Q)$ and construct all of them. Combining these results with Holm and Jørgensen's criterion for a maximal rigid subcategory (that is, a triangulation of the infinite-gon) to be cluster-tilting, we obtain a complete characterization of the cluster-tilting subcategories in $\mathscr{C}(Q)$; see (6.1.9), and a method to construct them all; see (6.1.8). Indeed, a strictly additive subcategory $\mathscr{T}$ of $\mathscr{C}(Q)$ is cluster-tilting if and only if ind$\mathscr{T}$ is a maximal τ_D-rigid set in $\mathscr{F}(Q)$ obtained by taking a locally finite section-generator of $\mathscr{F}(Q)$, or taking two section-generators with one almost contained in R^+_n and the other almost contained the R^-_{n+2} for some n.

Now, assume that Q is of type $\mathbb{A}^\infty_\infty$. The fundamental domain $\mathscr{F}(Q)$ consists of three connected components of $\Gamma_{D^b(\text{rep}(Q))}$, namely, the connecting component $\mathcal{C}_Q$ which is of shape $\mathbb{Z}\mathbb{A}^\infty_\infty$, and two regular components $\mathcal{L}, \mathcal{R}$ of $\Gamma_{\text{rep}(Q)}$ which are orthogonal of shape $\mathbb{Z}\mathbb{A}_\infty$. W shall choose the coordinate systems for $\mathcal{L}, \mathcal{R}$ to be related to the coordinate system for $\mathcal{C}_Q$ in such a way that the τ-rigidity of a pair of objects in two different components can be easily described; see (7.2.1). As a consequence, every maximal τ_D-rigid set in $\mathscr{F}(Q)$ without objects of $\mathcal{C}_Q$ is a union of a maximal τ_D-rigid set in $\mathcal{L}$ and a maximal τ_D-rigid set in $\mathcal{R}$ with at least one of them being *dense* in its own component; see (7.2.3).

To obtain the maximal τ_D-rigid sets in $\mathscr{F}(Q)$ intersecting $\mathcal{C}_Q$, we shall take a sectional chain $\mathcal{S}$ in $\mathcal{C}_Q$, which is necessarily of the form $\{X_n\}_{n \in \mathcal{I}}$ with $\mathcal{I}$ an interval of $\mathbb{Z}$ such that $X_n \prec X_{n+1}$ and X_n, X_{n+1} are connected by a path p_n, for all non-maximal $n \in \mathcal{I}$. For each non-maximal $n \in \mathcal{I}$, we define a finite wing $\mathcal{W}_{X_n, X_{n+1}}$, which lies in $\mathcal{L}$ or $\mathcal{R}$ depending on $s(p_n)$; see (7.1.2). Then, choosing a maximal τ-rigid set Θ_n in $\mathcal{W}_{X_n,X_{n+1}}$, we obtain an *addend* $\Theta = \bigcup_{n\in\mathcal{I}} \Theta_n$ to $\mathcal{S}$, such that $\mathcal{S} \cup \Theta$ is $\tau_{_D}$-rigid.

Consider two coordinate sets $I_{\mathcal{S}} = \{i_{_X} \mid X \in \mathcal{S}\}$ and $J_{\mathcal{S}} = \{j_{_X} \mid X \in \mathcal{S}\}$. We define $\mathcal{L}^+_{<I_{\mathcal{S}}}$ to be $\mathcal{L}^+_{<i_0}$ in case $I_{\mathcal{S}}$ has a minimal element i_0; and the empty set otherwise, moreover, define $\mathcal{L}^-_{>I_{\mathcal{S}}}$ to be $\mathcal{L}^-_{>i_1}$ in case $I_{\mathcal{S}}$ has a maximal element i_1; the empty set otherwise. In a similar fashion, we define two subquivers $\mathcal{R}^+_{<J_{\mathcal{S}}}$ and $\mathcal{R}^-_{>J_{\mathcal{S}}}$ of $\mathcal{R}$. Then, we choose maximal $\tau_{_D}$-rigid sets $\Phi_{\mathcal{L}}^{<I_{\mathcal{S}}}$, $\Phi_{\mathcal{L}}^{>I_{\mathcal{S}}}$, $\Phi_{\mathcal{R}}^{<J_{\mathcal{S}}}$ and $\Phi_{\mathcal{R}}^{>J_{\mathcal{S}}}$ in $\mathcal{L}^+_{<I_{\mathcal{S}}}$, $\mathcal{L}^-_{>I_{\mathcal{S}}}$, $\mathcal{R}^+_{<J_{\mathcal{S}}}$, and $\mathcal{R}^-_{>J_{\mathcal{S}}}$, respectively, such that $\Phi_{\mathcal{L}}^{<I_{\mathcal{S}}}$ is dense in $\mathcal{L}^+_{<I_{\mathcal{S}}}$ and $\Phi_{\mathcal{R}}^{>J_{\mathcal{S}}}$ is dense in $\mathcal{R}^-_{>J_{\mathcal{S}}}$ in case $\mathcal{S}$ has a minimal element, and $\Phi_{\mathcal{L}}^{>I_{\mathcal{S}}}$ is dense in $\mathcal{L}^-_{>I_{\mathcal{S}}}$ and $\Phi_{\mathcal{R}}^{<J_{\mathcal{S}}}$ is dense in $\mathcal{R}^+_{<J_{\mathcal{S}}}$ in case $\mathcal{S}$ has a maximal element. This yields a maximal $\tau_{_D}$-rigid set

$$\mathcal{S} \cup \Theta \cup \Phi_{\mathcal{L}}^{<I_{\mathcal{S}}} \cup \Phi_{\mathcal{L}}^{>I_{\mathcal{S}}} \cup \Phi_{\mathcal{R}}^{<J_{\mathcal{S}}} \cup \Phi_{\mathcal{R}}^{>J_{\mathcal{S}}}$$

in $\mathscr{F}(Q)$. More importantly, every maximal τ-rigid set containing objects of $\mathcal{C}_Q$ can be obtained in this way.

As shown above, we obtain a complete description of the maximal $\tau_{_D}$-rigid sets in $\mathscr{F}(Q)$. Using our methods to construct maximal τ-rigid sets in a standard Auslander-Reiten component of an Auslander-Reiten category, this description allows us to construct all the maximal $\tau_{_D}$-rigid sets in $\mathscr{F}(Q)$. Finally, combining these results with Liu and Paquette's criterion for a maximal rigid subcategory (that is, a triangulation of the infinite strip) to be cluster-tilting, we obtain a complete characterization of the cluster-tilting subcategories in $\mathscr{C}(Q)$ and an effective method to construct them all; see (7.3.11). Indeed, a strictly additive subcategory $\mathscr{T}$ of $\mathscr{C}(Q)$ is cluster-tilting if and only if ind$\mathscr{T}$ is a maximal $\tau_{_D}$-rigid set in $\mathscr{F}(Q)$ obtained by taking a section-generator $\mathcal{S}$ (not just a sectional chain) as follows:

$$\mathcal{S} \cup \Theta \cup \Phi_{\mathcal{L}}^{<I_{\mathcal{S}}} \cup \Phi_{\mathcal{L}}^{>I_{\mathcal{S}}} \cup \Phi_{\mathcal{R}}^{<J_{\mathcal{S}}} \cup \Phi_{\mathcal{R}}^{>J_{\mathcal{S}}},$$

where Θ is an addend to $\mathcal{S}$, $\Phi_{\mathcal{L}}^{<I_{\mathcal{S}}}$, $\Phi_{\mathcal{L}}^{>I_{\mathcal{S}}}$, $\Phi_{\mathcal{R}}^{<J_{\mathcal{S}}}$, $\Phi_{\mathcal{R}}^{>J_{\mathcal{S}}}$ are densely maximal $\tau_{_D}$rigid sets in $\mathcal{L}^+_{<I_{\mathcal{S}}}$, $\mathcal{L}^-_{>I_{\mathcal{S}}}$, $\mathcal{R}^+_{<J_{\mathcal{S}}}$, and $\mathcal{R}^-_{>J_{\mathcal{S}}}$, respectively.

Chapter 1

Preliminaries

Throughout this thesis, k stands for an algebraically closed field. In this chapter, we shall introduce some terminology of partially ordered sets, which will be used throughout this thesis. We also collect some notions and basic facts about k-linear categories, path categories and mesh categories. It comes mainly from [5], [43], and [55].

1.1 The partially ordered sets

In this section, we shall introduce some terminology of partially ordered sets, which will be frequently used later.

Let $(\mathbb{P}, \preceq)$ be a partially ordered set, also called a poset. If $a \preceq b$ and $a \neq b$, we write $a \prec b$ and call b a *cover* of a in $\mathbb{P}$. We say that b is a *minimal cover* of a in $\mathbb{P}$ provided that $a \prec b$ and there exists no element $c \in \mathbb{P}$ such that $a \prec c \prec b$.

A subset $\mathcal{S}$ of $\mathbb{P}$ is called a *chain* provided, for any $a, b \in \mathcal{S}$, that $a \preceq b$ or $b \preceq a$. A chain $\mathcal{S}$ in $\mathbb{P}$ is called *maximal* if there is no such a chain $\mathcal{S}'$ in $\mathbb{P}$ that $\mathcal{S} \subsetneq \mathcal{S}'$.

Let Σ be a subset of $\mathbb{P}$. A subset $\mathcal{T}$ of Σ is called *dense* in Σ if for any element $x \in \Sigma$, there is an element $a \in \mathcal{T}$ such that $x \preceq a$.

1.2 Linear categories

A *k-linear category* or simply *k-category* is a category in which the morphisms sets are k-vector spaces and the composition of morphisms is k-bilinear. Such a k-category is said to be *Hom-finite* if all the morphisms spaces are finite dimensional over k.

For the rest of this section, $\mathcal{A}$ stands for a Hom-finite k-category. Given an object X in $\mathcal{A}$, $\mathrm{End}_{\mathcal{A}}(X) = \mathrm{Hom}_{\mathcal{A}}(X, X)$ is a finite dimensional k-algebra, called the *endomorphism algebra* of X. Given an ideal $\mathcal{I}$ in $\mathcal{A}$, as defined in [5, A.3(3.1)], one defines a *quotient category* $\mathcal{A}/\mathcal{I}$ as follows. The objects are those of $\mathcal{A}$, and given objects X, Y, we have $\mathrm{Hom}_{\mathcal{A}/\mathcal{I}}(X, Y) = \mathrm{Hom}_{\mathcal{A}}(X, Y)/\mathcal{I}(X, Y)$, and the composition of morphisms is induced from the composition of morphisms in $\mathcal{A}$.

1.2.1 Definition. An object $X \in \mathcal{A}$ is called a *direct sum* of $X_1, \ldots, X_n \in \mathcal{A}$ if there exist morphisms $q_i : X_i \to X$, called *injections*, and morphisms $p_i : X \to X_i$, called *projections*, such that $\Sigma_{i=1}^n q_i p_i = 1_X$, and for $1 \le i, j \le n$,

$$p_i q_j = \begin{cases} 1_{X_i}, & \text{if } i = j; \\ 0, & \text{otherwise.} \end{cases}$$

In this case, write $X = X_1 \oplus \cdots \oplus X_n$.

A non-zero object $X \in \mathcal{A}$ is *indecomposable* provided that $X = X_1 \oplus X_2$ implies $X_1 = 0$ or $X_2 = 0$.

Given $X = X_1 \oplus \cdots \oplus X_r$ with injections $q_j : X_j \to X$ and $Y = Y_1 \oplus \cdots \oplus Y_s$ with projections $u_i : Y \to Y_i$, every morphism $f : X \to Y$ in $\mathcal{A}$ is identified with a matrix $f = (f_{ij})_{s \times r}$, where $f_{ij} = u_i f q_j \in \mathrm{Hom}_{\mathcal{A}}(X_j, Y_i)$, for all i, j. In particular,

$$\mathrm{Hom}_{\mathcal{A}}(X, Y) = \bigoplus_{i,j} \mathrm{Hom}_{\mathcal{A}}(X_i, Y_j).$$

1.2.2 Definition. A k-category $\mathcal{A}$ is called *additive* if the following are satisfied.

(1) There is a *zero object* 0, that is, $\mathrm{Hom}_{\mathcal{A}}(X, 0) = 0$ and $\mathrm{Hom}_{\mathcal{A}}(0, X) = 0$ for every object $X \in \mathcal{A}$.

(2) For any objects $X_1, \cdots, X_n \in \mathcal{A}$, the direct sum $X_1 \oplus \cdots \oplus X_n$ exists in $\mathcal{A}$.

1.2.3 Definition. An additive k-category is called a *Krull-Schmidt category* provided that every object decomposes into a finite direct sum of objects having local endomorphism rings.

Let $\mathcal{A}$ be a Hom-finite Krull-Schmidt k-category. Non-zero object decomposes into a finite direct sum of indecomposable objects; see, for example, [43, (4.2)].

To conclude this section, assume that $\mathcal{A}$ is an abelian k-category, as defined in [56]. We refer the notion of Yoneda Ext Groups $\mathrm{Ext}^n(X,Y)$, with $n \geq 1$, to [61, §3.4]. An abelian category $\mathcal{A}$ is called *hereditary* if $\mathrm{Ext}^2_{\mathcal{A}}(X,Y) = 0$, for any $X, Y \in \mathcal{A}$.

1.3 Quivers and path categories

The objective of this section is to recall some notions and terminology from quivers and path categories.

1.3.1 Definition. A *quiver* is a quadruple $Q = (Q_0, Q_1, s, t)$, where Q_0 is a set of vertices, Q_1 is a set of arrows, and $s, t : Q_1 \to Q_0$ are maps. Given an arrow $\alpha \in Q_1$, one calls $s(\alpha)$ its *source* and $t(\alpha)$ its *target*; and writes $\alpha : s(\alpha) \to t(\alpha)$.

Throughout this section, we fix such a quiver $Q = (Q_0, Q_1, s, t)$. The *underlying graph* of Q is obtained from Q by forgetting the orientation of the arrows. The quiver Q is said to be *connected* if its underlying graph is connected. One says that Q is *finite* if Q_0 and Q_1 are finite sets. Moreover, Q is called *locally finite* if, for any $a \in Q_0$, the number of arrows α with $s(\alpha) = a$, as well as, the number of arrows β with $t(\beta) = a$, is finite. In this thesis, we assume that all quivers are locally finite.

A vertex a in Q is called a *source vertex* if Q has no arrow α with $t(\alpha) = a$; and a *sink vertex* if Q has no arrow α with $s(\alpha) = a$.

Let $a, b \in Q_0$. A *path* p of *length* $l(p) \geq 1$ from a to b is a sequence of arrows denoted by $p = \alpha_{l(p)} \cdots \alpha_2\alpha_1$, where $\alpha_i \in Q_1$ for all $1 \leq i \leq l(p)$ and $s(\alpha_1) = a$, $t(\alpha_i) = s(\alpha_{i+1})$ for $1 \leq i < l(p)$ and $t(\alpha_{l(p)}) = b$. In this case, a is called a *predecessor* of b, and b is called a *successor* of a. In particular, if there exists an arrow $a \to b$, then a is said to be an *immediate predecessor* of b, and b is said to be an *immediate successor* of a. In the sequel, a path from a to b will be simply denoted by $a \rightsquigarrow b$. Moreover, with each vertex a one associates a *trivial path* ε_a, which is of length 0. A path of length greater than or equal to 1 is called a *cycle* if its source and target coincide. A quiver is called *acyclic* if it contains no cycles. The quiver Q is called *strongly locally finite* if it is locally finite and, for any $a, b \in Q_0$, Q contains only finitely many paths from a to b.

A quiver $Q' = (Q'_0, Q'_1, s', t')$ is called *subquiver* of $Q = (Q_0, Q_1, s, t)$ if $Q'_0 \subseteq Q_0$ and $Q'_1 \subseteq Q_1$ and $s' = s\,|_{Q'_1}$ and $t' = t\,|_{Q'_1}$. A subquiver Q' of Q is called *full* if $Q'_1 = \{\alpha \in Q_1 \mid s(\alpha) \in Q'_0 \text{ and } t(\alpha) \in Q'_0\}$, and *convex* provided that every path $x_0 \to x_1 \to \cdots \to x_n$ in Q with $x_0, x_n \in Q'$ lies entirely in Q'. In particular, a convex subquiver of Q is always full.

Let $\mathcal{T}$ be a set of vertices of Q. The *convex hull* of $\mathcal{T}$ is the minimal convex subquiver of Q containing $\mathcal{T}$, that is the full subquiver of Q generated by the vertices lying on a path in Q whose endpoints belong to $\mathcal{T}$.

With each arrow $\alpha : a \to b$ in Q, we associate a formal inverse $\alpha^{-1} : b \to a$, with the source $s(\alpha^{-1}) = b$ and the target $t(\alpha^{-1}) = a$. An *edge* in Q is an arrow or the inverse of an arrow. A trivial path is called a *trivial walk*. A *non-trivial walk* w in Q is a finite or infinite product of the form $\cdots c_{i+1}c_i \cdots$, where c_i are edges such that $t(c_i) = s(c_{i+1})$ for all i. Such a walk is called *reduced* if $c_{i+1} \neq c_i^{-1}$ for every i. We shall say that a vertex x appears in w if $x = s(c_i)$ or $x = t(c_i)$ for some i; and an arrow α appears in w if $c_j = \alpha$ or $c_j = \alpha^{-1}$ for some j. Moreover, one says that w is *simple* if every vertex appears in w at most once. Finally, we shall say that a subquiver Q' of a quiver Q is given by a simple walk w if Q'_0 consists of the vertices appearing in w and Q'_1 consists of the arrows appearing in w.

Given a quiver $Q = (Q_0, Q_1)$, one defines its *path category* $k[Q]$ over k as follows. Its objects are the vertices of Q, and given $a, b \in Q_0$, the set of morphisms from a to b is given by the k-vector space with a basis the set of all paths from a to b. The composition of morphisms is induced from the composition of paths. Clearly, $k[Q]$ is a k-category. Furthermore, the *path algebra* of Q over k is defined by

$$kQ = \bigoplus_{a,b \in Q_0} \mathrm{Hom}_{k[Q]}(a, b),$$

whose multiplication is induced from the composition of morphisms in $k[Q]$.

Recall that a k-algebra H is called *hereditary* if every submodule of a projective H-module is projective.It is well known that if Q is finite and acyclic, then the path algebra kQ is hereditary; see, for example, [5, (VII 1.7)]. For more details on algebras and modules, we refer the reader to [3], [61] and [56].

1.4 Translation quivers and mesh categories

In this section, we shall collect some notions and facts about translation quivers and mesh categories. Moreover, we shall introduce some notions and terminology in a translation quiver which is a wing or of shape $\mathbb{Z}\mathbb{A}$.

1.4.1 Definition. A *translation quiver* $\Gamma = (\Gamma_0, \Gamma_1, \tau)$ is given by an underlying quiver (Γ_0, Γ_1), which is locally finite without multiple arrows, together with a *translation*, that is an injective map $\tau : \Gamma_0' \to \Gamma_0$, where Γ_0' is a subset of Γ_0, such, for any $z \in \Gamma_0'$ and any $y \in \Gamma_0$, that $y \to z$ is an arrow if and only if $\tau z \to y$ is an arrow. The vertices in Γ_0 which do not belong to Γ_0' are called *projective*, those not belonging to the image of τ are called *injective*.

Throughout this section, let $\Gamma = (\Gamma_0, \Gamma_1, \tau)$ stand for a translation quiver. Given $x \in \Gamma_0$, the *τ-orbit* of x is the set of all vertices of the form $\tau^n x$ with $n \in \mathbb{Z}$. A connected full subquiver of Γ is called a *section* in Γ if it is acyclic, it meets each τ-orbit exactly once and it is convex in Γ.

1.4.2 Definition. A *section-generator* of Γ is a set of vertices whose convex hull is a section of Γ.

A path $x_0 \to x_1 \to \cdots \to x_n$ in Γ is called *sectional* if $x_i \ncong \tau x_{i-1}$, for all $0 < i \leq n$.

A translation quiver (Δ, τ_Δ) is a *translation subquiver* of (Γ, τ) provided that Δ is a subquiver of Γ and $\tau_\Delta x = \tau x$, whenever x is a vertex of Δ such that τx belongs to Δ.

Next, we will recall some facts on mesh categories. Recall first that $k[\Gamma]$ is the path category of the quiver Γ over k. Given a non-projective vertex $z \in \Gamma$, if $\alpha_i : \tau z \to y_i$, $i = 1, \ldots, r$ are the arrows starting in τz and $\beta_i : y_i \to z$, $i = 1, \ldots, r$ are the arrows ending in z, then $m_z = \sum_{i=1}^r \beta_i \alpha_i \in k[\Gamma]$ is called a *mesh relation*. The *mesh ideal* in $k[\Gamma]$ is the ideal generated by all the mesh relations. The *mesh category* $k(\Gamma)$ is the quotient category of $k[\Gamma]$ modulo the mesh ideal; see, [55, (2.1)]. There exists a canonical projection functor $p : k[\Gamma] \to k(\Gamma)$, acting identically on the vertices. Given $u \in k[\Gamma]$, we shall write $\bar{u}$ for the image of u under p.

Let Δ be a convex subquiver of Γ. Then Δ is a translation quiver and we denote its mesh category by $k(\Delta)$.

1.4.3 Lemma. *Let Γ be a translation quiver and let Δ be a convex subquiver of Γ. If $k\{\Delta\}$ denotes the full subcategory of $k(\Gamma)$ generated by the vertices of Δ, then there is an isomorphism $F : k(\Delta) \to k\{\Delta\}$ acting identically on objects. In particular, for any $x, y \in \Delta$, we have*

$$\mathrm{Hom}_{k(\Delta)}(x, y) \cong \mathrm{Hom}_{k(\Gamma)}(x, y).$$

Proof. Let $k\{\Delta\}$ be the full subcategory of $k(\Gamma)$ generated by the objects of Δ. Restricting the canonical projection $k[\Gamma] \to k(\Gamma)$ to the path category $k[\Delta]$, we obtain a full dense functor $F : k[\Delta] \to k\{\Delta\}$. It remains to show that $\mathrm{Ker} F = I_\Delta$, the mesh ideal of Δ.

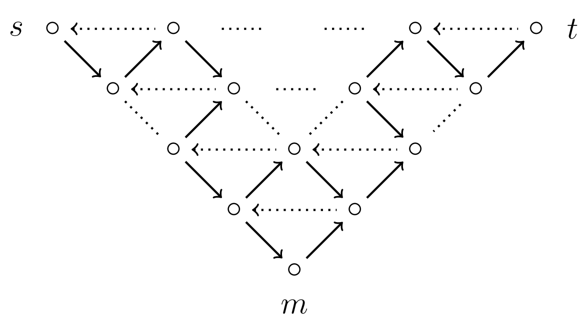
Indeed, let x, y be objects in Δ. Since Δ is convex in Γ, every mesh in Δ is a mesh in Γ. In particular, $I_\Delta(x, y) \subseteq I_\Gamma(x, y) = (\mathrm{Ker} F)(x, y)$. Conversely, let γ be a non-zero element in $(\mathrm{Ker} F)(x, y)$. Then $\gamma = \Sigma_{i=1}^r u_i m_i v_i$, where m_i is a mesh in $k[\Gamma]$ starting with x_i and ending with y_i, and $u_i \in \mathrm{Hom}_{k[\Gamma]}(y_i, y)$ and $v_i \in \mathrm{Hom}_{k[\Gamma]}(x, x_i)$. We may assume that the u_i and the v_i are non-zero. Then

Γ contains a path $x \rightsquigarrow x_i \rightsquigarrow y_i \rightsquigarrow y$, for each $1 \le i \le r$. Since Δ is convex in Γ, we see that $x_i, y_i \in \Delta$, and hence, $\gamma \in I_\Delta(x, y)$. This shows that $\mathrm{Ker} F = I_\Delta$. The proof of the lemma is completed.

Given a quiver $\Delta = (\Delta_0, \Delta_1)$ without oriented cycles, we can construct a translation quiver $\mathbb{Z}\Delta$ as follows. The set of vertices of $\mathbb{Z}\Delta$ is $\mathbb{Z} \times \Delta_0$; given an arrow $\alpha : x \to y$ in Δ, there are arrows $(n, \alpha) : (n, x) \to (n, y)$ and arrows $(n, \alpha') : (n+1, y) \to (n, x)$. Define $\tau(n, x) = (n+1, x)$, for any $x \in \Delta_0$. One says Δ is of type $\mathbb{A}$ if the underlying graph of Δ is $\mathbb{A}_n$, with $n \ge 1$, $\mathbb{A}_\infty$ or $\mathbb{A}_\infty^\infty$. In this case, $\mathbb{Z}\Delta$ will be simply written as $\mathbb{Z}\mathbb{A}$. A reduced walk w in $\mathbb{Z}\mathbb{A}$ is called *sectional* if the τ-orbits in w are pairwise distinct.

The following notion is well known; see, for example [55]. It shall play an important role in our later investigation.

1.4.4 Definition. A *wing* $\mathcal{W}$ is a translation quiver of the following shape



In such a wing $\mathcal{W}$, the vertex s is the unique source vertex, t is the unique sink vertex, while m is called the *wing vertex*. The unique path from s to m is a section formed by the projective vertices of $\mathcal{W}$, called the *left-most section*; and the unique path from m to t is a section formed by the injective vertices, called the *right-most section*. Observe that the left-most section and the right-most section have the same vertices, called the *rank* of $\mathcal{W}$. For convenience, we define a *wing of rank* 0 to be the empty set.

Let Γ be a translation quiver, which is a wing or of shape $\mathbb{Z}\mathbb{A}$. A *monomial mesh relation* in Γ is a path $\tau x \to y \to x$, where y is the only immediate predecessor of x in Γ. Given $x \in \Gamma$, one defines the *forward rectangle* R^x of x to be the full subquiver of $\varGamma$ generated by its successors y such that no path $x \rightsquigarrow y$ contains a monomial mesh relation. Dually, define the *backward rectangle* R_x of x in $\varGamma$. In particular, if $\varGamma$ is of shape $\mathbb{Z}\mathbb{A}_\infty^\infty$, then, by definition, R^x consists of all successors of x and R_x consists of all predecessors of x.

The following lemma is a slight extension of Lemma 1.1 in [48].

1.4.5 Lemma. *Let $\varGamma$ be a translation quiver which is a wing or of shape $\mathbb{Z}\mathbb{A}$. If $x, y \in \varGamma$, then $\mathrm{Hom}_{k(\varGamma)}(x, y) \neq 0$ if and only if $y \in R^x$ if and only if $x \in R_y$; and in this case, the class of every path from x to y forms a k-basis of $\mathrm{Hom}_{k(\varGamma)}(x, y)$.*

Proof. In case $\varGamma$ is of shape $\mathbb{Z}\mathbb{A}$, the statement is Lemma 1.1 in [48]. Let $\mathcal{W}$ be a wing of positive rank, which is considered as a convex subquiver of a translation quiver $\varGamma$ of shape $\mathbb{Z}\mathbb{A}_\infty$. Given $x, y \in \mathcal{W}$, by Lemma 1.4.3, we have

$$\mathrm{Hom}_{k(\mathcal{W})}(x, y) \cong \mathrm{Hom}_{k(\varGamma)}(x, y).$$

Since $\mathcal{W}$ is convex in $\varGamma$, by definition, y is in R^x in $\mathcal{W}$ if and only if y is in forward rectangle of x in $\varGamma$. Thus, the statement follows immediately from the previously considered case. The proof of the lemma is completed.

We conclude this section with more notions and terminology, which will be used later. Let $\varGamma$ stand for a translation quiver which is a wing or of shape $\mathbb{Z}\mathbb{A}_\infty$. In case $\varGamma$ is a wing of rank n, the vertices $t_i = \tau^{i-1}t$, where t is the unique sink vertex and $1 \leq i \leq n$, are called *quasi-simple*. In case $\varGamma$ is of shape $\mathbb{Z}\mathbb{A}_\infty$, then a vertex is called *quasi-simple* if it has only one immediate predecessor. Moreover, given a quasi-simple vertex s, there exists in $\varGamma$ a unique infinite sectional path starting in s, called the *ray* starting with s; and a unique infinite sectional path ending in s, called the *co-ray* ending with s.

The following statement is evident.

1.4.6 Lemma. *Let $\varGamma$ be a translation quiver which is a wing or of shape $\mathbb{Z}\mathbb{A}_\infty$, and let x be a vertex in $\varGamma$.*

(1) *There exists a sectional path $x_1 \to \cdots \to x_n = x$ with x_1 quasi-simple.*

(2) *There exists a sectional path $x = y_n \to \cdots \to y_1$ with y_1 quasi-simple.*

(3) *The convex hull* $\mathcal{W}_x$ *of* x_1, y_1 *in* Γ *is a wing of rank* n*, whose wing vertex is* x*.*

Let Γ be a translation quiver which is a wing or of shape $\mathbb{Z}\mathbb{A}_\infty$. Given a vertex $x \in \Gamma$, the rank of $\mathcal{W}_x$ is called the *quasi-length* of x, written as $\ell(x)$.

The following statement is easy to verify.

1.4.7 Lemma. *Let* Γ *be a translation quiver which is a wing or of shape* $\mathbb{Z}\mathbb{A}_\infty$*. There exists a partial order* $\preceq$ *over* Γ_0 *so that* $x \preceq y$ *if and only if* $\mathcal{W}_x \subseteq \mathcal{W}_y$*.*

Remark. Given $x, y \in \Gamma$, it is evident that $\mathcal{W}_x \subseteq \mathcal{W}_y$ if and only if $x \in \mathcal{W}_y$.

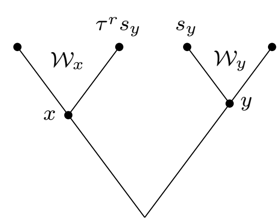
The following definition is important to our later investigation.

1.4.8 Definition. Let Γ be a translation quiver which is a wing or of shape $\mathbb{Z}\mathbb{A}_\infty$. Let $\mathcal{W}_x, \mathcal{W}_y$ be wings in Γ with source vertices s_x, s_y and sink vertices t_x, t_y, respectively. We shall say that $\mathcal{W}_x, \mathcal{W}_y$ are

(1) *comparable* if $\mathcal{W}_x \subseteq \mathcal{W}_y$ or $\mathcal{W}_y \subseteq \mathcal{W}_x$;

(2) *separable* if $\tau^r s_y = t_x$ for some $r \geq 2$, or $\tau^r s_x = t_y$ for some $r \geq 2$.

We illustrate Definition 1.4.8(2) by the following figure, where $r \geq 2$.



The following statement gives some properties of paths between two comparable vertices of Γ.

1.4.9 Lemma. *Let* Γ *be a translation quiver which is a wing or of shape* $\mathbb{Z}\mathbb{A}_\infty$*, and let* $x, y \in \Gamma$ *be comparable. If* Γ *has a path between* x *and* y*, then it is sectional and is the unique path in* Γ *between* x *and* y*.*

Proof. We shall consider only the case where $\mathcal{W}_x \subseteq \mathcal{W}_y$. In particular, $x \in \mathcal{W}_y$. Since $\mathcal{W}_y$ is convex in Γ, there exists a path in Γ between x and y if and only if there exists a path in $\mathcal{W}_y$ between x and y. Since y is the wing vertex of $\mathcal{W}_y$, the latter statement holds if and only if x lies on the left-most or the right-most section of $\mathcal{W}_y$; and in this case, $\mathcal{W}_y$ contains a unique path between x and y, which is sectional. The proof of the lemma is completed.

In view of Lemma 1.4.9, we have the following definition.

1.4.10 DEFINITION. Let Γ be a translation quiver which is a wing or of shape $\mathbb{Z}\mathbb{A}_\infty$. A chain $\mathcal{S} : x_1 \prec x_2 \cdots \prec x_m$ in Γ is called *sectional* if Γ contains a path between x_t and x_{t+1}, $t = 1, \ldots, m-1$.

The following statement will be used to characterize the τ-rigid pair, which will be introduced in Chapter 5.

1.4.11 PROPOSITION. *Let Γ be a translation quiver which is a wing or of shape $\mathbb{Z}\mathbb{A}_\infty$. Given $x, y \in \Gamma$, the following statements are equivalent.*

(1) *$\mathcal{W}_x, \mathcal{W}_y$ are comparable or separable.*

(2) *$y \notin R_{\tau x}$ and $y \notin R^{\tau^- x}$.*

(3) *$x \notin R_{\tau y}$ and $x \notin R^{\tau^- y}$.*

Proof. By definition, $y \in R_{\tau x}$ if and only if $\tau x \in R^y$ if and only if $x \in R^{\tau^- y}$. Therefore, $y \notin R_{\tau x}$ if and only $x \notin R^{\tau^- y}$. Similarly, $y \notin R^{\tau^- x}$ if and only if $x \notin R_{\tau y}$. This shows the equivalence of Statement (2) and Statement (3).

For proving the equivalence between Statements (1) and (2), we shall first consider the case where Γ is of shape $\mathbb{Z}\mathbb{A}_\infty$. Fix $x, y \in \Gamma$. Let s_x, s_y be the source vertices and t_x, t_y the sink vertices of the wings $\mathcal{W}_x$ and $\mathcal{W}_y$, respectively. Assume that the ray starting with s_x has the form

$$s_x = m_1 \to m_2 \to \cdots \to m_r \to m_{r+1} \to \cdots,$$

where $m_r = x$. Then the right-most section of $\mathcal{W}_x$ is

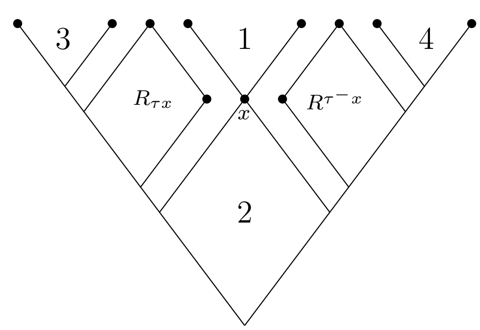
$$m_r \to \tau^{-1} m_{r-1} \to \cdots \to \tau^{-(r-i)} m_i \to \cdots \to \tau^{-(r-2)} m_2 \to \tau^{-(r-1)} m_1 = t_x.$$

On the other hand, since the above ray is a section, $y = \tau^n m_i$, for some $n \in \mathbb{Z}$ and some $i \geq 1$. In particular, $t_y = \tau^n t_x$.

Assume first that $1 \leq i \leq r$. Suppose that $-(r-i) \leq n \leq 0$. Then $y \in \mathcal{W}_x$, and hence, $\mathcal{W}_y, \mathcal{W}_x$ are comparable. In this situation, Statements (1) and (2) both hold. Suppose now that $n > 0$. Then y is a predecessor of τx, and in particular, $y \notin R^{\tau^- x}$ and $\mathcal{W}_x, \mathcal{W}_y$ are not comparable. If $y \notin R_{\tau x}$, then Γ contains a path $y \rightsquigarrow \tau s \rightarrow z \rightarrow s \rightsquigarrow \tau x$, where s is quasi-simple. In this case, $s = \tau^a m_1$ for some $a \geq 1$, and $\tau s = \tau^{-b} t_y$ for some $b \geq 0$. This yields $t_y = \tau^{b+1+a} s_x$ with $b + a + 1 \geq 2$, and hence, $\mathcal{W}_y, \mathcal{W}_x$ are separable. Conversely, if $\mathcal{W}_y, \mathcal{W}_x$ are separable, then $t_y = \tau^p s_t$ with $p \geq 2$. Thus, Γ contains a path $y \rightsquigarrow t_y \rightarrow z \rightarrow \tau^- t_y \rightsquigarrow \tau^{-p} t_y = s_x \rightsquigarrow x$. Hence, $y \notin R_{\tau x}$. This establishes the equivalence of Statements (1) and (2) in this situation. Similarly, Statements (1) and (2) are equivalent in case $n < -(r-i)$. Assume now that $i > r$. It is similar to show that Statements (1) and (2) are equivalent in this situation.

Now consider the case where Γ is a wing. It can be viewed as a convex subquiver of a translation quiver of shape $\mathbb{Z}\mathbb{A}_\infty$. Then the statement follows from the properties of forward rectangles and backward rectangles. The proof of the proposition is completed.

We shall give the following sketch to illustrate the above proposition in case Γ is a wing.



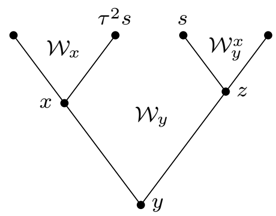
In the above figure, a vertex $y \in \Gamma$ lies in the region 1 if and only if $\mathcal{W}_y \subseteq \mathcal{W}_x$; and y lies in the region 2 if and only if $\mathcal{W}_x \subseteq \mathcal{W}_y$; and y lies in the region 3 or 4 if and only if $\mathcal{W}_y$ and $\mathcal{W}_x$ are separable.

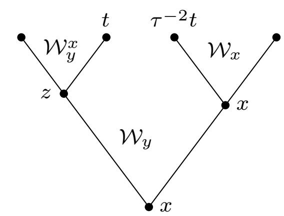
The following definition is essential for our later investigation.

1.4.12 Definition. Let Γ be a translation quiver which is a wing or of shape $\mathbb{Z}\mathbb{A}_\infty$. Let $x, y \in \Gamma$ with $x \prec y$ such that x, y are connected by a path. We define $\mathcal{W}_y^x$ to be the maximal wing contained in $\mathcal{W}_y$ so that $\mathcal{W}_x$ and $\mathcal{W}_y^x$ are separable.

Remark. If x, y with $x \prec y$ are connected by an arrow, then $\mathcal{W}_y^x = \emptyset$.

Definition 1.4.12 can be illustrated by the following figures.





Chapter 2

Auslander-Reiten Theory

The notion of Auslander-Reiten sequences was first introduced by M. Auslander and I. Reiten in [8, 9] in 1970s. Since then, it has been playing an important role in the representation theory of artin algebras. Later on, it has been developed in abelian categories; see [11, 13], and triangulated categories; see [31, 51]. Until now the Auslander-Reiten theory has been extended to Krull-Schmidt categories; see [12, 46].

2.1 Auslander-Reiten categories

The objective of this section is to recall the Auslander-Reiten theory in a Krull-Schmidt category from [9, 10, 12, 46].

Throughout this section, $\mathcal{A}$ stands for a Hom-finite Krull-Schmidt k-category. The *Jacobson radical* of $\mathcal{A}$ is the two-sided ideal $\text{rad}\mathcal{A}$ such, for all $X, Y \in \mathcal{A}$, that $\text{rad}(X,Y) = \{h \in \text{Hom}_{\mathcal{A}}(X,Y) \mid 1_X - g \circ h \text{ invertible for all } g \in \text{Hom}_{\mathcal{A}}(Y,X)\}$. Define $\text{rad}^2(X,Y)$ to be the k-subspace of $\text{rad}(X,Y)$ consisting of all finite sums of morphisms of the form gf, where $f \in \text{rad}(X,Z)$ and $g \in \text{rad}(Z,Y)$.

Let $f : X \to Y$ be a morphism in $\mathcal{A}$. One says that f is a *section* if there is a morphism $g : Y \to X$ such that $gf = 1_X$; and a *retraction* if there is a morphism $h : Y \to X$ such that $fh = 1_Y$. Moreover, f is called *irreducible* if f is neither a section nor a retraction while every factorization $f = hg$ implies that g is a section or h is a retraction. Finally, f is *left almost split* if f is not a section and every non-section morphism $g : X \to L$ factors through f; *left minimal* if any endomorphism $h : Y \to Y$ such that $hf = f$ is an automorphism; and *source morphism* if it is left minimal and left almost split. Dually, one says that f is *right almost split, right minimal* and a *sink* morphism. Note that a source morphism is originally called a *minimal left almost split morphism*, whereas a sink morphism is called a *minimal right almost split morphism* in [9, 10].

A sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ of morphisms in $\mathcal{A}$ is called a *short pseudo-exact sequence* if the following two conditions are satisfied:

(1) f is a *pseudo-kernel* of g, that is, for any object $M \in \mathcal{A}$, we have an exact sequence

$$\mathrm{Hom}_{\mathcal{A}}(M, X) \xrightarrow{f_*} \mathrm{Hom}_{\mathcal{A}}(M, Y) \xrightarrow{g_*} \mathrm{Hom}_{\mathcal{A}}(M, Z);$$

(2) g is a *pseudo-cokernel* of f, that is, for any object $N \in \mathcal{A}$, we have an exact sequence

$$\mathrm{Hom}_{\mathcal{A}}(Z, N) \xrightarrow{g^*} \mathrm{Hom}_{\mathcal{A}}(Y, N) \xrightarrow{f^*} \mathrm{Hom}_{\mathcal{A}}(X, N).$$

The next two definitions are quoted from [46].

2.1.1 Definition. A short pseudo-exact sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{A}$ with $Y \neq 0$ is called an *Auslander-Reiten sequence* if f is a source morphism and g is a sink morphism.

Remark. If $X \xrightarrow{f} Y \xrightarrow{g} Z$ is an Auslander-Reiten sequence in $\mathcal{A}$, then it is unique up to isomorphism for X and unique for Z. Write $\tau_{\mathcal{A}} Z = X$ and $\tau_{\mathcal{A}}^{-} X = Z$.

2.1.2 Definition. We shall say that $\mathcal{A}$ is an *Auslander-Reiten category* if, for each indecomposable object X in $\mathcal{A}$, the following two statements hold.

(1) Either X is the starting term of an Auslander-Reiten sequence or there is a source epimorphism $f : X \to Y$;

(2) Either X is the ending term of an Auslander-Reiten sequence or there is a sink monomorphism $g : Y \to X$.

The following result is quoted from [46, (1.5), (6.1)]. For more details on Auslander-Reiten theory in abelian categories and in triangulated categories, we refer the reader to [9] and [31].

2.1.3 Lemma. (1) *If $\mathcal{A}$ is abelian, then* $X \xrightarrow{f} Y \xrightarrow{g} Z$ *is an Auslander-Reiten sequence if and only if* $0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$ *is an almost split sequence as defined in [9].*

(2) *If $\mathcal{A}$ is triangulated with a shift functor* [1], *then* $X \xrightarrow{f} Y \xrightarrow{g} Z$ *is an Auslander-Reiten sequence in $\mathcal{A}$ if and only if it can be embedded in an Auslander-Reiten triangle* $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$.

The following lemma is well known; see, for example, [10, (V.1(1.7))].

2.1.4 Lemma. *Let $\mathcal{A}$ be an Auslander-Reiten category.*

(1) *If $f : X \to Y$ is a source morphism in $\mathcal{A}$, then X is indecomposable.*

(2) *If $g : Y \to Z$ is a sink morphism in $\mathcal{A}$, then Z is indecomposable.*

The following statement is well known; see, for example, [46, Section 1].

2.1.5 Lemma. *Let $\mathcal{A}$ be an Auslander-Reiten category. If $f : X \to Y$ is a source morphism or sink morphism in $\mathcal{A}$, then f is irreducible if and only if $f \neq 0$.*

We refer the following result to [12, (3.4)(3.8)].

2.1.6 Lemma. *Let $\mathcal{A}$ be an Auslander-Reiten category. If*

$$\begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix} : X \to \oplus_{i=1}^n Y_i$$

is a source morphism in $\mathcal{A}$ with $Y_i \neq 0$, then

$$\begin{bmatrix} f_{i_1} \\ \vdots \\ f_{i_s} \end{bmatrix}$$

is irreducible for any subset $\{f_{i_1}, \cdots, f_{i_s}\}$ *of* $\{f_1, \cdots, f_n\}$.

We conclude this section with the *Auslander-Reiten quiver* $\Gamma_{\mathcal{A}}$ of $\mathcal{A}$, which is a translation quiver defined as follows. The vertex set is a chosen complete set of representatives of the isomorphism classes of indecomposable objects in $\mathcal{A}$. Given vertices X, Y, the number of arrows from X to Y is the k-dimension of

$$\mathrm{Irr}(X, Y) := \mathrm{rad}(X, Y)/\mathrm{rad}^2(X, Y).$$

The translation $\tau_{\mathcal{A}}$, called the *Auslander-Reiten translation*, is such that $\tau_{\mathcal{A}} Z = X$ if and only if $\mathcal{A}$ has an Auslander-Reiten sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$.

One says that $X \in \mathcal{A}$ is *basic* if $X \cong X_1 \oplus \cdots \oplus X_t$, where $X_1, \ldots, X_t$ are pairwise different objects in $\Gamma_{\mathcal{A}}$; and in this case, $\{X_1, \ldots, X_t\}$ is called the *corresponding set* of X in $\Gamma_{\mathcal{A}}$.

Given two connected components Γ, Ω of $\Gamma_{\mathcal{A}}$, we write $\mathrm{Hom}_{\mathcal{A}}(\Gamma, \Omega) = 0$, if $\mathrm{Hom}_{\mathcal{A}}(M, N) = 0$ for all $M \in \Gamma$ and $N \in \Omega$; and say that Γ, Ω are *orthogonal* if $\mathrm{Hom}_{\mathcal{A}}(\Gamma, \Omega) = 0 = \mathrm{Hom}_{\mathcal{A}}(\Omega, \Gamma)$.

2.2 Standard Auslander-Reiten components

The main objective of this section is to study subcategories of an Auslander-Reiten category generated by the objects of a convex subquiver of a standard component of its Auslander-Reiten quiver.

Throughout this section, let $\mathcal{A}$ stand for an Auslander-Reiten category. Let $\Gamma_{\mathcal{A}}$ be the Auslander-Reiten quiver, and $\tau_{\mathcal{A}}$ be the Auslander-Reiten translation of $\mathcal{A}$. Let Δ be a convex subquiver of $\Gamma_{\mathcal{A}}$. We shall denote by $\mathcal{A}(\Delta)$ the full subcategory of $\mathcal{A}$, whose objects are the vertices of Δ; and by $\mathrm{add}\Delta$ the full subcategory of $\mathcal{A}$, whose objects are the finite direct sums of objects of Δ. Observe that Δ itself is a translation quiver with mesh category $k(\Delta)$. One says that Δ is *standard* if there exists an isomorphism $\phi : k(\Delta) \to \mathcal{A}(\Delta)$, acting identically on the objects; see, for example, [47, (1.2)]. A connected component of $\Gamma_{\mathcal{A}}$ is called *standard* if it is standard as a convex subquiver of $\Gamma_{\mathcal{A}}$.

2.2.1 Lemma. *Let $\mathcal{A}$ be an Auslander-Reiten category and Γ be a standard component of $\Gamma_{\mathcal{A}}$. If Δ is a convex subquiver of Γ, then Δ is also standard.*

Proof. Let $\phi : k(\Gamma) \to \mathcal{A}(\Gamma)$ be an isomorphism, acting identically on the objects. Let Δ be a convex subquiver of Γ. Denote by $k\{\Delta\}$ the full subcategory of $k(\Gamma)$ generated by the objects in Δ. Restricting ϕ yields an isomorphism $\phi_\Delta : k\{\Delta\} \to \mathcal{A}(\Delta)$. By Lemma 1.4.3, there is an isomorphism $F : k(\Delta) \to k\{\Delta\}$ acting identically on objects of Δ. Thus, the composite of F and ϕ_Δ is an isomorphism from $k(\Delta)$ to $\mathcal{A}(\Delta)$ acting identically on objects. The proof of the lemma is completed.

2.2.2 Lemma. *Let $\mathcal{A}$ be an Auslander-Reiten category and Γ be a standard component of $\Gamma_{\mathcal{A}}$. If Δ is a convex subquiver of Γ, then* addΔ *is an Auslander-Reiten category.*

Proof. First of all, every Auslander-Reiten sequence $X \longrightarrow Y \longrightarrow Z$ in $\mathcal{A}$ with X or Z in Γ is an Auslander-Reiten sequence in addΓ. Moreover, a source epimorphism $f : X \to Y$ in $\mathcal{A}$ with $X \in \Gamma$ is a source epimorphism in addΓ, a sink monomorphism $g : Y \to Z$ in $\mathcal{A}$ with $Z \in \Gamma$ is an sink monomorphism in addΓ. This is, addΓ is also an Auslander-Reiten category and $\Gamma_{\mathrm{add}\Gamma} = \Gamma$.

Now, assume that Δ is a convex subquiver of Γ. Let Δ' be the full subquiver of Γ generated by the objects $M \notin \Delta$. Let $\phi : \mathrm{add}\Delta \to \mathrm{add}\Gamma/\mathrm{add}\Delta'$ be the composite functor of the canonical embeding $\mathrm{add}\Delta \to \mathrm{add}\Gamma$ and the canonical projection $\mathrm{add}\Gamma \to \mathrm{add}\Gamma/\mathrm{add}\Delta'$. We claim that ϕ is an equivalence. Indeed, since the objects of addΔ' are zero objects in $\mathrm{add}\Gamma/\mathrm{add}\Delta'$, we see that ϕ is dense. Since ϕ is evidently full, it remains to prove that ϕ is faithful. Suppose that this is not the case. That is, there exists a non-zero morphism $f : X \to Y \in \mathrm{add}\Delta$ such that $\phi(f) = 0$. Then, $f = gh$ with morphisms $h : X \to Z$ and $g : Z \to Y$, where Z is a non-zero object of addΔ'. Write $Z = \oplus_{i=1}^n Z_i$, where $Z_1, \ldots, Z_n \in \Delta'$, and $g = (g_1, \ldots, g_n)$ and $h = (h_1, \ldots, h_n)^T$ with $g_i : Z_i \to Y$ and $h_i : X \to Z_i$, for $i = 1, \cdots, n$. Since $f \neq 0$, we have $g_i h_i \neq 0$ for some $1 \leq i \leq n$. Being standard, Γ contains a path $X \rightsquigarrow Z_i \rightsquigarrow Y$. Since Δ is convex in Γ, we have $Z_i \in \Delta$, which is a contradiction. This establishes our claim. By Proposition 2.9 in [46], it follows that addΔ is also an Auslander-Reiten category. The proof of the lemma is completed.

Let Γ be a translation quiver. Given a path p in the path category $k[\Gamma]$, we denote by $\bar{p}$ its image in the mesh category $k(\Gamma)$.

2.2.3 Lemma. *Let $\mathcal{A}$ be an Auslander-Reiten category, and let Δ be a convex subquiver of a standard component Γ of $\Gamma_{\mathcal{A}}$. If $\sigma : k(\Delta) \to \mathcal{A}(\Delta)$ is an isomorphism acting identically on objects, then $\sigma(\bar{\alpha})$ is in* $\mathrm{rad}(X, Y)$, *for any arrow $\alpha : X \to Y \in \Delta$.*

Proof. Let $\sigma : k(\Delta) \to \mathcal{A}(\Delta)$ be an isomorphism acting identically on objects. Let $\alpha : X \to Y$ be an arrow in Δ. Set $u = \sigma(\bar{\alpha})$. Since X, Y are indecomposable, we need only to show that u is not an isomorphism. Suppose that $v : Y \to X$ is a morphism in $\mathrm{add}\Delta$ such that $vu = 1_X$. Then there are some pairwise different paths $q_i : Y \rightsquigarrow X$ such that $v = \Sigma_{i=1}^n \lambda_i \, \sigma(\bar{q}_i)$, where $0 \neq \lambda_i \in k$. Thus, $\sigma(\bar{\varepsilon}_X) = 1_X = vu = \Sigma_{i=1}^n \sigma(\lambda_i \bar{q}_i \bar{\alpha})$. That is, $\bar{\varepsilon}_X = \Sigma_{i=1}^n \lambda_i \bar{q}_i \bar{\alpha}$, that is, $\varepsilon_X - \Sigma_{i=1}^n \lambda_i q_i \alpha \in I_\Delta$, which is absurd. The proof of the lemma is completed.

Given an object $X \in \Delta$, denote by X^+ the set of the arrows in Δ starting with X and by X^- the set of the arrows in Δ ending with X.

2.2.4 Lemma. *Let $\mathcal{A}$ be an Auslander-Reiten category, and let Δ be a convex subquiver of a standard component Γ of $\Gamma_{\mathcal{A}}$. If $\sigma : k(\Delta) \to \mathcal{A}(\Delta)$ is an isomorphism acting identically on objects, then the following statements hold.*

(1) *Let X be an object of Δ. Then* $\mathrm{add}\Delta$ *has a source morphism*

$$\sigma(X^+) = \begin{pmatrix} \sigma(\bar{\alpha}_1) \\ \vdots \\ \sigma(\bar{\alpha}_n) \end{pmatrix} : \; X \longrightarrow \textstyle\bigoplus_{i=1}^n Y_i \, ,$$

where $\alpha_i : X \to Y_i$, $i = 1, \ldots, n$, are the arrows of X^+.

(2) *Let Z be an object of Δ. Then* $\mathrm{add}\Delta$ *has a sink morphism*

$$\sigma(Z^-) = (\sigma(\bar{\beta}_1), \cdots, \sigma(\bar{\beta}_n)) : \; \textstyle\bigoplus_{i=1}^n Y_i \longrightarrow Z \, ,$$

where $\beta_i : Y_i \to Z$, $i = 1, \ldots, n$, are the arrows of Z^-.

(3) *If Δ has a mesh starting at X and ending at Z, then $\mathcal{A}$ has an Auslander-Reiten sequence $X \xrightarrow{\sigma(X^+)} Y \xrightarrow{\sigma(Z^-)} Z$.*

In particular, $\Gamma_{\mathrm{add}\Delta} = \Delta$, which is a translation subquiver of Γ.

Proof. Let $\sigma : k(\Delta) \to \mathcal{A}(\Delta)$ be an isomorphism acting identically on objects. We shall first prove Statement (3). Assume that Δ contains a mesh as follows.

$$\begin{array}{ccccc} & & Y_1 & & \\ & {\scriptstyle \alpha_1}\nearrow & \vdots & \searrow{\scriptstyle \beta_1} & \\ X & & & & Z \\ & {\scriptstyle \alpha_n}\searrow & \vdots & \nearrow{\scriptstyle \beta_n} & \\ & & Y_n & & \end{array} .$$

Set $\sigma(X^+) = (\sigma(\bar{\alpha}_1), \cdots, \sigma(\bar{\alpha}_n))^T$ and $\sigma(Z^-) = (\sigma(\bar{\beta}_1), \cdots, \sigma(\bar{\beta}_n))$. We claim that $\sigma(X^+)$ is a source morphism in $\mathcal{A}$ and $\sigma(Z^-)$ is a sink morphism in $\mathcal{A}$. First of all, by the property of Auslander-Reiten quiver, there is a source morphism $f : X \to \oplus_{i=1}^n Y_i$ in $\mathcal{A}$. Then $f \neq 0$; see, [46, (1.1)]. Write $f = (f_1, \cdots, f_n)$. By the equivalence, there are some paths $\gamma_{ij} : X \rightsquigarrow Y_i$, $j = 1, \cdots, n_j$, such that $f_i = \Sigma_{j=1}^{n_j} \lambda_{ij} \sigma(\bar{\gamma}_{ij})$, where $0 \neq \lambda_{ij} \in k$, for all $i = 1, \cdots, n$. Obviously, each γ_{ij} factors some arrows of X^+. It gives us that f_i factors through $\sigma(X^+)$, for all $i = 1, \cdots, n$. Hence, there is a morphism $h : \oplus_{i=1}^n Y_i \to \oplus_{i=1}^n Y_i$ such that $f = h\sigma(X^+)$. By Lemma 2.1.5, f is irreducible. By Lemma 2.2.3, $\delta(X^+)$ is not a section. Thus, h is a retraction. Hence, h is an automorphism. It follows that $\sigma(X^+)$ is a source morphism in $\mathcal{A}$. Similarly, we could show that $\sigma(Z^-)$ is a non-zero sink morphism in $\mathcal{A}$. Thus, our claim is true. By the property of Auslander-Reiten quiver, $\mathcal{A}$ has an Auslander-Reiten sequence $X \xrightarrow{\sigma(X^+)} \bigoplus_{i=1}^n Y_i \xrightarrow{g'} Z$ with Z indecomposable. Note that $\sigma(Z^-)\sigma(X^+) = \sigma(\Sigma_{i=1}^n \bar{\beta}_i \bar{\alpha}_i) = 0$. By the pseudo exactness, there is a morphism $h : Z \to Z$ such that $\sigma(Z^-) = hg'$. Thus we have the following commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{\sigma(X^+)} & \bigoplus_{i=1}^n Y_i & \xrightarrow{g'} & Z \\ \| & & \| & & \downarrow{\scriptstyle h} \\ X & \xrightarrow{\sigma(X^+)} & \bigoplus_{i=1}^t Y_i & \xrightarrow{\sigma(Z^-)} & Z. \end{array}$$

Since $\sigma(Z^-)$ is a non-zero sink morphism, by Lemma 2.1.5, it is also irreducible. On the other hand, g' is not a section. Thus, h is a retraction. Since Z is indecomposable, h is an automorphism. Therefore,

$$X \xrightarrow{\sigma(X^+)} \bigoplus_{i=1}^t Y_i \xrightarrow{\sigma(Z^-)} Z$$

is an Auslander-Reiten sequence in $\mathcal{A}$.

For proving statement (1), let X be an object of Δ such that $X^+ = \{\alpha_1, \cdots, \alpha_n\}$, where $\alpha_i : X \to Y_i$, for $i = 1, \cdots, n$. Denote by X^+_Γ the set of all the arrows in Γ starting at X. Thus, X^+ is a subset of X^+_Γ. By Statement (3), $\sigma(X^+_\Gamma)$ is a source morphism in $\mathcal{A}$. Set $\sigma(X^+) = (\sigma(\bar{\alpha}_1), \cdots, \sigma(\bar{\alpha}_n))^T$. By Lemma 2.1.6, $\sigma(X^+)$ is irreducible. Then, by similar discussion as the proof of Statement (1), $\sigma(X^+)$ is left almost split in addΔ and left minimal. Hence, $\sigma(X^+)$ is a source morphism in addΔ. The proof of Statement (2) is similar. Finally, it follows that $\Gamma_{\mathrm{add}\Delta} = \Delta$, which is a translation subquiver of Γ. The proof of this lemma is completed.

A connected subquiver Δ of a translation quiver Γ is called *sectional* if Δ meets each τ-orbit at most once.

2.2.5 LEMMA. *Let $\mathcal{A}$ be an Auslander-Reiten category, and let Δ be a finite convex sectional subquiver of a standard component of $\Gamma_{\mathcal{A}}$. If M is the direct sum of the objects in Δ, then* $\mathrm{End}_{\mathcal{A}}(M) \cong k\Delta$.

Proof. Write $H = \mathrm{End}_{\mathcal{A}}(M)$, where M is the direct sum of the objects of Δ. By Lemma 2.2.2 and Lemma 2.2.4, addΔ is an Auslander-Reiten category with $\Gamma_{\mathrm{add}\Delta} = \Delta$. Being a sectional subquiver of Γ, the translation quiver Δ contains no mesh. Therefore, $k[\Delta] = k(\Delta)$. Since Δ is standard by Lemma 2.2.1, we obtain $k[\Delta] \cong \mathcal{A}(\Delta)$. This yields the following isomorphisms:

$$\begin{aligned} k\Delta &= \textstyle\bigoplus_{X,Y\in\Delta_0} \mathrm{Hom}_{k[\Delta]}(X,Y) \\ &\cong \textstyle\bigoplus_{X,Y\in\Delta_0} \mathrm{Hom}_{\mathcal{A}(\Delta)}(X,Y) \\ &\cong \textstyle\bigoplus_{X,Y\in\Delta_0} \mathrm{Hom}_{\mathcal{A}}(X,Y) \\ &\cong H, \end{aligned}$$

where the first equation is the definition of a path algebra. The proof of the lemma is completed.

2.2.6 DEFINITION. Let Γ be a connected component of $\Gamma_{\mathcal{A}}$. A *wing* in Γ is a convex translation subquiver of Γ which is a wing.

In the rest of this section, we shall study the additive subcategory add$\mathcal{W}$ of $\mathcal{A}$ generated by the objects in a wing $\mathcal{W}$ in Γ.

2.2.7 Lemma. *Let $\mathcal{A}$ be an Auslander-Reiten category, and let $\mathcal{W}$ be a wing in a standard component of $\Gamma_{\mathcal{A}}$. If $X, Y \in \mathcal{W}$, then $\mathrm{Hom}_{\mathcal{A}}(X, Y) \neq 0$ if and only if $X \in R^Y$ if and only if $Y \in R_X$; and in this case, every path of irreducible morphisms in $\mathcal{A}$ from X to Y forms a k-basis of $\mathrm{Hom}_{\mathcal{A}}(X, Y)$.*

Proof. By Lemma 2.2.1, $\mathcal{W}$ is standard. Then there exists an isomorphism $\sigma : k(\mathcal{W}) \to \mathcal{A}(\mathcal{W})$, which acts identically on the objects. It follows that, for any $X, Y \in \mathcal{W}$, we have $\mathrm{Hom}_{k(\mathcal{W})}(X, Y) \cong \mathrm{Hom}_{\mathcal{A}(\mathcal{W})}(X, Y) \cong \mathrm{Hom}_{\mathcal{A}}(X, Y)$. It follows from Lemma 1.4.5 that $\mathrm{Hom}_{\mathcal{A}}(X, Y) \neq 0$ if and only if $X \in R^Y$ if and only if $Y \in R_X$. In this case, by Lemma 1.4.5 again, $\dim_k \mathrm{Hom}_{k(\mathcal{W})}(X, Y) = \dim_k \mathrm{Hom}_{\mathcal{A}}(X, Y) = 1$. Assume that we have a path of irreducible morphisms

$$X = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \longrightarrow X_{n-1} \xrightarrow{f_n} X_n = Y .$$

In particular, $f_i \neq 0$, for each $1 \leq i \leq n$. It is sufficient to show that $f_n \cdots f_1 \neq 0$. Since σ is an isomorphism, for each $1 \leq i \leq n$, there is a path $p_i : X_{i-1} \rightsquigarrow X_i \in \mathcal{W}$ such that $f_i = \lambda_i \sigma(\bar{p})$, for some $0 \neq \lambda_i \in k$. Since f_i is irreducible, by Lemma 2.2.3, we deduce that p_i is an arrow. Write $p_i = \alpha_i$, for $i = 1, \cdots, n$. By Lemma 1.4.5, we see that $\bar{\alpha}_n \cdots \bar{\alpha}_1$ forms a k-basis of $\mathrm{Hom}_{k(\mathcal{W})}(X, Y)$. It follows that $f_n \cdots f_1 = \lambda_1 \cdots \lambda_n \sigma(\bar{\alpha}_n) \cdots \sigma(\bar{\alpha}_1) \neq 0$. The proof of this lemma is completed.

2.2.8 Lemma. *Let $\mathcal{A}$ be an Auslander-Reiten category, and let $\mathcal{W}$ be a wing in a standard component of $\Gamma_{\mathcal{A}}$. Let $f : Y \to X$ be an irreducible morphism in* add$\mathcal{W}$. *If there is a non-zero morphism $u : Z \to Y$ such that $fu = 0$, then $Z \notin R^X$.*

Proof. Suppose that $Z \in R^X$. Since $u \neq 0$ and Lemma 2.2.7, we see that $Z \in R^Y$. Moreover, there is a path of irreducible morphisms

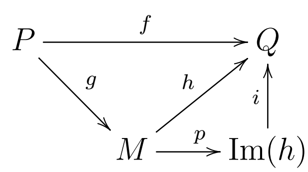
$$Y = Y_0 \xrightarrow{g_1} Y_1 \xrightarrow{g_2} \cdots \longrightarrow Y_{n-1} \xrightarrow{g_n} Y_n = Z$$

such that $g_n \cdots g_1 : Y \to Z$ forms a basis of $\mathrm{Hom}_{\mathcal{A}}(Y, Z)$. Thus, $u = \lambda g_n \cdots g_1$ for some $\lambda \in k$. Again by Lemma 2.2.7, we see that $f g_n \cdots g_1$ forms a basis of $\mathrm{Hom}_{\mathcal{A}}(Z, X)$, and hence it is not zero. Thus, $0 = fu = \lambda f g_n \cdots g_1$ gives us that $\lambda = 0$. It follows that $u = 0$, a contradiction. The proof of the lemma is completed.

For later use, we shall recall the following notions and terminology. Let H be a finite dimensional k-algebra. Denote by mod$H$ the category of finitely generated left H-modules and by proj$H$ the full subcategory of finitely generated projective left H-modules. The following statement is well known. For the reader's convenience, we shall include a short proof.

2.2.9 LEMMA. *Let H be a hereditary algebra and $f : P \to Q$ a morphism in* proj H. *Then f is irreducible in* proj$H$ *if and only if f is irreducible in* mod$H$.

Proof. We need only to show the necessity. Assume that $f : P \to Q$ is irreducible in proj H. Let $f = hg$, where $g : P \to M, h : M \to Q$ and $M \in \operatorname{mod} H$. Then we have the following commutative diagram.



where i is an injection, p is an epimorphism. Since H is hereditary and $\operatorname{Im}(h)$ is a submodule of Q, $\operatorname{Im}(h)$ is projective. Since f is irreducible in proj$H$, either pg is a section or i is a retraction. If pg is a section, then there is a morphism $s : \operatorname{Im}(h) \to P$ such that $spg = 1_P$ which yields g is a section. If i is a retraction, then there is a morphism $t : Q \to \operatorname{Im}(h)$ such that $it = 1_Q$. Since p is also a retraction, there is a morphism $p' : \operatorname{Im}(h) \to M$ such that $pp' = 1_{\operatorname{Im}(h)}$. Then $hp't = ipp't = 1_Q$, which yields that h is a retraction. Hence, f is irreducible in mod H. The proof of the lemma is completed.

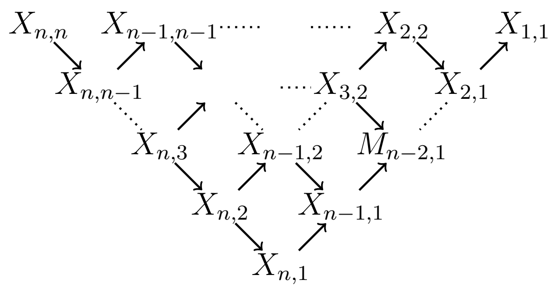
The following statement is crucial to our later investigation.

2.2.10 THEOREM. *Let $\mathcal{A}$ be an Auslander-Reiten category, and let $\mathcal{W}$ be a wing in a standard component of $\Gamma_{\mathcal{A}}$ with left-most section Δ. Set $H = \operatorname{End}_{\mathcal{A}}(M)$, where M is the direct sum of the objects on Δ. Then we have an equivalence*

$$\operatorname{Hom}_{\mathcal{A}}(M, -) : \operatorname{add}\mathcal{W} \to \operatorname{mod} H^{\mathrm{op}}$$

which induces an isomorphism $\mathcal{W} \to \Gamma_{H^{\mathrm{op}}}$ of translation quivers.

Proof. By Lemma 2.2.4, $\Gamma_{\operatorname{add}\mathcal{W}} = \mathcal{W}$. We shall need only to consider the case where $\mathcal{W}$ is of rank $n > 1$. Then, $\mathcal{W}$ can be depicted as follows.



The arrows are denoted by $\alpha_{i,j} : X_{i,j} \to X_{i,j-1}$, where $2 \le j \le i \le n$ and by $\beta_{i,j} : X_{i,j} \to X_{i-1,j}$, where $1 \le j < i \le n$. By Lemma 2.2.1, $\mathcal{W}$ is standard. Hence, there exists an isomorphism $\sigma : k(\mathcal{W}) \to \mathcal{A}(\mathcal{W})$, which acts identically on objects. By Lemma 2.2.4 $f_{i,j} = \sigma(\bar{\alpha}_{i,j}) : X_{i,j} \to X_{i,j-1}$ with $2 \le j \le i \le n$; and $g_{i,j} = \sigma(\bar{\beta}_{i,j}) : X_{i,j} \to X_{i-1,j}$ with $1 \le j < i \le n$ are irreducible morphisms in $\mathcal{A}$, which are fitted into the following Auslander-Reiten sequences:

$$X_{i,i} \xrightarrow{f_{i,i}} X_{i,i-1} \xrightarrow{g_{i,i-1}} X_{i-1,i-1}, \quad i = 2, \ldots, n.$$

and

$$X_{i,j} \xrightarrow{(f_{i,j}, g_{i,j})^T} X_{i,j-1} \oplus X_{i-1,j} \xrightarrow{(g_{i,j-1}, f_{i-1,j})} X_{i-1,j-1}\ ,\ 2 \le j < i \le n.$$

By the assumption, $M = X_{n,1} \oplus \cdots \oplus X_{n,n}$. Note that, $X_{n,t} \in R^{X_{i,j}}$ only if $t = i, i-1, \cdots, j$, for any $X_{i,j} \in \mathcal{W}$. For simplicity, we write $(-)^* = \operatorname{Hom}_{\mathcal{A}}(M, -)$. We shall split our proof into several statements.

(1) *The morphism* $f^*_{i,j} : X^*_{i,j} \to X^*_{i,j-1}$ *is a monomorphism, for* $2 \le j \le i \le n$. Indeed, assume that $h \in X^*_{i,j}$ such that $f^*_{i,j}(h) = 0$. We may write $h = (h_1, \cdots, h_n)$, where $h_m \in \operatorname{Hom}_A(X_{n,t}, X_{i,j})$, for all $t = 1, \ldots, n$. In particular, $f_{i,j} \circ h_t = 0$, for all $t = 1, \ldots, n$. Assume that $h_p \neq 0$, for some $p = 1, \cdots, n$. By Lemma 2.2.7, we have $X_{n,p} \in R^{X_{i,j}}$. Thus, $j \le p \le i$. In this case, $X_{n,p}$ is also in $R^{X_{i,j-1}}$. By Lemma 2.2.8, it follows that $h_p = 0$, a contradiction. Thus, $h = 0$ and hence, $f^*_{i,j}$ is a monomorphism.

(2) *The morphism* $g^*_{i,j} : X^*_{i,j} \to X^*_{i-1,j}$ *is an epimorphism, for* $1 \le j < i \le n$. Indeed, let $h \in X^*_{i-1,j}$. We may write $h = (h_1, \cdots, h_n)$ where $h_t \in \operatorname{Hom}_{\mathcal{A}}(X_{n,t}, X_{i-1,j})$, for all $1 \le t \le n$. Suppose that $h_p \neq 0$. Then by Lemma 2.2.7, we have $X_{n,p} \in R^{X_{i-1,j}}$. Thus, we see that $p = i-1, \cdots, j$. By Lemma

2.2.7, we see that $g_{i,j} \circ g_{i+1,j} \cdots \circ g_{n,j} \circ f_{n,j+1} \circ \cdots \circ f_{n,p}$ forms a k-basis of $\mathrm{Hom}_{\mathcal{A}}(X_{n,p}, X_{i-1,j})$. Thus, we have the following equations

$$\begin{aligned} h_p &= \lambda g_{i,j} \circ g_{i+1,j} \cdots \circ g_{n,j} \circ f_{n,j+1} \circ \cdots \circ f_{n,p} \\ &= g^*_{i,j}(\lambda g_{i+1,j} \cdots \circ g_{n,j} \circ f_{n,j+1} \circ \cdots \circ f_{n,p}), \end{aligned}$$

for some $\lambda \in k$. Hence, $g^*_{i,j}$ is an epimorphism.

(3) $\mathrm{Ker}(g^*_{i,j}) \subseteq \mathrm{Im}(f^*_{i,j+1})$, *for* $1 \leq j < i \leq n$. Indeed, let $u = (u_1, \cdots, u_n)$ be a morphism from M to $X_{i,j}$ such that $g_{i,j}(u) = 0$, where $u_t : X_{n,t} \to X_{i,j}$ for $1 \leq t \leq n$. Hence, we have $g_{i,j}u_t = 0$, for $1 \leq t \leq n$. Fix p with $1 \leq p \leq n$. If $u_p = 0$, then, trivially, the statement is true. Assume that $u_p \neq 0$. By Lemma 2.2.8, we see that $X_{n,p} \in R^{X_{i,j}} \backslash R^{X_{i-1,j}}$. Thus, $p = i$. In this case, by Lemma 2.2.7, we see that $f_{i,j+1} \cdots g_{i+1,i} \cdots g_{n-1,i}g_{n,i}$ forms a k-basis of $\mathrm{Hom}_{\mathcal{A}}(X_{n,i}, X_{i,j})$. Hence, $u_i = \lambda f_{i,j+1} \cdots g_{i+1,i} \cdots g_{n-1,i}g_{n,i}$ for some $\lambda \in k$. Clearly, u_i factors through $f_{i,j}$. Thus, we have

$$u = (0, \cdots, u_i, \cdots, 0) \in \mathrm{Im}(f^*_{i,j+1}).$$

(4) $\mathrm{Ker}(g^*_{i,j-1}, f^*_{i-1,j}) \subseteq \mathrm{Im}(f^*_{i,j}, g^*_{i,j})^T$, *for* $2 \leq j < i \leq n$. Indeed, let

$$\begin{bmatrix} u_1 & u_2 & \cdots & u_n \\ v_1 & v_2 & \cdots & v_n \end{bmatrix}$$

be a morphism from $M \to X_{i,j-1} \oplus X_{i-1,j}$, where $u_t : X_{n,t} \to X_{i,j-1}$ and $v_t : X_{n,t} \to X_{i-1,j}$, for $t = 1, \cdots, n$, such that

$$(g_{i,j-1}, f_{i-1,j}) \circ \begin{bmatrix} u_1 & u_2 & \cdots & u_n \\ v_1 & v_2 & \cdots & v_n \end{bmatrix} = 0.$$

Hence, $g_{i,j-1}u_t + f_{i-1,j}v_t = 0$, for $t = 1, \cdots, n$. It is sufficient to show that each $\begin{bmatrix} u_t \\ v_t \end{bmatrix}$ factors trough $\begin{bmatrix} f_{i,j} \\ g_{i,j} \end{bmatrix}$. For this purpose, we assume that $\begin{bmatrix} u_p \\ v_p \end{bmatrix} \neq 0$, for some $1 \leq p \leq n$. Thus, by Lemma 2.2.7, we see that $X_{n,p}$ is in $R^{X_{i,j-1}}$ or in $R^{X_{i-1,j}}$. Thus, $p = i, i-1, \cdots, j, j-1$. Consider $p = i$ or $j-1$. In this case, $X_{n,p} \notin R^{X_{i-1,j}}$. By Lemma 2.2.7, we have $v_p = 0$. It follows that $g_{i,j-1}u_p = 0$. By Statement (3), there is $(h_1, \cdots, h_n) : M \to X_{i,j}$, where $h_t : X_{n,t} \to X_{i,j}$, for $t = 1, \cdots, n$, such that $(0, \cdots, u_p, \cdots, 0) = f^*_{i,j}(h_1, \cdots, h_p, \cdots, h_n)$. In particular, $u_p = f_{i,j}h_p$. Moreover, by Lemma 2.2.7, $g_{i,j}h_p = 0$. Thus,

$$\begin{bmatrix} u_p \\ v_p \end{bmatrix} = \begin{bmatrix} f_{i,j}h_p \\ g_{i,j}h_p \end{bmatrix} \in \mathrm{Im} \begin{bmatrix} f^*_{i,j} \\ g^*_{i,j} \end{bmatrix}.$$

Now consider $p = i-1, \cdots, j$. By Lemma 2.2.7 agin, $f_{i,j}f_{i,j+1}\cdots f_{i,p}g_{i+1,p}\cdots g_{n,p}$ forms a k-basis of $\mathrm{Hom}_{\mathcal{A}}(X_{n,p}, X_{i,j-1})$ and $g_{i,j}f_{i,j+1}\cdots f_{i,p}g_{i+1,p}\cdots g_{n,p}$ forms a k-basis of $\mathrm{Hom}_{\mathcal{A}}(X_{n,p}, X_{i-1,j})$. Hence, we have

$$u_p = \lambda_p f_{i,j}f_{i,j+1}\cdots f_{i,p}g_{i+1,p}\cdots g_{n,p} \text{ and } v_p = \mu_p g_{i,j}f_{i,j+1}\cdots f_{i,p}g_{i+1,p}\cdots g_{n,p},$$

for some $\lambda_p, \mu_p \in k$. Then the following equations hold.

$$\begin{aligned} 0 &= g_{i,j-1}u_p + f_{i-1,j}v_p \\ &= \lambda_p g_{i,j-1}f_{i,j}f_{i,j+1}\cdots f_{i,p}g_{i+1,p}\cdots g_{n,p} \\ &+ \mu_p f_{i-1,j}g_{i,j}f_{i,j+1}\cdots f_{i,p}g_{i+1,p}\cdots g_{n,p} \\ &= (\lambda_p g_{i,j-1}f_{i,j} + \mu_p f_{i-1,j}g_{i,j})f_{i,j+1}\cdots f_{i,p}g_{i+1,p}\cdots g_{n,p} \\ &= (\lambda_p - \mu_p)g_{i,j-1}f_{i,j}f_{i,j+1}\cdots f_{i,p}g_{i+1,p}\cdots g_{n,p}, \end{aligned}$$

where the last equation follows from that $g_{i,j-1}f_{i,j} + f_{i-1,j}g_{i,j} = 0$. Note that $f_{i,j+1}\cdots f_{i,p}g_{i+1,p}\cdots g_{n,p}$ forms a basis of $\mathrm{Hom}_{\mathcal{A}}(X_{n,p}, X_{i,j})$, and in particular it is non zero. This gives us $\lambda_p = \mu_p$. Thus,

$$\begin{bmatrix} u_p \\ v_p \end{bmatrix} = \begin{bmatrix} f^*_{i,j} \\ g^*_{i,j} \end{bmatrix} (\lambda_p f_{i,j+1}\cdots f_{i,p}g_{i+1,p}\cdots g_{n,p}) \in \mathrm{Im} \begin{bmatrix} f^*_{i,j} \\ g^*_{i,j} \end{bmatrix}.$$

(5) *The functor* $(-)^*$ *maps every Auslander-Reiten sequence in* $\mathrm{add}\mathcal{W}$ *to Auslander-Reiten sequences in* $\mathrm{mod}H^{\mathrm{op}}$.

Indeed, first of all, since $g_{i,i-1}f_{i,i} = 0$, by the pseudo-exactness, we have $g^*_{i,i-1}f^*_{i,i} = 0$. Then it follows immediately from Statement (1), (2) and (3) that, for each $1 \leq i \leq n$, we have a short exact sequence

$$0 \longrightarrow X^*_{i,i} \xrightarrow{f^*_{i,i}} X^*_{i,i-1} \xrightarrow{g^*_{i,i-1}} X^*_{i-1,i-1} \longrightarrow 0 .$$

Again, by Statement (1) and (2), we see that $(f^*_{i,j}, g^*_{i,j})^T$ is a monomorphism and $(g^*_{i,j-1}, f^*_{i-1,j})$ is an epimorphism. Since $(g_{i,j-1}, f_{i-1,j}) \circ (f_{i,j}, g_{i,j})^T = 0$ by the pseudo-exactness, we have $(g^*_{i,j-1}, f^*_{i-1,j}) \circ (f^*_{i,j}, g^*_{i,j})^T = 0$. By Statement (4), we see that, for each pair (i,j) with $2 \leq j < i \leq n$, we have a short exact sequence

$$0 \longrightarrow X^*_{i,j} \xrightarrow{(f^*_{i,j}, g^*_{i,j})^T} X^*_{i,j-1} \oplus X^*_{i-1,j} \xrightarrow{(g^*_{i,j-1}, f^*_{i-1,j})} X^*_{i-1,j-1} \longrightarrow 0.$$

Secondly, we shall show that all the above exact sequences are Auslander-Reiten sequences. By Lemma 2.2.5, we know that $H = k\Delta$, and consequently,

$H^{\text{op}} = k\Delta^{\text{op}}$. Hence, the Auslander-Reiten quiver of H^{op} is a wing having the same rank as $\mathcal{W}$. In view of Proposition 2.3 in [43], $X^*_{n,n}, X^*_{n,n-1}, \cdots, X^*_{n,1}$ are the non-isomorphic indecomposable projective modules of modH^{op} and the morphisms $f^*_{n,n}, f^*_{n,n-1}, \cdots, f^*_{n,1}$ are irreducible monomorphism in proj H^{op}. Moreover, by Lemma 2.2.9, we see that $f^*_{n,n}, f^*_{n,n-1}, \cdots, f^*_{n,1}$ are also irreducible monomorphisms in modH^{op}. This gives us a path of irreducible monomorphisms

$$X^*_{n,n} \xrightarrow{f^*_{n,n}} X^*_{n,n-1} \xrightarrow{f^*_{n,n-1}} \cdots \longrightarrow X^*_{n,2} \xrightarrow{f^*_{n,2}} X^*_{n,1},$$

with $X^*_{n,n}$ simple. Since $X^*_{n,n}$ is projective simple and $f^*_{n,n}$ is irreducible, we see that $f^*_{n,n}$ is a source morphism. This gives us an Auslander-Reiten sequence

$$0 \longrightarrow X^*_{n,n} \xrightarrow{f^*_{n,n}} X^*_{n,n-1} \xrightarrow{g^*_{n,n-1}} X^*_{n-1,n-1} \longrightarrow 0.$$

In particular, $g^*_{n,n-1}$ is irreducible and $X^*_{n-,n-1}$ is indecomposable as well as simple. Since $f^*_{n,n-1}$ is a monomorphism and $g^*_{n,n-1}$ is an epimorphism, we have $X^*_{n,n-2} \ncong X^*_{n-1,n-1}$. Hence, $\begin{bmatrix} f^*_{n,n-1} \\ g^*_{n,n-1} \end{bmatrix}$ is irreducible. By the description of $\Gamma_{\text{mod}H^{\text{op}}}$, we see that $\begin{bmatrix} f^*_{n,n-1} \\ g^*_{n,n-1} \end{bmatrix}$ is a source morphism. Therefore,

$$0 \longrightarrow X^*_{n,n-1} \xrightarrow{\begin{bmatrix} f^*_{n,n-1} \\ g^*_{n,n-1} \end{bmatrix}} X^*_{n,n-2} \oplus X^*_{n-1,n-1} \xrightarrow{(g^*_{n,n-2}, f^*_{n-1,n-1})} X^*_{n-1,n-2} \longrightarrow 0$$

is an Auslander-Reiten sequence. In particular, $X^*_{n-1,n-2}$ is indecomposable and $g^*_{n,n-2}$ is irreducible. Thus, by induction on j where $n-1 \geq j \geq 2$, we have that

$$0 \longrightarrow X^*_{n,j} \xrightarrow{\begin{bmatrix} f^*_{n,j} \\ g^*_{n,j} \end{bmatrix}} X^*_{n,j-1} \oplus X^*_{n-1,j} \xrightarrow{(g^*_{n,j-1}, f^*_{n-1,j})} X^*_{n-1,j-1} \longrightarrow 0$$

is Auslander-Reiten sequence, for $n-1 \geq j \geq 2$. In particular, we obtain a path of irreducible monomorphisms in modH^{op}

$$X^*_{n-1,n-1} \xrightarrow{f^*_{n-1,n-1}} X^*_{n-1,n-2} \xrightarrow{f^*_{n-1,n-2}} \cdots \longrightarrow X^*_{n-1,2} \xrightarrow{f^*_{n-1,2}} X^*_{n-1,1}$$

with $X^*_{n-1,n-1}$ simple. Thus, we complete the proof of Statement (5) by doing the same discussion.

(6) *The functor* $(-)^*$: add$\mathcal{W} \to$ mod$H^{\rm op}$ *is an equivalence.* Indeed, by Statement (5), we have an isomorphism of translation quivers $G : \mathcal{W} \to \mathit{\Gamma}_{H^{\rm op}}$, where $G(X_{i,j}) = X^*_{i,j}$ and $G(\alpha_{i,j}) = \alpha^*_{i,j}$ and $G(\beta_{i,j}) = \beta^*_{i,j}$ for any $X_{i,j} \in \mathcal{W}$ and $\alpha_{i,j}, \beta_{i,j} \in \mathcal{W}$. Now we define a functor $\sigma_H : k(\mathit{\Gamma}_{\rm H^{op}}) \to {\rm ind}H^{\rm op}$, acting identically on objects, such that $\sigma_H(\bar{\alpha}^*_{i,j}) = f^*_{i,j}$ and $\sigma_H(\bar{\beta}^*_{i,j}) = g^*_{i,j}$, for all $\alpha^*_{i,j}, \beta^*_{i,j} \in \mathit{\Gamma}_{\rm H^{op}}$. Since $\mathit{\Gamma}_{\rm H^{op}}$ is also standard, we see that σ_H is an isomorphism. Thus, we have the following commutative diagram

$$\begin{array}{ccc} k(\mathcal{W}) & \xrightarrow{k(G)} & k(\mathit{\Gamma}_{H^{\rm op}}) \\ \big\downarrow{\scriptstyle \sigma} & & \big\downarrow{\scriptstyle \sigma_H} \\ \mathcal{A}(\mathcal{W}) & \xrightarrow{(-)^*} & {\rm ind}H^{\rm op} \end{array},$$

where $k(G)$ is an equivalence of mesh categories induced by G. It follows that ${\rm Hom}_{\mathcal{A}}(M,-) : \mathcal{A}(\mathcal{W}) \to {\rm ind}H^{\rm op}$ is an equivalence. Moreover, it is natural to define an additive functor ${\rm Hom}_{\mathcal{A}}(M,-)$: add$\mathcal{W} \to$ mod$H^{\rm op}$. By additivity, ${\rm Hom}_{\mathcal{A}}(M,-)$ is dense and fully faithful. Hence, ${\rm Hom}_{\mathcal{A}}(M,-)$ is an equivalence. From the proof, we see that ${\rm Hom}_{\mathcal{A}}(M,-)$ induces an isomorphism from $\mathcal{W}$ to $\mathit{\Gamma}_{H^{\rm op}}$. The proof of the theorem is completed.

Chapter 3

Cluster Categories

The aim of this chapter is to recall briefly the definition of the cluster category $\mathscr{C}(Q)$ associated with a strongly locally finite quiver Q. In the finite case, it was defined by Buan, Marsh, Reineke, Reiten and Todorov in [20], by taking a particular orbit category of the derived category of finite dimensional representations of Q. In the infinite Dynkin case, Liu-Paquette and Yang have showed in [48] and [62] that the same construction yields a triangulated category, whose cluster tilting subcategories form a cluster structure as defined in [21]. More recently, Št'ovíčěk and Roosmalen proved in [59] that the same result holds in the general strongly locally finite case.

3.1 Cluster categories in the general sense

The objective of this section is to recall from [21] and [48] some basic notions and terminology for general cluster categories.

Throughout this section, $\mathcal{A}$ shall stand for a Hom-finite Krull-Schmidt triangulated category with a shift functor [1]. Denote by $D = \mathrm{Hom}_k(-, k)$ the standard duality for the category of finite dimensional k-vector spaces. Let $\mathscr{T}$ be a full subcategory of $\mathcal{A}$ and X be an object in $\mathcal{A}$. A morphism $f : X \to T$ with $T \in \mathscr{T}$ is called a *left $\mathscr{T}$-approximation* of X if f induces an epimorphism

$$\mathrm{Hom}_{\mathcal{A}}(f, M) : \mathrm{Hom}_{\mathcal{T}}(T, M) \to \mathrm{Hom}_{\mathcal{A}}(X, M),$$

for any $M \in \mathscr{T}$; a morphism $g : T \to X$ with $T \in \mathscr{T}$ is called a *right $\mathscr{T}$-approximation* of X if g induces an epimorphism

$$\mathrm{Hom}_{\mathcal{A}}(M, g) : \mathrm{Hom}_{\mathcal{A}}(M, T) \to \mathrm{Hom}_{\mathcal{A}}(M, X),$$

for any $M \in \mathscr{T}$. One says that $\mathscr{T}$ is *covariantly finite* in $\mathcal{A}$ if every object in $\mathcal{A}$ admits a left $\mathscr{T}$-approximation; dually, one says that $\mathscr{T}$ is *contravariantly finite* in $\mathcal{A}$ if every object in $\mathcal{A}$ admits a right $\mathscr{T}$-approximation; and *functorially finite* in $\mathcal{A}$ if it is covariantly and contravariantly finite in $\mathcal{A}$.

A *Serre functor* for $\mathcal{A}$ is an auto-equivalence $\mathbb{S}$ of $\mathcal{A}$ such that, for any objects $X, Y \in \mathcal{A}$, there exists a natural isomorphism $\mathrm{Hom}_{\mathcal{A}}(X, Y) \cong D\mathrm{Hom}_{\mathcal{A}}(Y, \mathbb{S}X)$. If $\mathcal{A}$ has a Serre functor $\mathbb{S}$, then it is an Auslander-Reiten category whose Auslander-Reiten translation is given by $\mathbb{S} \circ [-1]$; see [53]. Moreover, one says that $\mathcal{A}$ is 2-*Calabi-Yau* if [2] is a Serre functor.

Now let $\mathcal{A}$ be a 2-Calabi-Yau triangulated category. One says that a full subcategory of $\mathcal{A}$ is *strictly additive* if it is closed under isomorphisms, taking finite direct sums and taking direct summands. Let $\mathscr{T}$ be a strictly additive subcategory of $\mathcal{A}$. In particular, $\mathscr{T}$ is Krull-Schmidt. The *quiver* of $\mathscr{T}$ is defined to be the underlying quiver of its Auslander-Reiten quiver. Moreover, given an indecomposable object M of $\mathscr{T}$, denote by $\mathscr{T}_M$ the full additive subcategory of $\mathscr{T}$ generated by the indecomposable objects not isomorphic to M. Observe that $\mathscr{T}_M$ is also strictly additive in $\mathcal{A}$.

3.1.1 Definition. [21] Let $\mathcal{A}$ be a 2-Calabi-Yau triangulated k-category. A non-empty collection $\mathfrak{C}$ of strictly additive subcategories of $\mathcal{A}$ is called a *cluster structure* if, for each subcategory $\mathscr{T} \in \mathfrak{C}$ and each indecomposable object $M \in \mathscr{T}$, the following conditions are verified.

(1) There exists a unique (up to isomorphism) indecomposable object M^* of $\mathcal{A}$, with $M^* \ncong M$, such that the additive subcategory $\mu_M(\mathscr{T})$ of $\mathcal{A}$ generated by $\mathscr{T}_M$ and M^* belongs to $\mathfrak{C}$.

(2) There exist two exact triangles in $\mathcal{A}$ as follows :

$$M \xrightarrow{f} N \xrightarrow{g} M^* \longrightarrow M[1] \text{ and } M^* \xrightarrow{u} L \xrightarrow{v} M \longrightarrow M^*[1]$$

where f, u are minimal left $\mathscr{T}_M$-approximations, and g, v are minimal right $\mathscr{T}_M$-approximations in $\mathcal{A}$.

(3) The quiver of $\mathscr{T}$ contains no oriented cycle of length one or two, from which the quiver of $\mu_M(\mathscr{T})$ is obtained by the Fomin-Zelevinsky mutation at M as described in [24, (1.1)].

Let $\mathcal{A}$ be a 2-Calabi-Yau triangulated category with a strictly additive subcategory $\mathscr{T}$. Given $X \in \mathcal{A}$, write $\mathrm{Hom}_{\mathcal{A}}(\mathscr{T}, X[1]) = 0$ if $\mathrm{Hom}_{\mathcal{A}}(Y, X[1]) = 0$ for any $Y \in \mathscr{T}$. One says that $\mathscr{T}$ is *weakly cluster-tilting* provided, for every $X \in \mathcal{A}$, that $\mathrm{Hom}_{\mathcal{A}}(\mathscr{T}, X[1]) = 0$ if and only if $X \in \mathscr{T}$; and *cluster-tilting* provided that $\mathscr{T}$ is weakly cluster-tilting and functorially finite in $\mathcal{A}$.

3.1.2 Definition. [48] A 2-Calabi-Yau triangulated k-category is called a *cluster category* if its cluster-tilting subcategories form a cluster structure.

3.2 Derived categories of finite dimensional representations of quivers

The objective of this section is to recall briefly the derived category of finite dimensional representations of a quiver. For more details, we refer to [10, 5, 15].

Throughout this section, Q stands for a connected locally finite quiver without infinite paths. Under this assumption, by König's Lemma; see [42], Q is strongly locally finite.

3.2.1 Definition. A *k-representation* M of Q consists of

(1) a family of k-vector spaces $M(a)$ with $a \in Q_0$;

(2) a family of k-linear maps $M(\alpha) : M(a) \to M(b)$ with $\alpha : a \to b$ in Q_1.

Given two representations M, N of Q, a *morphism* $f : M \to N$ is a family $\{f_a : M(a) \to N(a)\}_{a \in Q_0}$ of k-linear maps such that for each arrow $\alpha : a \to b$

in Q_1, we have $N(\alpha)f_a = f_b M(\alpha)$. Let $f : M \to N$ and $g : N \to L$ be two morphisms of representations of Q. Their composition gf is defined to be the family $\{g_a f_a : M(a) \to L(a)\}_{a \in Q_0}$. This yields the category of k-representations of Q, denoted by $\mathrm{Rep}(Q)$. Moreover, one says that a representation M is *finite dimensional* if $\Sigma_{a \in Q_0} \dim_k M(a)$ is finite. We shall denote by $\mathrm{rep}(Q)$ the full subcategory of $\mathrm{Rep}(Q)$ of finite dimensional representations.

Let $a \in Q_0$. The *simple representation* S_a at a is defined by $S_a(a) = k\varepsilon_a$ and $S_a(b) = 0$ for all vertices $b \neq a$. The *projective representation* P_a at a is defined as follows. For any vertex $b \in Q_0$, $P_a(b)$ is a k-vector space spanned by the paths from a to b; and for each arrow $\alpha : b \to c$, $P_a(\alpha) : P_a(b) \to P_a(c)$ is a k-linear map sending every path p to αp. Finally, the *injective representation* I_a at a is a k-representation such that for each vertex $b \in Q_0$, $I_a(b)$ is a k-vector spaces spanned by the paths from b to a; and for each arrow $\alpha : b \to c$, $I_a(\alpha) : I_a(b) \to I_a(c)$ is a k-linear map sending every path $p\alpha$ to p and vanishing on the paths which not factor through α.

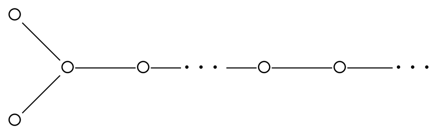
Since Q has no infinite paths, $\mathrm{rep}(Q)$ is a Hom-finite Krull-Schmidt hereditary abelian k-category; see [26, (8.2)] and has Auslander-Reiten sequences; see, [15, (3.7)]. That is, $\mathrm{rep}(Q)$ is an Auslander-Reiten category, whose Auslander-Reiten translation is denoted by τ_Q. We define the Auslander-Reiten quiver $\Gamma_{\mathrm{rep}(Q)}$ of $\mathrm{rep}(Q)$ in such a way that its vertex set contains the indecomposable projective representations P_a, the indecomposable injective representations I_a and the simple representations S_a, for all $a \in Q_0$. A connected component of $\Gamma_{\mathrm{rep}(Q)}$ is called *preprojective* if it contains some P_a with $a \in Q_0$, *preinjective* if it contains some I_a with $a \in Q_0$, and *regular* if it contains neither P_a nor I_a, for any $a \in Q_0$. Since Q is assumed to be connected, $\Gamma_{\mathrm{rep}(Q)}$ has a unique preprojective component and a unique preinjective component; see [15, 26].

A quiver Q is said to be of *infinite Dynkin type* if its underlying graph is one of the following infinite graphs.

$\mathbb{A}_\infty$: $\circ$——$\circ$—$\cdots$—$\circ$——$\circ$——$\cdots$

$\mathbb{A}_\infty^\infty$: $\cdots$—$\circ$——$\circ$—$\cdots$—$\circ$——$\circ$——$\cdots$

$\mathbb{D}_\infty$:



In this case, the Auslander-Reiten quiver $\Gamma_{\mathrm{rep}(Q)}$ of $\mathrm{rep}(Q)$ has been explicitly described by Bautista, Liu and Paquette in [15, 48].

3.2.2 Theorem. *Let Q be a quiver of infinite Dynkin type with no infinite path.*

(1) *Every connected component of $\Gamma_{\mathrm{rep}(Q)}$ is standard.*

(2) *The preprojective component $\mathcal{P}$ is of shape $\mathbb{N}Q^{\mathrm{op}}$ and the preinjective component $\mathcal{I}$ is of shape $\mathbb{N}^{-}Q^{\mathrm{op}}$ such that $\mathrm{Hom}_{\mathrm{rep}(Q)}(\mathcal{I}, \mathcal{P}) = 0$.*

(3) *Every regular component $\mathcal{R}$ is of shape $\mathbb{Z}\mathbb{A}_\infty$ such that $\mathrm{Hom}_{\mathrm{rep}(Q)}(\mathcal{I}, \mathcal{R}) = 0$ and $\mathrm{Hom}_{\mathrm{rep}(Q)}(\mathcal{R}, \mathcal{P}) = 0$.*

(4) *There are r regular components, where*

 (*a*) *$r = 0$ if Q is of type $\mathbb{A}_\infty$;*

 (*b*) *$r = 1$ if Q is of type $\mathbb{D}_\infty$;*

 (*c*) *$r = 2$ if Q is of type $\mathbb{A}_\infty^\infty$, and in this case, the two regular components are orthogonal.*

Now we shall study the derived category $D^b(\mathrm{rep}(Q))$ of $\mathrm{rep}(Q)$. For more details about derived categories, we refer to [49, 60]. As usual, considering an object $M \in \mathrm{rep}(Q)$ as a stalk complex concentrated at degree 0, we shall regard $\mathrm{rep}(Q)$ as a full subcategory of $D^b(\mathrm{rep}(Q))$. It is well known that $D^b(\mathrm{rep}(Q))$ is a Hom-finite Krull-Schmidt triangulated category having Auslander-Reiten triangles; see [15, 31]. That is, $D^b(\mathrm{rep}(Q))$ is an Auslander-Reiten category, whose Auslander-Reiten translation is denoted by $\tau_{_D}$. Observe that $D^b(\mathrm{rep}(Q))$ admits a Serre functor $\mathbb{S} = \tau_{_D} \circ [1]$; see [53]. One defines the Auslander-Reiten quiver $\Gamma_{D^b(\mathrm{rep}(Q))}$ of $D^b(\mathrm{rep}(Q))$ such that its vertices are the shifts of the vertices of $\Gamma_{\mathrm{rep}(Q)}$. In this way, $\Gamma_{\mathrm{rep}(Q)}$ becomes a full translation subquiver of $\Gamma_{D^b(\mathrm{rep}(Q))}$. Recall that $\Gamma_{D^b(\mathrm{rep}(Q))}$ has a *connecting* component $\mathcal{C}_Q$ obtained by gluing the preprojective component $\mathcal{P}$ of $\Gamma_{\mathrm{rep}(Q)}$ with the shift by -1 of the preinjective component $\mathcal{I}$ of

$\Gamma_{\text{rep}(Q)}$ in such a way that each arrow $a \to b \in Q_1$ induces an arrow $I_a[-1] \to P_b$ in $\mathcal{C}_Q$; see [15, 31].

The following statement gives some properties of connected components of $\Gamma_{D^b(\text{rep}(Q))}$, for which we refer to [15, Section 7].

3.2.3 THEOREM. *Let Q be an infinite Dynkin quiver without infinite paths.*

(1) *Every connected components of $\Gamma_{D^b(\text{rep}(Q))}$ is standard.*

(2) *The connecting component $\mathcal{C}_Q$ of $\Gamma_{D^b(\text{rep}(Q))}$ is of shape $\mathbb{Z}Q^{op}$.*

(3) *The connected components of $\Gamma_{D^b(\text{rep}(Q))}$ are the shifts of $\mathcal{C}_Q$ and the shifts of the regular components of $\Gamma_{\text{rep}(Q)}$.*

3.3 Cluster categories associated with a quiver

The objective of this section is to recall from [20, 48] the cluster category associated with a quiver. For this purpose, let Q be a connected locally finite quiver with no infinite path. Recall that rep(Q) is an Auslander-Reiten category, whose Auslander-Reiten quiver is $\Gamma_{\text{rep}(Q)}$ and whose Auslander-Reiten translation is τ_Q. The derived category $D^b(\text{rep}(Q))$ is an Auslander-Reiten category, whose Auslander-Reiten translation is merely an auto-equivalence of $D^b(\text{rep}(Q))$. In order to obtain Auslander-Reiten category whose Auslander-Reiten translation is an automorphism, we shall choose a skeleton $\mathscr{D}^b(Q)$ of $D^b(\text{rep}(Q))$, that is the additive subcategory of $D^b(\text{rep}(Q))$ generated by the objects which are the shifts of the objects in $\Gamma_{\text{rep}(Q)}$. Then $\mathscr{D}^b(Q)$ is an Auslander-Reiten category, whose Auslander-Reiten quiver $\Gamma_{\mathscr{D}^b(Q)}$ coincides with the Auslnader-Reiten quiver of $D^b(\text{rep}(Q))$, and whose Auslander-Reiten translation τ_D is an automorphism of $\mathscr{D}^b(Q)$. Considering the automorphism $F = \tau_D^{-1} \circ [1]$ of $\mathscr{D}^b(Q)$, one defines the orbit category

$$\mathscr{C}(Q) := \mathscr{D}^b(Q)/F$$

as follows. The objects are the same as those of $\mathscr{D}^b(Q)$; for any pair of objects $X, Y \in \mathscr{C}(Q)$, the morphisms are given by

$$\text{Hom}_{\mathscr{C}(Q)}(X, Y) = \oplus_{i \in \mathbb{Z}} \text{Hom}_{\mathscr{D}^b(Q)}(X, F^i Y).$$

The composition of morphisms is given by $(g_i)_{i\in\mathbb{Z}} \circ (f_i)_{i\in\mathbb{Z}} = (h_i)_{i\in\mathbb{Z}}$, where $h_i = \Sigma_{p+q=i} F^p(g_p) f_q$. The canonical projection functor

$$\pi : \mathscr{D}^b(Q) \to \mathscr{C}(Q)$$

is defined by $X \mapsto X$ and $f \mapsto (f_i)_{i\in\mathbb{Z}}$ where $f_0 = f$ and $f_i = 0$ while $i \neq 0$.

The following result is due to [20, 39, 48, 59, 62].

3.3.1 THEOREM. *Let Q be a connected locally finite quiver without infinite paths.*

(1) *The orbit category $\mathscr{C}(Q)$ is a cluster category.*

(2) *The canonical projection functor $\pi : \mathscr{D}^b(Q) \to \mathscr{C}(Q)$ sends indecomposable objects to indecomposable objects.*

(3) *The canonical projection functor $\pi : \mathscr{D}^b(Q) \to \mathscr{C}(Q)$ sends exact triangles in $D^b(\mathrm{rep}(Q))$ to exact triangles in $\mathscr{C}(Q)$.*

(4) *The canonical projection functor $\pi : \mathscr{D}^b(Q) \to \mathscr{C}(Q)$ sends Auslander-Reiten triangles in $D^b(\mathrm{rep}(Q))$ to Auslander-Reiten triangles in $\mathscr{C}(Q)$.*

REMARK. Denote by $\tau_{\mathscr{C}}$ the Auslander-Reiten translation of $\mathscr{C}(Q)$. In view of Theorem 3.3.1(4), we have the following observation. If $X \in \Gamma_{\mathrm{rep}(Q)}$ which is non-projective, then $\tau_Q X = \tau_D X = \tau_{\mathscr{C}} X$; and for any $Y \in \Gamma_{\mathscr{D}^b(Q)}$, we have $\tau_D Y = \tau_{\mathscr{C}} Y$, while for any $Z \in \mathscr{C}(Q)$, we have $\tau_{\mathscr{C}} Z = Z[1]$.

3.3.2 DEFINITION. Let Q be a connected locally finite quiver without infinite paths. The *fundamental domain* $\mathscr{F}(Q)$ of the cluster category $\mathscr{C}(Q)$ is defined as follows. If Q is of finite Dynkin type, then $\mathscr{F}(Q)$ is the full subquiver of $\Gamma_{\mathscr{D}^b(Q)}$ generated by the representations in $\Gamma_{\mathrm{rep}(Q)}$ and the shifts by -1 of the injective representation in $\Gamma_{\mathrm{rep}(Q)}$; otherwise, $\mathscr{F}(Q)$ is the subquiver of $\Gamma_{\mathscr{D}^b(Q)}$ consisting of the connecting component $\mathcal{C}_Q$ of $\Gamma_{\mathscr{D}^b(Q)}$ and the regular components of $\Gamma_{\mathrm{rep}(Q)}$.

REMARK. (1) It is well known that every indecomposable object of $\mathscr{C}(Q)$ is isomorphic to a unique object in $\mathscr{F}(Q)$. In particular, we shall define the Auslander-Reiten quiver $\Gamma_{\mathscr{C}(Q)}$ of $\mathscr{C}(Q)$ so that its vertices are the vertices of $\mathscr{F}(Q)$.

(2) If Q is an infinite quiver without infinite paths, then the canonical functor $\pi : \mathscr{D}^b(Q) \to \mathscr{C}(Q)$ induces a translation-quiver-isomorphism $\pi : \mathscr{F}(Q) \to \mathit{\Gamma}_{\mathscr{C}(Q)}$, acting identically on the underlying quiver. We shall say that the image of the connecting component in $\mathscr{F}(Q)$ under π is the *connecting component* of $\mathit{\Gamma}_{\mathscr{C}(Q)}$; and the image of a regular component in $\mathscr{F}(Q)$ under π is a *regular component* of $\mathit{\Gamma}_{\mathscr{C}(Q)}$. Note, however, that none of the connected components of $\mathit{\Gamma}_{\mathscr{C}(Q)}$ is standard in $\mathscr{C}(Q)$.

The following statement describes the morphisms in $\mathscr{C}(Q)$ between objects in fundamental domain in case Q is of infinite Dynkin type.

3.3.3 LEMMA. *Let Q be an infinite Dynkin quiver with no infinite path. Then, for any two objects $X, Y \in \mathscr{F}(Q)$, we have*

$$\mathrm{Hom}_{\mathscr{C}(Q)}(X,Y) \cong \mathrm{Hom}_{\mathscr{D}^b(Q)}(X,Y) \oplus D\mathrm{Hom}_{\mathscr{D}^b(Q)}(Y, \tau_{_D}^2 X).$$

Proof. Now let $X, Y \in \mathscr{F}(Q)$. There exists an integer $n \geq 0$ such that $M = \tau_{_D}^{-n} X$ and $N = \tau_{_D}^{-n} Y$ are representations. Since $\tau_{_D}$ is an equivalence, in view of Lemma 2.6(1) in [48], we have

$$\begin{aligned}
\mathrm{Hom}_{\mathscr{C}(Q)}(X,Y) &\cong \mathrm{Hom}_{\mathscr{C}(Q)}(\tau_{_D}^{-n} X, \tau_{_D}^{-n} Y) \\
&= \mathrm{Hom}_{\mathscr{C}(Q)}(M,N) \\
&\cong \mathrm{Hom}_{\mathscr{D}^b(Q)}(M,N) \oplus D\mathrm{Hom}_{\mathscr{D}^b(Q)}(N, \tau_{_D}^2 M) \\
&\cong \mathrm{Hom}_{\mathscr{D}^b(Q)}(\tau_{_D}^n M, \tau_{_D}^n N) \oplus D\mathrm{Hom}_{\mathscr{D}^b(Q)}(\tau_{_D}^n N, \tau_{_D}^{2+n} M) \\
&= \mathrm{Hom}_{\mathscr{D}^b(Q)}(X,Y) \oplus D\mathrm{Hom}_{\mathscr{D}^b(Q)}(Y, \tau_{_D}^2 X).
\end{aligned}$$

The proof of the lemma is completed.

We conclude this section with a description of the Auslander-Reiten quiver $\mathit{\Gamma}_{\mathscr{C}(Q)}$ of $\mathscr{C}(Q)$ in case Q is of infinite Dynkin type; see [48, (2.9)].

3.3.4 THEOREM. *Let Q be an infinite Dynkin quiver without infinite paths. Then $\mathit{\Gamma}_{\mathscr{C}(Q)}$ consists of the connecting component of shape $\mathbb{Z}Q^{op}$ and r regular component of shape $\mathbb{Z}\mathbb{A}_\infty$, where*

(1) *$r = 0$, in case Q is of type $\mathbb{A}_\infty$;*

(2) *$r = 1$, in case Q is of type $\mathbb{D}_\infty$;*

(3) *$r = 2$, in case Q is of type $\mathbb{A}_\infty^\infty$; and in this case, the two regular components are orthogonal.*

Chapter 4

Coordinate systems for some special translation quivers

In this chapter, we shall introduce a coordinate system for a translation quiver which is a wing, or of shape $\mathbb{Z}\mathbb{A}_\infty$ or $\mathbb{Z}\mathbb{A}_\infty^\infty$, in order to characterize wings, sections and section-generators in these types of translation quivers. The results will be used later to study the τ-rigidity theory, which is an essential topic of this thesis.

4.1 Coordinate system for a wing

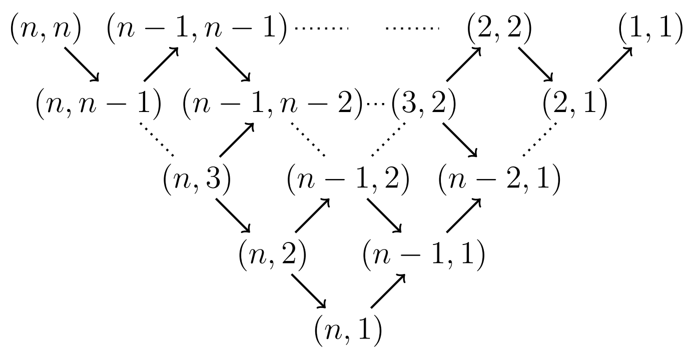
The objective of this section is to introduce a coordinate system for a wing, which will enable us to describe sub-wings, sections and section-generators in such a wing.

Throughout this section, let $\mathcal{W}$ stand for a wing of rank n, whose translation is written as τ. Let T be the unique sink vertex of $\mathcal{W}$. The quasi-simple vertices of $\mathcal{W}$ are $T_i = \tau^{i-1}T$, $i = 1, \ldots, n$. For each $1 \le i \le n$, denote by R_i^+ the longest sectional path in $\mathcal{W}$ starting with T_i, and by R_i^- the longest sectional path ending with T_i.

4.1.1 Lemma. *For any vertex $X \in \mathcal{W}$, there exists a unique pair (i_X, j_X) of integers with $n \ge i_X \ge j_X \ge 1$ such that $R^+_{i_X} \cap R^-_{j_X} = X$.*

Proof. Let $X \in \mathcal{W}_0$. Clearly, $X \in R^+_{i_X}$ for some unique $1 \le i_X \le n$. Then, $R^+_{i_X}$ has a subpath $T_{i_X} \to \cdots \to X$ of length $l \ge 0$. Observe that $\mathcal{W}$ has a sectional path $X \to \cdots \to T_{j_X}$ of length l, for some $1 \le j_X \le n$. Thus, $X \in R^-_{j_X}$, and hence, $X = R^+_{i_X} \cap R^-_{j_X}$. Observing that $T_{j_X} = \tau^{-l} T_{i_X} = \tau^{i_X - l - 1} T = T_{i_X - l}$, we see that $j_X = i_X - l$. Therefore, $1 \le j_X \le i_X \le n$. The proof of the lemma is completed.

Writing $X = (i_X, j_X)$, we obtain a coordinate system for $\mathcal{W}$ as follows.



Observe that the quasi-simple vertices are (i, i), $i = 1, \ldots, n$. In particular, (n, n) is the source vertex and $(1, 1)$ is the sink vertex.

The following easy statement describes the arrows, the translation and the paths in $\mathcal{W}$.

4.1.2 Lemma. *Let X, Y be vertices in $\mathcal{W}$.*

1. *There exists an arrow $X \to Y$ in $\mathcal{W}$ if and only if $(i_Y, j_Y) = (i_X, j_X - 1)$ or $(i_Y, j_Y) = (i_X - 1, j_X)$.*
2. *$X = \tau Y$ if and only if $(i_X, j_X) = (i_Y + 1, j_Y + 1)$.*
3. *There exists a path $p : X \rightsquigarrow Y$ in $\mathcal{W}$ if and only if $i_X \ge i_Y$ and $j_X \ge j_Y$. In this case, $l(p) = (i_X - i_Y) + (j_X - j_Y)$.*

Proof. Statement (1) and Statement (2) are obvious from our coordinate system. Since a path is a composition of arrows, Statement (3) follows from the Statement (1). The proof of the lemma is completed.

The following statement is a description of the sectional paths in $\mathcal{W}$.

4.1.3 Lemma. *Let $p : X_1 \to X_2 \to \cdots \to X_m$ be a sectional path in $\mathcal{W}$. Then exactly one of the following statements is true.*

(1) $(i_{X_t}, j_{X_t}) = (i_{X_1}, j_{X_1} - t + 1)$, *for* $t = 1, \ldots, m$.

(2) $(i_{X_t}, j_{X_t}) = (i_{X_1} - t + 1, j_{X_1})$, *for* $t = 1, \ldots, m$.

Proof. Write $(i_{X_1}, j_{X_1}) = (s, t)$. By Lemma 4.1.2(1), the statement is evident for $m = 2$. Assume now that $m > 2$. By the induction hypothesis, we may assume that $i_{X_t} = s$ and $j_{X_t} = r - t + 1$, for all $1 \le t \le m - 1$. In view of Lemma 4.1.2(1), we see that $(i_{X_m}, j_{X_m}) = (i_{X_{m-1}}, j_{X_{m-1}} - 1) = (s, r - m + 1)$ or

$$(i_{X_m}, j_{X_m}) = (i_{X_{m-1}} - 1, j_{X_{m-1}}) = (i - 1, j - n + 2) = (i_{X_{m-2}} - 1, j_{X_{m-2}} - 1).$$

Since p is sectional, by Lemma 4.1.2(2), the second case does not occur. The proof of the lemma is completed.

As an application of the preceding statement, we obtain the following description of a sub-wing in $\mathcal{W}$ in terms of the coordinates.

4.1.4 Lemma. *Let X, M be vertices in $\mathcal{W}$.*

(1) *The source vertex of $\mathcal{W}_X$ is (i_X, i_X), and the sink vertex is (j_X, j_X).*

(2) $M \in \mathcal{W}_X$ *if and only if* $i_X \ge i_M$ *and* $j_M \ge j_X$.

(3) *The quasi-length of $X \in \mathcal{W}$ is given by $\ell(X) = i_X - j_X + 1$.*

Proof. Let S_X be the source vertex of $\mathcal{W}_X$. Then there is a sectional path $S_X = X_1 \to X_2 \to \cdots \to X_m = X$ in $\mathcal{W}$. Since X_1 is quasi-simple, $i_{X_1} = j_{X_1}$. Since $i_{X_m} \ge j_{X_m}$ by Lemma 4.1.1, we deduce from Lemma 4.1.3 that $(i_{X_m}, j_{X_m}) = (i_{X_1}, j_{X_1} - m + 1)$. In particular, $i_{X_1} = i_{X_m} = i_X$. Then, $S_X = (i_X, i_X)$ and $\ell(X) = m = i_X - j_X + 1$. Similarly, we see that the sink vertex $T_X = (j_X, j_X)$.

Now, let $M \in \mathcal{W}_X$. Then $M \preceq X$ if and only if $M \in \mathcal{W}_X$. This is, by Lemma 1.4.6, equivalent to the existence of a path $S_X = (i_X, i_X) \rightsquigarrow M$ and a path $M \rightsquigarrow T_X = (j_X, j_X)$. By Lemma 4.1.2(3), this is equivalent to $i_X \ge i_M$, $i_X \ge j_M$ and $i_M \ge j_X$, $j_M \ge j_X$. That is, $i_X \ge \max\{i_M, j_M\}$ and $\min\{i_M, j_M\} \ge j_X$. Since $i_M \ge j_M$, the latter condition is equivalent to $i_X \ge i_M$ and $j_M \ge j_X$. The proof of the lemma is completed.

4.1.5 Remark. Let $X, Y \in \mathcal{W}$. By Lemma 4.1.2(2) and Lemma 4.1.4(3), we see that X, Y are in the same τ-orbit if and if only $\ell(X) = \ell(Y)$. Moreover, $1 \le \ell(X) \le n$, for any $X \in \mathcal{W}$

Recall from Lemma 1.4.7 that the vertex set $\mathcal{W}_0$ is a partially ordered set. The following statement characterizes the partial order $\preceq$ in terms of the coordinates.

4.1.6 Lemma. *If $X, M \in \mathcal{W}_0$, then $M \preceq X$ if and only if $i_{_M} \le i_{_X}$ and $j_{_M} \ge j_{_X}$.*

Proof. Let $X, M \in \mathcal{W}_0$. Then $M \preceq X$ if and only if $M \in \mathcal{W}_X$. The statement follows immediately from Lemma 4.1.4(2). The proof of the lemma is completed.

Recall that we have defined two wings in $\mathcal{W}$ to be separable in Definition 1.4.8(2). The following statement gives a description of two separable wings in terms of the coordinates.

4.1.7 Lemma. *If $X, Y \in \mathcal{W}$, then $\mathcal{W}_X, \mathcal{W}_Y$ are separable if and only if $j_{_X} \ge i_{_Y} + 2$ or $j_{_Y} \ge i_{_X} + 2$; and in this case, $\mathcal{W}_M, \mathcal{W}_N$ are separable for any $M \in \mathcal{W}_X$ and $N \in \mathcal{W}_Y$.*

Proof. Let X, Y be vertices in $\mathcal{W}$. Let S_X, S_Y be the source vertices in $\mathcal{W}_X$ and in $\mathcal{W}_Y$, respectively, while let T_X, T_Y be the sink vertices in $\mathcal{W}_X$ and in $\mathcal{W}_Y$, respectively. Clearly, we have $S_X = (i_{_X}, i_{_X})$ and $T_X = (j_{_X}, j_{_X})$ while $S_Y = (i_{_Y}, i_{_Y})$ and $T_Y = (j_{_Y}, j_{_Y})$. By definition, $\mathcal{W}_X, \mathcal{W}_Y$ are separable if and only if $\tau^t S_Y = T_X$ for some $t \ge 2$, or $\tau^r S_X = T_Y$ for some $r \ge 2$. Indeed, by Lemma 4.1.2(2), the first condition is equivalent to $j_{_X} \ge i_{_Y} + 2$ and the second condition is equivalent to $j_{_Y} \ge i_{_X} + 2$. The rest of the statement is trivially to see. The proof of the lemma is completed.

Let $X, Y \in \mathcal{W}$ with $X \prec Y$. Recall from Definition 1.4.12 that if X, Y are connected by a path, then we defined a wing $\mathcal{W}_Y^X$ in $\mathcal{W}_Y$. The following statement describes $\mathcal{W}_Y^X$ in terms of the coordinates.

4.1.8 Lemma. *Let $X, Y \in \mathcal{W}$ with $X \prec Y$. If X, Y are connected by a path p, then $\mathcal{W}_Y^X = \emptyset$ in case $\ell(p) = 1$; and otherwise, $\mathcal{W}_Y^X = \mathcal{W}_Z$, where $Z \in \mathcal{W}_Y$ with*

$$(i_{_Z}, j_{_Z}) = \begin{cases} (j_{_X} - 2, j_{_Y}), & \text{if } s(p) = X; \\ (i_{_X}, i_{_Y} + 2), & \text{if } t(p) = X. \end{cases}$$

In particular, $\mathcal{W}_Y^X$ is a wing of rank $\ell(Y) - \ell(X) - 1$.

Proof. Assume that X, Y are connected by a path p in $\mathcal{W}$. If $l(p) = 1$, then p is an arrow. Thus, $\mathcal{W}_Y^X = \emptyset$. Now consider $l(p) > 1$. Suppose that $s(p) = X$. That is, $p : X \rightsquigarrow Y$. By Lemma 1.4.9, p is sectional. Since $X \prec Y$, by Lemma 4.1.3 and Lemma 4.1.6, $i_{_X} = i_{_Y}$ and $j_{_X} > j_{_Y}$. By Lemma 4.1.2(3), $l(p) = j_{_X} - j_{_Y} \geq 2$. Hence, by Lemma 4.1.1, $Z = (j_{_X} - 2, j_{_Y}) \in \mathcal{W}_0$. Moreover, since $i_{_Z} = i_{_X} - 2 < i_{_Y}$ and $j_{_Z} = j_{_Y}$, by Lemma 4.1.6, $\mathcal{W}_Z \subseteq \mathcal{W}_Y$, and by Lemma 4.1.7, $\mathcal{W}_Z, \mathcal{W}_X$ are separable. This shows that $\mathcal{W}_Z \subseteq \mathcal{W}_Y^X$.

Now let $M \in \mathcal{W}_Y^X$. In particular, $M \in \mathcal{W}_Y$, by Lemma 4.1.6, $i_{_M} \leq i_{_Y}$ and $j_{_M} \geq j_{_Y}$. Since $\mathcal{W}_X, \mathcal{W}_Y^X$ are separable, by Lemma 4.1.7, $\mathcal{W}_X, \mathcal{W}_M$ are separable. By Lemma 4.1.7, consider first $j_{_M} \geq i_{_X} + 2$. Then $j_{_M} > i_{_Y}$. Hence, by Lemma 4.1.6, $M \notin \mathcal{W}_Y$, contradiction. Hence, $i_{_M} \leq j_{_X} - 2$. By Lemma 4.1.6, $\mathcal{W}_M \subseteq \mathcal{W}_Z$. Hence, $\mathcal{W}_Y^X \subseteq \mathcal{W}_Z$. Thus, $\mathcal{W}_Y^X = \mathcal{W}_Z$. We deduce from Lemma 4.1.4(3) that $\ell(Z) = \ell(Y) - \ell(X) - 1$. That is, $\mathcal{W}_Y^X$ is of rank $\ell(Y) - \ell(X) - 1$. The case that $t(p) = X$ is similar to show. The proof of the lemma is completed.

In the rest of this section, we study sections and section-generators in $\mathcal{W}$. We shall start with the following lemma.

4.1.9 Lemma. *Let X, Y be two vertices in $\mathcal{W}$.*

(1) *If X, Y are comparable, then $X \prec Y$ if and only if $\ell(X) < \ell(Y)$.*

(2) *There is an edge X — Y in $\mathcal{W}$ if and only if $X \prec Y$ with $\ell(Y) = \ell(X) + 1$ or $Y \prec X$ with $\ell(Y) = \ell(X) - 1$.*

Proof. Assume that X, Y are comparable. It is easy to see that $X \prec Y$ if and only if the rank of $\mathcal{W}_X$ is smaller than the rank of $\mathcal{W}_Y$. The latter is equivalent to $\ell(X) < \ell(Y)$. This shows Statement (1).

For proving the necessity of Statement (2), we consider only the case where there is an arrow $X \to Y$ in $\mathcal{W}$. By Lemma 4.1.2(1), $(i_{_Y}, j_{_Y}) = (i_{_X} - 1, j_{_X})$ or $(i_{_Y}, j_{_Y}) = (i_{_X}, j_{_X} - 1)$. In the first case, $Y \prec X$ with $\ell(Y) = \ell(X) - 1$; and in the second case, $X \prec Y$ with $\ell(Y) = \ell(X) + 1$.

For proving the sufficiency of Statement (2), we consider only the case where $X \prec Y$ with $\ell(Y) = \ell(X) + 1$. Then, $1 = \ell(Y) - \ell(X) = (i_{_Y} - i_{_X}) + (j_{_X} - j_{_Y})$.

Since $i_Y \geq i_X$ and $j_X \geq j_Y$, we have $i_Y = i_X$ and $j_X = j_Y + 1$ or $i_Y = i_X + 1$ and $j_X = j_Y$. By Lemma 4.1.2 (1), $\mathcal{W}$ has an arrow $X \to Y$ in the first case; and an arrow $Y \to X$ in the second case. The proof of the lemma is completed.

Recall that $\mathcal{W}_0$ is a poset. By abuse of language, we shall say that $\mathcal{W}$ is a poset. Let $\mathcal{S}$ be a subset of $\mathcal{W}_0$. We shall write $\ell(\mathcal{S}) = \{\ell(X) \mid X \in \mathcal{S}\}$. As a subset of integers, $\ell(\mathcal{S})$ is a totally ordered set.

4.1.10 Lemma. *Given a chain $\mathcal{S}$ in $\mathcal{W}$, we have an isomorphism of posets*

$$\ell : \mathcal{S} \to \ell(\mathcal{S}) : X \mapsto \ell(X).$$

Proof. It is evident that the map ℓ is surjective. Given $X, Y \in \mathcal{S}$, by Lemma 4.1.9(1), $X \prec Y$ if and only if $\ell(X) < \ell(Y)$. This implies that ℓ is injective and preserves the order. The proof of the lemma is completed.

The following statement describes the maximal chains in $\mathcal{W}$.

4.1.11 Lemma. *A chain $\mathcal{S}$ in $\mathcal{W}$ is maximal if and only if $\ell(\mathcal{S}) = \{1, 2, \ldots, n\}$.*

Proof. Let $\mathcal{S}$ be a chain in $\mathcal{W}$. Assume first that $\ell(\mathcal{S}) = \{1, 2, \ldots, n\}$. Let $Y \in \mathcal{W}$ be such that $\mathcal{S} \cup \{Y\}$ is a chain in $\mathcal{W}$. By assumption, there is a vertex $X \in \mathcal{S}$ such that $\ell(X) = \ell(Y)$. If $X \neq Y$, then $X \prec Y$ or $Y \prec X$. By Lemma 4.1.10, $\ell(X) < \ell(Y)$ in the first case and $\ell(X) > \ell(Y)$ in the second case, a contradiction. Thus, $X = Y$. Therefore, $\mathcal{S}$ is a maximal chain.

Conversely, assume that $\mathcal{S}$ is a maximal chain in $\mathcal{W}$. Suppose on the contrary that $m \in \{1, 2, \ldots, n\}$ but $m \notin \ell(S)$. Observe that the wing vertex Z of $\mathcal{W}$ is a maximal element in $\mathcal{W}$. By the maximality of $\mathcal{S}$, we see that $Z \in \mathcal{S}$, and hence $n = \ell(Z) \in \ell(\mathcal{S})$. Now, let N be the minimal element in $\mathcal{S}$. We claim that $\ell(N) = 1$. Otherwise, $\ell(N) = t > 1$. Let S be the source vertex of $\mathcal{W}_N$. In particular, $S \preceq N$. Since $\ell(S) = 1 < t = \ell(N)$, we have $S \prec N$. Hence, $\mathcal{S} \cup \{S\}$ is a chain, a contradiction. Therefore, $\{1, n\} \subseteq \ell(\mathcal{S})$. In particular, $1 < m < n$.

Now let s be the maximal integer in $\ell(\mathcal{S})$ such that $s < m$ and let t be the minimal integer in $\ell(\mathcal{S})$ such that $m < t$. In particular, t is a minimal cover of s in $\ell(\mathcal{S})$. By Lemma 4.1.10, there exist $X, Y \in \mathcal{S}$ such that Y is a minimal cover of X with $\ell(X) = s$ and $\ell(Y) = t$. By Lemma 4.1.6, $i_X < i_Y$ and $j_X \geq j_Y$, or else, $i_X \leq i_Y$ and $j_X > j_Y$.

In the first case, by Lemma 4.1.4(3), $i_Y \geq j_Y - m + \ell(Y)$. By Lemma 4.1.1, The vertex M with $(i_M, j_M) = (i_Y, j_Y - m + \ell(Y))$ belongs to $\mathcal{W}$. By Lemma 4.1.6, $M \prec Y$. Moreover, since $i_M = i_Y > i_X$ and $j_M = j_Y - m + \ell(Y) < j_Y - \ell(X) + \ell(Y) = j_X - (i_Y - i_X) < j_X$, by Lemma 4.1.6 again, $X \prec M$. Since Y is a minimal cover of X, we see that $\mathcal{S} \cup \{M\}$ is a chain in $\mathcal{W}$. Since $\ell(M) = m \notin \ell(\mathcal{S})$, we have $M \notin \mathcal{S}$, a contradiction. Similarly, we shall obtain a contradiction in the second case. Thus, $\ell(\mathcal{S}) = \{1, 2, \cdots, n\}$. The proof of the lemma is completed.

Recall that a reduced walk w is called sectional if the τ-orbits in w are pairwise distinct.

4.1.12 LEMMA. *Let X_1 — X_2 — $\cdots$ — X_m be a sectional walk in $\mathcal{W}$.*

(1) *Either $\ell(X_i) = \ell(X_1) + i - 1$, for all $1 \leq i \leq m$ or $\ell(X_i) = \ell(X_1) - i + 1$, for all $1 \leq i \leq m$.*

(2) *For $1 \leq i, j \leq m$, we have $\ell(X_i) < \ell(X_j)$ if and only if $X_i \prec X_j$.*

Proof. We first show Statement (1) by induction. It is trivial when $m = 1$. Now we may assume that $m > 1$ and $\ell(X_i) = \ell(X_1) + i - 1$ for $1 \leq i \leq m-1$. There is an edge X_{m-1} — X_m. By Lemma 4.1.9(2), we know that $\ell(X_m) = \ell(X_{m-1}) - 1$ or $\ell(X_m) = \ell(X_{m-1}) + 1$. Assume that $\ell(X_m) = \ell(X_{m-1}) + 1$. Then, $\ell(X_m) = \ell(X_{m-2})$. Thus, by Remark 4.1.5, X_m and X_{m-2} are in the same τ-orbit. Since w is reduced, $X_m \neq X_{m-2}$. It is a contradiction, since w is a sectional walk. Therefore, $\ell(X_m) = \ell(X_1) + m - 1$. Thus, we establish Statement (1). Statement (2) follows easily from Statement (1) and Lemma 4.1.9(2). The proof of the lemma is completed.

REMARK: Let $p:\ X_1$ — X_2 — $\cdots$ —X_m, with $m \leq n$, be a sectional path in $\mathcal{W}$. Since the X_i are in different τ-orbits, by Remark 4.1.5, the $\ell(X_i)$ are pairwise distinct. Hence, p satisfies Lemma 4.1.12.

4.1.13 LEMMA. *If $p : X_1 \to X_2 \to \cdots \to X_m$, with $m \leq n$, is a sectional path in $\mathcal{W}$, then p is the unique sectional walk from X_1 to X_m in $\mathcal{W}$.*

Proof. Let $p: X_1 \xrightarrow{\alpha_1} X_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{m-1}} X_m$ be a sectional path in $\mathcal{W}$. We shall consider only the case where $\ell(X_m) > \ell(X_1)$. In view of Lemma 4.1.12(1), we deduce that $\ell(X_i) = \ell(X_1) + (i-1)$, for $i = 1, \ldots, m$. Moreover, by Lemma 4.1.3, there is some integer i such that $i = i_{X_p}$ for $1 \le p \le m$. We shall show the lemma by induction. It is trivial when $m = 1$. We assume that $m > 1$ and the statement holds for $m-1$. Assume that

$$X_1 = Y_1 \overset{c_1}{\text{------}} Y_2 \overset{c_2}{\text{------}} \cdots \overset{c_{r-1}}{\text{------}} Y_r = X_m$$

is a sectional walk in $\mathcal{W}$ from X_1 to X_m. Applying Lemma 4.1.12(1) again, we obtain $\ell(Y_i) = \ell(Y_1) + (i-1)$, for $i = 1, \ldots, r$. In particular, $r = m$ and $\ell(X_{m-1}) = \ell(Y_{m-1})$. Moreover, by Lemma 4.1.12(2), $X_1 \prec Y_{m-1}$, and then $i_{Y_{m-1}} \ge i_{X_1} = i$. We claim that c_{m-1} is an arrow. Otherwise, it is the inverse of an arrow $Y_m \to Y_{m-1}$. Thus, $Y_{m-1} = \tau^- X_{m-1}$. Then, $i_{Y_{m-1}} = i_{X_{m-1}} - 1 < i$, which is a contradiction. Hence, c_{m-1} is an arrow. Since $\ell(Y_{m-1}) = \ell(X_{m-1})$, we obtain $c_{m-1} = \alpha_{m-1}$. Moreover, by the induction hypothesis, we have $c_p = \alpha_p$, for $p = 1, \ldots, m-1$. The proof of the lemma is completed.

The following is a description of sections in $\mathcal{W}$.

4.1.14 PROPOSITION. *Let Σ be a full subquiver of $\mathcal{W}$. The following statements are equivalent.*

(1) *Σ is a section in $\mathcal{W}$.*

(2) *Σ_0 is a maximal chain in $\mathcal{W}$.*

(3) *Σ is given by a sectional walk $X_1 — X_2 — \cdots — X_n$, with $\ell(X_t) = t$ for $t = 1, \ldots, n$.*

Proof. Suppose that Σ is a section. Since Σ meets every τ-orbit exactly once, by Remark 4.1.5, we may assume that $\Sigma_0 = \{X_1, X_2, \cdots, X_n\}$ with $\ell(X_t) = t$, for $t = 1, \ldots, n$. Indeed, for each $t \in \{1, 2, \cdots, n-1\}$, Σ contains a walk $X_t = Y_0 — Y_1 — \cdots — Y_r = X_{t+1}$, which is sectional since Σ is a section. By Lemma 4.1.12(1), $\ell(X_{t+1}) = \ell(X_t) + r$, and since $\ell(X_{t+1}) = t + 1$, we obtain $r = 1$. That is, Γ contains an edge $X_t — X_{t+1}$ for every $t \in \{1, 2, \cdots, n-1\}$. This shows that Σ is given by a sectional walk as stated in Statement (3).

Assume now that Σ_0 is a maximal chain in Γ. By Lemma 4.1.11, we have $\ell(\Sigma_0) = \{1, 2, \cdots, n\}$. Then in view of Lemma 4.1.10, Σ_0 may be written as

$$X_1 \prec X_2 \prec \cdots \prec X_n$$

with $\ell(X_t) = t$, for $t = 1, \ldots, n.$. By Lemma 4.1.9(2), Γ contains an edge X_t — X_{t+1}, for each $t \in \{1, 2, \cdots, n-1\}$. Thus, Σ is given by a sectional walk as stated in Statement (3).

Finally, assume that Σ can be given by a sectional walk $X_1 — X_2 — \cdots — X_n$ with $\ell(X_t) = t$, for $t = 1, \ldots, n$. In view of Lemma 4.1.12(2), Σ_0 is a chain in $\mathcal{W}$. Since $\ell(\Sigma_0) = \{1, 2, \cdots, n\}$, by Lemma 4.1.11, Σ_0 is a maximal chain in $\mathcal{W}$. Thus, Statement (2) holds. It remains to show that Statement (1) holds. Indeed, Σ meets every τ-orbit exactly once. Let, moreover,

$$p: \ X_s = Y_0 \to Y_1 \to \cdots \to Y_r = X_t$$

be a non-trivial path in $\mathcal{W}$ with $X_s, X_t \in \Sigma$. We may consider only the case where $\ell(X_s) < \ell(X_t)$. Suppose that p is not sectional. Then there is a path from $\tau^- X_s$ to X_t. By Lemma 4.1.2(3), $i_{X_s} - 1 = i_{\tau^- X_s} \geq i_{X_t}$ and $j_{X_s} - 1 = j_{\tau^- X_s} \geq j_{X_t}$. By Lemma 4.1.6, X_s and X_t are not comparable. On the other hand, Σ contains a subwalk $w : X_s — X_{s+1} — \cdots — X_t$, which is a sectional walk. By Lemma 4.1.12(2), $X_s \prec X_t$, a contradiction. Hence, p is sectional. By the uniqueness stated in Lemma 4.1.13, $p = w$. In particular, p lies in Σ. Thus, Σ is a section in $\mathcal{W}$. The proof of the proposition is completed.

Recall that a set of vertices of $\mathcal{W}$ is a section-generator of $\mathcal{W}$ if its convex hull is a section in $\mathcal{W}$. The following statement gives a description of section-generators in $\mathcal{W}$ and also provides a way to obtain them. We refer the notion of a sectional chain in $\mathcal{W}$ to Definition 1.4.10.

4.1.15 PROPOSITION. *Let $\mathcal{S}$ be a set of vertices in $\mathcal{W}$. Then the following statements are equivalent.*

(1) *$\mathcal{S}$ is a section-generator of $\mathcal{W}$.*

(2) *$\mathcal{S}$ is a sectional chain such that $\{1, n\} \subseteq \ell(\mathcal{S})$.*

(3) *$\mathcal{S}$ is a subset of vertices of a section Σ in $\mathcal{W}$ containing all the source vertices and all the sink vertices of Σ.*

Proof. Assume that $\mathcal{S}$ is a section-generator of $\mathcal{W}$. That is, its convex hull Σ is a section in $\mathcal{W}$. In particular, $\mathcal{S} \subseteq \Sigma_0$. By Proposition 4.1.14(2), $\mathcal{S}$ is a chain. Then we may assume that $\mathcal{S}$ is of the form $X_1 \prec X_2 \prec \cdots \prec X_m$. By Lemma 4.1.10, we have $\ell(X_1) < \cdots < \ell(X_m)$. On the other hand, by Proposition 4.1.14(3), Σ is given by a sectional walk $M_1 — M_2 — \cdots — M_n$ with $\ell(M_t) = t$ with $1 \le t \le n$. By the definition of the convex hull, M_1 lies on a path p in Σ from X_s to X_r for some $1 \le s, r \le m$. Since p is a sectional walk, by Lemma 4.1.12(1), $\ell(X_s) \le \ell(M_1) \le \ell(X_r)$ or $\ell(X_r) \le \ell(M_1) \le \ell(X_s)$. Hence, $\ell(X_s) = \ell(M_1) = 1$ or $\ell(X_r) = \ell(M_1) = 1$. In either case, since $\ell(X_1)$ is the smallest in $\ell(\mathcal{S})$, we see $\ell(X_1) = 1$. In a similar fashion, we may show that $\ell(X_m) = \ell(M_n) = n$.

We claim, for each $1 \le t < m$, that $\mathcal{W}$ contains a path between X_t and X_{t+1}. Indeed, being a section, Σ contains a sectional walk

$$X_t = Y_0 — Y_1 — \cdots — Y_i = X_{t+1},$$

where $i \ge 1$. If $i = 1$, then our claim is evident. Assume that $i > 1$. Since $\ell(X_t) < \ell(X_{t+1})$, by Lemma 4.1.12(1), we have $\ell(X_t) \le \ell(Y_1) \le \ell(X_{t+1})$. On the other hand, since Σ is the convex hull of $\mathcal{S}$, we see that Y_1 lies on a path p in Σ between two vertices $M, N \in \mathcal{S}$. We may assume that $\ell(M) \le \ell(N)$. Then in view of Proposition 4.1.14(3), $\ell(M) \le \ell(Y_1) \le \ell(N)$. By Lemma 4.1.10, $\ell(X_{t+1})$ is a minimal cover of $\ell(X_t)$ in $\ell(\mathcal{S})$. Thus, $\ell(M) \le \ell(X_t) \le \ell(Y_1) \le \ell(X_{t+1}) \le \ell(N)$. In view of Proposition 4.1.14(3), Σ contains a sectional walk

$$M — \cdots — X_t — \cdots — Y_1 — \cdots — X_{t+1} — \cdots — N.$$

By Lemma 4.1.13, this walk coincides with p or p^{-1}. In particular, there is a path between X_t and X_{t+1}. Hence, $\mathcal{S}$ is a sectional chain. This shows that Statement (1) implies Statement (2).

Now assume that $\mathcal{S}$ is a sectional chain such that $\{1, n\} \subseteq \ell(\mathcal{S})$. Then we may assume that $\mathcal{S}$ is of the form $X_1 \prec X_2 \prec \cdots \prec X_m$. In view of Lemma 4.1.10, we see that $\ell(X_1) = 1$ and $\ell(X_m) = n$. By Lemma 1.4.9, assume that ρ_p is the sectional path between X_p and X_{p+1}, for each $1 \le p < m$. In particular, ρ_p is a sectional walk $X_p = X_{p,0} — X_{p,1} — \cdots — X_{p,t_p} = X_{p+1}$. Since $X_p \prec X_{p+1}$, by Lemma 4.1.9(1), we have $\ell(X_p) < \ell(X_{p+1})$, and hence, by Lemma 4.1.12 (1), the $\ell(X_p) = \ell(X_{p,0}) < \ell(X_{p,1}) < \cdots < \ell(X_{p,t_p}) = \ell(X_{p+1})$. Therefore, the walks ρ_p with $1 \leq p \leq m$ can be composed to form a sectional walk $\varSigma$. By Proposition 4.1.14(3), $\varSigma$ is a section in $\mathcal{W}$. Since the ρ_p are paths, we see that $\mathcal{S} = \{X_1, \ldots, X_m\}$ contains all the sink vertices and the source vertices of Σ. This shows that Statement (2) implies Statement (3). Moreover, since $\varSigma$ is generated by the paths ρ_p, we see that $\varSigma$ is contained in the convex hull of $\mathcal{S}$. Since $\varSigma$ is convex, it is the convex hull of $\mathcal{S}$. Thus, $\mathcal{S}$ is a section-generator. This shows that Statement (2) implies Statement (1).

Finally, assume that $\mathcal{S} = \{X_1, \cdots, X_m\}$ is a subset of vertices of a section $\varSigma$ in $\mathcal{W}$ containing all the source vertices and all the sink vertices of $\varSigma$. Since $\mathcal{S} \subseteq \varSigma$, by Proposition 4.1.14(2), $\mathcal{S}$ is a chain. Again, by Proposition 4.1.14(3), $\varSigma$ is given by a sectional walk of form $M_1 — M_2 — \cdots — M_n$ with $\ell(M_1) = 1$ and $\ell(M_n) = n$. Therefore, M_1 and M_n are sink or source vertices in $\varSigma$. Apparently, $M_1, M_n \in \mathcal{S}$. In particular, the $\ell(X_p)$ with $1 \leq p \leq m$ are pairwise distinct. We may assume that $\ell(X_1) < \ell(X_2) < \cdots < \ell(X_m)$. In this case, $X_1 = M_1$ and $X_m = M_n$. Hence, $\ell(X_1) = 1$ and $\ell(X_m) = n$. Fix p with $1 \leq p < m$. Being connected, $\varSigma$ contains a reduced walk

$$\rho_p : X_p = Y_0 — Y_1 — \cdots — Y_{r-1} — Y_r = X_{p+1}.$$

By Lemma 4.1.12, $\ell(X_p) = \ell(Y_0) < \ell(Y_1) < \cdots < \ell(Y_r) = \ell(X_{p+1})$. If ρ_p is not a path or the inverse of a path, then $r > 1$ and some Y_j with $1 \leq j < r$ is a sink vertex or a source vertex in $\varSigma$. Observing that $\ell(Y_j) \neq \ell(X_i)$ for all $1 \leq i \leq m$, we see that $Y_j \notin \mathcal{S}$, a contradiction. Therefore, $\mathcal{S}$ is a sectional chain. The proof of the proposition is completed.

4.2 Coordinate system for a translation quiver of shape $\mathbb{Z}\mathbb{A}_\infty$

The main objective of this section is to introduce a coordinate system for a translation quiver of shape $\mathbb{Z}\mathbb{A}_\infty$, which is slightly different from the one considered in [34] and has been implicitly used in [48]. It allows us to describe some essential combinatorial notions such as the partial order, the section-generators and the sectional chains in such a translation quiver.

Throughout this section, $\mathit{\Gamma}$ stands for a translation quiver of shape $\mathbb{Z}\mathbb{A}_\infty$ with translation τ. Fixing a quasi-simple vertex S_0, we obtain the quasi-simple vertices $S_i = \tau^i S_0$ with $i \in \mathbb{Z}$ in $\mathit{\Gamma}$. For each $i \in \mathbb{Z}$, denote by R_i^+ the ray in $\mathit{\Gamma}$ starting with S_i, that is, the unique infinite sectional path starting with S_i; and by R_i^- the co-ray ending with S_i, that is, the unique infinite sectional path ending with S_i. Then, $\mathit{\Gamma}_0 = \bigcup_{i\in\mathbb{Z}} (R_i^+)_0 = \bigcup_{j\in\mathbb{Z}} (R_j^-)_0$.

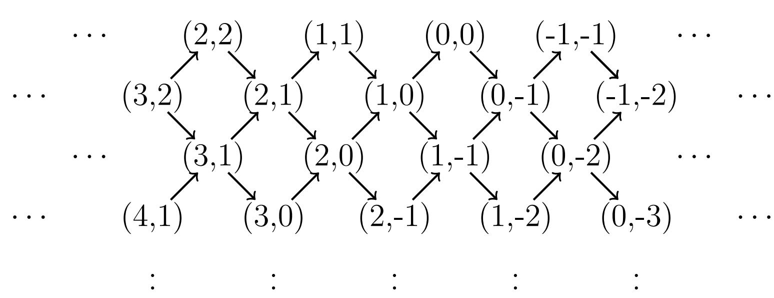
4.2.1 LEMMA. *Writing $\mathbb{Z}_\mathit{\Gamma} = \{(i,j) \in \mathbb{Z}\times\mathbb{Z} \mid i \geq j\}$, we obtain a bijection*

$$\Phi : \mathit{\Gamma}_0 \to \mathbb{Z}_\mathit{\Gamma} : X \mapsto (i_X, j_X)$$

so that $R^+_{i_X} \cap R^-_{j_X} = \{X\}$, for every $X \in \mathit{\Gamma}_0$.

Proof. Let $X \in \mathit{\Gamma}_0$. There exists a unique integer i_X such that $X \in R^+_{i_X}$. Then, $R^+_{i_X}$ has a subpath $\tau^{i_X} S_0 \to \cdots \to X$ of length $l \geq 0$. Observe that $\mathit{\Gamma}$ has a sectional path $X \to \cdots \to \tau^{j_X} S_0$ of length l. Thus, $X \in R^-_{j_X}$, and hence, $X = R^+_{i_X} \cap R^-_{j_X}$. Since $\tau^{j_X} S_0 = \tau^{-l}(\tau^{i_X} S_0) = \tau^{i_X - l} S_0$, we see that $j_X = i_X - l \leq i_X$. Therefore, $(i_X, j_X) \in \mathbb{Z}_\mathit{\Gamma}$ with $i_X \geq j_X$. In particular, Φ is injective. Assume conversely that $(i,j) \in \mathbb{Z}_\mathit{\Gamma}$. Since $l = j - i \geq 0$, the ray R_i^+ has a subpath $\tau^i S_0 \to \cdots \to Y$ of length l. Then, $\mathit{\Gamma}$ contains a sectional path $Y \to \cdots \to \tau^{i+l} S_0 = \tau^j S_0$. That is, $Y \in \mathit{\Gamma}_0$ is such that $\Phi(Y) = (i,j)$. The proof of the lemma is completed.

In view of Lemma 4.2.1, for every $X \in \mathit{\Gamma}_0$, we shall write $X = (i_X, j_X)$. This yields a coordinate system for $\mathit{\Gamma}$ as follows.



The following statement is an explanation of the coordinates of a vertex in terms of the rays and co-rays in $\mathit{\Gamma}$.

4.2.2 LEMMA. *Let i, j be integers. If X is a vertex of Γ, then*

(1) $i_X = i$ *if and only if* $X \in R_i^+$;

(2) $j_X = j$ *if and only if* $X \in R_j^-$.

The following statements is similar to Lemma 4.1.2, whose proof is omitted.

4.2.3 LEMMA. *Let X, Y be two vertices in Γ.*

(1) *There exists an arrow $\alpha : X \to Y$ in Γ if and only if $(i_Y, j_Y) = (i_X, j_X - 1)$ or $(i_Y, j_Y) = (i_X - 1, j_X)$.*

(2) *$X = \tau Y$ if and only if $(i_X, j_X) = (i_Y + 1, j_Y + 1)$.*

(3) *There exists a path $p : X \rightsquigarrow Y$ in $\mathcal{W}$ if and only if $i_X \geq i_Y$ and $j_X \geq j_Y$. In this case, $l(p) = (i_X - i_Y) + (j_X - j_Y)$.*

The following result describes the sectional paths in Γ, whose proof can be translated word-by-word from the proof of Lemma 4.1.3.

4.2.4 LEMMA. *Let $p : X_1 \to X_2 \to \cdots \to X_n$ be a sectional path in Γ. Then one of the following statements is true.*

(1) *$(i_{X_t}, j_{X_t}) = (i_{X_1}, j_{X_1} - t + 1)$, for $t = 1, \ldots, n$.*

(2) *$(i_{X_t}, j_{X_t}) = (i_{X_1} - t + 1, j_{X_1})$, for $t = 1, \ldots, n$.*

Recall that every vertex $X \in \Gamma$ is the wing vertex of a wing $\mathcal{W}_X$ in Γ as defined in Definition 1.4.6. The following statement is similar to Lemma 4.1.4, whose proof will be omitted.

4.2.5 LEMMA. *Let X, M be two vertices in Γ.*

(1) *The source vertex of $\mathcal{W}_X$ is (i_X, i_X), and the sink vertex is (j_X, j_X).*

(2) *$M \in \mathcal{W}_X$ if and only if $i_X \geq i_M$ and $j_M \geq j_X$.*

(3) *The quasi-length of* $X \in \Gamma$ *is given by* $\ell(X) = i_X - j_X + 1$.

4.2.6 Remark. By Lemma 4.2.3(2) and Lemma 4.2.5(3), we see that $X, Y \in \Gamma$ belong to the same τ-orbit if and if only $\ell(X) = \ell(Y)$. Observe that the $\ell(X)$, with $X \in \Gamma$, may run over all the positive integers.

Recall from Lemma 1.4.7 that there is a partial order $\preceq$ over Γ_0. By abuse of language, we shall say that Γ is partially ordered by $\preceq$. The following statement is an analogue to Lemma 4.1.6, whose proof is omitted.

4.2.7 Lemma. *If* $X, M \in \Gamma_0$*, then* $M \preceq X$ *if and only if* $i_M \leq i_X$ *and* $j_M \geq j_X$.

We describe the separability of two wings in Γ as defined Definition 1.4.8 in terms of the coordinates in the following statement, which is similar to Lemma 4.1.7.

4.2.8 Lemma. *Given* $X, Y \in \Gamma_0$*, the wings* $\mathcal{W}_X, \mathcal{W}_Y$ *are separable if and only if* $j_X \geq i_Y + 2$ *or* $j_Y \geq i_X + 2$*; and in this case,* $\mathcal{W}_M, \mathcal{W}_N$ *are separable for any* $M \in \mathcal{W}_X$ *and* $N \in \mathcal{W}_Y$.

Let $X, Y \in \Gamma$ with $X \prec Y$. Recall from Definition 1.4.12 that if X, Y are connected by a path, then we have defined a wing $\mathcal{W}_Y^X$ in $\mathcal{W}_Y$. The following statement describes $\mathcal{W}_Y^X$ in terms of the coordinates which is similar to Lemma 4.1.8.

4.2.9 Lemma. *Let* $X, Y \in \Gamma$ *with* $X \prec Y$. *If* X, Y *are connected by a path* p*, then* $\mathcal{W}_Y^X = \emptyset$ *in case* $\ell(p) = 1$*; and otherwise,* $\mathcal{W}_Y^X = \mathcal{W}_Z$*, where* $Z \in \mathcal{W}_Y$ *with*

$$(i_Z, j_Z) = \begin{cases} (j_X - 2, j_Y), & \text{if } s(p) = X; \\ (i_X, i_Y + 2), & \text{if } t(p) = X. \end{cases}$$

In particular, $\mathcal{W}_Y^X$ *is a wing of rank* $\ell(Y) - \ell(X) - 1$.

The partial order $\preceq$ over Γ plays an essential role in this section. Compare the following statement with Lemma 4.1.9.

4.2.10 Lemma. *Let X, Y be two vertices in Γ.*

(1) *If X, Y are comparable, then $X \prec Y$ if and only if $\ell(X) < \ell(Y)$.*

(2) *There is an edge X — Y in Γ if and only if $X \prec Y$ with $\ell(Y) = \ell(X) + 1$ or $Y \prec X$ with $\ell(Y) = \ell(X) - 1$.*

Next, we shall study the chains in Γ. The following observation is evident.

4.2.11 Lemma. *Every vertex in Γ is a cover of only finitely many vertices.*

Given a set $\mathcal{S}$ of vertices in Γ, we shall write $\ell(\mathcal{S}) = \{\ell(X) \mid X \in \mathcal{S}\}$. The following statement is similar to Lemma 4.1.10.

4.2.12 Lemma. *Given a chain $\mathcal{S}$ in Γ, we have an isomorphism of posets*

$$\ell : \mathcal{S} \to \ell(\mathcal{S}) : X \mapsto \ell(X).$$

The following statement describes the infinite chains in Γ.

4.2.13 Lemma. *An infinite chain $\mathcal{S}$ in Γ is of the form*

$$X_1 \prec X_2 \prec \cdots \prec X_n \prec \cdots .$$

Proof. Let $\mathcal{S}$ be an infinite chain. By Lemma 4.2.12, $\ell(\mathcal{S})$ is an infinite set of positive integers, and hence, it is of the following form

$$\ell(X_1) < \ell(X_2) < \cdots < \ell(X_n) < \cdots .$$

Applying Lemma 4.2.12 again, $\mathcal{S}$ is of the form as stated in the lemma. The proof of the lemma is completed.

The following statement describes the maximal chains in Γ. By $\mathbb{N}^+$ we denote the set of the positive integers. Compare it with Lemma 4.1.11.

4.2.14 Lemma. *A chain $\mathcal{S}$ in Γ is maximal if and only if $\ell(\mathcal{S}) = \mathbb{N}^+$.*

Proof. Let $\mathcal{S}$ be a chain in Γ. Assume first that $\ell(\mathcal{S}) = \mathbb{N}^+$. Suppose that there is a vertex Y such that $\mathcal{S} \cup \{Y\}$ is a chain in Γ. Then there is a vertex $X \in \mathcal{S}$ such that $\ell(X) = \ell(Y)$. If $X \neq Y$, then $X \prec Y$ or $Y \prec X$. By Lemma 4.2.12, $\ell(X) < \ell(Y)$ in the first case and $\ell(X) > \ell(Y)$ in the second case, a contradiction. Thus, $X = Y$. Therefore, $\mathcal{S}$ is a maximal chain in Γ.

Conversely, assume that $\mathcal{S}$ is a maximal chain in Γ. We claim that $\mathcal{S}$ is an infinite chain and contains a minimal element of quasi-length 1. Suppose first that $\mathcal{S}$ has a maximal element X. Take $Z = (i_X + 1, j_X)$. By Lemma 4.2.7, $X \prec Z$, and hence, $Z \notin \mathcal{S}$, which is a contradiction. Thus, $\mathcal{S}$ is infinite. By Lemma 4.2.13, $\mathcal{S}$ has a minimal element Y. Suppose secondly that $\ell(Y) > 1$. Let S be the source vertex of $\mathcal{W}_Y$. Then, $S \preceq Y$. Since $\ell(S) = 1 < \ell(Y)$, we have $S \prec Y$, and in particular $S \notin \mathcal{S}$, which is a contradiction. Our claim is true.

Suppose now on the contrary that there is an integer n such that $n \notin \ell(\mathcal{S})$. In particular, $n > 1$. Since $\mathcal{S}$ is infinite, by Lemma 4.2.12, there are $M, N \in \mathcal{S}$ such that $\ell(M) < n < \ell(N)$. It is evident that we may assume that $\ell(N)$ is the minimal cover of $\ell(M)$ in $\ell(\mathcal{S})$. Then by Lemma 4.2.12, N is a minimal cover of M in $\mathcal{S}$. By Lemma 4.2.7, $i_M < i_N$ and $j_M \geq j_N$, or else, $i_M \leq i_N$ and $j_M > j_N$. In the first case, by Lemma 4.2.5(3), $i_N \geq j_N - n + \ell(N)$. By Lemma 4.2.1, Γ contains a vertex L with $(i_L, j_L) = (i_N, j_N - n + \ell(N)) \in \Gamma$. By Lemma 4.2.7, $L \prec N$. Moreover, since $i_L = i_N > i_M$ and $j_L = j_N - n + \ell(N) \leq j_N - \ell(M) + \ell(N) = j_M - (i_N - i_M) < j_M$, By Lemma 4.2.7, $M \prec L$. Since N is a minimal cover of M, we see that $\mathcal{S} \cup \{L\}$ is a chain in Γ. Since $\ell(L) = n \notin \ell(\mathcal{S})$, we have $L \notin \mathcal{S}$, a contradiction. Similarly, we shall obtain a contradiction in the second case. Thus, $\ell(\mathcal{S}) = \mathbb{N}^+$. The proof of the lemma is completed.

The following definition gives us two convex subquivers of Γ, which will play an important role on the study of cluster categories of type $\mathbb{Z}\mathbb{A}_\infty^\infty$.

4.2.15 Definition. Given an integer n, define

$$\Gamma^+_{<n} = \bigcup\nolimits_{i<n} R_i^+ \text{ and } \Gamma^-_{>n} = \bigcup\nolimits_{j>n} R_j^- .$$

Remark. Let m, n be integers.

(1) If $m > n$, then $\Gamma^+_{<n} \subseteq \Gamma^+_{<m}$ and $\Gamma^-_{>m} \subseteq \Gamma^-_{>n}$.

(2) If $X \in \Gamma$, then $X \in \Gamma^+_{<n}$ if and only if $i_{_X} < n$, whereas $X \in \Gamma^-_{>n}$ if and only if $j_{_X} > n$.

(3) It will be useful to observe that $\Gamma^+_{<n}$ is the full subquiver of Γ generated by the successors of the quasi-simple vertex S_{n-1}, and $\Gamma^-_{>n}$ is the full subquiver of Γ generated by the predecessors of the quasi-simple vertex S_{n+1}.

The following easy lemma describes the intersection of two subquivers defined in Definition 4.2.15.

4.2.16 LEMMA. *Let m, n be two integers.*

(1) *If $m \leq n+1$ then $\Gamma^+_{<m} \cap \Gamma^-_{>n} = \emptyset$.*

(2) *If $m \geq n+2$, then $\Gamma^+_{<m} \cap \Gamma^-_{>n} = \mathcal{W}_Z$ with $(i_{_Z}, j_{_Z}) = (m-1, n+1)$.*

Proof. Let $X \in \Gamma^+_{<m} \cap \Gamma^-_{>n}$. Thus, $n < j_{_X} \leq i_{_X} < m$. We deduce from Lemma 4.2.1 that $m \geq n+2$. This shows Statement (1). Consider the vertex $Z \in \Gamma$ with $(i_{_Z}, j_{_Z}) = (m-1, n+1)$. Obviously, $X \in \mathcal{W}_Z$. Hence, $\Gamma^+_{<m} \cap \Gamma^-_{>n} \subseteq \mathcal{W}_Z$. On the other hand, $Z \in \Gamma^+_{<m} \cap \Gamma^-_{>n}$. The proof of the lemma is completed.

The following easy statement follows immediately from Lemmas 4.2.2 and 4.2.7.

4.2.17 LEMMA. *Consider a ray R_i^+ and a co-ray R_j^- in Γ.*

(1) *The vertices in R_i^+ form a chain such, for $X, Y \in R_i^+$, that $X \prec Y$ if and only if $j_{_X} > j_{_Y}$.*

(2) *The vertices in R_j^- form a chain such, for $X, Y \in R_j^-$, that $X \prec Y$ if and only if $i_{_X} < i_{_Y}$.*

4.2.18 DEFINITION. A set of vertices in Γ is called *locally finite* if it contains at most finitely many vertices of each of the rays R_i^+ and each of the co-rays R_j^- in Γ.

The following statement collects some properties of infinite chains in Γ. We refer the notion of the density in a poset to Section 1.4.

4.2.19 Lemma. *Let $\mathcal{S}$ be an infinite chain in Γ, and let i, j be two integers.*

(1) *If $\mathcal{S}$ is locally finite, then $\mathcal{S}$ is dense in Γ.*

(2) *If $\mathcal{S}$ has infinitely many vertices of R_i^+, then $\mathcal{S}$ is a dense subset of $\Gamma^+_{<i+1}$.*

(3) *If $\mathcal{S}$ has infinitely many vertices of R_j^-, then $\mathcal{S}$ is a dense subset of $\Gamma^-_{>j-1}$.*

Proof. Assume that $\mathcal{S}$ contains finitely many vertices of each ray and each co-ray in Γ and X_1 is the minimal element in $\mathcal{S}$. Since $\mathcal{S} \cap R^+_{i_{X_1}}$ and $\mathcal{S} \cap R^-_{j_{X_1}}$ both are finite, the chain $S_2 = \mathcal{S}\backslash(R^+_{i_{X_1}} \cup R^-_{j_{X_1}})$ is infinite. Then, take X_2 to be the minimal element in $\mathcal{S}_2$. Thus, we have $X_1 \prec X_2$ with $j_{X_2} < j_{X_1} \le i_{X_1} < i_{X_2}$. By induction, since $\mathcal{S}$ is infinite, we obtain an infinite subchain

$$X_1 \prec X_2 \prec \cdots \prec X_n \prec \cdots$$

of $\mathcal{S}$ such that $i_{X_n} < i_{X_{n+1}}$ and $j_{X_n} > j_{X_{n+1}}$, for $n \ge 1$. Now let $M \in \Gamma$. There is some X_n such that $i_M < i_{X_n}$ and $j_M < j_{X_n}$. That is, $M \prec X_n$. This shows Statement (1).

Assume now that $\mathcal{S}$ contains infinitely many vertices of R_i^+. In particular, $\mathcal{S} \cap R_i^+$ is an infinite chain. Let $X \in \mathcal{S}$. Then, by Lemma 4.2.11, there is a vertex $Y \in \mathcal{S} \cap R_i^+$ such that $X \prec Y$. Thus, by Lemma 4.2.7, $i_X \le i_Y = i$. Hence, $\mathcal{S}$ is a subset of $\Gamma^+_{<i+1}$. Now let $M \in \Gamma^+_{<i+1}$. Being infinite, $\mathcal{S} \cap R_i^+$ contains an object Z such that $j_Z < j_M$. Moreover, since $i_M \le i_Z$ and $j_M > j_Z$, by Lemma 4.2.7, $M \prec Z$. This shows Statement (2). The proof of Statement (3) is similar.The proof of the lemma is completed.

The following statement will be used in the study of cluster-tilting subcategories of cluster categories of type $\mathbb{A}_\infty^\infty$.

4.2.20 Lemma. *Let $\mathcal{T}$ be a set of vertices of $\Gamma^+_{<n}$(respectivley, $\Gamma^-_{>n}$) for some $n \in \mathbb{Z}$. Then $\mathcal{T}$ is dense in $\Gamma^+_{<n}$(respectively, $\Gamma^-_{>n}$) if and only if $\mathcal{T}$ contains infinitely many vertices of R^+_{n-1}(respectively, R^-_{n+1}).*

Proof. We shall consider only the case where $\mathcal{T} \subseteq \Gamma^+_{<n}$. The sufficiency follows from Lemma 4.2.19(2). Assume now that $\mathcal{T}$ contains finitely many vertices of R^+_{n-1}. Then there is a vertices $X \in R^+_{n-1}$ which does not belong to $\mathcal{T}$. Apparently X has no cover in $\mathcal{T}$. Hence, $\mathcal{T}$ is not dense. The proof of the lemma is completed.

In the following, we shall focus on the sections and section-generators in Γ. Comparing the following statement with Lemma 4.1.12, we shall omit the proof.

4.2.21 Lemma. *Let $X_1 — X_2 — \cdots — X_n$ be a sectional walk in Γ, where $n \in \mathbb{Z}$.*

(1) *Either $\ell(X_i) = \ell(X_1) - i + 1$ for all $1 \le i \le n-1$ or $\ell(X_i) = \ell(X_1) + i - 1$ for all $1 \le i \le n$.*

(2) *$\ell(X_i) < \ell(X_j)$ if and only if $X_i \prec X_j$, for all $i, j \ge 1$.*

The following statement describes the sectional paths in Γ, whose proof is similar to that of Lemma 4.1.13.

4.2.22 Lemma. *If $p : X_1 \to X_2 \to \cdots \to X_m$ is a sectional path in Γ, then p is the unique sectional walk from X_1 to X_m in Γ.*

The following statement is a characterization of sections of Γ. Compare it to Proposition 4.1.14.

4.2.23 Proposition. *Let Σ be a full subquiver of Γ. The following statements are equivalent.*

(1) *Σ is a section in Σ.*

(2) *Σ_0 is a maximal chain in Γ.*

(3) *Σ is given by an infinite sectional walk as follows:*

$$X_1 — X_2 — \cdots — X_n — \cdots$$

with $\ell(X_1) = 1$.

Proof. Assume first that Σ is a section in Γ. Since Σ meets every τ-orbit exactly once, we may write $\Sigma_0 = \{X_1, X_2, \ldots, X_n, \ldots\}$ with $\ell(X_n) = n$ for all $n \ge 1$. For each n, Σ contains a walk $X_n = Y_0 — Y_1 — \cdots — Y_r = X_{n+1}$, which is sectional since Σ is a section. By Lemma 4.2.21(1), $\ell(X_{n+1}) = \ell(X_n) + r$, and since $\ell(X_{n+1}) = n+1$, we obtain $r = 1$. That is, Γ contains an edge $X_n — X_{n+1}$ for every n. This shows that Σ is given by an infinite sectional walk as stated in Statement (3).

Assume now that Σ_0 is a maximal chain in Γ. By Lemma 4.2.14, $\ell(\Sigma_0) = \mathbb{N}^+$. Thus, Σ_0 is an infinite chain of the form

$$X_1 \prec X_2 \prec \cdots \prec X_n \prec \cdots$$

with $\ell(X_n) = n$, for all $n \geq 1$. By Lemma 4.2.10(2), we see that Γ contains an edge $X_n — X_{n+1}$, for each $n \geq 1$. Thus, Σ is given by an infinite sectional walk as stated in Statement (3).

Assume now that Σ is given by an infinite sectional walk

$$X_1 — X_2 — \cdots — X_n — \cdots$$

with $\ell(X_1) = 1$. By Lemma 4.2.21(1), we have $\ell(X_n) = n$ for $n \geq 1$. By Lemma 4.2.21(2) and Lemma 4.2.14, Σ_0 is a maximal chain in Γ. This shows Statement (2). It remains to shows Statement (3). Indeed, Σ meets every τ-orbit in Γ exactly once. Now, consider a non-trivial path $p: X_m = Y_0 \to Y_1 \to \cdots \to Y_r = X_n$ in Γ for some m, n. We shall consider only the case where $\ell(X_m) < \ell(X_n)$. Suppose that p is not sectional. Then there is a path from $\tau^- X_m$ to X_n. By Lemma 4.2.3, $i_{\tau^- X_m} \geq i_{X_n}$ and $j_{\tau^- X_m} \geq j_{X_n}$. Since $i_{X_m} - 1 = i_{\tau^- X_m}$ and $j_{X_m} - 1 = j_{\tau^- X_m}$, we see that $i_{X_m} > i_{X_n}$ and $j_{X_m} > j_{X_n}$. That is, X_m and X_n are not comparable, a contradiction. Hence, p is sectional. Since Σ contains a sectional subwalk $w : X_m — X_{m+1} — \cdots — X_n$, by the uniqueness stated in Lemma 4.2.22, $p = w$. In particular, p lies in Σ. That is, Σ is a section in Γ. The proof of the proposition is completed.

We refer the notion of a section-generator of Γ to Definition 1.4.2 and the notion of a sectional chain in Γ to Definition 1.4.10. The following statement gives a description of section-generators in Γ.

4.2.24 PROPOSITION. *Let $\mathcal{S}$ be a set of vertices of Γ. The following statements are equivalent.*

(1) *$\mathcal{S}$ is a section-generator of Γ.*

(2) *$\mathcal{S}$ is an infinite sectional chain containing a vertex of quasi-length* 1.

(3) *$\mathcal{S}$ is an infinite set of vertices of a section Σ of Γ containing all the source vertices and all the sink vertices of Σ.*

Proof. Assume that $\mathcal{S}$ is a section-generator of Γ. That is, the convex hull Σ of $\mathcal{S}$ is a section in Γ. By proposition 4.2.23(3), Σ is given by a sectional walk starting with a vertex M of quasi-length 1. Moreover, by Proposition 4.2.23(2), Σ is a chain in Γ. Since $\mathcal{S} \subseteq \Sigma$, we see that $\mathcal{S}$ is also a chain in Γ. Assume that $\mathcal{S}$ has a maximal element Y. By Lemma 4.2.7, $\mathcal{S}$ is contained in $\mathcal{W}_Y$. Since $\mathcal{W}_Y$ is convex, Σ is contained in $\mathcal{W}_Y$. Therefore, Σ is finite by Lemma 4.2.11, which is absurd. Thus, $\mathcal{S}$ is an infinite chain. Write

$$\mathcal{S} : X_1 \prec X_2 \prec \cdots \prec X_n \prec \cdots .$$

By definition, M lies on a path p in Σ from X_s to X_r for some $s, r \geq 1$. Since p is a sectional walk, by Lemma 4.2.21(2), we have $\ell(X_s) \leq \ell(M) \leq \ell(X_r)$ or $\ell(X_r) \leq \ell(M) \leq \ell(X_s)$. Hence, $\ell(X_s) = \ell(M) = 1$ or $\ell(X_r) = \ell(M) = 1$. In either case, since $\ell(X_1)$ is the smallest in $\ell(\mathcal{S})$, we see $\ell(X_1) = 1$.

We claim, for each $t \geq 1$, that Γ contains a path between X_t and X_{t+1}. Indeed, being a section, Σ contains a sectional walk

$$X_t = Y_0 — Y_1 — \cdots — Y_i = X_{t+1},$$

where $i \geq 1$. Since $\ell(X_t) < \ell(X_{t+1})$, by Lemma 4.2.21(1), we have $\ell(X_t) \leq \ell(Y_1) \leq \ell(X_{t+1})$. Since Σ is the convex hull of $\mathcal{S}$, on the other hand, Y_1 lies on a path p in Σ between two vertices $M, N \in \mathcal{S}$. We may assume that $\ell(M) \leq \ell(N)$. Then $\ell(M) \leq \ell(Y_1) \leq \ell(N)$. Since $\ell(X_{t+1})$ is a minimal cover of $\ell(X_t)$ in $\ell(\mathcal{S})$, we obtain $\ell(M) \leq \ell(X_t) \leq \ell(Y_1) \leq \ell(X_{t+1}) \leq \ell(N)$. In view of Proposition 4.2.23 and Lemma 4.2.21, we see that Σ contains a sectional walk

$$M — \cdots — X_t — \cdots — Y_1 — \cdots — X_{t+1} — \cdots — N.$$

By Lemma 4.1.13, this walk coincides with p or p^{-1}. In particular, there is a path between X_t and X_{t+1}. Hence, $\mathcal{S}$ is a sectional chain. This shows that Statement (1) implies Statement (2).

Now assume that $\mathcal{S}$ is an infinite chain, with $1 \in \ell(\mathcal{S})$, of the form

$$X_1 \prec X_2 \prec \cdots \prec X_n \prec \cdots .$$

In view of Lemma 4.2.12, we have $\ell(X_1) = 1$. By Lemma 1.4.9, assume that p_n is the sectional path between X_n and X_{n+1}, for each $n \geq 1$. In particular, p_n is a sectional walk $X_n = X_{n,0} — X_{n,1} — \cdots — X_{n,t_n} = X_{n+1}$. Since $X_n \prec X_{n+1}$, by Lemma 4.2.10(1), we have $\ell(X_n) < \ell(X_{n+1})$, and hence, by Lemma 4.2.21(1), the $\ell(X_n) = \ell(X_{n,0}) < \ell(X_{n,1}) < \cdots < \ell(X_{n,t_n}) = \ell(X_{n+1})$. Therefore, the walks p_n with $n \geq 1$ can be composed to form an infinite sectional walk Σ with $\ell(X_1) = 1$. By Proposition 4.2.23(3), Σ is a section in Γ. Since the p_n are paths, we see that $\mathcal{S} = \{X_1, \ldots, X_n, \ldots\}$ contains all the sink vertices and the source vertices of Σ. This shows that Statement (2) implies Statement (3).

Finally, suppose that $\mathcal{S} = \{X_1, \cdots, X_n, \cdots\}$ is a subset of vertices of a section Σ in Γ containing all the source vertices and sink vertices of Σ. Since Σ is a section, by Proposition 4.2.23(3), Σ is given by an infinite sectional walk of form $M_1 — M_2 — \cdots — M_n — \cdots$ with $\ell(M_1) = 1$. Therefore, M_1 is sink or source vertex in Σ. Apparently, $M_1 \in \mathcal{S}$. In particular, the $\ell(X_n)$ with $n \geq 1$ are pairwise distinct. We may assume that $\ell(X_1) < \ell(X_2) < \cdots < \ell(X_n) < \cdots$. In this case, $X_1 = M_1$. Hence, $\ell(X_1) = 1$. Fix n with $n \geq 1$. Being connected, Σ contains a reduced walk $p_n : X_n = Y_0 — Y_1 — \cdots — Y_{r-1} — Y_r = X_{n+1}$. By Lemma 4.2.21(1), $\ell(X_n) = \ell(Y_0) < \ell(Y_1) < \cdots < \ell(Y_r) = \ell(X_{n+1})$. If p_n is not a path or the inverse of a path, then $r > 1$ and some Y_j with $1 \leq j < r$ is a sink vertex or a source vertex in Σ. Observing that $\ell(Y_j) \neq \ell(X_i)$ for all $i \geq 1$, we see that $Y_j \notin \mathcal{S}$, a contradiction. Thus, p_n is a path in Γ for all $n \geq 1$, and hence, Σ is contained in the convex hull of $\mathcal{S}$. Being convex, Σ is the convex hull of $\mathcal{S}$. Hence, $\mathcal{S}$ is a section-generator. This shows that Statement (3) implies Statement (1). The proof of the proposition is completed.

The following result states some properties of section-generators of Γ.

4.2.25 COROLLARY. *Let $\mathcal{S}$ be a section-generator of Γ and Σ be its convex hull.*

(1) *If $\mathcal{S}$ contains at most finitely many vertices of each ray and each co-ray, then Σ has no infinite path.*

(2) *If $\mathcal{S}$ contains infinitely many vertices of some ray R_i^+, then Σ has an infinite path p such that p lies in R_i^+.*

(3) *If $\mathcal{S}$ contains infinitely many vertices of some co-ray R_j^-, then Σ has an infinite path p such that p lies in R_i^-.*

Proof. By Proposition 4.2.24(2), $\mathcal{S}$ is infinite. By Proposition 4.2.23, Σ is given by an infinite sectional walk of the form $M_1 — M_2 — \cdots — M_n — \cdots$ with $\ell(M_1) = 1$. Moreover, by Lemma 4.2.21(1) and (2), Σ is such that $M_n \prec M_{n+1}$ for $n \geq 1$.

Assume first that $\mathcal{S}$ contains at most finitely many vertices of each ray and each co-ray. Suppose on the contrary that Σ contains an infinite path p starting or ending at some M_n in Σ. We shall only consider the first case. Since Σ is a section, p is a sectional path. Then p entirely lies in R_i^+, for some $i \in \mathbb{Z}$. By Lemma 4.2.12, there are finitely many $X \in \mathcal{S}$ such that $X \prec M_n$. Since $\mathcal{S}$ is infinite, p contains infinitely many vertices of $\mathcal{S}$, which is a contradiction. Therefore, p is finite. This shows Statement (1). Suppose now that $\mathcal{S} \cap R_i^+$ is infinite. Then, we may write $\mathcal{S} \cap R_i^+$ as follows.

$$N_1 \prec N_2 \prec \cdots \prec N_n \prec \cdots .$$

Observe that there is a $p_n : N_n \rightsquigarrow N_{n+1}$ lying entirely in R_i^+, for $n \geq 1$. Then, the p_n can be composed to an infinite path p which lies in R_i^+. Moreover, by the definition of convex hull, p is contained in Σ. This shows Statement (2). Statement (3) is similar to show. The proof of the corollary is completed.

We conclude this section by introducing infinite co-wings in Γ, which will be used in Chapter 7.

4.2.26 Definition. Given a quasi-simple vertex $S \in \Gamma$, we define the *infinite co-wing* $\mathcal{W}(S)$ with *co-wing vertex* S to be the full subquiver of Γ generated by the vertices X for which there exists a path $N \rightsquigarrow X \rightsquigarrow M$, where M belongs to the ray starting with S and N belongs to the co-ray ending with S.

We give a description of infinite co-wings in Γ in terms of the coordinates.

4.2.27 LEMMA. *Let S be a quasi-simple vertex of $\varGamma$. If $X \in \varGamma$, then $X \in \mathcal{W}(S)$ if and only if $i_X \geq i_S = j_S \geq j_X$ if and only if $S \in \mathcal{W}_X$.*

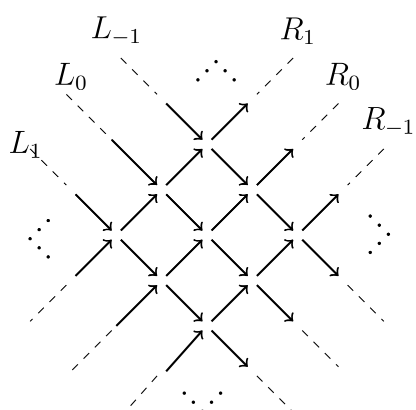
Proof. Since S is quasi-simple, $i_S = j_S$. Thus, $R^+_{i_S}$ is the ray starting with S and $R^-_{i_S}$ is the co-ray ending with S. Now assume that $X \in \mathcal{W}(S)$. By definition, there exists a path $N \rightsquigarrow X \rightsquigarrow M$, where $M \in R^+_{i_S}$ and $N \in R^-_{i_S}$. In particular, $i_M = j_N = i_S$. By Lemma 4.2.3 (3), $j_X \leq j_N$ and $i_X \geq i_M$. Thus, $i_X \geq i_S = j_S \geq j_X$.

Conversely, suppose that $X \in \varGamma$ such that $i_X \geq i_S = j_S \geq j_X$. By Lemma 4.2.1, the vertices N, M with $(i_N, j_N) = (i_X, i_S)$ and $(i_M, j_M) = (i_S, j_X)$ belong to $\varGamma$. Clearly, $M \in R^+_{i_S}$ and $N \in R^-_{i_S}$. In view of Lemma 4.1.2(3), $\varGamma$ contains a path $N \rightsquigarrow X \rightsquigarrow M$. Therefore, $X \in \mathcal{W}(S)$. The rest of the statement follows immediately from Lemma 4.2.7. The proof of the lemma is completed.

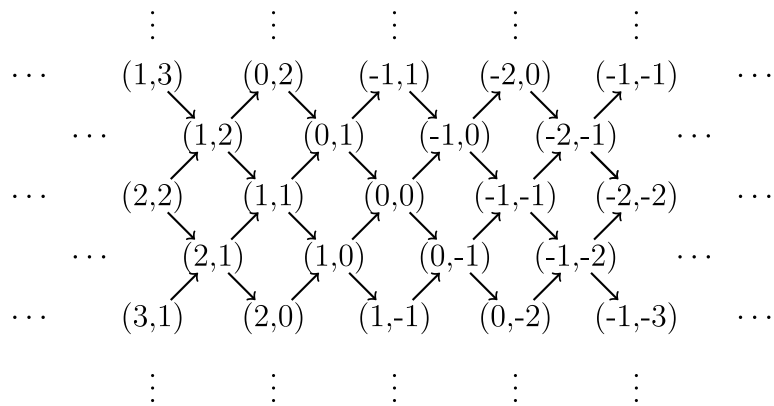
4.3 Coordinate system for a translation quiver of shape $\mathbb{Z}\mathbb{A}_\infty^\infty$

The objective of this section is to introduce a coordinate system for a translation quiver of shape $\mathbb{Z}\mathbb{A}_\infty^\infty$ in order to describe its sections and section-generators.

Throughout this section, let $\varGamma$ stand for a translation quiver of shape $\mathbb{Z}\mathbb{A}_\infty^\infty$, whose translation is written as τ. We fix two double infinite sectional paths R_0 and L_0 in $\varGamma$, whose intersection consists of one vertex. Writing $R_i = \tau^i R_0$ and $L_i = \tau^i L_0$ for $i \in \mathbb{Z}$, we can picture $\varGamma$ as follows.



Thus, for any vertex $X \in \Gamma$, there is a unique pair (i_X, j_X) of integers such that $X = L_{i_X} \cap R_{j_X}$. In the sequel, for convenience, we shall identify X with (i_X, j_X). In this way, Γ is endowed with a coordinate system as follows.



The following easy statement relates the coordinates of a vertex and the double infinite sectional paths L_i and R_j in Γ.

4.3.1 Lemma. *If $X \in \Gamma$, then the following statements hold.*

(1) *If $i \in \mathbb{Z}$, then $i_X = i$ if and only if $X \in L_i$;*

(2) *If $j \in \mathbb{Z}$, then $j_X = j$ if and only if $X \in R_j$.*

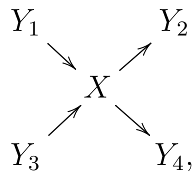
Using the coordinate system for Γ, the arrows and the translation can be described as follows.

4.3.2 Lemma. *Let X, Y be two vertices in Γ.*

(1) *There exists an arrow $X \to Y$ if and only if $(i_Y, j_Y) = (i_X, j_X - 1)$ or $(i_Y, j_Y) = (i_X - 1, j_X)$.*

(2) *$X = \tau Y$ if and only if $(i_X, j_X) = (i_Y + 1, j_Y + 1)$.*

Remark. In view of Lemma 4.3.2(1), every vertex $X \in \Gamma$ has exactly four neighbors as follows:



where $(i_{Y_1}, j_{Y_1}) = (i_X, j_X + 1)$, $(i_{Y_2}, j_{Y_2}) = (i_X - 1, j_X)$, $(i_{Y_3}, j_{Y_3}) = (i_X + 1, j_X)$ and $(i_{Y_4}, j_{Y_4}) = (i_X, j_X - 1)$.

The following statement is an easy consequence of Lemma 4.3.2(1).

4.3.3 Lemma. *Given vertices $X, Y \in \mathit{\Gamma}$, there exists a path $X \rightsquigarrow Y$ if and only if $i_X \ge i_Y$ and $j_X \ge j_Y$; and in this case, the path is sectional if and only if $i_X = i_Y$ or $j_X = j_Y$.*

The following statement describes the sectional paths in $\mathit{\Gamma}$.

4.3.4 Lemma. *Let $X_1 \to X_2 \to \cdots \to X_n$ be a sectional path in $\mathit{\Gamma}$. Then*

(1) $(i_{X_p}, j_{X_p}) = (i_{X_1}, j_{X_1} - p + 1)$, *for* $p = 1, \ldots, n$; *or*

(2) $(i_{X_p}, j_{X_p}) = (i_{X_1} - p + 1, j_{X_1})$, *for* $p = 1, \ldots, n$.

Proof. Write $(i_{X_1}, j_{X_1}) = (s, t)$. By Lemma 4.3.2(1), the statement is evident for $n = 2$. Assume now that $n > 2$. By the induction hypothesis, we may assume that $i_{X_p} = s$ and $j_{X_p} = t - p + 1$, for all $1 \le p \le n - 1$. In view of Lemma 4.3.2(1), we see that $(i_{X_n}, j_{X_n}) = (i_{X_{n-1}}, j_{X_{n-1}} - 1) = (s, t - n + 1)$ or

$$(i_{X_n}, j_{X_n}) = (i_{X_{n-1}} - 1, j_{X_{n-1}}) = (s - 1, t - n + 2) = (i_{X_{n-2}} - 1, j_{X_{n-2}} - 1).$$

Since this path is sectional, by Lemma 4.3.2(2), the second case does not occur. The proof of the lemma is completed.

4.3.5 Definition. Given a vertex $X \in \mathit{\Gamma}$, its *level* $\ell(X)$ is defined by

$$\ell(X) = i_X - j_X.$$

The following statement is an easy consequence of Lemma 4.3.2(2).

4.3.6 Lemma. *Two vertices $X, Y \in \Gamma$ lie in the same τ-orbit if and only if $\ell(X) = \ell(Y)$.*

Remark. Observe that the levels of vertices in the double infinite sectional path L_0 run over the integers. Thus, in view of Lemma 4.3.2(2), we have a bijection between the levels of vertices in Γ and the integers.

The following statement is easy to verify.

4.3.7 Lemma. *The vertex set of Γ is partially ordered in such a way that $X \preceq Y$ if and only if $i_X \le i_Y$ and $j_X \ge j_Y$.*

Remark. For convenience we shall say, by abuse of language, that Γ is a poset.

The partial order introduced in Lemma 4.3.7 plays an essential role in this section.

4.3.8 Lemma. *Let X, Y be two vertices in Γ.*

(1) *If X, Y are comparable, then $X \prec Y$ if and only if $\ell(X) < \ell(Y)$.*

(2) *There is an edge $X — Y$ in Γ if and only if $X \prec Y$ with $\ell(Y) = \ell(X) + 1$ or $Y \prec X$ with $\ell(Y) = \ell(X) - 1$.*

Proof. By definition, $\ell(Y) - \ell(X) = (i_Y - i_X) + (j_X - j_Y)$. If $X \prec Y$, then $\ell(Y) - \ell(X) > 0$. Assume now that X, Y comparable with $\ell(X) < \ell(Y)$. We claim that $X \prec Y$. Otherwise, $Y \preceq X$. In view of the equation, we see that $\ell(Y) - \ell(X) \le 0$, a contradiction. This shows Statement (1).

For proving the necessity of Statement (2), we consider only the case where there is an arrow $X \to Y$ in Γ. By Lemma 4.3.2(1), $(i_Y, j_Y) = (i_X - 1, j_X)$ or $(i_Y, j_Y) = (i_X, j_X - 1)$. In the first case, $Y \prec X$ with $\ell(Y) = \ell(X) - 1$; and in the second case, $X \prec Y$ with $\ell(Y) = \ell(X) + 1$.

For proving the sufficiency of Statement (2), we consider only the case where $X \prec Y$ with $\ell(Y) = \ell(X) + 1$. Then, $1 = \ell(Y) - \ell(X) = (i_Y - i_X) + (j_X - j_Y)$. Since $i_Y \ge i_X$ and $j_X \ge j_Y$, we have $i_Y = i_X$ and $j_X = j_Y + 1$ or $i_Y = i_X + 1$ and $j_X = j_Y$. By Lemma 4.3.2(1), Γ has an arrow $X \to Y$ in the first case; and an arrow $Y \to X$ in the second case. This shows Statement (2). The proof of the lemma is completed.

The following statement gives some properties of paths between comparable vertices.

4.3.9 Lemma. *Let $X, Y \in \Gamma$ be comparable. If Γ has a path p between X and Y, then p is sectional and is the unique path between X and Y.*

Proof. We shall only prove the case where Γ has a path p from X to Y. Write

$$p : X = M_1 \to \cdots \to M_{r-1} \to M_r = Y$$

with $r \geq 1$. By Lemma 4.3.3, we have $i_X \geq i_{M_t} \geq i_Y$ and $j_X \geq j_{M_t} \geq j_Y$ for $t = 1, \ldots, r$. Assume first that $X \preceq Y$. Then $i_X \leq i_Y$ and $j_X \geq j_Y$. This implies that either $i_{M_t} = i_X$, for $t = 1, \ldots, r$, or $j_{M_t} = j_X$ for $t = 1, \ldots, r$. By Lemma 4.3.2(1), in the first case, we have $(i_{M_t}, j_{M_t}) = (i_X, j_X - t + 1)$, for $t = 1, \ldots, r$; and in the second case, we have $(i_{M_t}, j_{M_t}) = (i_X - t + 1, j_X)$, for $t = 1, \ldots, r$. In particular, p is the unique path from X to Y, and it is sectional by Lemma 4.3.4. It is similar to show that the lemma holds in case $Y \preceq X$. The proof of the lemma is completed.

Given a set $\mathcal{S}$ of vertices of Γ, we shall write $\ell(\mathcal{S}) = \{\ell(X) \,|\, X \in \mathcal{S}\}$. The following statement is similar to Lemma 4.1.10 and Lemma 4.2.12, whose proof will be omitted.

4.3.10 Lemma. *Given a chain $\mathcal{S}$ in Γ, we have an isomorphism of posets*

$$\ell : \mathcal{S} \to \ell(\mathcal{S}) : X \mapsto \ell(X).$$

Observe that Γ contains infinite chains; for instance, given any integer i, the vertices of R_i form an infinite chain. The following statements collect some properties of infinite chains in Γ.

4.3.11 Lemma. *If $\mathcal{S}$ is an infinite chain in Γ, then it is of one of the following three forms:*

(1) $X_1 \prec X_2 \prec \cdots \prec X_n \prec \cdots$.

(2) $\cdots \prec X_n \prec \cdots \prec X_2 \prec X_1$.

(3) $\cdots \prec X_{n-1} \prec X_n \prec X_{n+1} \prec \cdots$.

Proof. By Lemma 4.3.10, $\ell(\mathcal{S})$ is an infinite set of integers. Thus, $\ell(\mathcal{S})$ is of one of the following form

(1) $\ell(X_1) < \ell(X_2) < \cdots < \ell(X_n) < \cdots$.

(2) $\cdots < \ell(X_n) < \cdots < \ell(X_2) < \ell(X_1)$.

(3) $\cdots < \ell(X_{n-1}) < \ell(X_n) < \ell(X_{n+1}) < \cdots$.

Applying Lemma 4.3.10 again, $\mathcal{S}$ is of one of the forms stated in the lemma. The proof of the lemma is completed.

REMARK. A chain $\mathcal{S}$ in Γ is called *double infinite* if it is of the form $\cdots \prec X_{n-1} \prec X_n \prec X_{n+1} \prec \cdots$. That is, $\mathcal{S}$ has neither maximal nor minimal element.

The following statement is a description of maximal chains in Γ.

4.3.12 LEMMA. *A chain $\mathcal{S}$ is maximal in Γ if and only if $\ell(\mathcal{S}) = \mathbb{Z}$.*

Proof. Let $\mathcal{S}$ be a chain such that $\ell(\mathcal{S}) = \mathbb{Z}$. Assume that there is a vertex Y such that $\mathcal{S} \cup \{Y\}$ is a chain in Γ. Then there is a vertex $X \in \mathcal{S}$ such that $\ell(X) = \ell(Y)$. If $X \neq Y$, then $X \prec Y$ or $Y \prec X$. By Lemma 4.3.10, $\ell(X) < \ell(Y)$ in the first case and $\ell(X) > \ell(Y)$ in the second case, a contradiction. Thus, $X = Y$. The necessity is established.

Conversely, assume that $\mathcal{S}$ is a maximal chain in Γ. Suppose first that $\mathcal{S}$ has a minimal element X. Take $Z = (i_X - 1, j_X)$. Clearly, $Z \prec X$, and hence, $Z \notin \mathcal{S}$. Then, we obtain a chain $\mathcal{S} \cup \{Z\}$, which contradicts the maximality of $\mathcal{S}$. Similarly, we can show that $\mathcal{S}$ has no maximal element. Thus, $\mathcal{S}$ is a double infinite chain. Suppose on the contrary that there is an integer n such that $n \notin \ell(S)$. By Lemma 4.3.10, $\ell(\mathcal{S})$ has neither a lower bound nor an upper bound, and hence, there are $X, Y \in \mathcal{S}$ such that $\ell(X) < n < \ell(Y)$. It is evident that we may assume that Y is a minimal cover of X in $\mathcal{S}$. Then, either $i_X < i_Y$ and $j_X \geq j_Y$, or else, $i_X \leq i_Y$ and $j_X > j_Y$. In the first case, consider the vertex $M = (i_Y, j_Y - n + \ell(Y)) \in \Gamma$. Then, $M \prec Y$. Moreover, since $i_M = i_Y > i_X$ and $j_M = j_Y - n + \ell(Y) \leq j_Y - \ell(X) + \ell(Y) = j_X - (i_Y - i_X) < j_X$, we have $X \prec M$.

As a consequence, $\mathcal{S} \cup \{M\}$ is a chain. Since $\ell(M) = n \notin \ell(\mathcal{S})$, we have $M \notin \mathcal{S}$, a contradiction to the maximality of $\mathcal{S}$. Similarly, we shall obtain a contradiction in the second case. Hence, $\ell(\mathcal{S}) = \mathbb{Z}$. The proof of the lemma is completed.

4.3.13 Definition. Let $\mathcal{S}$ be a chain in Γ. We shall write

$$I_{\mathcal{S}} = \{i \in \mathbb{Z} \mid \mathcal{S} \cap L_i \neq \emptyset\} = \{i_{_X} \mid X \in \mathcal{S}\}.$$

and

$$J_{\mathcal{S}} = \{j \in \mathbb{Z} \mid \mathcal{S} \cap R_j \neq \emptyset\} = \{j_{_X} \mid X \in \mathcal{S}\}.$$

Let $\mathcal{S}$ be a chain in Γ with $X \in \Gamma$. By the definition of the partial order, we clearly see that if X is a minimal element in Γ, then $i_{_X}$ is a minimal element in $I_{\mathcal{S}}$ and $j_{_X}$ is a maximal element in $J_{\mathcal{S}}$; if X is a maximal element in Γ, then $i_{_X}$ is a maximal element in $I_{\mathcal{S}}$ and $j_{_X}$ is a minimal element in $J_{\mathcal{S}}$. We shall now state more properties of $I_{\mathcal{S}}$ and $J_{\mathcal{S}}$.

4.3.14 Lemma. *Let $\mathcal{S}$ be an infinite chain in Γ, and let i, j be integers.*

(1) *If $\mathcal{S} \cap L_i$ is a double infinite chain, then $\mathcal{S} \subseteq L_i$, that is, $I_{\mathcal{S}} = \{i\}$.*

(2) *If $\mathcal{S} \cap R_j$ is a double infinite chain, then $\mathcal{S} \subseteq R_j$, that is, $J_{\mathcal{S}} = \{j\}$.*

(3) *If $\mathcal{S} \cap L_i$ is an infinite chain having a minimal (respectively, maximal) element, then i is the largest (respectively, smallest) integer in $I_{\mathcal{S}}$, while $J_{\mathcal{S}}$ has no minimal (respectively, maximal) element.*

(4) *If $\mathcal{S} \cap R_j$ is an infinite chain having a minimal (respectively, maximal) element, then j is the smallest (respectively, largest) integer in $J_{\mathcal{S}}$, while $I_{\mathcal{S}}$ has no maximal (respectively, minimal) element.*

Proof. Assume that $\mathcal{S} \cap L_i$ is a double infinite chain. By Lemma 4.3.10, $\ell(\mathcal{S} \cap L_i)$ is an infinite set of integers having neither minimal nor maximal element. Thus, for any $Z \in \mathcal{S}$, there exist $X, Y \in \mathcal{S} \cap L_i$ such that $\ell(X) \leq \ell(Z) \leq \ell(Y)$. By Lemma 4.3.10, $X \preceq Z \preceq Y$. Therefore, $i = i_{_X} \leq i_{_Z} \leq i_{_Y} = i$, and hence, $i_{_Z} = i$. Hence, $I_{\mathcal{S}} = \{i\}$. This establishes Statement (1). Similarly, we can prove Statement (2).

For proving Statement (3), we shall consider only the case where $\mathcal{S} \cap L_i$ is an infinite chain with a minimal element. That is, $\mathcal{S} \cap L_i$ is of the form

$$X_1 \prec X_2 \prec \cdots \prec X_n \prec \cdots .$$

Since the X_i have the same i-coordinate, $j_{X_1} > j_{X_2} > \cdots > j_{X_n} > \cdots$. In particular, $J_{\mathcal{S} \cap L_i}$ has no minimal element, and consequently, neither does $J_{\mathcal{S}}$.

Moreover, since $\mathcal{S} \cap L_i$ has no maximal element, by Lemma 4.3.10, neither does $\ell(\mathcal{S} \cap L_i)$. Thus, for any $Y \in \mathcal{S}$, there is some X_n such that $\ell(Y) \leq \ell(X_n)$. By Lemma 4.3.10, $Y \preceq X_n$, and hence, $i_Y \leq i_{X_n} = i$. This implies that i is the maximal element in $I_{\mathcal{S}}$. This establishes Statement (3). Similarly, we can show Statement (4). The proof of the lemma is completed.

4.3.15 Lemma. *Let $\mathcal{S}$ be a double infinite chain in Γ.*

(1) *If $I_{\mathcal{S}}$ has a minimal* (*respectively, maximal*) *element i, then $\mathcal{S} \cap L_i$ is a chain having no minimal* (*respectively, maximal*) *element.*

(2) *If $J_{\mathcal{S}}$ has a minimal* (*respectively, maximal*) *element j, then $\mathcal{S} \cap R_j$ is a chain having no maximal* (*respectively, minimal*) *element.*

Proof. For proving Statement (1), we shall only prove the case where $I_{\mathcal{S}}$ has a minimal element i. Suppose on the contrary that $\mathcal{S} \cap L_i$ has a minimal element X. We claim that X is a minimal element in $\mathcal{S}$. Let $Y \in \mathcal{S}$ be such that $Y \prec X$. That is, $i_Y \leq i_X = i$ and $j_Y \geq j_X$. Since i is the minimal in $I_{\mathcal{S}}$, we have $i_Y = i$. That is, $Y \in \mathcal{S} \cap L_i$. By the minimality of X, we have $Y = X$, a contradiction. This establishes Statement (1). The proof of Statement (2) is similar. The proof of the lemma is completed.

4.3.16 Lemma. *Let $\mathcal{S}$ be a chain in Γ.*

(1) *If $i_0, i_1 \in I_{\mathcal{S}}$ with $i_0 < i_1$, then i_1 is the minimal cover of i_0 in $I_{\mathcal{S}}$ if and only if there are $X, Y \in \mathcal{S}$, with Y a minimal cover of X, such that $i_X = i_0$ and $i_Y = i_1$.*

(2) *If $j_0, j_1 \in J_{\mathcal{S}}$ with $j_0 < j_1$, then j_1 is the minimal cover j_0 in $J_{\mathcal{S}}$ if and only if there are $X, Y \in \mathcal{S}$, with Y a minimal cover of X, such that $j_Y = j_0$ and $j_X = j_1$.*

Proof. We shall prove only Statement (1). Let $i_0, i_1 \in I_{\mathcal{S}}$ with $i_0 < i_1$. Assume first that $X, Y \in \mathcal{S}$, with Y a minimal cover of X, such that $i_{_X} = i_0$ and $i_{_Y} = i_1$. Let $i \in I_{\mathcal{S}}$, that is, $\mathcal{S} \cap L_i$ contains a vertex Z. Observe that either $Z \preceq X$ or $Y \preceq Z$. In the first case, $i = i_{_Z} \leq i_{_X} = i_0$; and in the second case, $i = i_{_Z} \geq i_{_Y} = i_1$. Thus, i_1 is the minimal cover of i_0 in $I_{\mathcal{S}}$.

Conversely, assume that i_1 is the minimal cover of i_0 in $I_{\mathcal{S}}$. By definition, $\mathcal{S} \cap L_{i_0}$ and $\mathcal{S} \cap L_{i_1}$ are non-empty. We claim that $\mathcal{S} \cap L_{i_0}$ has a maximal element X. Indeed, if this was not the case, $\mathcal{S} \cap L_{i_0}$ is either a right infinite chain or a double infinite chain. By Lemma 4.3.14, i_0 is the largest integer in $I_{\mathcal{S}}$, which is contrary to the fact that $i_0 < i_1$. Thus, our claim is true. Similarly, $\mathcal{S} \cap L_{i_1}$ has a minimal element Y. Since X, Y are comparable with $i_{_X} = i_0 < i_1 = i_{_Y}$, we see that $X \prec Y$. Given a vertex $Z \in \mathcal{S}$, if $X \prec Z \prec Y$, then $i_0 = i_{_X} \leq i_{_Z} \leq i_{_Y} = i_1$. Since X is maximal in $\mathcal{S} \cap L_{i_0}$ and Y is minimal in $\mathcal{S} \cap L_{i_1}$, we see that $Z \notin \mathcal{S} \cap L_{i_0}$ and $Z \notin \mathcal{S} \cap L_{i_1}$. That is, $i_{_Z} \neq i_0$ and $i_{_Z} \neq i_1$. This yields that $i_0 < i_{_Z} < i_1$, which is a contradiction. Hence, Y is a minimal cover of X in $\mathcal{S}$. The proof of the lemma is completed.

From now on, we shall start to study sections and section-generators in Γ. First recall that a reduced walk $X_1 — X_2 — \cdots —X_n$ in a translation quiver is called sectional if the τ-orbits of X_i with $1 \leq i \leq n$ are pairwise different.

4.3.17 LEMMA. *Let $X_1 — X_2 — \cdots —X_n$ be a sectional walk in Γ.*

(1) *Either $\ell(X_i) = \ell(X_1) + i - 1$ for all $1 \leq i \leq n$, or $\ell(X_i) = \ell(X_1) - i + 1$ for all $1 \leq i \leq n$.*

(2) *For any $1 \leq i, j \leq n$, we have $\ell(X_i) < \ell(X_j)$ if and only if $X_i \prec X_j$.*

Proof. We show Statement (1) by induction. It is trivial when $n = 1$. We may assume that $\ell(X_i) = \ell(X_1) + i - 1$ for $1 \leq i \leq n - 1$. Consider n. There is an edge $X_{n-1} — X_n$. By Lemma 4.3.8 (2), we know that $\ell(X_n) = \ell(X_1) + n - 1$ or $\ell(X_n) = \ell(X_n) + n - 2$. Since $\ell(X_{n-2}) = \ell(X_n) + n - 2$, by assumption, we have $\ell(X_n) = \ell(X_n) + n - 1$. Statement (2) follows from Statement (1) and Lemma 4.3.8(2). The proof of the lemma is completed.

REMARK: Let $p : X_1 \to X_2 \to \cdots \to X_n$ be a sectional path in $\mathit{\Gamma}$. Since the X_i are in different τ-orbits, by Lemma 4.3.6, the $\ell(X_i)$ with $1 \le i \le n$ are pairwise distinct. Hence, p satisfies Lemma 4.3.17.

4.3.18 LEMMA. *If $p : X_1 \to X_2 \to \cdots \to X_n$ is a sectional path in $\mathit{\Gamma}$, then p is the unique sectional walk from X_1 to X_n in $\mathit{\Gamma}$.*

Proof. Let $X_1 \xrightarrow{\alpha_1} X_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha^{n-1}} X_n$ be a sectional path in $\mathit{\Gamma}$. We shall consider only the case where $\ell(X_n) > \ell(X_1)$. In view of Lemma 4.3.17(1), we deduce that $\ell(X_i) = \ell(X_1) + (i-1)$, for $i = 1, \ldots, n$. Moreover, by Lemma 4.3.3 and 4.3.4, there is some integer i such that $i = i_{X_t}$ for $1 \le t \le n$. We shall show the lemma by induction. It is trivial when $n = 1$. We assume that $n > 1$ and the statement holds for $n-1$. Assume that

$$X_1 = Y_1 \overset{c_1}{\text{———}} Y_2 \overset{c_2}{\text{———}} \cdots \overset{c_{r-1}}{\text{———}} Y_r = X_n$$

is a sectional walk in $\mathit{\Gamma}$ from X_1 to X_n. Applying Lemma 4.3.17(1) again, we obtain $\ell(Y_i) = \ell(Y_1) + (i-1)$, for $i = 1, \ldots, r$. In particular, $r = n$ and $\ell(X_{n-1}) = \ell(Y_{n-1})$. Moreover, by Lemma 4.3.17(2), $X_1 \prec Y_{n-1}$, and then $i_{Y_{n-1}} \ge i_{X_1} = i$. We claim that c_{n-1} is an arrow. Otherwise, it is the inverse of an arrow $X_n \to Y_{n-1}$. Then, $Y_{n-1} = \tau^- X_{n-1}$. Then, $i_{Y_{n-1}} = i_{X_{n-1}} - 1 < i$ which is a contradiction. Hence, c_{n-1} is an arrow. Since $\ell(Y_{n-1}) = \ell(X_{n-1})$, we obtain $c_{n-1} = \alpha_{n-1}$. By the induction hypothesis, we have $c_t = \alpha_t$, for $t = 1, \ldots, n-1$. The proof of the lemma is completed.

The following statement describes the sections in $\mathit{\Gamma}$.

4.3.19 PROPOSITION. *Let $\mathit{\Sigma}$ be a full subquiver of $\mathit{\Gamma}$. The following statements are equivalent.*

(1) *$\mathit{\Sigma}$ is a section in $\mathit{\Gamma}$.*

(2) *$\mathit{\Sigma}_0$ is a maximal chain in $\mathit{\Gamma}$.*

(3) *$\mathit{\Sigma}$ is given by a double infinite sectional walk as follows:*

$$\cdots \;—\; X_{n-1} \;—\; X_n \;—\; X_{n+1} \;—\; \cdots,$$

with $\ell(X_{n+1}) = \ell(X_n) + 1$ for all $n \in \mathbb{Z}$.

Proof. Assume first that Σ is a section in Γ. Since Σ meets every τ-orbit exactly once, we may write $\Sigma_0 = \{\cdots, X_n, X_{n+1}, \cdots\}$ with $\ell(X_n) = n$ for all $n \in \mathbb{Z}$. For each n, Σ contains a walk $X_n = Y_0 — Y_1 — \cdots — Y_r = X_{n+1}$, which is sectional since Σ is a section. By Lemma 4.3.17(1), $\ell(X_{n+1}) = \ell(X_n) + r$, and since $\ell(X_{n+1}) = n+1$, we obtain $r = 1$. That is, Γ contains an edge $X_n — X_{n+1}$ for every n. This shows that Σ is given by a double infinite sectional walk as stated in Statement (3).

Assume now that Σ_0 is a maximal chain in Γ. By Lemma 4.3.12, $\ell(\Sigma_0) = \mathbb{Z}$. Thus, Σ_0 is a double infinite chain of the form

$$\cdots \prec X_{n-1} \prec X_n \prec X_{n+1} \prec \cdots$$

with $\ell(X_n) = n$, for all $n \in \mathbb{Z}$. By Lemma 4.3.8(2), we see that Γ contains an edge $X_n — X_{n+1}$, for each $n \in \mathbb{Z}$. Thus, Σ is given by a double infinite sectional walk as stated in Statement (3).

Finally, assume that Σ is given by a double infinite sectional walk

$$\cdots — X_{n-1} — X_n — X_{n+1} — \cdots,$$

with $\ell(X_n) = n$ for $n \in \mathbb{Z}$. In view of Lemma 4.3.17(2), Σ_0 is a double infinite chain in Γ. Since $\ell(\Sigma_0) = \mathbb{Z}$, by Lemma 4.3.12, Σ_0 is a maximal chain in Γ. That is, Statement (2) holds. It remains to show that Statement (1) holds. Indeed, Σ meets every τ-orbit in Γ exactly once. Now, consider a non-trivial path $p: X_m = Y_0 \to Y_1 \to \cdots \to Y_r = X_n$ in Γ for some $m, n \in \mathbb{Z}$. We shall consider only the case where $\ell(X_m) < \ell(X_n)$. Suppose that p is not sectional. Then there is a path from $\tau^- X_m$ to X_n. By Lemma 4.3.2, $i_{\tau^- X_m} \geq i_{X_n}$ and $j_{\tau^- X_m} \geq j_{X_n}$. Since $i_{X_m} - 1 = i_{\tau^- X_m}$ and $j_{X_m} - 1 = j_{\tau^- X_m}$, we see that $i_{X_m} > i_{X_n}$ and $j_{X_m} > j_{X_n}$. That is, X_m and X_n are not comparable, a contradiction. Hence, p is sectional. Since Σ contains a sectional subwalk $w : X_m — X_{m+1} — \cdots — X_n$, by the uniqueness stated in Lemma 4.3.18, $p = w$. In particular, p lies in Σ. That is, Σ is a section in Γ. The proof of the proposition is completed.

The following statement is about the sections in Γ containing no infinite path.

4.3.20 Lemma. *A section of Γ contains no infinite path if and only if it passes L_i and R_j for all $i, j \in \mathbb{Z}$.*

Proof. Let Σ be a section of Γ. Assume that Σ passes every L_i and every R_j, where $i, j \in \mathbb{Z}$. That is, $I_{\Sigma_0} = J_{\Sigma_0} = \mathbb{Z}$. Suppose on the contrary that Σ contains an infinite path p. In view of Lemma 4.3.3, we may assume that p is a subpath of some L_i. That is, $\Sigma \cap L_i$ is an infinite chain. By Lemma 4.3.14(1) and (3), i is either minimal or maximal in I_{Σ_0}, a contradiction. The sufficiency is established.

Conversely, assume that Σ contains no infinite path. By Proposition 4.3.19, Σ_0 is a maximal chain in Γ, and by Lemma 4.3.12, $\ell(\Sigma_0) = \mathbb{Z}$. Given $i \in I_\Sigma$, we claim that $i+1, i-1 \in I_{\Sigma_0}$. Indeed, by the convexity of Σ in Γ, we see that $\Sigma \cap L_i$ is a finite chain. Let Y_0 be the maximal element in $\Sigma \cap L_i$. By Proposition 4.3.19(3), there is a vertex $Y_1 \in \Sigma$ such that $\ell(Y_1) = \ell(Y_0)+1$. Then, $Y_0 \prec Y_1$. In particular, $Y_1 \notin L_i$. That is, $i_{_{Y_1}} \neq i$. Since $i_{_{Y_1}} \geq i_{_{Y_0}}$, we obtain $i_{_{Y_1}} > i$ and $j_{_{Y_1}} \leq j_{_{Y_0}}$. Since $1 = \ell(Y_1) - \ell(Y_0) = (i_{_{Y_1}} - i_{_{Y_0}}) + (j_{_{Y_0}} - j_{_{Y_1}})$, we obtain $i_{_{Y_1}} = i_{_{Y_0}} + 1 = i+1$. Thus, $i+1 \in I_{\Sigma_0}$. Considering the minimal element of $\Sigma \cap L_i$, we see that $i-1 \in I_{\Sigma_0}$. This establishes our claim. As a consequence, $I_{\Sigma_0} = \mathbb{Z}$. In a similar fashion, we can show that $J_{\Sigma_0} = \mathbb{Z}$. The proof of the lemma is completed.

We give the following definition. Compare it with Definition 1.4.10.

4.3.21 Definition. A chain $\mathcal{S}$ in Γ is called a *sectional chain* if any two vertices $X, Y \in \mathcal{S}$, with Y a minimal cover of X in $\mathcal{S}$, are connected by a path in Γ.

The following is a description of a sectional chain in Γ.

4.3.22 Proposition. *A set $\mathcal{S}$ of vertices of Γ is a sectional chain if and only if its convex hull is a connected subquiver Σ of a section of Γ such that $\mathcal{S}$ is contained in Σ_0 and contains all the sink vertices and all the source vertices of Σ.*

Proof. Suppose that $\mathcal{S}$ is a sectional chain in Γ. By Lemma 4.3.11, we can write $\mathcal{S} = \{X_n\}_{n \in \mathcal{I}}$, where $\mathcal{I}$ is some interval of $\mathbb{Z}$, such that $X_n \prec X_{n+1}$ for every non-maximal $n \in \mathcal{I}$. In view of Lemma 4.3.9, X_n, X_{n+1} are connected by a sectional path p_n, for each non-maximal $n \in \mathcal{I}$. In particular, p_n is a sectional walk $X_n = X_{n,0} — X_{n,1} — \cdots — X_{n,t_n} = X_{n+1}$. Since $X_n \prec X_{n+1}$, by Lemma 4.3.8(1), we have $\ell(X_n) < \ell(X_{n+1})$, and hence, by Lemma 4.3.17(1),

$$\ell(X_n) = \ell(X_{n,0}) < \ell(X_{n,1}) < \cdots < \ell(X_{n,t_n}) = \ell(X_{n+1}).$$

Therefore, the $X_{n,j}$ with $n \in \mathcal{I}$ and $1 \le j \le t_n$ generate a connected subquiver Σ of Γ. In view of Proposition 4.3.19(3), Σ is a subquiver of a section in Γ. In particular, Σ is convex. On the other hand, Σ is contained in the convex hull of $\mathcal{S}$. Therefore, Σ is the convex hull of $\mathcal{S}$. Since the p_n are paths, we see that $\mathcal{S} = \{X_n\}_{n\in\mathcal{I}}$ contains all the sink vertices and all the source vertices of Σ. The necessity is established.

Conversely, let $\mathcal{S}$ be a subset of vertices of Σ of Γ, containing all the source vertices and sink vertices of Σ. Since Σ is a subquiver of a section, in view of Proposition 4.3.19, $\mathcal{S}$ is a chain. By Lemma 4.3.10, we may write $\mathcal{S} = \{X_n\}_{n\in\mathcal{I}}$, where $\mathcal{I}$ is some interval of $\mathbb{Z}$ such that $\ell(X_n) < \ell(X_{n+1})$ for each non-maximal $n \in \mathcal{I}$. Fix $n \in \mathcal{I}$. Since Σ is a connected subquiver of a section of Γ, it contains a sectional walk $p_n : X_n = Y_0 — Y_1 — \cdots — Y_{r-1} — Y_r = X_{n+1}$. Since $\ell(X_n) < \ell(X_{n+1})$, by Lemma 4.3.17(1), $\ell(X_n) = \ell(Y_0) < \ell(Y_1) < \cdots < \ell(Y_r) = \ell(X_{n+1})$. If p_n is neither a path nor the inverse of a path, then $r > 1$ and some Y_s with $1 < s < r$ is a sink vertex or a source vertex in Σ. Observing that $\ell(Y_s) \ne \ell(X_n)$ for all $n \in \mathcal{I}$, we see that $Y_s \notin \mathcal{S}$, a contradiction. The proof of the proposition is completed.

The following is a description of section-generators in Γ.

4.3.23 Proposition. *A set of vertices in Γ is a section-generator if and only if it is a double infinite sectional chain.*

Proof. Let $\mathcal{S}$ be a set of vertices in Γ. Assume first that the convex hull Σ of $\mathcal{S}$ is a section. Since $\mathcal{S} \subseteq \Sigma_0$, by Proposition 4.3.19(2), $\mathcal{S}$ is a chain. Suppose that $\mathcal{S}$ has a maximal element X. By Lemma 4.3.10, $\ell(X)$ is the maximal element in $\ell(\mathcal{S})$. Let $M \in \Sigma$. Then, M lies on a path p in Σ between two vertices $Y, Z \in \mathcal{S}$. We may assume that $\ell(Y) \le \ell(Z)$. Since Σ is a section, p is a sectional walk. By Lemma 4.3.17(1), $\ell(Y) \le \ell(M) \le \ell(Z)$, and thus, $\ell(M) \le \ell(X)$. That is, $\ell(X)$ is the maximal element of $\ell(\Sigma)$, a contradiction to Proposition 4.3.19(2). Thus, $\mathcal{S}$ has no maximal element. Similarly, $\mathcal{S}$ has no minimal element. Thus, $\mathcal{S}$ is a double infinite chain. Assume now that $X, Y \in \mathcal{S}$ such that Y is a minimal cover of X. Then Σ contains a sectional walk

$$X = X_1 — X_2 — \cdots — X_n = Y.$$

We claim that there is a path between X and Y. This is evident if $i = 1$. Assume that $i > 1$. Then, $\ell(X) \leq \ell(X_2) \leq \ell(Y)$ by Lemma 4.3.17(1) . Moreover, X_2 lies on a path p in Σ between two vertices $M, N \in \mathcal{S}$. We may assume that $\ell(M) \leq \ell(N)$. Then $\ell(M) \leq \ell(X_2) \leq \ell(N)$. Since $\ell(Y)$ is a minimal cover of $\ell(X)$ in $\ell(\mathcal{S})$ by Lemma 4.3.8 , we obtain $\ell(M) \leq \ell(X) \leq \ell(X_2) \leq \ell(Y) \leq \ell(N)$. In view of Proposition 4.3.19(3), Σ contains a sectional walk

$$M — \cdots — X — \cdots — X_2 — \cdots — Y — \cdots — N.$$

By Lemma 4.3.18, this walk coincides with p or p^{-1}. In particular, there is a path between X and Y. Hence, $\mathcal{S}$ is a sectional chain.

Conversely, assume that $\mathcal{S}$ is a double infinite chain of the form

$$\cdots \prec X_1 \prec X_2 \prec \cdots \prec X_n \prec \cdots$$

such that X_n and X_{n+1} are connected by a path p_n, for every $n \in \mathbb{Z}$. By Lemma 4.3.10, we have

$$\cdots < \ell(X_1) < \ell(X_2) < \cdots < \ell(X_n) < \cdots .$$

By Lemma 4.3.9, p_n is a sectional path, which is a sectional walk of the form

$$X_n = X_{n,0} — X_{n,1} — \cdots — X_{n,t_n} = X_{n+1}.$$

In particular, the full subquiver Σ of Γ generated by the vertices $X_{n,i}$ with $n \in \mathbb{Z}$ and $1 \leq i \leq t_n$ is connected and contained in the convex hull of $\mathcal{S}$. On the other hand, since $\ell(X_n) < \ell(X_{n+1})$, in view of Lemma 4.3.17, we have $\ell(X_{n,i+1}) = \ell(X_{n,i}) + 1$ and $X_{n,i} \prec X_{n,i+1}$, for $0 \leq i \leq t_n - 1$. Thus, Σ_0 is a chain with $\ell(\Sigma_0) = \mathbb{Z}$. By Lemma 4.3.12, Σ_0 is a maximal chain in Γ. By Proposition 4.3.19, Σ is a section in Γ. Being convex, Σ is the convex hull of $\mathcal{S}$. The proof of the proposition is completed.

Remark. We can easily deduce from Proposition 4.3.22 that $\mathcal{S}$ is a section-generator of Γ if and only if its is a subset of vertices of a section Σ in Γ containing all the sink vertices and all the source vertices of Σ.

The following result states a property of sectional chains in Γ.

4.3.24 Proposition. *Let $\mathcal{S}$ be a sectional chain in Γ, and let Σ be its convex hull in Γ. Then Σ is a section in Γ without infinite paths if and only if neither of $I_{\mathcal{S}}$ and $J_{\mathcal{S}}$ has an upper or lower bound.*

Proof. Assume first that Σ is a section containing no infinite path. Then $\mathcal{S}$, by definition, is a section-generator of Γ. By Proposition 4.3.23, $\mathcal{S}$ is a double infinite chain. Suppose on the contrary that $I_{\mathcal{S}}$ has a smallest integer i_0. Since Σ is a section, by Lemma 4.3.19, Σ_0 is a chain and $\mathcal{S} \subseteq \Sigma_0$. In particular, $I_{\mathcal{S}} \subseteq I_{\Sigma_0}$ and $J_{\mathcal{S}} \subseteq J_{\Sigma_0}$. We shall show that i_0 is also the smallest integer I_{Σ_0}. Indeed, let $X \in \Sigma$. By definition, there exist $M, N \in \mathcal{S}$ such that X lies on a path p between M and N. Since Σ is a section, p is a sectional path lying entirely in Σ. By Lemma 4.3.18, p is a sectional walk. In view of Lemma 4.3.17(2), either $M \preceq X \preceq N$ or $N \preceq X \preceq M$. We may assume that the first case occurs. Since $i_{_M} \in I_{\mathcal{S}}$, we obtain $i_0 \le i_{_M} \le i_{_X}$. Thus, i_0 is indeed the smallest integer in I_{Σ_0}. On the other hand, since Σ contains no infinite path, $I_{\Sigma_0} = \mathbb{Z}$ by Lemma 4.3.20, a contradiction. Similarly, we shall obtain a contradiction if $I_{\mathcal{S}}$ has a largest integer. In a similar fashion, we can show that $J_{\mathcal{S}}$ has neither upper bound nor lower bound.

Conversely, assume that neither of $I_{\mathcal{S}}$ and $J_{\mathcal{S}}$ has an upper or lower bound. Then, by the definition of the partial order over Γ we see that $\mathcal{S}$ has neither a minimal element nor a maximal element, that is, $\mathcal{S}$ is a double infinite chain. Since $\mathcal{S}$ is also a sectional chain, by Proposition 4.3.23, $\mathcal{S}$ is a section-generator of Γ, that is, Σ is a section in Γ. By Lemma 4.3.19, Σ_0 is a chain. Since $\mathcal{S} \subseteq \Sigma_0$, we have $I_{\mathcal{S}} \subseteq I_{\Sigma_0}$ and $J_{\mathcal{S}} \subseteq J_{\Sigma_0}$. Suppose that Σ contains an infinite path p. Being sectional, p is contained in some L_i or in some R_j. Consider only the first case. In particular, $\Sigma_0 \cap L_i$ is infinite. By Lemma 4.3.14(3), i is the largest or the smallest integer in I_{Σ_0}, a contradiction. The proof of the proposition is completed.

We shall conclude this section by the following two lemmas, which will be used to characterize the τ-rigidity in Chapter 5. Recall that, given $X \in \Gamma$, denote R^X the forward rectangle of X and R_X the backward rectangle of X. The following statement describes R^X and R_X in terms of the coordinates.

4.3.25 Lemma. *Let X, Y be vertices in Γ. The following statements hold.*

(1) $Y \in R_X$ *if and only* $i_Y \geq i_X$ *and* $j_Y \geq j_X$.

(2) $Y \in R^X$ *if and only if* $i_Y \leq i_X$ *and* $j_Y \leq j_X$.

Proof. We shall prove only (1). Consider a vertex $Y \in \Gamma$. By definition, $Y \in R_X$ if and only if Y is a predecessor of X. By Lemma 4.3.3, the latter is equivalent to $i_Y \geq i_X$ and $j_Y \geq j_X$. The proof of the lemma is completed.

We have the following useful observation. Compare it with Lemma 1.4.11.

4.3.26 Lemma. *Let* $X, Y \in \Gamma$. *The following statements are equivalent.*

(1) X, Y *are comparable.*

(2) $X \notin R_{\tau Y}$ *and* $X \notin R^{\tau^- Y}$.

(3) $Y \notin R_{\tau X}$ *and* $Y \notin R^{\tau^- X}$.

Proof. First, we shall show the equivalence of Statements (2) and (3). By definition, $X \in R_{\tau Y}$ if and only if $\tau Y \in R^X$ if and only if $Y \in R^{\tau^- X}$. Similarly, $X \in R^{\tau^- Y}$ if and only if $Y \in R_{\tau X}$. It remains to show the equivalence of Statements (1) and (2).

Assume first that $X \preceq Y$. That is, $i_X \leq i_Y$ and $j_X \geq j_Y$. Hence, we have $i_X < i_Y + 1 = i_{\tau Y}$ and $j_X > j_Y - 1 = j_{\tau^- Y}$. By Lemma 4.3.25, the first inequality shows that $X \notin R_{\tau Y}$ and the second inequality shows that $X \notin R^{\tau^- Y}$. Similarly, if $Y \preceq X$, then $Y \notin R_{\tau X}$ and $Y \notin R^{\tau^- X}$, and by the equivalence of Statements (2) an (3), we also have $X \notin R_{\tau Y}$ and $X \notin R^{\tau^- Y}$. This shows that Statement (1) implies Statement (2).

Assume, conversely, that X, Y not comparable. In particular, $i_Y \neq i_X$ and $j_Y \neq j_X$. Suppose first that $i_Y > i_X$. Then $j_Y > j_X$. Thus, we have $i_Y \geq i_X + 1 = i_{\tau X}$ and $j_Y \geq j_X + 1 = j_{\tau X}$. That is, $Y \in R_{\tau X}$, and in particular, Statement (2) does not hold. Similarly, if $i_X > i_Y$, then $X \in R_{\tau Y}$, that is, $\tau Y \in R^X$. This implies that $Y \in R^{\tau^- X}$. This shows that Statement (2) implies Statement (1). The proof of the lemma is completed.

Chapter 5

The τ-rigidity theory in an Auslander-Reiten category

The aim of this chapter is to study the τ-rigidity theory in an Auslander-Reiten category, one of the essential topics in this thesis. The τ-rigidity theory first appeared in the representation theory of artin algebras in connection with tilting modules; see, for example, [45, 58], while a systematic study in the representation theory was first carried out by Adachi, Iyama and Reiten in their introduction of τ-tilting theory; see [1]. Observe that the τ-tilting theory was actually inspired by the cluster tilting theory; see, for example, [20].

Throughout this chapter, let $\mathcal{A}$ stand for an Auslander-Reiten category, whose Auslander-Reiten quiver is denoted by $\Gamma_{\mathcal{A}}$ and whose Auslander-Reiten translation will be written as $\tau_{\mathcal{A}}$.

5.1 The τ-rigidity

The objective of this section is to introduce the notion of $\tau_{\mathcal{A}}$-rigidity in $\mathcal{A}$. We shall start with the following definition.

5.1.1 Definition. An object $X \in \mathcal{A}$ is called *$\tau_{\mathcal{A}}$-rigid* if $\mathrm{Hom}_{\mathcal{A}}(X, \tau_{\mathcal{A}} X) = 0$, and an additive subcategory of $\mathcal{A}$ is called *$\tau_{\mathcal{A}}$-rigid* if each of its objects is $\tau_{\mathcal{A}}$-rigid.

Since every non-zero object of $\mathcal{A}$ is a finite direct sum of objects in $\Gamma_{\mathcal{A}}$, it is natural for us to introduce the notion of $\tau_{\mathcal{A}}$-rigid set in $\Gamma_{\mathcal{A}}$ as follows.

5.1.2 DEFINITION. (1) A pair (X, Y) of objects in $\Gamma_{\mathcal{A}}$ is called a *$\tau_{\mathcal{A}}$-rigid pair* if $X \oplus Y$ is a $\tau_{\mathcal{A}}$-rigid object in $\mathcal{A}$.

(2) A set $\mathcal{T}$ of objects in $\Gamma_{\mathcal{A}}$ is called *$\tau_{\mathcal{A}}$-rigid* if every pair (X, Y) in $\mathcal{T} \times \mathcal{T}$ is $\tau_{\mathcal{A}}$-rigid.

(3) Given a subquiver Δ of $\Gamma_{\mathcal{A}}$, a $\tau_{\mathcal{A}}$-rigid set $\mathcal{T}$ of objects in Δ is called *maximal $\tau_{\mathcal{A}}$-rigid in Δ* provided, for any $X \in \Delta$, that $\mathcal{T} \cup \{X\}$ is $\tau_{\mathcal{A}}$-rigid if and only if $X \in \mathcal{T}$.

REMARK. Let Δ be a subquiver of $\Gamma_{\mathcal{A}}$. A maximal $\tau_{\mathcal{A}}$-rigid set in Δ is not necessarily a maximal $\tau_{\mathcal{A}}$-rigid set in $\Gamma_{\mathcal{A}}$.

Let $\mathscr{T}$ be an additive subcategory of $\mathcal{A}$. We shall denote by $\operatorname{ind}\mathscr{T}$ the set of objects of $\Gamma_{\mathcal{A}}$ which lie in $\mathscr{T}$. The following statement is evident.

5.1.3 LEMMA. *An additive subcategory $\mathscr{T}$ of $\mathcal{A}$ is $\tau_{\mathcal{A}}$-rigid if and only if $\operatorname{ind}\mathscr{T}$ is a $\tau_{\mathcal{A}}$-rigid set in $\Gamma_{\mathcal{A}}$.*

Now, let Δ be a convex subquiver of a standard component of $\Gamma_{\mathcal{A}}$. By Lemma 2.2.2, $\operatorname{add}\Delta$ is an Auslander-Reiten category, whose Auslander-Reiten translation will be denoted by τ_{Δ}. The following easy observation is important for our later investigation.

5.1.4 LEMMA. *Let $\mathcal{A}$ be an Auslander-Reiten category, and let Δ be a convex subquiver of a standard component Γ of $\Gamma_{\mathcal{A}}$. If $X, Y \in \Delta$ are $\tau_{\mathcal{A}}$-rigid, then the pair (X, Y) is $\tau_{\mathcal{A}}$-rigid if and only if it is τ_{Δ}-rigid.*

Proof. Let X, Y be two $\tau_{\mathcal{A}}$-rigid objects lying in Δ. By Lemma 2.2.4, $\Gamma_{\operatorname{add}\Delta}$ is a translation subquiver of Γ. In particular, for any object $X \in \Delta$, either $\tau_{\Delta} X = \tau_{\mathcal{A}} X$, or else, $\tau_{\Delta} X = 0$. Thus, the necessity is trivial.

Assume that (X, Y) is a τ_{Δ}-rigid pair in $\operatorname{add}\Delta$. If $\operatorname{Hom}_{\mathcal{A}}(X, \tau_{\mathcal{A}} Y) \neq 0$, then $\tau_{\mathcal{A}} Y \neq 0$ and $\tau_{\mathcal{A}} Y \notin \Delta$. Being standard, Γ contains a path $X \rightsquigarrow \tau_{\mathcal{A}} Y$, and hence, a path $X \rightsquigarrow \tau_{\mathcal{A}} Y \rightsquigarrow Y$. Since Δ is convex in Γ, we obtain $\tau_{\mathcal{A}} Y \in \Delta$, a contradiction. Hence, $\operatorname{Hom}_{\mathcal{A}}(X, \tau_{\mathcal{A}} Y) = 0$. Similarly, $\operatorname{Hom}_{\mathcal{A}}(Y, \tau_{\mathcal{A}} X) = 0$.

Since $\text{Hom}_{\mathcal{A}}(X, \tau_{_\mathcal{A}} X) = \text{Hom}_{\mathcal{A}}(Y, \tau_{_\mathcal{A}} Y) = 0$ by the assumption, $X \oplus Y$ is a $\tau_{_\mathcal{A}}$-rigid object, that is, (X, Y) is a $\tau_{_\mathcal{A}}$-rigid pair in $\mathcal{A}$. The proof of the lemma is completed.

Remark. Let Δ be a convex subquiver of a standard component of $\Gamma_{\mathcal{A}}$. If every object in Δ is $\tau_{_\mathcal{A}}$-rigid, then the $\tau_{_\mathcal{A}}$-rigid sets in Δ are the $\tau_{_\Delta}$-rigid sets in Δ.

We shall remark that the τ-rigidity theory is closely related to the tilting theory over a finite dimensional hereditary algebra and the cluster tilting theory in a cluster category; see, for example, [32, 35, 20]. Indeed, consider a path algebra $H = kQ$, where Q a finite acyclic quiver with n vertices. It is well known that the category mod$H$ of finitely generated H-modules is an Auslander-Reiten category; see [9]. We denote by Γ_H its Auslander-Reiten quiver and by $\tau_{_H}$ its Auslander-Reiten translation. A module M in mod$H$ is called *tilting* if it is $\tau_{_H}$-rigid with n non-isomorphic indecomposable direct summands; see [32]. We refer to [18] for the definition of a tilting module over a general finite dimensional k-algebra.

The following statement is well known; see, for example, [35, (14)].

5.1.5 Lemma. *A basic module in* mod$H$ *is tilting if and only if its corresponding set in* Γ_H *is maximal* $\tau_{_H}$*-rigid.*

Next, we recall the τ-rigidity theory in cluster categories. Fix Q a locally finite quiver without infinite paths. Recall that the skeleton $\mathscr{D}^b(Q)$ of the derived category $D^b(\text{rep}(Q))$, chosen in Section 3.3, is an Auslander-Reiten category, whose Auslander-Reiten quiver is denoted by $\Gamma_{\mathscr{D}^b(Q)}$ and the Auslander-Reiten translation is denoted by $\tau_{_D}$. Furthermore, the cluster category $\mathscr{C}(Q)$ is an Auslander-Reiten category, whose Auslander-Reiten quiver $\Gamma_{\mathscr{C}(Q)}$ has as vertices the objects of the fundamental domain $\mathscr{F}(Q)$.

An object X in $\mathscr{C}(Q)$ is said to be *rigid* if $\text{Hom}_{\mathscr{C}(Q)}(X, X[1]) = 0$; and a pair of objects (X, Y) in $\mathscr{C}(Q)$ is called a *rigid pair* in $\mathscr{C}(Q)$ if $X \oplus Y$ is rigid. A set of objects in $\mathscr{C}(Q)$ is *rigid* if every pair of its objects is rigid.

The following result is very important to our study.

5.1.6 Lemma. *Let Q be an infinite Dynkin quiver with no infinite path. If X, Y are indecomposable objects in $\mathscr{C}(Q)$, then the following statements are equivalent.*

(1) *The pair (X, Y) is rigid in $\mathscr{C}(Q)$.*

(2) *The pair (X, Y) is $\tau_{\mathscr{C}}$-rigid in $\mathscr{C}(Q)$*

(3) *The pair (X, Y) is τ_{D}-rigid in $\mathscr{D}^b(Q)$.*

Proof. Since $\mathscr{C}(Q)$ is 2-Calabi-Yau, the equivalence of Statement (1) and Statement (2) is trivial. Let X, Y be two indecomposable objects of $\mathscr{C}(Q)$. We may assume that X, Y are in the fundamental domain $\mathscr{F}(Q)$. Suppose first that X and $\tau_D Y$ are representations. By Lemma 2.6(1) in [48], we have

$$\begin{aligned}
\mathrm{Hom}_{\mathscr{C}(Q)}(X, Y[1]) &\cong \mathrm{Hom}_{\mathscr{C}(Q)}(X, \tau_{\mathscr{C}} Y) \\
&\cong \mathrm{Hom}_{\mathscr{C}(Q)}(X, \tau_D Y) \\
&\cong \mathrm{Hom}_{\mathscr{D}^b(Q)}(X, \tau_D Y) \oplus D\mathrm{Hom}_{\mathscr{D}^b(Q)}(\tau_D Y, \tau_D^2 X) \\
&\cong \mathrm{Hom}_{\mathscr{D}^b(Q)}(X, \tau_D Y) \oplus D\mathrm{Hom}_{\mathscr{D}^b(Q)}(Y, \tau_D X).
\end{aligned}$$

Since $\mathscr{C}(Q)$ is 2-Calabi-Yau in which every indecomposable object is rigid; see [48, Corollary 2.10], (X, Y) is rigid in $\mathscr{C}(Q)$ if and only if $\mathrm{Hom}_{\mathscr{C}(Q)}(X, Y[1]) = 0$. In view of the above isomorphisms, this is equivalent to $\mathrm{Hom}_{\mathscr{D}^b(Q)}(X, \tau_D Y) = 0$ and $\mathrm{Hom}_{\mathscr{D}^b(Q)}(Y, \tau_D X) = 0$. Since every connected component in $\mathscr{F}(Q)$ is standard, every object in $\mathscr{F}(Q)$ is τ_D-rigid. Thus, the latter condition is equivalent to (X, Y) being τ_D-rigid in $\mathscr{F}(Q)$.

In general, there is an integer $n \geq 0$ such that $\tau_{\mathscr{C}}^{-n} X$ and $\tau_{\mathscr{C}}^{-n+1} Y$ are representations. Set $M = \tau_{\mathscr{C}}^{-n} X$ and $N = \tau_{\mathscr{C}}^{-n} Y$. Since $\tau_{\mathscr{C}}$ is an equivalence, (X, Y) is a rigid pair in $\mathscr{C}(Q)$ if and only if (M, N) is a rigid pair in $\mathscr{C}(Q)$. As we have just shown, this is equivalent to (M, N) is a τ_D-rigid pair in $\mathscr{F}(Q)$. Since τ_D is an equivalence, the latter is equivalent to (X, Y) being τ_D-rigid pair in $\mathscr{F}(Q)$. The proof of the lemma is completed.

Given a strictly additive subcategory $\mathscr{T}$ of $\mathscr{C}(Q)$, we shall denote by $\mathrm{ind}\mathscr{T}$ the set of objects of $\Gamma_{\mathscr{C}(Q)}$ which lie in $\mathscr{T}$. Observe that the objects in $\mathrm{ind}\mathscr{T}$ form a complete set of representatives of the isomorphism classes of the indecomposable objects of $\mathscr{T}$.

5.1.7 PROPOSITION. *Let Q be an infinite Dynkin quiver with no infinite path, and let $\mathscr{T}$ be a strictly additive subcategory of $\mathscr{C}(Q)$. The following statements are equivalent.*

(1) *$\mathscr{T}$ is weakly cluster-tilting in $\mathscr{C}(Q)$.*

(2) ind$\mathscr{T}$ *is a maximal rigid set in $\Gamma_{\mathscr{C}(Q)}$.*

(3) ind$\mathscr{T}$ *is a maximal τ_D-rigid set in $\mathscr{F}(Q)$.*

Proof. By Lemma 2.11 in [48], $\mathscr{T}$ is weakly cluster-tilting in $\mathscr{C}(Q)$ if and only if $\mathscr{T}$ is maximal rigid in $\mathscr{C}(Q)$. Since $\mathscr{C}(Q)$ is Krull-Schmidt, the equivalence of Statement (1) and Statement (2) holds. The equivalence of Statement (2) and Statement (3) follows immediately from Lemma 5.1.6. The proof of the proposition is completed.

5.2 Maximal τ-rigid sets in a standard wing

Throughout this section, assume that $\mathcal{A}$ is an Auslander-Reiten category such that its Auslander-Reiten quiver is a standard wing of positive rank n, which is denoted by $\mathcal{W}$. The Auslander-Reiten translation of $\mathcal{A}$ will be simply written as τ. Our objective of this section is to study the maximal τ-rigid sets in $\mathcal{W}$.

As seen in Lemma 1.4.6, each object $X \in \mathcal{W}$ is a wing vertex of a unique wing $\mathcal{W}_X$ of rank $\ell(X)$ in $\mathcal{W}$. In Definition 1.4.8, we have defined two wings $\mathcal{W}_X, \mathcal{W}_Y$ in $\mathcal{W}$ to be comparable or separable. These notions enable us to describe the τ-rigidity of a pair of objects in $\mathcal{W}$.

5.2.1 LEMMA. *If $X, Y \in \mathcal{W}$, then (X, Y) is a τ-rigid pair if and only if $\mathcal{W}_X, \mathcal{W}_Y$ are comparable or separable.*

Proof. Let X, Y be objects in $\mathcal{W}$. Since $\mathcal{W}$ is standard, by Lemma 2.2.7, $\mathrm{Hom}_{\mathcal{A}}(X, \tau Y) \neq 0$ if and only if $\tau Y \in R^X$ if and only if $Y \in R^{\tau^- X}$. Similarly, $\mathrm{Hom}_{\mathcal{A}}(Y, \tau X) \neq 0$ if and only if $Y \in R_{\tau X}$. Since $X \notin R^{\tau^- X}$, we have $\mathrm{Hom}_{\mathcal{A}}(X, \tau X) = 0$, and also, $\mathrm{Hom}_{\mathcal{A}}(Y, \tau Y) = 0$. Therefore, (X, Y) is a τ-rigid pair if and only if $\mathrm{Hom}_{\mathcal{A}}(X, \tau Y) = 0$ and $\mathrm{Hom}_{\mathcal{A}}(Y, \tau X) = 0$, if and only if $Y \notin R^{\tau^- X}$ and $Y \notin R_{\tau X}$. By Proposition 1.4.11, the latter is equivalent to $\mathcal{W}_X, \mathcal{W}_Y$ are comparable or separable. The proof of the lemma is completed.

5.2.2 Remark. (1) In view of Lemma 5.2.1, every object in $\mathcal{W}$ is τ-rigid.

(2) Let $\mathcal{W}_X$ be a wing in $\mathcal{W}$. By Lemma 2.2.2, add$\mathcal{W}_X$ is an Auslander-Reiten category with $\Gamma_{\mathrm{add}\mathcal{W}_X} = \mathcal{W}_X$, whose Auslander-Reiten translation is written as $\tau_{_X}$. By Lemma 5.1.4, a set of objects in $\mathcal{W}_X$ is τ-rigid if and only if it is $\tau_{_X}$-rigid.

Recall that $\mathcal{W}_0$ is a poset in such a way that $X \preceq Y$ if and only if $\mathcal{W}_X \subseteq \mathcal{W}_Y$. As an immediate consequence of Lemma 5.2.1, we obtain the following statement.

5.2.3 Corollary. *Every chain $\mathcal{S}$ in $\mathcal{W}$ is a τ-rigid set.*

The following lemma is useful in our future investigation.

5.2.4 Lemma. *Let $X, Z \in \mathcal{W}$ with $X \notin \mathcal{W}_Z$. Then (X, Z) is a τ-rigid pair if and only if (X, Y) is a τ-rigid pair, for every $Y \in \mathcal{W}_Z$.*

Proof. We shall only need to prove the necessity. Assume that (X, Z) is a τ-rigid pair. Since $X \notin \mathcal{W}_Z$, by Lemma 5.2.1, $\mathcal{W}_Z \subseteq \mathcal{W}_X$, or else, $\mathcal{W}_X, \mathcal{W}_Z$ are separable. Let $Y \in \mathcal{W}_Z$. In particular, $\mathcal{W}_Y \subseteq \mathcal{W}_Z$. If $\mathcal{W}_Z \subseteq \mathcal{W}_X$, then $\mathcal{W}_Y \subseteq \mathcal{W}_X$. If $\mathcal{W}_X, \mathcal{W}_Z$ are separable, then by Lemma 4.1.7, $\mathcal{W}_X, \mathcal{W}_Y$ are separable. By Lemma 5.2.1, (X, Y) is a τ-rigid pair. The proof of the lemma is completed.

From now on, we shall study the maximal τ-rigid sets in $\mathcal{W}$. Given a set $\mathcal{T}$ of objects in $\mathcal{W}$, denote by $|\mathcal{T}|$ its cardinality.

5.2.5 Lemma. *A τ-rigid set $\mathcal{T}$ in $\mathcal{W}$ is maximal τ-rigid if and only if $|\mathcal{T}| = n$.*

Proof. By Theorem 2.2.10, there is an isomorphism $F : \mathrm{add}\mathcal{W} \to \mathrm{mod}H$, where $H = k\vec{\mathbb{A}}_n$ the path algebra of a linearly oriented quiver of type $\mathbb{A}_n$. In particular, F induces a translation quiver isomorphism $F' : \mathcal{W} \to \Gamma_H$. Then, $\mathcal{T}_H = F'(\mathcal{T})$ is a $\tau_{_H}$-rigid set in Γ_H, which is maximal $\tau_{_H}$-rigid if and only if $\mathcal{T}$ is maximal τ-rigid in $\mathcal{W}$. Now, by Lemma 5.1.5, $\mathcal{T}_H$ is maximal $\tau_{_H}$-rigid in Γ_H if and only if $\oplus_{M \in \mathcal{T}_H} M$ is tilting in mod$H$. The latter is equivalent to $|\mathcal{T}_H| = n$. The proof of the lemma is completed.

The following result states more properties of a maximal τ-rigid set in $\mathcal{W}$.

5.2.6 LEMMA. *Let $\mathcal{T}$ be a maximal τ-rigid set in $\mathcal{W}$.*

(1) *The wing vertex of $\mathcal{W}$ is in $\mathcal{T}$.*

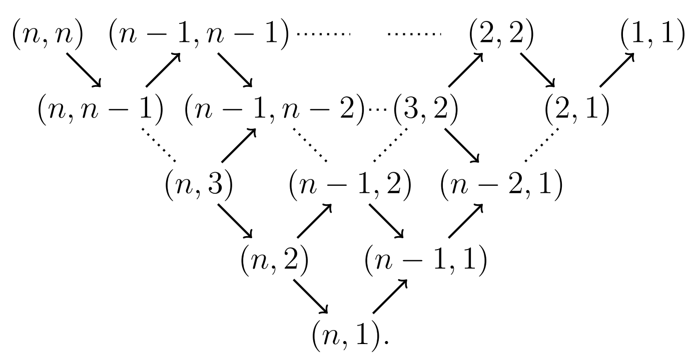
(2) *If $X \in \mathcal{T}$, then $\mathcal{T} \cap \mathcal{W}_X$ is maximal τ-rigid in $\mathcal{W}_X$, and consequently, $|\mathcal{T} \cap \mathcal{W}_X| = \ell(X)$.*

Proof. Let M be the wing vertex of $\mathcal{W}$. Given any $X \in \mathcal{T}$, since $\mathcal{W}_X \subseteq \mathcal{W}_M$, we deduce from Lemma 5.2.1 that (X, M) is a τ-rigid pair. Hence, $\mathcal{T} \cup \{M\}$ is τ-rigid. By the τ-rigid maximality of $\mathcal{T}$, we obtain $M \in \mathcal{T}$.

Now, we fix $X \in \mathcal{T}$. Let $Y \in \mathcal{W}_X$ be such that $(\mathcal{T} \cap \mathcal{W}_X) \cup \{Y\}$ is τ-rigid. We claim that (Z, Y) is a τ-rigid pair for any $Z \in \mathcal{T}$. Indeed, the claim is obvious if $Z \in \mathcal{W}_X$, and otherwise, since (Z, X) is τ-rigid, it follows from Lemma 5.2.4. That is, $\mathcal{T} \cup \{Y\}$ is τ-rigid. By the τ-rigid maximality of $\mathcal{T}$, we obtain $Y \in \mathcal{T}$, and hence, $Y \in \mathcal{T} \cap \mathcal{W}_X$. This shows that $\mathcal{T} \cap \mathcal{W}_X$ is maximal τ-rigid in $\mathcal{W}_X$. By Lemma 5.2.5, $|\mathcal{T} \cap \mathcal{W}_X| = \ell(X)$. The proof of the lemma is completed.

REMARK. If $n = 1$, then $\mathcal{W}$ is the only maximal τ-rigid set in $\mathcal{W}$. If $n = 2$, then $\mathcal{W}$ has exactly two maximal τ-rigid sets, namely, the vertex set of the left-most section and that of right-most section.

Our main objective is to show that the maximal τ-rigid sets in $\mathcal{W}$ can be constructed from section-generators of $\mathcal{W}$, which are defined in Definition 1.4.2. For this purpose, we shall need to recall the coordinate system for $\mathcal{W}$. As seen in Section 4.1, we shall identify an object $X \in \mathcal{W}$ with a pair (i_X, j_X) of integers, where $n \geq i_X \geq j_X \geq 1$. In this way, $\mathcal{W}$ can be pictured as follows:



5.2.7 Proposition. *Let $\mathcal{T}$ be a maximal τ-rigid set in $\mathcal{W}$. Then every maximal chain in $\mathcal{T}$ is a section-generator of $\mathcal{W}$.*

Proof. Let $\mathcal{S}$ be a maximal chain in $\mathcal{T}$. If the rank n of $\mathcal{W}$ is equal to 1, then trivially our statement is true. Now assume $n \geq 2$. By Lemma 5.2.6(1), the wing vertex M of $\mathcal{W}$ lies in $\mathcal{T}$. Since M is the maximal element in $\mathcal{W}$, we may enumerate the objects of $\mathcal{S}$ in such a way that $X_1 \prec X_2 \prec \cdots \prec X_m$, where $m \geq 2$ and $X_m = M$. If $\ell(X_1) > 1$, then $|\mathcal{T} \cap \mathcal{W}_{X_1}| > 1$ by Lemma 5.2.6 (2). Thus, $\mathcal{T}$ contains some X_0 with $\ell(X_0) < \ell(X_1)$. This yields a chain $\mathcal{S} \cup \{X_0\}$ in $\mathcal{T}$, a contradiction. This shows that $\ell(X_1) = 1$.

By Proposition 4.1.15(2), it remains to show that $\mathcal{W}$ has a path between X_p, X_{p+1}, for every $1 \leq p < m$. If $\mathcal{W}$ has no path connecting X_p and X_{p+1}, for some $1 \leq p < m$, then by Lemma 4.1.2(3) and Lemma 4.1.6, $i_{X_p} < i_{X_{p+1}}$ and $j_{X_p} > j_{X_{p+1}}$. In view of Lemma 4.1.1, $i_{X_p} \geq j_{X_p} > j_{X_{p+1}}$, and hence, $\mathcal{W}$ contains the object $Y \in \mathcal{W}$ with $(i_Y, j_Y) = (i_{X_p}, j_{X_{p+1}})$. Then, $X_p \prec Y \prec X_{p+1}$ by Lemma 4.1.6. Since $\mathcal{S}$ is a maximal chain in $\mathcal{T}$, we obtain $Y \notin \mathcal{T}$.

Let $Z \in \mathcal{T}$. We claim that (Y, Z) is τ-rigid. Suppose that $\mathcal{W}_Y, \mathcal{W}_Z$ are not comparable. Then, $X_{p+1} \notin \mathcal{W}_Z$, $Z \notin \mathcal{W}_Y$ and $Z \notin \mathcal{W}_{X_p}$. In particular, $Z \notin \{X_p, X_{p+1}\}$. By Lemma 5.2.1, it suffices to show that $\mathcal{W}_Y, \mathcal{W}_Z$ are separable. If $\mathcal{W}_{X_{p+1}}, \mathcal{W}_Z$ are separable, then $\mathcal{W}_Y$ and $\mathcal{W}_Z$ are separable by Lemma 4.1.7.

It remains to consider the case where $Z \in \mathcal{W}_{X_{p+1}}$. Then $j_Z \geq j_{X_{p+1}}$. Since $\mathcal{S}$ is a maximal chain in $\mathcal{T}$, we see that $X_p \notin \mathcal{W}_Z$. Since (Z, X_p) is τ-rigid, $\mathcal{W}_Z$ and $\mathcal{W}_{X_p}$ are separable. By Lemma 4.1.7, consider first $j_{X_p} \geq i_Z + 2$. Then $i_Y = i_{X_p} \geq j_{X_p} > i_Z \geq j_Z \geq j_{X_{p+1}} = j_Y$. This implies that $\mathcal{W}_Z \subseteq \mathcal{W}_Y$, a contradiction. Thus, $i_{X_p} \leq j_Z - 2$, and hence, $i_Y = i_{X_p} \leq j_Z - 2$. This shows that $\mathcal{W}_Z, \mathcal{W}_Y$ are separable. Thus, our claim is true. Hence, $\mathcal{T} \cup \{Y\}$ is τ-rigid which contradicts the maximality of $\mathcal{T}$. Therefore, $\mathcal{W}$ has a path between X_p, X_{p+1}, for every $1 \leq p < m$. The proof of the proposition is completed.

We now give some properties of a section-generator $\mathcal{S}$ of $\mathcal{W}$. First, by Proposition 4.1.15, $\mathcal{S}$ is a sectional chain. That is, $\mathcal{S}$ is of form $X_1 \prec X_2 \prec \cdots \prec X_m$, with $m \leq n$, such that X_t, X_{t+1} are connected by a path in $\mathcal{W}$, for $t = 1, 2, \ldots, m-1$. As seen in Definition 1.4.12, every pair (X_t, X_{t+1}) determines a wing $\mathcal{W}_{X_{t+1}}^{X_t}$ in $\mathcal{W}$, for $t = 1, \ldots, m-1$.

5.2.8 Lemma. *Let $\mathcal{S}$ be a section-generator of $\mathcal{W}$ of form $X_1 \prec X_2 \prec \cdots \prec X_m$.*

(1) *The wings $\mathcal{W}^{X_t}_{X_{t+1}}$ with $1 \le t < m$ are pairwise separable.*

(2) *If $M \in \mathcal{W}^{X_t}_{X_{t+1}}$ with $1 \le t < m$, then $M \notin \mathcal{S}$ and $\mathcal{S} \cup \{M\}$ is τ-rigid.*

Proof. Assume that $\mathcal{W}^{X_t}_{X_{t+1}} \neq \emptyset$ and $\mathcal{W}^{X_p}_{X_{p+1}} \neq \emptyset$ with $1 \le t < p < m$. Since $X_{t+1} \preceq X_p$, we obtain $\mathcal{W}_{X_{t+1}} \subseteq \mathcal{W}_{X_p}$. By Definition 1.4.12, $\mathcal{W}^{X_t}_{X_{t+1}} \subseteq \mathcal{W}_{X_{t+1}}$ and $\mathcal{W}^{X_p}_{X_{p+1}}, \mathcal{W}_{X_p}$ are separable. Since $\mathcal{W}^{X_t}_{X_{t+1}} \subseteq \mathcal{W}_{X_p}$, Statement (1) follows from Lemma 4.1.7.

Now let $X_p \in \mathcal{S}$ with $1 \le p \le m$ and let $M \in \mathcal{W}^{X_t}_{X_{t+1}}$ with $1 \le t < m$. If $t < p$, then $\mathcal{W}_{X_{t+1}} \subseteq \mathcal{W}_{X_p}$. We deduce from Definition 1.4.12 that $\mathcal{W}_M \subseteq \mathcal{W}^{X_t}_{X_{t+1}} \subsetneq \mathcal{W}_{X_p}$. Hence, $M \neq X_p$, and (M, X_p) is τ-rigid by Lemma 5.2.1. If $t \ge p$, then $X_p \in \mathcal{W}_{X_t}$. By definition, $\mathcal{W}^{X_t}_{X_{t+1}}, \mathcal{W}_{X_t}$ are separable. Thus, by Lemma 4.1.7, $\mathcal{W}_{X_p}, \mathcal{W}_M$ are separable. Therefore, $M \neq X_p$, and (X_p, M) is a τ-rigid pair by Lemma 5.2.1. Moreover, since $\mathcal{S}$ is τ-rigid by Corollary 5.2.3, $\mathcal{S} \cup \{M\}$ is τ-rigid. The proof of the lemma is completed.

The following statement is our main result of this section.

5.2.9 Theorem. *Let $\mathcal{T}$ be a set of objects of $\mathcal{W}$. Then $\mathcal{T}$ is maximal τ-rigid if and only if there exists a section-generator*

$$\mathcal{S} : X_1 \prec X_2 \prec \cdots \prec X_m$$

of $\mathcal{W}$ such that

$$\mathcal{T} = \mathcal{S} \cup \Theta_1 \cup \cdots \cup \Theta_{m-1},$$

where Θ_t is a maximal τ-rigid set in $\mathcal{W}^{X_t}_{X_{t+1}}$, for $t = 1, \ldots, m-1$.

Proof. Assume that $\mathcal{S} : X_1 \prec X_2 \prec \cdots \prec X_m$ is a section-generator of $\mathcal{W}$ and Θ_t is a maximal τ-rigid set in $\mathcal{W}^{X_t}_{X_{t+1}}$, for $t = 1, \ldots, m-1$. By Proposition 4.1.15(2) and Lemma 4.1.10, $\ell(X_1) = 1$ and $\ell(X_m) = n$. Set

$$\mathcal{T} = \mathcal{S} \cup \Theta_1 \cup \cdots \cup \Theta_{m-1}.$$

In view of Lemma 5.2.8 and Lemma 5.2.1, $\mathcal{T}$ is a τ-rigid set in $\mathcal{W}$. By Lemma 5.2.8(2), the Θ_t with $1 \le t < m$ are pairwise disjoint. Moreover, by Lemma 5.2.5

and Lemma 4.1.8, we have $|\Theta_t| = \ell(X_{t+1}) - \ell(X_t) - 1$. Thus,

$$\begin{aligned} |\mathcal{T}| &= |\mathcal{S}| + |\Theta_1| + \cdots + |\Theta_{m-1}| \\ &= m + (\ell(X_2) - \ell(X_1) - 1) + \cdots + (\ell(X_m) - \ell(X_{m-1}) - 1) \\ &= \ell(X_m) - \ell(X_1) + 1 \\ &= n. \end{aligned}$$

By Lemma 5.2.5, $\mathcal{T}$ is maximal τ-rigid in $\mathcal{W}$.

Conversely, assume that $\mathcal{T}$ is a maximal τ-rigid set in $\mathcal{W}$. Let

$$\mathcal{S} : X_1 \prec X_2 \prec \cdots \prec X_m$$

be a maximal chain in $\mathcal{T}$. By Proposition 5.2.7, $\mathcal{S}$ is a section-generator. Set $\Theta_t = \mathcal{T} \cap \mathcal{W}^{X_t}_{X_{t+1}}$, for each $1 \le t < m$. Let $1 \le t < m$. We claim that

$$\mathcal{T} \cap \mathcal{W}_{X_{t+1}} = (\mathcal{T} \cap \mathcal{W}_{X_t}) \cup (\mathcal{T} \cap \mathcal{W}^{X_t}_{X_{t+1}}) \cup \{X_{t+1}\}.$$

Let $M \in \mathcal{T} \cap \mathcal{W}_{X_{t+1}}$. Since (M, X_t) is a τ-rigid pair, by Lemma 5.2.1, $\mathcal{W}_M, \mathcal{W}_{X_t}$ are comparable or separable. This gives us that $M \preceq X_t$ or $X_t \prec M \preceq X_{t+1}$ or $M \in \mathcal{W}^{X_t}_{X_{t+1}}$. Since $\mathcal{S}$ is a maximal chain in $\mathcal{T}$, we have $M \preceq X_t$ or $M = X_{t+1}$ or $M \in \mathcal{W}^{X_t}_{X_{t+1}}$. Hence, our claim is true. Since $\mathcal{T}$ is maximal τ-rigid, by Lemma 5.2.6(2), we have

$$|\Theta_t| = |\mathcal{T} \cap \mathcal{W}_{X_{t+1}}| - |\mathcal{T} \cap \mathcal{W}_{X_t}| - 1 = \ell(X_{t+1}) - \ell(X_t) - 1,$$

which is equal to the rank of $\mathcal{W}^{X_t}_{X_{t+1}}$. Thus, by Lemma 5.2.5, Θ_t is a maximal τ-rigid set in $\mathcal{W}^{X_t}_{X_{t+1}}$. By sufficiency, $\mathcal{S} \cup \Theta_1 \cup \cdots \cup \Theta_{m-1} \subseteq \mathcal{T}$ is maximal τ-rigid in $\mathcal{W}$. Hence,

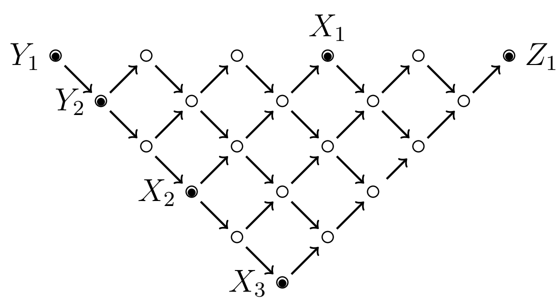
$$\mathcal{T} = \mathcal{S} \cup \Theta_1 \cup \cdots \cup \Theta_{m-1}.$$

The proof of the theorem is completed.

Remark. We should point out that Theorem 5.2.9 enables us to construct all the maximal τ-rigid sets in $\mathcal{W}$ by induction. As we have seen, this is trivial if $\mathcal{W}$ is of rank one or two. Given a wing of rank $n > 2$, applying Proposition 4.1.15, we are able to obtain all the section-generators of $\mathcal{W}$. Next, let $\mathcal{S}$ be a section-generator of $\mathcal{W}$ of the form $X_1 \prec X_2 \prec \cdots \prec X_m$. For each $1 \le t < m$, in view of Lemma 4.1.8, $\mathcal{W}_{X_{t+1}}^{X_t}$ is a wing of rank smaller than n, and by the induction hypothesis, its maximal τ-rigid sets can be found using Theorem 5.2.9.

In particular, consider the hereditary algebra $H = k\vec{\mathbb{A}}_n$, where $\vec{\mathbb{A}}_n$, with $n \geq 1$, a linearly oriented quiver of type $\mathbb{A}_n$. Since $\mathit{\Gamma}_H$ is a standard wing of rank n, in view of Lemma 5.1.5, we see that Theorem 5.2.9 provides a method to construct all the basic tilting modules in mod$H$.

Example. Let $H = k\vec{\mathbb{A}}_6$. The Auslander-Reiten quiver $\mathit{\Gamma}_H$ is shown as below. We easily see that $\{X_1, X_2, X_3\}$ is a section-generator of $\mathit{\Gamma}_H$ such that $X_1 \prec X_2 \prec X_3$, and $\{Y_1, Y_2\}$ is maximal τ_H-rigid in $\mathcal{W}_{X_2}^{X_1}$, and $\{Z_1\}$ is maximal τ_H-rigid in $\mathcal{W}_{X_3}^{X_2}$. By Theorem 5.2.9, $X_1 \oplus X_2 \oplus X_3 \oplus Y_1 \oplus Y_2 \oplus Z_1$ is a tilting module.



5.3 Maximal τ-rigid sets in a standard component of shape $\mathbb{Z}\mathbb{A}_\infty$

Throughout this section, $\mathcal{A}$ stands for an Auslander-Reiten category, whose Auslander-Reiten translation will be simply written as τ. Let $\mathit{\Gamma}$ be a standard component of $\mathit{\Gamma}_{\mathcal{A}}$ of shape $\mathbb{Z}\mathbb{A}_\infty$. In this section, we shall first characterize the maximal τ-rigid sets in $\mathit{\Gamma}$ and then give a method to construct all of them.

For this purpose, recall first from Lemma 1.4.6 that each object X in $\mathit{\Gamma}$ is a wing vertex of a unique wing $\mathcal{W}_X$ of rank $\ell(X)$ in $\mathit{\Gamma}$. In Definition 1.4.8, we have defined the notion of comparable or separable wings. In terms of these notions, the following statement describes the τ-rigidity of a pair of objects of $\mathit{\Gamma}$. We shall omit its proof since it is similar to the proof of Lemma 5.2.1.

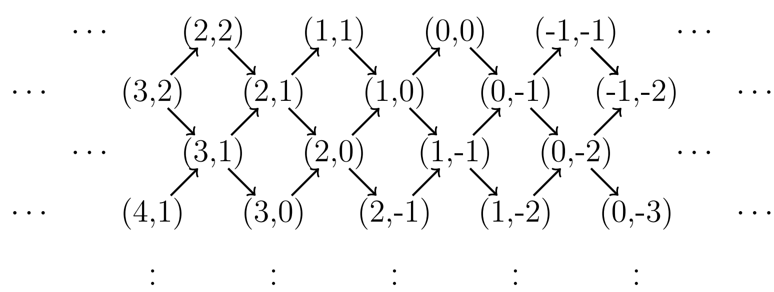
5.3.1 Lemma. *If $X, Y \in \Gamma$, then (X, Y) is a τ-rigid pair if and only if $\mathcal{W}_X, \mathcal{W}_Y$ are comparable or separable.*

We shall give the following remark. Compare it with Remark 5.2.2.

5.3.2 Remark. (1) In view of Lemma 5.3.1, every object in Γ is τ-rigid.

(2) Let $\mathcal{W}_X$ be a wing in Γ. By Lemma 2.2.2, add$\mathcal{W}_X$ is an Auslander-Reiten category with $\Gamma_{\text{add}\mathcal{W}_X} = \mathcal{W}_X$, whose Auslander-Reiten translation is written as τ_X. By Lemma 5.1.4, a set of objects in $\mathcal{W}_X$ is τ-rigid if and only if it is τ_X-rigid.

Next, we shall recall a coordinate system for Γ as defined in Section 4.2. Indeed, fix a quasi-simple object S in Γ. Then the quasi-simple objects in Γ are $S_i = \tau^i S$, with $i \in \mathbb{Z}$. The ray starting with S_i is denoted by R_i^+, and the co-ray ending with S_i is denoted by R_i^-. Given an object $X \in \Gamma$, by Lemma 4.2.1, there is a unique pair (i_X, j_X) of integers with $i_X \geq j_X$, such that $X = R^+_{i_X} \cap R^-_{j_X}$. For simplicity, we write $X = (i_X, j_X)$. In this way, Γ can be pictured as follows.



Moreover, by Lemma 1.4.7, $(\Gamma_0, \preceq)$ is a partially ordered set; and, as seen in Definition 4.2.15, every integer n determines two convex subquivers $\Gamma^+_{<n}$ and $\Gamma^-_{>n}$ of Γ. We have the following observation.

5.3.3 Corollary. (1) *A chain $\mathcal{S}$ in Γ is a τ-rigid set.*

(2) *If $M \in \Gamma^+_{<n}$ and $N \in \Gamma^-_{>n}$ for some $n \in \mathbb{Z}$, then (M, N) is τ-rigid.*

Proof. Statement (1) follows immediately from Lemma 5.3.1. Now fix $n \in \mathbb{Z}$. Let $M \in \Gamma^+_{<n}$ and $N \in \Gamma^-_{>n}$, that is, $i_M \leq n - 1$ and $j_N \geq n + 1$. Therefore,

$i_{_M} \le j_{_N} - 2$. By Lemma 4.2.8, $\mathcal{W}_M, \mathcal{W}_N$ are separable. By Lemma 5.3.1, (M, N) is τ-rigid. The proof of the corollary is completed.

Now we shall state some properties of τ-rigid sets in Γ.

5.3.4 Lemma. *Let $\mathcal{T}$ be a τ-rigid set in Γ.*

(1) *There exists at most one integer i such that $\mathcal{T}$ contains infinitely many objects of R_i^+. If such i exists and $\mathcal{T} \cup \{M\}$ is τ-rigid for some $M \in \Gamma$, then $M \in \Gamma^+_{<i+1}$ or $M \in \Gamma^-_{>i+1}$.*

(2) *There exists at most one integer j such that $\mathcal{T}$ contains infinitely many objects of R_j^-. If such j exists and $\mathcal{T} \cup \{M\}$ is τ-rigid for some $M \in \Gamma$, then $M \in \Gamma^-_{>j-1}$ or $M \in \Gamma^+_{<j-1}$.*

Proof. We shall prove only Statement (1). The proof of Statement (2) is similar. Assume that there are $i, j \in \mathbb{Z}$ such that $\mathcal{T}$ contains infinitely many objects of each of R_i^+ and R_j^+. We may assume that $j > i$. By Lemma 4.2.17, both $\mathcal{T} \cap R_i^+$ and $\mathcal{T} \cap R_j^+$ are infinite chains. Then there exists $X \in \mathcal{T} \cap R_j^+$ with $j_{_X} < i$. Moreover, there exists $Y \in \mathcal{T} \cap R_i^+$ such that $j_{_Y} < j_{_X}$. That is, $j = i_{_X} > i = i_{_Y} > j_{_X} > j_{_Y}$. Hence, in view of Lemma 4.2.7 and Lemma 4.2.8, $\mathcal{W}_X, \mathcal{W}_Y$ are neither comparable nor separable. By Lemma 5.3.1, (X, Y) is not τ-rigid, which is a contradiction. Therefore, there exists at most one $i \in \mathbb{Z}$ such that $\mathcal{T}$ contains infinitely many objects of ray R_i^+.

Now assume that $\mathcal{T} \cap R_i^+$ is infinite. Let $M \in \Gamma$ such that $\mathcal{T} \cup \{M\}$ is τ-rigid. Since $\mathcal{T} \cap R_i^+$ is infinite, by Lemma 4.2.11, there is an object $X \in \mathcal{T} \cap R_i^+$ such that X is not less than M. Since (X, M) is τ-rigid, by Lemma 5.3.1, $\mathcal{W}_M, \mathcal{W}_X$ are comparable or separable. In the first case, by Lemma 4.2.7, $i_{_M} \le i_{_X} \le i$, that is, $M \in \Gamma^+_{<i+1}$. In the second case, by Lemma 4.2.8, $i_{_M} \le j_{_X} - 2 \le i_{_X} - 2 < i$ or $j_{_M} \ge i_{_X} + 2 = i + 2$. That is, $M \in \Gamma^+_{<i+1}$ or $M \in \Gamma^-_{>i+1}$. The proof of the lemma is completed.

We introduce the following notion to simplify the later statements.

5.3.5 Definition. A convex subquiver Δ of Γ is called *wing-complete* provided the wing $\mathcal{W}_X$ is contained in Δ, for all $X \in \Delta$.

Observe that every wing in Γ is wing-complete. Moreover, given an integer n, the subquivers $\Gamma^+_{<n}$ and $\Gamma^-_{>n}$ are wing-complete. In the following, we shall give some properties of τ-rigid sets in a wing-complete subquiver of Γ. The proof is similar to that of Lemma 5.2.4.

5.3.6 Lemma. *Let Δ be a wing-complete subquiver of Γ. If $X, Z \in \Delta$ with $X \notin \mathcal{W}_Z$, then (X, Z) is τ-rigid if and only if (X, Y) is τ-rigid, for all $Y \in \mathcal{W}_Z$.*

Given a wing-complete subquiver Δ of Γ, a τ-rigid set $\mathcal{T}$ in Δ is called *locally maximal* if $\mathcal{T} \cap \mathcal{W}_X$ is maximal τ-rigid in $\mathcal{W}_X$, for all $X \in \mathcal{T}$. By Lemma 5.2.5, this is equivalent to say that $|\mathcal{T} \cap \mathcal{W}_X| = \ell(X)$, for all $X \in \mathcal{T}$.

The following statement gives a property of a maximal τ-rigid set in a wing-complete subquiver of Γ, whose proof can be translated word-by-word from that of Lemma 5.2.6(2).

5.3.7 Lemma. *If Δ is a wing-complete subquiver of Γ, then every maximal τ-rigid set in Δ is locally maximal.*

In the following, we shall focus on the maximal τ-rigid sets in $\Gamma^+_{<n}$ and those in $\Gamma^-_{>n}$, for some $n \in \mathbb{Z}$. This is very important for our later study.

5.3.8 Lemma. *Let n be an integer.*

(1) *A maximal τ-rigid set in $\Gamma^+_{<n}$ is infinite.*

(2) *A maximal τ-rigid set in $\Gamma^-_{>n}$ is infinite.*

Proof. We shall only prove Statement (1). Let $\mathcal{T}$ be a maximal τ-rigid set in $\Gamma^+_{<n}$. Assume that $\mathcal{T}$ is finite. Then there is $X \in R^+_{n-1}$ with $j_X < j_M$, for any $M \in \mathcal{T}$. Thus, $X \in \Gamma^+_{<n}$ and $X \notin \mathcal{T}$. Since $i_M \leq i_X = i$, by Lemma 4.2.7, $\mathcal{W}_M \subsetneq \mathcal{W}_X$, for any $M \in \mathcal{T}$. By Lemma 5.3.1, $\mathcal{T} \cup \{X\}$ is τ-rigid, a contradiction. Hence, $\mathcal{T}$ is infinite. The proof of the lemma is completed.

5.3.9 Lemma. *Let n be an integer.*

(1) *If $\mathcal{T}$ is a maximal τ-rigid set in $\Gamma^+_{<n}$, then $\mathcal{T}$ contains infinitely many objects of R^+_i, for some $i < n$.*

(2) *If $\mathcal{T}$ is a maximal τ-rigid set in $\Gamma^-_{>n}$, then $\mathcal{T}$ contains infinitely many objects of R^-_j, for some $j > n$.*

Proof. We shall only prove Statement (1). Let $\mathcal{T}$ be a maximal τ-rigid set in $\Gamma^+_{<n}$. Suppose that $\mathcal{T}$ contains only finitely many objects of R^+_i, for each $i < n$. Let $i_0 < n$ be the largest integer such that $\mathcal{T} \cap R^+_{i_0} \neq \emptyset$. Take $N_0 \in \mathcal{T} \cap R^+_{i_0}$ with $\ell(N_0)$ being maximal. We claim that N_0 has no cover in $\mathcal{T}$. Suppose on the contrary that $N_0 \prec N$ for some $N \in \mathcal{T}$. Then, $j_{_N} \le j_{_{N_0}} \le i_{_{N_0}} \le i_{_N} < n$. Since $i_0 = i_{_{N_0}}$, by the maximality of i_0, we have $i_{_N} = i_0$, that is, $N \in R^+_{i_0}$. Since N_0 is the object in $R^+_{j_0}$ with the biggest quasi-length, in view of Lemma 4.2.5(3), $j_{_N} = j_{_{N_0}}$. It follows that $N = N_0$, a contradiction. Our claim is true. Now let $i_1 < j_{_{N_0}}$ be the biggest integer such that $\mathcal{T} \cap R^+_{i_1} \neq \emptyset$. The existence of i_1 is deduced from Lemma 5.3.8(1). Again, take the object N_1 in $\mathcal{T} \cap R^+_{i_1}$ with the biggest quasi-length. By similar discussion, N_1 has no cover in $\mathcal{T}$.

Consider now the object M with $(i_{_M}, j_{_M}) = (i_0, j_{_{N_1}})$. Since $n > i_0 \ge j_{_{N_1}}$, by Lemma 4.2.1, $M \in \Gamma^+_{<n}$. Observe that $N_0 \prec M$ and $N_1 \prec M$. Thus, $M \notin \mathcal{T}$. Let $Z \in \mathcal{T}$. If $Z \in \mathcal{W}_M$, then by Lemma 5.3.1, (Z, M) is τ-rigid. Suppose now that $Z \notin \mathcal{W}_M$. In particular, $Z \notin \mathcal{W}_{N_1}$. Since N_1 has no cover in $\mathcal{T}$, we have $N_1 \notin \mathcal{W}_Z$. That is, $\mathcal{W}_Z, \mathcal{W}_{N_1}$ are not comparable. Since (N_1, Z) is τ-rigid, $\mathcal{W}_{N_1}, \mathcal{W}_Z$ are separable with $i_{_Z} + 2 \le j_{_{N_1}}$. Since $j_{_{N_1}} = j_{_M}$, we see that $\mathcal{W}_M, \mathcal{W}_Z$ are separable. Hence, (M, Z) is τ-rigid. This shows that $\mathcal{T} \cup \{M\}$ is τ-rigid, which is a contradiction to the τ-rigid maximality of $\mathcal{T}$. Therefore, $\mathcal{T}$ contains infinitely many objects of R^+_i, for some $i < n$. The proof of the lemma is completed.

The following statement is a characterization of maximal τ-rigid sets in $\Gamma^+_{<n}$.

5.3.10 PROPOSITION. *Let $\mathcal{T}$ be a locally maximal τ-rigid set in $\Gamma^+_{<n}$, which contains infinitely many objects of R^+_i for some $i < n$.*

(1) *If $i = n - 1$, then $\mathcal{T}$ is maximal τ-rigid in $\Gamma^+_{<n}$.*

(2) *If $i = n - 2$, then $\mathcal{T}$ is maximal τ-rigid in $\Gamma^+_{<n}$ and contained in $\Gamma^+_{<n-1}$.*

(3) *If $i < n-2$, then $\mathcal{T}$ is maximal τ-rigid in $\Gamma^+_{<n}$ if and only if*

$$\mathcal{T} = (\mathcal{T} \cap \Gamma^+_{<i+1}) \cup (\mathcal{T} \cap \mathcal{W}_Z),$$

where $Z = (n-1, i+2) \in \mathcal{T}$.

Proof. Let $M \in \Gamma^+_{<n}$ such that $\mathcal{T} \cup \{M\}$ is τ-rigid. By Lemma 5.3.4(1), $M \in \Gamma^+_{<i+1}$ or $M \in \Gamma^-_{>i+1}$.

If $i = n-1$ or $i = n-2$, then by Lemma 4.2.16(1), $\Gamma^+_{<n} \cap \Gamma^-_{>i+1} = \emptyset$. That is, $M \in \Gamma^+_{<i+1}$. In particular, $\mathcal{T}$ is contained in $\Gamma^+_{<i+1}$. Since $\mathcal{T} \cap R^+_i$ is an infinite chain, by Lemma 4.2.19(2), there is an object $X \in \mathcal{T} \cap R^+_i$ such that $M \prec X$. That is $M \in \mathcal{W}_X$. In particular, $(\mathcal{T} \cap \mathcal{W}_X) \cup \{M\}$ is τ-rigid. By the local maximality of $\mathcal{T}$, we see that $M \in \mathcal{T}$. Therefore, $\mathcal{T}$ is maximal τ-rigid in $\Gamma^+_{<n}$. This shows Statement (1) and (2).

Consider now that $i < n-2$. By Lemma 4.2.16(2), $\Gamma^+_{<n} \cap \Gamma^-_{>i+1} = \mathcal{W}_Z$ with $(i_Z, j_Z) = (n-1, i+2)$. Thus, $M \in \Gamma^+_{<i+1}$ or $M \in \mathcal{W}_Z$. In particular, we have

$$\mathcal{T} = (\mathcal{T} \cap \Gamma^+_{<i+1}) \cup (\mathcal{T} \cap \mathcal{W}_Z).$$

Assume that $Z \in \mathcal{T}$. Thus, in view of Lemma 4.2.19(2), M has a cover X in $\mathcal{T}$. That is, $M \in \mathcal{W}_X$. Then, similarly, by the local maximality of $\mathcal{T}$, we deduce that $M \in \mathcal{T}$. Hence, $\mathcal{T}$ is maximal τ-rigid in $\Gamma^+_{<n}$. Conversely, assume that $\mathcal{T}$ is maximal τ-rigid in $\Gamma^+_{<n}$. It remains to show that $Z \in \mathcal{T}$. We may assume that $M \in \mathcal{T}$. Then $\mathcal{W}_M, \mathcal{W}_Z$ are comparable in case $M \in \mathcal{T} \cap \Gamma^+_{<i+1}$; and $\mathcal{W}_M \subseteq \mathcal{W}_Z$ in case $M \in \mathcal{W}_Z$. This shows that (M, Z) is a τ-rigid pair. By the τ-rigid maximality of $\mathcal{T}$, we see that $Z \in \mathcal{T}$. The proof of the proposition is completed.

We shall state a characterization of maximal τ-rigid sets in $\Gamma^-_{>n}$ without proof.

5.3.11 Proposition. *Let $\mathcal{T}$ be a locally maximal τ-rigid set in $\Gamma^-_{>n}$, containing infinitely many objects of R^-_j, for some $j > n$.*

(1) *If $j = n+1$, then $\mathcal{T}$ is maximal τ-rigid in $\Gamma^-_{>n}$.*

(2) *If $j = n+2$, then $\mathcal{T}$ is maximal τ-rigid in $\Gamma^-_{>n}$ and contained in $\Gamma^-_{>n+1}$.*

(3) *If $j > n+2$, then $\mathcal{T}$ is maximal τ-rigid in $\Gamma^-_{>n}$ if and only if*

$$\mathcal{T} = (\mathcal{T} \cap \Gamma^-_{>j-1}) \cup (\mathcal{T} \cap \mathcal{W}_Z),$$

where $Z = (j-2, n+1) \in \mathcal{T}$.

Given an integer n, let $\mathcal{T}_1$ be a maximal τ-rigid set in $\Gamma^+_{<n}$ and $\mathcal{T}_2$ be a maximal τ-rigid set in $\Gamma^-_{>n}$. The following gives us a sufficient and necessary condition for $\mathcal{T}_1 \cup \mathcal{T}_2$ being a maximal τ-rigid set in Γ. We refer the notion of the density in a poset to Section 1.4.

5.3.12 Proposition. *Let $\mathcal{T}_1$ be a maximal τ-rigid set in $\Gamma^+_{<n}$ and $\mathcal{T}_2$ be a maximal τ-rigid set in $\Gamma^-_{>n}$, for some $n \in \mathbb{Z}$. Then $\mathcal{T}_1 \cup \mathcal{T}_2$ is a maximal τ-rigid set in Γ if and only if $\mathcal{T}_1$ is dense in $\Gamma^+_{<n}$ or $\mathcal{T}_2$ is dense in $\Gamma^-_{>n}$.*

Proof. Set $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2$. By Corollary 5.3.3(2), $\mathcal{T}$ is a τ-rigid set in Γ. By Lemma 5.3.9, $\mathcal{T}_1 \cap R^+_{i_0}$ and $\mathcal{T}_2 \cap R^-_{j_0}$ both are infinite, for some $i_0, j_0 \in \mathbb{Z}$ with $i_0 < n < j_0$.

For proving sufficiency, we may assume that $\mathcal{T}_1$ is dense. By Lemma 4.2.20, we have $i_0 = n-1$. Let $M \in \Gamma$ such that $\mathcal{T} \cup \{M\}$ is τ-rigid. Since $\mathcal{T}_1 \cup \{M\}$ is τ-rigid, by Lemma 5.3.4, we have $M \in \Gamma^+_{<i_0+1}$ or $M \in \Gamma^-_{>i_0+1}$. That is, $M \in \Gamma^+_{<n}$ or $M \in \Gamma^-_{>n}$. By the maximality of $\mathcal{T}_1$ and $\mathcal{T}_2$, the first case implies that $M \in \mathcal{T}_1$ and the second case implies that $M \in \mathcal{T}_2$. Therefore, $M \in \mathcal{T}$, and $\mathcal{T}$ is maximal τ-rigid in Γ.

Conversely, assume that $\mathcal{T}$ is maximal τ-rigid in Γ but $\mathcal{T}_1$ is not dense in $\Gamma^+_{<n}$ and $\mathcal{T}_2$ is not dense in $\Gamma^-_{>n}$. Then, by Lemma 4.2.20, we have $i_0 < n-1$ and $j_0 > n+1$. Assume that $i_0 = n-2$. By Proposition 5.3.10(2), $\mathcal{T}_1$ is contained in $\Gamma^+_{<n-1}$. If $j_0 = n+2$, then by Proposition 5.3.11(2), $\mathcal{T}_2$ is contained in $\Gamma^-_{>n+1}$. Observe that the quasi-simple object S with $(i_S, j_S) = (n, n)$ belongs to $\Gamma^+_{<n+1} \cap \Gamma^-_{>n-1}$. Hence, by Corollary 5.3.3(2), $\mathcal{T} \cup \{S\}$ is τ-rigid, which contradicts the maximality of $\mathcal{T}$. If $j_0 > n+2$, then, by Proposition 5.3.11(3) $\mathcal{T}$ contains an object $Z \in \Gamma$ with $(i_Z, j_Z) = (j_0 - 2, n+1)$ having no cover in $\mathcal{T}$. Take the object X with $(i_X, j_X) = (j_0 - 2, n)$. Observe that $X \in \Gamma^-_{>n-1}$ and $Z \prec X$. Hence, $X \notin \mathcal{T}_2$. Let $M \in \mathcal{T}$. If $M \in \mathcal{T}_1$, then $M \in \Gamma^+_{<n-1}$. By Corollary 5.3.3(2), (M, X) is τ-rigid. If $M \in \mathcal{T}_2$, then, by Proposition 5.3.11(3), $M \in \mathcal{W}_Z \subseteq \mathcal{W}_X$ or $M \in \Gamma^-_{>j_0-1}$. This shows that (M, X) is τ-rigid. Thus, $\mathcal{T} \cup \{X\}$ is τ-rigid, which contradicts the τ-rigid maximality of $\mathcal{T}$. If $i_0 < n-2$, then we have similar discussion. This shows the necessity. The proof of the proposition is completed.

From now on, we come back to our main objective of this section, which is to study the maximal τ-rigid sets in Γ.

5.3.13 Proposition. *Let $\mathcal{T}$ be a maximal τ-rigid set in Γ. Then $\mathcal{T}$ contains infinitely many objects of some ray R_i^+ if and only if $\mathcal{T}$ contains infinitely many objects of some coray R_j^-, where $j \geq i+2$.*

Proof. Assume that $\mathcal{T}$ contains infinitely many objects of R_i^+, for some $i \in \mathbb{Z}$. Let $X \in \mathcal{T}$. By Lemma 5.3.4(2), we have $X \in \Gamma^+_{<i+1}$ or $X \in \Gamma^-_{>i+1}$. Since$\Gamma^+_{<i+1}$ and $\Gamma^-_{>i+1}$ are disjoint, $\mathcal{T} = (\mathcal{T} \cap \Gamma^+_{<i+1}) \cup (\mathcal{T} \cap \Gamma^-_{>i+1})$. In view of Corollary 5.3.3(2), we deduce that $\mathcal{T} \cap \Gamma^-_{>i+1}$ is maximal τ-rigid in $\Gamma^-_{>i+1}$. By Lemma 5.3.9(2), $\mathcal{T} \cap \Gamma^-_{>i+1}$ contains infinitely many objects of some co-ray R_j^-, with $j > i+1$. Similarly, we can show the other direction. The proof of the proposition is completed.

Let $\mathcal{T}$ be a set of objects in Γ with $Z \in \mathcal{T}$. We say that Z is a *maximal* object in $\mathcal{T}$ if Z has no cover in $\mathcal{T}$.

5.3.14 Proposition. *Let $\mathcal{T}$ be a maximal τ-rigid set in Γ. If $Z \in \Gamma$, then Z is a maximal object in $\mathcal{T}$ if and only if $\mathcal{T}$ contains infinitely many objects of each of the co-ray $R^-_{i_Z+2}$ and the ray $R^+_{j_Z-2}$.*

Proof. Let Z be an object in Γ. In particular, $i_Z \geq j_Z$. Suppose that $\mathcal{T}$ contains infinitely many objects of each of $R^-_{i_Z+2}$ and $R^+_{j_Z-2}$. Let N be an arbitrary object in $\mathcal{T}$. We deduce from Lemma 5.3.4 that $N \in \Gamma^+_{<j_Z-1}$ or $N \in \Gamma^-_{>i_Z+1}$ or $N \in \Gamma^+_{<i_Z+1} \cap \Gamma^-_{>j_Z-1}$. That is, $N \in \Gamma^+_{<j_Z-1}$ or $N \in \Gamma^-_{>i_Z+1}$ or $N \preceq Z$ by Lemma 4.2.16(2). In particular, Z has no cover in $\mathcal{T}$. It remains to show that $Z \in \mathcal{T}$. Indeed, by Corollary 5.3.3, (N, Z) is τ-rigid. Hence, $\mathcal{T} \cup \{Z\}$ is τ-rigid. By the τ-rigid maximality of $\mathcal{T}$, we have $Z \in \mathcal{T}$.

Conversely, assume that $Z \in \mathcal{T}$ is a maximal object. Let $M \in \mathcal{T}$. Since (Z, M) is τ-rigid, by Lemma 5.3.1, $\mathcal{W}_M \subseteq \mathcal{W}_Z$ or $\mathcal{W}_M, \mathcal{W}_Z$ are separable. In the first case, we have $i_Z \geq i_M \geq j_M \geq j_Z$. In second case, we have $j_M \geq i_Z + 2$ or $i_M \leq j_Z - 2$. Thus, we see that $M \in \Gamma^-_{>j_Z-1}$ or $M \in \Gamma^+_{<j_Z-1}$. Therefore, we have

$$\mathcal{T} = (\mathcal{T} \cap \Gamma^-_{>j_Z-1}) \cup (\mathcal{T} \cap \Gamma^+_{<j_Z-1}).$$

Since $\mathcal{T}$ is maximal τ-rigid in Γ, and $\Gamma^-_{>j_Z-1}, \Gamma^+_{<j_Z-1}$ are disjoint, in view of Corollary 5.3.3(2), $\mathcal{T} \cap \Gamma^-_{>j_Z-1}$ is maximal τ-rigid in $\Gamma^-_{>j_Z-1}$ and $\mathcal{T} \cap \Gamma^+_{<j_Z-1}$ is maximal τ-rigid in $\Gamma^+_{<j_Z-1}$. By Lemma 5.3.9, $\mathcal{T} \cap \Gamma^-_{>j_Z-1}$ contains infinitely many of some co-ray R^-_j with $j > j_Z - 1$ and $\mathcal{T} \cap \Gamma^+_{<j_Z-1}$ contains infinitely many of some ray R^+_i with $i < j_Z - 1$. Since $\mathcal{T}$ is maximal τ-rigid in Γ, by Proposition 5.3.12, $\mathcal{T} \cap \Gamma^-_{>j_Z-1}$ is dense in $\Gamma^-_{>j_Z-1}$ or $\mathcal{T} \cap \Gamma^+_{<j_Z-1}$ is dense in $\Gamma^+_{<j_Z-1}$. By Lemma 4.2.20, $j = j_Z$ or $i = j_Z - 2$. In case $j = j_Z$, since $\mathcal{T} \cap R^-_{j_Z}$ is infinite, $\mathcal{T}$ has an object X lying in $R^-_{j_Z}$ such that $Z \prec X$, which is a contradiction. Hence, $i = j_Z - 2$. That is, $\mathcal{T}$ contains infinitely many objects of the ray $R^+_{j_Z-2}$.

Observe that we also have

$$\mathcal{T} = (\mathcal{T} \cap \Gamma^+_{<i_Z+1}) \cup (\mathcal{T} \cap \Gamma^-_{>i_Z+1}).$$

In a similar fashion, we can show that $\mathcal{T}$ contains infinitely many objects of the co-ray $R^-_{i_Z+2}$. The proof of the proposition is completed.

The following statement gives some properties of a maximal τ-rigid set in Γ.

5.3.15 Corollary. *If $\mathcal{T}$ is a maximal τ-rigid set in Γ, then $\mathcal{T}$ contains at most one maximal object and at least one infinite maximal chain.*

Proof. Let $\mathcal{T}$ be a maximal τ-rigid set in Γ. It follows from Proposition 5.3.14 and Lemma 5.3.4 that $\mathcal{T}$ has at most one maximal object.

For proving the second part, we first claim that $\mathcal{T}$ contains an infinite chain. Indeed, if $\mathcal{T}$ has a maximal object Z, then, by Proposition 5.3.14, $\mathcal{T} \cap R^+_{j_Z+2}$ is infinite, which is an infinite chain. Otherwise, every object of $\mathcal{T}$ has a cover in $\mathcal{T}$; and consequently, $\mathcal{T}$ contains an infinite chain. Our claim is established. Now let $\mathcal{S}$ be an infinite chain in $\mathcal{T}$. By Lemma 4.2.13, $\mathcal{S}$ is of the following form:

$$X_1 \prec X_2 \prec \cdots \prec X_n \prec \cdots.$$

Since Γ is interval-finite, we may assume without loss of generality that X_n is a minimal cover of X_{n-1} for every $n > 1$. Thus, by Lemma 4.2.12,

$$\ell(X_1) < \ell(X_2) < \cdots < \ell(X_n) < \cdots.$$

Now assume that $Y \in \mathcal{T}$ is such that $\mathcal{S} \cup \{Y\}$ is a chain. Then, we have $\ell(X_n) < \ell(Y) \leq \ell(X_{n+1})$ for some n. This gives us that $X_n \prec Y \preceq X_{n+1}$.

By the assumption, $Y = X_{n+1}$. Therefore, $\mathcal{S}$ is a maximal chain in $\mathcal{T}$. The proof of the corollary is completed.

Recall that a set of objects in Γ is locally finite if it contains only finitely many objects of every ray and every co-ray in Γ.

5.3.16 Lemma. *Let $\mathcal{T}$ be a maximal τ-rigid set in Γ. The following statements are equivalent.*

(1) *$\mathcal{T}$ is dense in Γ.*

(2) *$\mathcal{T}$ is locally finite in Γ.*

(3) *$\mathcal{T}$ has an infinite chain which is locally finite in Γ.*

Proof. First, assume that $\mathcal{T}$ is dense in Γ. Suppose on the contrary that $\mathcal{T}$ is not locally finite. Then we may assume that $\mathcal{T}$ contains infinitely many objects of some ray R_i^+. By Lemma 5.3.4, every object in $\mathcal{T}$ lies in $\Gamma^+_{<i+1}$ or $\Gamma^-_{>i+1}$. As a consequence, the quasi-simple object S with $(i_S, j_S) = (i+1, i+1)$ has no cover in $\mathcal{T}$. That is, $\mathcal{T}$ is not dense in Γ. This contradiction shows that Statement (1) implies (2).

Now, by Corollary 5.3.15, $\mathcal{T}$ contains an infinite chain $\mathcal{S}$. Assume that $\mathcal{T}$ is locally finite. In particular, $\mathcal{S}$ is locally finite. Hence, Statement (2) implies Statement (3). Finally, assume that $\mathcal{T}$ contains an infinite chain $\mathcal{S}$ which is locally finite . By Lemma 4.2.19(1), $\mathcal{S}$ is dense in Γ, and so is $\mathcal{T}$. This shows that Statement (3) implies Statement (1). The proof of the lemma is completed.

The following statement gives a complete description of the maximal τ-rigid sets in Γ.

5.3.17 Theorem. *If $\mathcal{T}$ is a locally maximal τ-rigid set in Γ, then it is maximal τ-rigid in Γ if and only if one of the following situations occurs.*

(1) *$\mathcal{T}$ contains an infinite chain which is locally finite.*

(2) *$\mathcal{T}$ contains infinitely many objects of some ray R_i^+ and infinitely many objects of some co-ray R_j^- with $j \geq i+2$; and it contains the object $Z = (j-2, i+2)$ whenever $j \geq i+4$.*

Proof. Let $\mathcal{T}$ be a locally maximal τ-rigid set in Γ. Assume that $\mathcal{T}$ is maximal τ-rigid in Γ. By Corollary 5.3.15, $\mathcal{T}$ contains an infinite chain $\mathcal{S}$. Suppose that Statement (1) does not hold. Then we may assume that $\mathcal{S} \cap R_j^-$ is infinite, for some j. Then $\mathcal{T} \cap R_j^-$ is infinite, and by Proposition 5.3.13, $\mathcal{T} \cap R_i^+$ is infinite, for some integer i with $i \leq j-2$. Moreover, in case $j \geq i+4$, by Propostion 5.3.14, the object $Z = (j-2, i+2)$ is a maximal object in $\mathcal{T}$. This shows the necessity.

For proving the sufficiency, let $M \in \Gamma$ such that $\mathcal{T} \cup \{M\}$ is τ-rigid. We claim that M has a cover in $\mathcal{T}$. If Statement (1) holds, then our claim follows from Lemma 4.2.19(1). Now assume that Statement (2) holds. Since both $\mathcal{T} \cap R_i^+$ and $\mathcal{T} \cap R_j^-$ are infinite chains, by Lemma 5.3.4, we have $M \in \Gamma^+_{<i+1}$ or $M \in \Gamma^-_{>j-1}$ or $M \in \Gamma^+_{<j-1} \cap \Gamma^-_{>i+1}$. In case $M \in \Gamma^+_{<i+1}$ or $M \in \Gamma^-_{>j-1}$, our claim follows from Lemma 4.2.19(2) and (3). Otherwise, $M \in \Gamma^+_{<j-1} \cap \Gamma^-_{>i+1}$. By Lemma 4.2.16(2), we have $\Gamma^+_{<j-1} \cap \Gamma^-_{>i+1} = \mathcal{W}_Z$ where $Z = (j-2, i+2)$. In particular, $j \geq i+4$, and hence, by assumption, $Z \in \mathcal{T}$. Obviously, Z is a cover of M in $\mathcal{T}$. Our claim is established. Now let X be a cover of M in $\mathcal{T}$. That is, $M \in \mathcal{W}_X$. Since $(\mathcal{T} \cap \mathcal{W}_X) \cup \{M\}$ is τ-rigid, by the local maximality of $\mathcal{T}$, we have $M \in \mathcal{T} \cap \mathcal{W}_X$, and hence $M \in \mathcal{T}$. Therefore, $\mathcal{T}$ is maximal τ-rigid in Γ. The proof of the theorem is completed.

The following statement shows some properties of the infinite maximal chains in a maximal τ-rigid set in Γ. For this, we refer the notion of section-generators of Γ to Definition 1.4.2. Let $\mathcal{S}$ be a section-generator of Γ. By Proposition 4.2.24(2), $\mathcal{S}$ is an infinite sectional chain of form

$$X_1 \prec X_2 \prec \cdots \prec X_n \prec \cdots$$

such that there is a path between X_n and X_{n+1}, for all $n \geq 1$. For each n, we define a wing $\mathcal{W}^{X_n}_{X_{n+1}}$ in Γ as indicated in Definition 1.4.12.

5.3.18 Proposition. *Let $\mathcal{T}$ be a maximal τ-rigid set in Γ. If*

$$\mathcal{S} : X_1 \prec X_2 \prec \cdots \prec X_n \prec \cdots$$

is an infinite maximal chain in $\mathcal{T}$, then

(1) *$\mathcal{S}$ is a section-generator of Γ;*

(2) *$\mathcal{T} \cap \mathcal{W}^{X^n}_{X_{n+1}}$ is maximal τ-rigid in $\mathcal{W}^{X^n}_{X_{n+1}}$, for every $n \geq 1$.*

Proof. Let $\mathcal{S} : X_1 \prec X_2 \prec \cdots \prec X_n \prec \cdots$ be an infinite maximal chain in $\mathcal{T}$. If $\ell(X_1) > 1$, then $|\mathcal{T} \cap \mathcal{W}_{X_1}| > 1$ by Lemma 5.3.7. Thus, $\mathcal{T}$ contains some X_0 with $\ell(X_0) < \ell(X_1)$. This yields a chain $\mathcal{S} \cup \{X_0\}$ in $\mathcal{T}$, a contradiction. This shows that $\ell(X_1) = 1$.

By Proposition 4.2.24, it remains to show that Γ has a path between X_n, X_{n+1}, for every $n \geq 1$. Let $n \geq 1$. Consider the wing $\mathcal{W}_{X_{n+1}}$. By Lemma 5.3.7, $\mathcal{T}$ is locally maximal. Then $\mathcal{T} \cap \mathcal{W}_{X_{n+1}}$ is maximal τ-rigid in $\mathcal{W}_{X_{n+1}}$. In particular,

$$\mathcal{S} \cap \mathcal{W}_{X_{n+1}} : X_1 \prec X_2 \cdots \prec X_{n+1}$$

is a maximal chain in $\mathcal{T} \cap \mathcal{W}_{X_{n+1}}$. By Proposition 5.2.7, $\mathcal{S} \cap \mathcal{W}_{X_{n+1}}$ is a section-generator of $\mathcal{W}_{X_{n+1}}$. By Proposition 4.1.15, it is a sectional chain. Hence, X_n and X_{n+1} is connected by a path in Γ. Statement (1) is established.

For proving Statement (2), we claim that

$$\mathcal{T} \cap \mathcal{W}_{X_{n+1}} = (\mathcal{T} \cap \mathcal{W}_{X_n}) \cup (\mathcal{T} \cap \mathcal{W}^{X_n}_{X_{n+1}}) \cup \{X_{n+1}\},$$

for $n \geq 1$. Now let $M \in \mathcal{T} \cap \mathcal{W}_{X_{n+1}}$. Since (M, X_n) is a τ-rigid pair, by Lemma 5.3.1, $\mathcal{W}_M, \mathcal{W}_{X_n}$ are comparable or separable. In the first case, we have $M \in \mathcal{W}_{X_n}$ or $M = X_{n+1}$ since $\mathcal{S}$ is a maximal chain in $\mathcal{T}$; in the second case, we have $M \in \mathcal{W}^{X_n}_{X_{n+1}}$. Observe that $\mathcal{T}$ is locally maximal and $\mathcal{W}_{X_n}, \mathcal{W}^{X_n}_{X_{n+1}}$ are separable, we have

$$|\mathcal{T} \cap \mathcal{W}^{X^n}_{X_{n+1}}| = |\mathcal{T} \cap \mathcal{W}_{X_{n+1}}| - |\mathcal{T} \cap \mathcal{W}_{X_n}| - 1 = \ell(X_{n+1}) - \ell(X_n) - 1,$$

which is equal to the rank of $\mathcal{W}^{X_n}_{X_{n+1}}$ by Lemma 4.2.9. Therefore, by Lemma 5.2.5, $\mathcal{T} \cap \mathcal{W}^{X^n}_{X_{n+1}}$ is a maximal τ-rigid set in $\mathcal{W}^{X_n}_{X_{n+1}}$. The proof of the proposition is completed.

The following statement is essential to our main purpose, which is to construct the maximal τ-rigid sets in Γ. Compare it with Lemma 5.2.8.

5.3.19 Lemma. *Let Γ have a section-generator*

$$\mathcal{S} : X_1 \prec X_2 \prec \cdots \prec X_n \prec \cdots .$$

(1) *The wings* $\mathcal{W}_{X_{t+1}}^{X_t}$ *with* $t \ge 1$ *are pairwise separable.*

(2) *If* $M \in \mathcal{W}_{X_{t+1}}^{X_t}$*, then* $M \notin \mathcal{S}$*. Moreover,* $\mathcal{W}_M, \mathcal{W}_{X_p}$ *are separable in case* $p \le t$*; and* $\mathcal{W}_M \subseteq \mathcal{W}_{X_p}$ *in case* $p > t$*.*

Proof. The proof of Statement (1) is similar to that of Lemma 5.2.8. For proving Statement (2), let $X_p \in \mathcal{S}$ and let $M \in \mathcal{W}_{X_{t+1}}^{X_t}$ with $p, t \ge 1$. If $p \le t$, then $X_p \in \mathcal{W}_{X_t}$. By definition, $\mathcal{W}_{X_{t+1}}^{X_t}, \mathcal{W}_{X_t}$ are separable. Therefore, $\mathcal{W}_M, \mathcal{W}_{X_p}$ are separable. If $p > t$, then $\mathcal{W}_{X_{t+1}} \subseteq \mathcal{W}_{X_p}$. We deduce from Definition 1.4.12 that $\mathcal{W}_M \subseteq \mathcal{W}_{X_{t+1}}^{X_t} \subsetneq \mathcal{W}_{X_p}$. The proof of the lemma is completed.

The following definition is essential to our main result of this section.

5.3.20 Definition. Let Γ have a section-generator

$$\mathcal{S} : X_1 \prec X_2 \prec \cdots \prec X_n \prec \cdots .$$

If Θ_n is a maximal τ-rigid set in the wing $\mathcal{W}_{X_{n+1}}^{X_n}$ for each $n \ge 1$, then the union $\Theta = \bigcup_{n=1}^{\infty} \Theta_n$ is called an *addend* to $\mathcal{S}$ in Γ.

5.3.21 Example. Given an integer n, the ray R_n^+ and the co-ray R_n^- both are section-generators. And both of them admit only the empty addend in Γ.

5.3.22 Definition. Given an infinite chain $\mathcal{S}$ in Γ, we say that $\mathcal{S}$ is *almost contained in* R_i^+ if all but finitely many objects of $\mathcal{S}$ are contained in some ray R_i^+; *almost contained in* R_j^- if all but finitely many objects of $\mathcal{S}$ are contained in some co-ray R_j^-.

5.3.23 Lemma. *Let* $\mathcal{S}$ *be an infinite chain in* Γ*. Then* $\mathcal{S}$ *is almost contained in some ray* R_i^+ *(respectively, co-ray* R_j^-*) if and only if* $\mathcal{S} \cap R_i^+$ *(respectively,* R_j^-*) is infinite.*

Proof. We shall only show the sufficiency. Assume that $\mathcal{S} \cap R_i^+$ is infinite, for some integer i. In view of Lemma 4.2.13, the chain $\mathcal{S} \cap R_i^+$ has a minimal element X. Let $Y \in \mathcal{S}$ such that $X \prec Y$. Being infinite, $\mathcal{S} \cap R_i^+$ contains an object Z such that $Y \prec Z$. Obviously, $i = i_{_X} \le i_{_Y} \le i_{_Z} = i$. Hence, $i_{_Y} = i$, that is,

$Y \in R_i^+$. By Lemma 4.2.11, $\mathcal{S}$ contains finitely many object covered by X. Thus, $\mathcal{S}$ is almost contained in R_i^+. The proof of the lemma is completed.

Let Δ be a convex subquiver of Γ with a set $\mathcal{T}$ of objects in Δ. Then $\mathcal{T}$ is called *densely maximal* τ-rigid in Δ if it is dense in Δ and maximal τ-rigid in Δ. The following statement provides a way to construct a τ-rigid set in Γ.

5.3.24 THEOREM. *Let $\mathcal{S}$ be a section-generator of Γ, and let $\Theta_{\mathcal{S}}$ be an addend to $\mathcal{S}$ in Γ.*

(1) *The set $\mathcal{S} \cup \Theta_{\mathcal{S}}$ is locally maximal τ-rigid.*

(2) *If $\mathcal{S}$ is almost contained in some ray R_i^+, then $\mathcal{S} \cup \Theta_{\mathcal{S}}$ is densely maximal τ-rigid in $\Gamma^+_{<i+1}$.*

(3) *If $\mathcal{S}$ is almost contained in some co-ray R_j^-, then $\mathcal{S} \cup \Theta_{\mathcal{S}}$ is densely maximal τ-rigid in $\Gamma^-_{>j-1}$.*

(4) *If $\mathcal{S}$ is locally finite, then $\mathcal{S} \cup \Theta_{\mathcal{S}}$ is densely maximal τ-rigid in Γ.*

Proof. Assume that $\mathcal{S}$ is of form $X_1 \prec X_2 \prec \cdots \prec X_n \prec \cdots$ and $\Theta_{\mathcal{S}} = \bigcup_{n=1}^{\infty} \Theta_n$, where Θ_n is a maximal τ-rigid set in $\mathcal{W}^{X_n}_{X_{n+1}}$ for all $n \geq 1$. By Lemma 5.2.5 and Lemma 4.2.9, we have $|\Theta_n| = \ell(X_{n+1}) - \ell(X_n) - 1$, for all $n \geq 1$. Set $\mathcal{T} = \mathcal{S} \cup \Theta_{\mathcal{S}}$. In view of Lemma 5.3.19 and Lemma 5.3.1, $\mathcal{T}$ is a τ-rigid set.

Now we claim that $|\mathcal{T} \cap \mathcal{W}_{X_n}| = \ell(X_n)$, for all $n \geq 1$. Since $\ell(X_1) = 1$, then our claim is true. Assume that the claim is true for $n-1$. Now consider $n \geq 2$. By Lemma 5.3.19(2), we see that

$$\begin{aligned} \mathcal{T} \cap \mathcal{W}_{X_n} &= \Theta_1 \cup \cdots \cup \Theta_{n-1} \cup \{X_1, \cdots, X_n\} \\ &= \Theta_{n-1} \cup (\mathcal{T} \cap \mathcal{W}_{X_{n-1}}) \cup \{X_n\}. \end{aligned}$$

Since $\mathcal{W}^{X_{n-1}}_{X_n}$ and $\mathcal{W}_{X_{n-1}}$ are disjoint and $\Theta_{n-1} \subseteq \mathcal{W}^{X_{n-1}}_{X_n}$, so are Θ_{n-1} and $\mathcal{W}_{X_{n-1}}$. Thus, we have

$$|\mathcal{T} \cap \mathcal{W}_{X_n}| = \ell(X_n) - \ell(X_{n-1}) - 1 + \ell(X_{n-1}) + 1 = \ell(X_n).$$

Hence, our claim is true. Now let $M \in \Theta_n$, for some $n \geq 1$. Since $\mathcal{W}_M \subseteq \mathcal{W}^{X_n}_{X_{n+1}}$, in view of Lemma 5.3.19(2), we have

$$|\mathcal{T} \cap \mathcal{W}_M| = |\mathcal{T} \cap \mathcal{W}^{X_n}_{X_{n+1}} \cap \mathcal{W}_M| = |\Theta_n \cap \mathcal{W}_M| = \ell(M),$$

where the last equality follows from Lemma 5.2.6(2) since Θ_n is a maximal τ-rigid set in $\mathcal{W}^{X_n}_{X_{n+1}}$. This shows that $\mathcal{T}$ is locally maximal τ-rigid. Statement (1) is established.

Next, Statement (2) follows from Proposition 5.3.10(1) and Lemma 4.2.19(2) while Statement (3) follows from Proposition 5.3.11(1) and Lemma 4.2.19(3). Finally, Statement (4) follows from Theorem 5.3.17(1) and Lemma 4.2.19(1). The proof of the theorem is completed.

The following statement is our main result of this section.

5.3.25 THEOREM. *Let Γ be a standard component of $\Gamma_{\mathcal{A}}$ of shape $\mathbb{Z}\mathbb{A}_\infty$. A set $\mathcal{T}$ of objects of Γ is maximal τ-rigid if and only if one of the following situations occurs.*

(1) $\mathcal{T} = \mathcal{S} \cup \Theta_{\mathcal{S}}$, *where $\mathcal{S}$ is a locally finite section-generator of Γ with an addend $\Theta_{\mathcal{S}}$.*

(2) $\mathcal{T} = \mathcal{S} \cup \Theta_{\mathcal{S}} \cup \mathcal{S}' \cup \Theta_{\mathcal{S}'}$, *where $\mathcal{S}$ is a section-generator of Γ almost contained in some R_i^+ with an addend $\Theta_{\mathcal{S}}$, and $\mathcal{S}'$ is a section-generator almost contained in R_j^-, for some $i+2 \le j \le i+3$, with an addend $\Theta'_{\mathcal{S}}$.*

(3) $\mathcal{T} = \mathcal{S} \cup \Theta_{\mathcal{S}} \cup \mathcal{S}' \cup \Theta_{\mathcal{S}'} \cup \Theta$, *where $\mathcal{S}$ is a section-generator of Γ almost contained in some R_i^+ with an addend $\Theta_{\mathcal{S}}$, and $\mathcal{S}'$ is a section-generator almost contained in R_j^-, for some $j \ge i+4$, with an addend $\Theta'_{\mathcal{S}}$, and Θ is a maximal τ-rigid set in $\mathcal{W}_Z$ with $Z = (j-2, i+2)$.*

Proof. Let $\mathcal{T}$ be a set of objects in Γ. If Statement (1) occurs, then by Theorem 5.3.24(4), $\mathcal{T}$ is maximal τ-rigid in Γ. Assume that Statement (2) occurs. By Theorem 5.3.24(2) and (3), $\mathcal{S} \cup \Theta_{\mathcal{S}}$ is a τ-rigid set in $\Gamma^+_{<i+1}$ and $\mathcal{S}' \cup \Theta_{\mathcal{S}'}$ is a τ-rigid set in $\Gamma^-_{>j-1}$. Since $j \ge i+2$, by Corollary 5.3.3(2),

$$\mathcal{T} = \mathcal{S} \cup \Theta_{\mathcal{S}} \cup \mathcal{S}' \cup \Theta_{\mathcal{S}'}$$

is τ-rigid. Moreover, by Theorem 5.3.24(1), $\mathcal{T}$ is locally maximal. Since $j \le i+3$, in view of Theorem 5.3.17(2), $\mathcal{T}$ is maximal τ-rigid in Γ. Now assume that Statement (3) occurs. Similarly, $\mathcal{S} \cup \Theta_{\mathcal{S}} \cup \mathcal{S}' \cup \Theta_{\mathcal{S}'}$ is τ-rigid and locally maximal. Since Θ is a maximal τ-rigid set in $\mathcal{W}_Z$, by Lemma 5.2.6, we see that Z belongs to Θ and Θ is locally maximal. Since $\mathcal{W}_Z \subseteq \Gamma^+_{<j-1} \cap \Gamma^-_{>i+1}$, $\mathcal{S} \cup \Theta_{\mathcal{S}} \subseteq \Gamma^+_{<i+1}$ and $\mathcal{S}' \cup \Theta_{\mathcal{S}'} \subseteq \Gamma^-_{j-1}$, by Corollary 5.3.3(2), $\mathcal{T} = \mathcal{S} \cup \Theta_{\mathcal{S}} \cup \mathcal{S}' \cup \Theta_{\mathcal{S}'} \cup \Theta$ is τ-rigid. Observing that $\mathcal{T}$ is locally maximal, by Theorem 5.3.17(2), $\mathcal{T}$ is maximal τ-rigid in Γ. This shows the sufficiency.

Conversely, let $\mathcal{T}$ be maximal τ-rigid in Γ. By Lemma 5.3.7, $\mathcal{T}$ is locally maximal. In view of Theorem 5.3.17, first we assume that $\mathcal{T}$ contains an infinite maximal chain

$$\mathcal{S} : X_1 \prec X_2 \prec \cdots \prec X_n \prec \cdots$$

which is locally finite in Γ. By Proposition 5.3.18(1) and (2), $\mathcal{S}$ is a locally finite section-generator of Γ and $\Theta_n = \mathcal{T} \cap \mathcal{W}^{X_n}_{X_{n+1}}$ is maximal τ-rigid in $\mathcal{W}^{X_n}_{X_{n+1}}$. By definition, $\Theta_{\mathcal{S}} = \bigcup_{n=1}^{\infty} \Theta_n$ is an addend to $\mathcal{S}$ in Γ. Therefore, by the sufficiency, $\mathcal{S} \cup \Theta_{\mathcal{S}}$ is a maximal τ-rigid set in Γ. Moreover, since $\mathcal{S} \cup \Theta_{\mathcal{S}} \subseteq \mathcal{T}$, we have $\mathcal{T} = \mathcal{S} \cup \Theta_{\mathcal{S}}$.

Second assume that $\mathcal{T}$ contains infinitely many objects of R_i^+ and infinitely many objects of R_j^-, where $j \geq i + 2$. Then we deduce that $\mathcal{T}$ contains two infinite maximal chains $\mathcal{S}$ and $\mathcal{S}'$ such that $\mathcal{T} \cap R_i^+ \subseteq \mathcal{S}$ and $\mathcal{T} \cap R_j^- \subseteq \mathcal{S}'$. In view of Proposition 5.3.18(1) and Lemma 5.3.23, $\mathcal{S}$ is a section-generator of Γ which is almost contained in R_i^+ and $\mathcal{S}'$ is a section-generator of Γ which is almost contained in R_j^-. Write

$$\mathcal{S} : X_1 \prec X_2 \prec \cdots \prec X_n \prec \cdots$$

and

$$\mathcal{S}' : Y_1 \prec Y_2 \prec \cdots \prec Y_n \prec \cdots .$$

Set $\Theta_t = \mathcal{T} \cap \mathcal{W}^{X_t}_{X_{t+1}}$, for each $t \geq 1$ and $\Theta'_s = \mathcal{T} \cap \mathcal{W}^{Y_s}_{Y_{s+1}}$, for each $s \geq 1$. In view of Proposition 5.3.18(2), $\Theta_{\mathcal{S}} = \bigcup_{t=1}^{\infty} \Theta_t$ is an addend to $\mathcal{S}$ in Γ and $\Theta_{\mathcal{S}'} = \bigcup_{s=1}^{\infty} \Theta'_s$ is an addend to $\mathcal{S}'$ in Γ. In case $i + 2 \leq j \leq i + 3$, by the sufficiency

$$\mathcal{T}' = \mathcal{S} \cup \Theta_{\mathcal{S}} \cup \mathcal{S}' \cup \Theta_{\mathcal{S}'}$$

is maximal τ-rigid in Γ. Since $\mathcal{T}' \subseteq \mathcal{T}$, we have $\mathcal{T} = \mathcal{T}'$. In case $j \geq i + 4$, by Theorem 5.3.17(2), the object Z with $(i_Z, j_Z) = (j - 2, i + 2)$ belongs to $\mathcal{T}$. Set $\Theta = \mathcal{T} \cap \mathcal{W}_Z$, and by the local maximality of $\mathcal{T}$, we have Θ is a maximal τ-rigid set in $\mathcal{W}_Z$. Again, by the sufficiency,

$$\mathcal{T}'' = \mathcal{S} \cup \Theta_{\mathcal{S}} \cup \mathcal{S}' \cup \Theta_{\mathcal{S}'} \cup \Theta$$

is maximal τ-rigid in Γ. Since $\mathcal{T}'' \subseteq \mathcal{T}$, we have $\mathcal{T} = \mathcal{T}''$. The proof of the theorem is completed.

REMARK. In view of Proposition 5.3.16, a set of objects in Γ is densely maximal τ-rigid in Γ if and only if it is of the form as stated in Theorem 5.3.25(1).

5.3.26 REMARK. We should point out that our results enable us to construct all the maximal τ-rigid sets in Γ. Indeed, using Proposition 4.2.24 and Corollary 4.2.25, we are able to construct all the section-generators in Γ, which are locally finite or almost contained in some ray or some co-ray. By applying Theorem 5.2.9, we are able to construct all the maximal τ-rigid set in a finite wing; and moreover, in view of Define 5.3.20, we are also able to construct all possible addends to a given section-generator of Γ.

To conclude this section, we shall give a method to construct all the maximal τ-rigid sets in $\Gamma^+_{<n}$ or in $\Gamma^-_{>n}$, respectively, for some $n \in \Gamma$. This will be used in Chapter 7.

5.3.27 PROPOSITION. *Let Γ be a standard component of $\Gamma_{\mathcal{A}}$ of shape $\mathbb{Z}\mathbb{A}_\infty$. Given an integer n, a set $\mathcal{T}$ of objects in $\Gamma^+_{<n}$ is maximal τ-rigid in $\Gamma^+_{<n}$ if and only if one of the following situations occurs.*

(1) $\mathcal{T} = \mathcal{S} \cup \Theta_{\mathcal{S}}$, *where $\mathcal{S}$ is a section-generator of Γ, which is almost contained in R^+_i with $n-2 \le i \le n-1$, and $\Theta_{\mathcal{S}}$ is an addend to $\mathcal{S}$ in Γ.*

(2) $\mathcal{T} = \mathcal{S} \cup \Theta_{\mathcal{S}} \cup \Theta$, *where $\mathcal{S}$ is a section-generator of Γ, which is almost contained in R^+_i with $i \le n-3$, $\Theta_{\mathcal{S}}$ is an addend to $\mathcal{S}$, and Θ is maximal τ-rigid set in $\mathcal{W}_Z$ with $(i_z, j_z) = (n-1, i+2)$.*

Proof. Let $\mathcal{T}$ be a set of objects in $\Gamma^+_{<n}$. Let $\mathcal{S}$ be a section-generator of Γ almost contained in R^+_i for some $i < n$. In particular, $\mathcal{S} \cap R^+_i$ is infinite. Let $\Theta_{\mathcal{S}}$ be an addend to $\mathcal{S}$ in Γ. By Theorem 5.3.24(1), $\mathcal{S} \cup \Theta_{\mathcal{S}}$ is locally maximal τ-rigid and contained in $\Gamma^+_{<i+1}$. Assume first that Statement (1) occurs. That is, $\mathcal{T} = \mathcal{S} \cup \Theta_{\mathcal{S}}$ with $i = n-1$ or $i = n-2$. By Proposition 5.3.10(1) and (2), $\mathcal{T}$ is maximal τ-rigid in $\Gamma^+_{<n}$. Assume now that Statement (2) occurs. That is, $\mathcal{T} = \mathcal{S} \cup \Theta_{\mathcal{S}} \cup \Theta$ with $n \ge i+3$, where Θ is a maximal τ-rigid set in $\mathcal{W}_Z$ with

$(i_z, j_z) = (n-1, i+2)$. Since $\mathcal{S} \cup \Theta_{\mathcal{S}} \subseteq \Gamma^+_{<i+1}$ and $\Theta \subseteq \Gamma^-_{>i+1}$, by Corollary 5.3.3(2), $\mathcal{T}$ is τ-rigid. By Lemma 5.3.7, Θ is locally maximal, and, hence, $\mathcal{T}$ is locally maximal. Observe that $\mathcal{T} = (\mathcal{T} \cap \Gamma^+_{<i+1}) \cup (\mathcal{T} \cap \mathcal{W}_Z)$. By Proposition 5.3.10(3), $\mathcal{T}$ is maximal τ-rigid in $\Gamma^+_{<n}$. The sufficiency is established.

Conversely, let $\mathcal{T}$ be a maximal τ-rigid set in $\Gamma^+_{<n}$. Consider the co-ray R^-_{n+1}. We claim that $\mathcal{T} \cup R^-_{n+1}$ is maximal τ-rigid in Γ. Since R^-_{n+1} is a section, in particular, it is a section-generator of Γ and has only the empty addend. By Theorem 5.3.24(2), R^-_{n+1} is densely maximal τ-rigid in $\Gamma^-_{>n}$. By Proposition 5.3.12, our claim is true.

Since $\mathcal{T} \cup R^-_{n+1}$ is not locally finite, in view of Theorem 5.3.25 and Lemma 5.3.16, $\mathcal{T} \cup R^-_{n+1}$ satisfies Statement (2) or (3) in Theorem 5.3.25. Consider the first case. That is,

$$\mathcal{T} \cup R^-_{n+1} = \mathcal{S}' \cup \Theta_{\mathcal{S}'} \cup \mathcal{S} \cup \Theta_{\mathcal{S}},$$

where $\Theta_{\mathcal{S}'}$ is an addend to a section-generator $\mathcal{S}'$ which is almost contained in some R^-_j and $\Theta_{\mathcal{S}}$ is an addend to a section-generator $\mathcal{S}$ which is almost contained in some R^+_i with $i+3 \geq j \geq i+2$. In particular, $\mathcal{T} \cup R^-_{n+1}$ contains infinitely many objects of R^-_j. Since $\mathcal{T} \cup R^-_{n+1}$ contains infinite may objects of R^-_{n+1}, by Lemma 5.3.4(2), we have $j = n+1$. Thus, $i = j-2 = n-1$ or $i = j-3 = n-2$. By Theorem 5.3.24(2) and (3), $\mathcal{S} \cup \Theta_{\mathcal{S}} \subseteq \Gamma^+_{<i+1} \subseteq \Gamma^+_{<n}$ and $\mathcal{S}' \cup \Theta_{\mathcal{S}'} \subseteq \Gamma^-_{>n}$. Therefore, we have $\mathcal{T} = \mathcal{S} \cup \Theta_{\mathcal{S}}$. That is, $\mathcal{T}$ verifies Statement (1) in the proposition. Considering the second, similarly we can show that $\mathcal{T}$ verifies Statement (2) in the proposition. This establishes the necessity. The proof of the proposition is completed.

Remark. A set of objects in $\Gamma^+_{<n}$ is densely maximal τ-rigid in $\Gamma^+_{<n}$ only if it is of the form as stated in Proposition 5.3.27(1) with $i = n-1$. In view of Remark 5.3.26, we are able to construct all the maximal τ-rigid sets in $\Gamma^+_{<n}$.

The following statement is similar to show.

5.3.28 Proposition. *Let Γ be a standard component of $\Gamma_{\mathcal{A}}$ of shape $\mathbb{Z}\mathbb{A}_\infty$. Given an integer n, a set $\mathcal{T}$ of objects in $\Gamma^-_{>n}$ is maximal τ-rigid in $\Gamma^-_{>n}$ if and only if one of the following situations occurs.*

(1) *$\mathcal{T} = \mathcal{S} \cup \Theta_{\mathcal{S}}$, where $\mathcal{S}$ is a section-generator of Γ, which is almost contained in R^-_j with $n < j < n+3$, with an addend $\Theta_{\mathcal{S}}$.*

(2) $\mathcal{T} = \mathcal{S} \cup \Theta_{\mathcal{S}} \cup \Theta$, *where* $\mathcal{S}$ *is a section-generator of* Γ, *which is almost contained in* R_j^- *with* $j \le n+3$, *with an addend* $\Theta_{\mathcal{S}}$; *and* Θ *is maximal* τ*-rigid set in* $\mathcal{W}_Z$ *with* $(i_z, j_z) = (j-2, n+1)$.

Remark. A set of objects in $\Gamma_{>n}^-$ is densely maximal τ-rigid in $\Gamma_{>n}^-$ only if it is of the form as stated in Proposition 5.3.28(1) with $j = n+1$.

5.4 Maximal τ-rigid sets in a standard component of shape $\mathbb{Z}\mathbb{A}_\infty^\infty$

The main objective of this section is to study maximal the τ-rigid sets in a standard component of shape $\mathbb{Z}\mathbb{A}_\infty^\infty$ of $\Gamma_{\mathcal{A}}$, where τ is the Auslander-Reiten translation of $\mathcal{A}$. For this purpose, we shall fix throughout this section a standard component Γ of $\Gamma_{\mathcal{A}}$ of shape $\mathbb{Z}\mathbb{A}_\infty^\infty$. Recall that we have defined a coordinate system for Γ in Section 4.3. This yields a partial order over Γ; see Lemma 4.3.7. As shown in the following statement, the τ-rigidity in Γ is completely determined by this order.

5.4.1 Lemma. *If* $X, Y \in \Gamma$, *then* (X, Y) *is a* τ*-rigid pair in* Γ *if and only if* X, Y *are comparable.*

Proof. Let X, Y be objects in Γ. Since Γ is standard, by Lemma 2.2.7, we have $\mathrm{Hom}_{\mathcal{A}}(X, \tau Y) \neq 0$ if and only if $\tau Y \in R^X$ if and only if $Y \in R^{\tau^- X}$. Similarly, $\mathrm{Hom}_{\mathcal{A}}(Y, \tau X) \neq 0$ if and only if $Y \in R_{\tau X}$. Since $X \notin R^{\tau^- X}$, we have $\mathrm{Hom}_{\mathcal{A}}(X, \tau X) = 0$, and also, $\mathrm{Hom}_{\mathcal{A}}(Y, \tau Y) = 0$. Therefore, (X, Y) is a τ-rigid pair if and only if $\mathrm{Hom}_{\mathcal{A}}(X, \tau Y) = 0$ and $\mathrm{Hom}_{\mathcal{A}}(Y, \tau X) = 0$, if and only if $Y \notin R^{\tau^- X}$ and $Y \notin R_{\tau X}$. By Lemma 4.3.26, the latter is equivalent to X, Y are comparable . The proof of the lemma is completed.

5.4.2 Remark. In particular, Lemma 5.4.1 implies that every object in Γ is τ-rigid.

Immediately, we are able to characterize the maximal τ-rigid sets in Γ.

5.4.3 Proposition. *Let Γ be a standard component of $\Gamma_{\mathcal{A}}$ of shape $\mathbb{Z}\mathbb{A}_{\infty}^{\infty}$. A set $\mathcal{T}$ of objects in Γ is maximal τ-rigid if and only if it is the vertex set of a section in Γ.*

Proof. Let $\mathcal{T}$ be a set of objects of Γ. In view of Lemma 5.4.1, $\mathcal{T}$ is maximal τ-rigid in Γ if and only if it is a maximal chain in Γ. By Proposition 4.3.19, the latter is equivalent to $\mathcal{T}$ being the vertex set of a section in Γ. The proof of the proposition is completed.

Chapter 6

Cluster-tilting subcategories of a cluster category of type $\mathbb{A}_\infty$

The objective of this chapter is to give a characterization of the cluster-tilting subcategories of a cluster category of type $\mathbb{A}_\infty$, and to provide a method to construct all of them. Throughout this chapter, let Q be a quiver of type $\mathbb{A}_\infty$ with no infinite paths.

6.1 Cluster-tilting subcategories of a cluster category of type $\mathbb{A}_\infty$

Recall that the skeleton $\mathscr{D}^b(Q)$ of $D^b(\mathrm{rep}(Q))$ chosen in Section 3.3, is an Auslander-Reiten category, whose Auslander-Reiten translation is written as $\tau_{_D}$. The connecting component $\mathcal{C}_Q$ of $\Gamma_{\mathscr{D}^b(Q)}$ is standard of shape $\mathbb{Z}\mathbb{A}_\infty$, which is obtained by gluing the preprojective component of $\Gamma_{\mathrm{rep}(Q)}$ and the shift by -1 of the preinjective component of $\Gamma_{\mathrm{rep}(Q)}$. The cluster category $\mathscr{C}(Q)$ is an Auslander-Reiten category, whose Auslander-Reiten translation is denoted by $\tau_{_\mathscr{C}}$. Since $\mathcal{C}_Q$ is the fundamental domain for $\mathscr{C}(Q)$, the canonical functor $\pi : \mathscr{D}^b(Q) \to \mathscr{C}(Q)$ induces a translation-quiver-isomorphism $\pi : \mathcal{C}_Q \to \Gamma_{\mathscr{C}(Q)}$, which acts identically on the underlying quiver. In particular, $\Gamma_{\mathscr{C}(Q)}$ is of shape $\mathbb{Z}\mathbb{A}_\infty$.

Observe that $\mathcal{C}_Q$ is a standard component of shape $\mathbb{Z}\mathbb{A}_\infty$. By applying Theorem 5.3.17 and Theorem 5.3.25, we are able to characterize and construct all the maximal τ_D-rigid sets in $\mathcal{C}_Q$. Recall that Holm and Jørgensen have given a geometric characterization of the functorial finiteness of maximal rigid subcategories of $\mathscr{C}(Q)$ in [34]. By using these results, in this section, we shall give a characterization of cluster-tilting subcategories of $\mathscr{C}(Q)$, and more importantly, provide a method to construct all of them.

Let Γ be a translation quiver of shape $\mathbb{Z}\mathbb{A}$. Given a vertex $X \in \Gamma$, we have the forward rectangle R^X and the backward rectangle R_X in Γ defined in Section 1.2. These enable us to describe the morphisms between objects in $\Gamma_{\mathscr{C}(Q)}$ in the following statement.

6.1.1 LEMMA. *Let Q be a quiver of type $\mathbb{A}_\infty$ with no infinite path. Given $X, Y \in \Gamma_{\mathscr{C}(Q)}$, we have*

$$\mathrm{Hom}_{\mathscr{C}(Q)}(X,Y) = \begin{cases} k, & \text{if } Y \in R^X \cup R_{\tau^2_{\mathscr{C}}X}; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Let $X, Y \in \Gamma_{\mathscr{C}(Q)}$. Since $\pi : \mathcal{C}_Q \to \Gamma_{\mathscr{C}(Q)}$ is an isomorphism of translation quivers acting identically on the underlying quiver, we have $Y \in R^X \cup R_{\tau^2_{\mathscr{C}}X} \subseteq \Gamma_{\mathscr{C}(Q)}$ if and only if $Y \in R^X \cup R_{\tau^2_D X} \subseteq \mathcal{C}_Q$. Since $\mathcal{C}_Q$ is standard, by Lemma 1.4.5, the latter is equivalent to

$$\mathrm{Hom}_{\mathscr{D}^b(Q)}(X,Y) \neq 0 \text{ or } \mathrm{Hom}_{\mathscr{D}^b(Q)}(Y, \tau_D^2 X) \neq 0.$$

That is,

$$\mathrm{Hom}_{\mathscr{D}^b(Q)}(X,Y) \oplus D\mathrm{Hom}_{\mathscr{D}^b(Q)}(Y, \tau_D^2 X) \neq 0.$$

In view of Lemma 3.3.3, it is equivalent to $\mathrm{Hom}_{\mathscr{C}(Q)}(X,Y) \neq 0$. Moreover, by Proposition 2.12 in [48], $\mathrm{Hom}_{\mathscr{C}(Q)}(X,Y)$ is at most of dimension 1. The proof of the lemma is completed.

On the other hand, the cluster category used by Holm and Jørgensen in [34] is constructed as follows. Consider the polynomial k-algebra $R = k[t]$, which is a DG-algebra with zero differential and t placed in homological degree 1. Let $D^f(R)$ be the derived category of DG-modules over R with finite dimensional homology over k. This is a Hom-finite Krull-Schmidt 2-Calabi-Yau triangulated category with a Serre functor $\mathbb{S} = [2]$, where [1] is the shift functor of $D^f(R)$ as a triangulated category. In particular, $D^f(R)$ is an Auslander-Reiten category, whose Auslander-Reitn quiver $\Gamma_{D^f(R)}$ is of shape $\mathbb{Z}\mathbb{A}_\infty$ with Auslander-Reiten translation $\tau_{_R} = \mathbb{S} \circ [-1] = [1]$. Given an object $X \in \Gamma_{D^f(R)}$, the backward rectangle $R_{\tau_R^- X}$ and the forward rectangle $R^{\tau_R X}$ in $\Gamma_{D^f(R)}$ are precisely $H^+(X)$ and $H^-(X)$, respectively, as defined in Definition 2.1 in [34]. Given $X, Y \in \Gamma_{D^f(R)}$, by Proposition 2.2 in [34], we obtain

$$\mathrm{Hom}_{D^f(R)}(X,Y) = \begin{cases} k, & \text{if } Y \in R^X \cup R_{\tau_R^2 X}; \\ 0, & \text{otherwise.} \end{cases}$$

The following statement says that the two cluster categories $\mathscr{C}(Q)$ and $D^f(R)$ are indeed equivalent.

6.1.2 Lemma. *Let Q be a quiver of type $\mathbb{A}_\infty$ without infinite paths. Then there is an equivalence from $\mathscr{C}(Q)$ to $D^f(R)$, which commutes with the Auslander-Reiten translations.*

Proof. First there is a translation-quiver-isomorphism $\phi : \Gamma_{D^f(R)} \to \Gamma_{\mathscr{C}(Q)}$. In particular, $\phi(\tau_{_R} X) = \tau_{_\mathscr{C}} \phi(X)$, for all $X \in \Gamma_{D^f(R)}$. Given $X, Y \in \Gamma_{D^f(R)}$, we have

$$\begin{aligned} \mathrm{Hom}_{D^f(R)}(X,Y) \neq 0 \quad & \text{if and only if} \quad Y \in R^X \cup R_{\tau_R^2 X} \\ & \text{if and only if} \quad \phi(Y) \in R^{\phi(X)} \cup R_{\tau_{\mathscr{C}}^2 \phi(X)} \end{aligned}$$

In view of Lemma 6.1.1, we see that ϕ induces an isomorphism

$$\phi_{X,Y} : \mathrm{Hom}_{D^f(R)}(X,Y) \to \mathrm{Hom}_{\mathscr{C}(Q)}(\phi(X), \phi(Y)).$$

It is easy to check that $\phi_{X,Y}$ is natural in both X and Y. It is evident that ϕ induces an equivalence from $D^f(R)$ to $\mathscr{C}(Q)$. The proof of the lemma is completed.

Now for our main purpose, we shall need to recall some geometric notions and terminology from [34]. Let $\mathfrak{A}_\infty$ stand for an ∞-gon with marked points, that is the upper half plane in the plane $\mathbb{R}^2$ with marked points which are denoted by $n \in \mathbb{Z}$, as shown below.

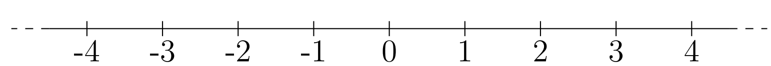


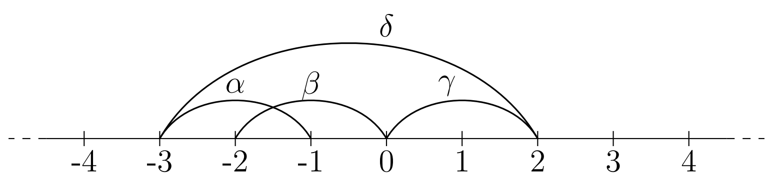
Figure 6.1: An ∞-gon with marked points.

A *simple curve* in $\mathfrak{A}_\infty$ is a curve which does not cross itself and joins two (maybe identical) marked points, called *endpoints* of the curve. A simple curve is called *internal* if it intersects the boundary of $\mathfrak{A}_\infty$ only at the endpoints. Two distinct simple curves in $\mathfrak{A}_\infty$ are said to *cross* if they have a common point which is not an endpoint of any of the two curves.

Given two marked points m, n with $m-n \geq 2$, the isotopy class of the internal simple curves in $\mathfrak{A}_\infty$ connecting m and n is called the *arc* in $\mathfrak{A}_\infty$ with end-points m, n, which is denoted by $[m, n]$ with $m > n$. We shall denote by $\operatorname{arc}(\mathfrak{A}_\infty)$ the set of all arcs in $\mathfrak{A}_\infty$, that is,

$$\operatorname{arc}(\mathfrak{A}_\infty) = \{[m, n] \mid (m, n) \in \mathbb{Z} \times \mathbb{Z} \text{ with } m \geq n + 2\}.$$

Two arcs $\alpha = [m, n]$ and $\beta = [p, q]$ in $\mathfrak{A}_\infty$ are said to be *crossing* if every curve in α crosses each of the curves in β. It is easy to see that this is equivalent to the condition that either $m > p > n > q$ or $p > m > q > n$. In the following figure, only α, β are crossing.



6.1.3 Definition. A *triangulation* of $\mathfrak{A}_\infty$ is a maximal set of pairwise non-crossing arcs in $\mathfrak{A}_\infty$.

Let $\mathbb{S}$ be a set of arcs of $\mathfrak{A}_\infty$. One says that $\mathbb{S}$ is *locally finite* if every marked point of $\mathfrak{A}_\infty$ is an endpoint of at most finitely many arcs in $\mathbb{S}$. Given a marked point n in $\mathfrak{A}_\infty$, if both the set of arcs $[m, n] \in \mathbb{S}$ with $m > n$ and that of arcs $[n, p] \in \mathbb{S}$ with $n > p$ are infinite, then the union of these two sets is called a *fountain* in $\mathbb{S}$ with *fountain base* n. Moreover, given two marked points m, n in $\mathfrak{A}_\infty$ with $m > n$, if both the set of arcs $[p, m] \in \mathbb{S}$ with $p > m$ and that of arcs $[n, q] \in \mathbb{S}$ with $n > q$ are infinite, then the union of these two sets is called a *splitting fountain* in $\mathbb{S}$.

The following figures show three types of triangulations of $\mathfrak{A}_\infty$.

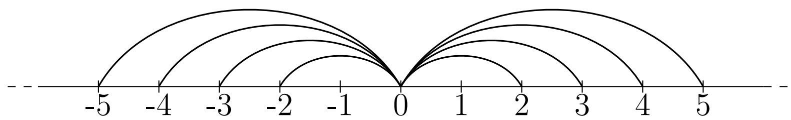


Figure 6.2: A triangulation of $\mathfrak{A}_\infty$ having a fountain

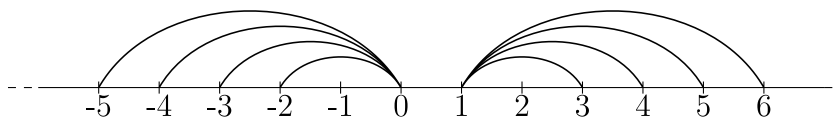


Figure 6.3: A triangulation of $\mathfrak{A}_\infty$ having a splitting fountain

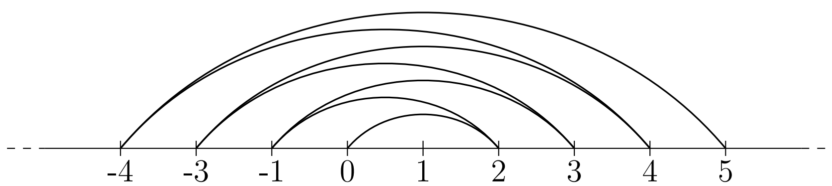


Figure 6.4: A locally finite triangulation of $\mathfrak{A}_\infty$

Note that $\Gamma_{\mathscr{C}(Q)}$ is a translation quiver of shape $\mathbb{Z}\mathbb{A}_\infty$. As we did in Section 4.2, we may give a coordinate system for $\Gamma_{\mathscr{C}(Q)}$. Indeed, the quasi-simple objects in $\Gamma_{\mathscr{C}(Q)}$ will be denoted as S_n with $n \in \mathbb{Z}$ such that $\tau_{\mathscr{C}} S_n = S_{n+1}$. The ray starting with S_n is denoted by R_n^+, and the co-ray ending with S_n is denoted by R_n^-. Given an object $X \in \Gamma_{\mathscr{C}(Q)}$, there is a unique pair of integers (i_X, j_X) with $i_X \geq j_X$, such that $R_{i_X}^+ \cap R_{j_X}^- = \{X\}$.

6.1.4 Lemma. *There is a bijection*

$$\Psi : \Gamma_{\mathscr{C}(Q)} \to \operatorname{arc}(\mathfrak{A}_\infty) : X \mapsto \alpha_X = [i_X + 1, j_X - 1].$$

Proof. By definition, $\operatorname{arc}(\mathfrak{A}_\infty) = \{[m,n] \mid m - n \geq 2\}$. Clearly, we have a bijection

$$\theta : \mathbb{Z}_\Gamma \to \operatorname{arc}(\mathfrak{A}_\infty) : (i,j) \mapsto [i+1, j-1],$$

where $\mathbb{Z}_{\Gamma} = \{(i,j) \in \mathbb{Z} \times \mathbb{Z} \mid i \geq j\}$. Thus, the statement follows from Lemma 4.2.1. The proof of the lemma is completed.

Given a set $\mathcal{T}$ of objects in $\Gamma_{\mathscr{C}(Q)}$, denote by $\operatorname{arc}(\mathcal{T})$ the image of $\mathcal{T}$ under Ψ. Then immediately, we have the following statement.

6.1.5 Lemma. *Let $\mathcal{T}$ be a set of objects in $\Gamma_{\mathscr{C}(Q)}$.*

(1) $\operatorname{arc}(\mathcal{T})$ *is locally finite if and only both $\mathcal{T} \cap R_i^+$ and $\mathcal{T} \cap R_i^-$ are finite, for every integer i.*

(2) $\operatorname{arc}(\mathcal{T})$ *has a fountain if and only if there is some i such that both $\mathcal{T} \cap R_{i-1}^+$ and $\mathcal{T} \cap R_{i+1}^-$ are infinite.*

(3) $\operatorname{arc}(\mathcal{T})$ *has a splitting fountain if and only if $\mathcal{T} \cap R_i^+$ and $\mathcal{T} \cap R_j^-$ are infinite for some integers i, j with $j > i + 2$.*

Proof. Given an integer i, we see that

$$\operatorname{arc}(\mathcal{T} \cap R_{i-1}^+) = \{[i,p] \in \operatorname{arc}(\mathcal{T}) \mid i - 1 > p\}$$

and

$$\operatorname{arc}(\mathcal{T} \cap R_{i+1}^-) = \{[q,i] \in \operatorname{arc}(\mathcal{T}) \mid q > i + 1\}.$$

Firstly, $\operatorname{arc}(\mathcal{T})$ is locally finite if and only if both $\{[i,p] \in \operatorname{arc}(\mathcal{T}) \mid i - 1 > p\}$ and $\{[q,i] \in \operatorname{arc}(\mathcal{T}) \mid q > i+1\}$ are finite for all $i \in \mathbb{Z}$ if and only if $\operatorname{arc}(\mathcal{T} \cap R_{i-1}^+)$ and $\operatorname{arc}(\mathcal{T} \cap R_{i+1}^-)$ are finite for all $i \in \mathbb{Z}$. Since Ψ is a bijection, the latter is equivalent to $\mathcal{T} \cap R_{i-1}^+$ and $\mathcal{T} \cap R_{i+1}^-$ are finite, for all $i \in \mathbb{Z}$. This establishes Statement (1).

Secondly, $\operatorname{arc}(\mathcal{T})$ has a fountain if and only if both $\{[i,p] \in \operatorname{arc}(\mathcal{T}) \mid i-1 > p\}$ and $\{[q,i] \in \operatorname{arc}(\mathcal{T}) \mid q > i+1\}$ are infinite for some i if and only if $\operatorname{arc}(\mathcal{T} \cap R_{i-1}^+)$ and $\operatorname{arc}(\mathcal{T} \cap R_{i+1}^-)$ are infinite for some i if and only if $\mathcal{T} \cap R_{i-1}^+$ and $\mathcal{T} \cap R_{i+1}^-$ are infinite for some i. This establishes Statement (2). Similarly, we can establish Statement (3). The proof of this lemma is completed.

Let $\mathscr{T}$ be a strictly additive subcategory of $\mathscr{C}(Q)$. Denote by $\operatorname{ind}\mathscr{T}$ the set of objects in $\Gamma_{\mathscr{C}(Q)}$ which lie in $\mathscr{T}$. Moreover, denote by $\operatorname{arc}(\mathscr{T})$ the set of arcs $\Psi(X)$ with $X \in \operatorname{ind}\mathscr{T}$.

Observe that the maximal rigid subcategories of $\mathscr{C}(Q)$ are precisely the 1-maximal orthogonal subcategories as defined in [34, (4.1)]. We have the following result.

6.1.6 PROPOSITION. *Let Q be a quiver of type $\mathbb{A}_\infty$ without infinite paths. A strictly additive subcategory $\mathscr{T}$ of $\mathscr{C}(Q)$ is weakly cluster-tilting if and only if* $\mathrm{arc}(\mathscr{T})$ *is a triangulation of $\mathfrak{A}_\infty$.*

Proof. By Lemma 2.1 in [48], $\mathscr{T}$ is weakly cluster-tilting if and only if $\mathrm{ind}\mathscr{T}$ is a maximal rigid set in $\varGamma_{\mathscr{C}(Q)}$. Since $\mathscr{C}(Q) \cong D^f(R)$ by Lemma 6.1.2, the proposition follows immediately from Theorem 4.3 stated in [34]. The proof of the proposition is completed.

The following statement gives a criterion for a weakly cluster-tilting subcategory of $\mathscr{C}(Q)$ to be a cluster-tilting subcategory.

6.1.7 THEOREM ([34]). *Let Q be a quiver of type $\mathbb{A}_\infty$ without infinite paths. A weakly cluster-tilting subcategory $\mathscr{T}$ of $\mathscr{C}(Q)$ is a cluster-tilting subcategory if and only if* $\mathrm{arc}(\mathscr{T})$ *is a triangulation of $\mathfrak{A}_\infty$ which is locally finite or has a fountain.*

Proof. The statement follows immediately from Theorem 4.4 in [34] and Lemma 6.1.2. The proof of the theorem is completed.

Recall that $\pi : \mathcal{C}_Q \to \varGamma_{\mathscr{C}(Q)}$ is an isomorphism of translation quivers acting identically on the underlying quiver. Let $\mathcal{S}$ be a section-generator of $\varGamma_{\mathscr{C}(Q)}$. Observe that $\mathcal{S}$ is also a section-generator of $\mathcal{C}_Q$. Let Θ be an addend to $\mathcal{S}$ in $\mathcal{C}_Q$ as defined in Definition 5.3.20. Considering Θ as a set of objects in $\varGamma_{\mathscr{C}(Q)}$, we shall call it an *addend* to $\mathcal{S}$ in $\varGamma_{\mathscr{C}(Q)}$.

The following statement is our main result of this section.

6.1.8 THEOREM. *Let $\mathscr{C}(Q)$ be the cluster category associated with a quiver Q of type $\mathbb{A}_\infty$ without infinite paths. A strictly additive subcategory $\mathscr{T}$ of $\mathscr{C}(Q)$ is cluster-tilting if and only if one of the following situations occurs.*

(1) $\mathrm{ind}\mathscr{T} = \mathcal{S} \cup \Theta_{\mathcal{S}}$, *where $\mathcal{S}$ is a locally finite section-generator of $\varGamma_{\mathscr{C}(Q)}$ with an addend $\Theta_{\mathcal{S}}$.*

(2) $\operatorname{ind}\mathscr{T} = \mathcal{S} \cup \Theta_{\mathcal{S}} \cup \mathcal{S}' \cup \Theta_{\mathcal{S}'}$, *where* $\mathcal{S}$ *is a section-generator of* $\Gamma_{\mathscr{C}(Q)}$ *which is almost contained in some ray* R_i^+ *and has an addend* $\Theta_{\mathcal{S}}$, *while* $\mathcal{S}'$ *is a section-generator of* $\Gamma_{\mathscr{C}(Q)}$ *which is almost contained in the co-ray* R_{i+2}^- *and has an addend* $\Theta_{\mathcal{S}'}$.

Proof. Let $\mathscr{T}$ be a strictly additive subcategory of $\mathscr{C}(Q)$. Observe that $\operatorname{ind}\mathscr{T}$ is a set of objects in $\Gamma_{\mathscr{C}(Q)}$, as well as, a set of objects in $\mathcal{C}_Q$.

Assume first that $\operatorname{ind}\mathscr{T}$ verifies Statement (1). Considered as a set of objects in $\mathcal{C}_Q$, $\operatorname{ind}\mathscr{T}$ satisfies Statement (1) stated in Theorem 5.3.25. Therefore, $\operatorname{ind}\mathscr{T}$ is a maximal τ_D-rigid set in $\mathcal{C}_Q$. By Proposition 5.1.7, $\mathscr{T}$ is a weakly cluster-tilting subcategory of $\mathscr{C}(Q)$. By Proposition 6.1.6, $\operatorname{arc}(\mathscr{T})$ is a triangulation of $\mathfrak{A}_\infty$. Moreover, by Statement (1), $\operatorname{ind}\mathscr{T}$ contains a locally finite section-generator $\mathcal{S}$ of $\Gamma_{\mathscr{C}(Q)}$. By Lemma 5.3.16, $\operatorname{ind}\mathscr{T}$ is locally finite in $\Gamma_{\mathscr{C}(Q)}$. That is, $\operatorname{ind}\mathscr{T} \cap R_i^+$ and $\operatorname{ind}\mathscr{T} \cap R_i^-$ are finite, for $i \in \mathbb{Z}$. By Lemma 6.1.5(1), $\operatorname{arc}(\mathscr{T})$ is locally finite, and by Theorem 6.1.7, $\mathscr{T}$ is a cluster-tilting subcategory of $\mathscr{C}(Q)$.

Assume next that $\operatorname{ind}\mathscr{T}$ verifies Statement (2). Considered as a set of objects in $\mathcal{C}_Q$, $\operatorname{ind}\mathscr{T}$ satisfies Statement (2) with $j = i + 2$ stated in Theorem 5.3.25. Therefore, $\operatorname{ind}\mathscr{T}$ is a maximal τ_D-rigid set in $\mathcal{C}_Q$. By Proposition 5.1.7, $\mathscr{T}$ is a weakly cluster-tilting subcategory of $\mathscr{C}(Q)$. By Proposition 6.1.6, $\operatorname{arc}(\mathscr{T})$ is a triangulation of $\mathfrak{A}_\infty$. Moreover, by Statement (2), $\operatorname{ind}\mathscr{T}$ contains section-generators $\mathcal{S}$ and $\mathcal{S}'$ of $\Gamma_{\mathscr{C}(Q)}$ with $\mathcal{S}$ almost contained in some ray R_i^+ and $\mathcal{S}'$ almost contained in co-ray R_{i+2}^-. Since $\mathcal{S}$ and $\mathcal{S}'$ are infinite, $\operatorname{ind}\mathscr{T} \cap R_i^+$ and $\operatorname{ind}\mathscr{T} \cap R_{i+2}^-$ are infinite. By Lemma 6.1.5(2), $\operatorname{arc}(\mathscr{T})$ has a fountain, and by Theorem 6.1.7, $\mathscr{T}$ is cluster-tilting in $\mathscr{C}(Q)$. This establishes the sufficiency.

Conversely, assume that $\mathscr{T}$ is cluster-tilting in $\mathscr{C}(Q)$. As a set of objects $\mathcal{C}_Q$, by Proposition 5.1.7, $\operatorname{ind}\mathscr{T}$ is maximal τ_D-rigid. That is, $\operatorname{ind}\mathscr{T}$ verifies one of the three conditions stated in Theorem 5.3.25.

Suppose that $\operatorname{ind}\mathscr{T}$ verifies Statement (2) stated in Theorem 5.3.25 with $j > i + 2$. That is, $\operatorname{ind}\mathscr{T}$ contains section-generators $\mathcal{S}$ and $\mathcal{S}'$ of $\mathcal{C}_Q$ with $\mathcal{S}$ almost contained in the ray R_i^+ and $\mathcal{S}'$ almost contained in the co-ray R_j^-. Since $\mathcal{S}$ and $\mathcal{S}'$ are infinite, $\operatorname{ind}\mathscr{T} \cap R_i^+$ and $\operatorname{ind}\mathscr{T} \cap R_j^-$ are infinite. By Lemma 6.1.5(3), $\operatorname{arc}(\mathscr{T})$ has a splitting fountain. By Theorem 6.1.7, $\mathscr{T}$ is not cluster-tilting, a contradiction. Similarly, we shall obtain a contradiction if $\operatorname{ind}\mathscr{T}$ verifies Statement (3) stated in Theorem 5.3.25. Therefore, $\operatorname{ind}\mathscr{T}$ verifies either Statement (1) or Statement (2) with $j = i + 2$ stated in Theorem 5.3.25. Considered as a set of objects in $\Gamma_{\mathscr{C}(Q)}$, ind$\mathscr{T}$ verifies Statement (1) or Statement (2) stated in the theorem. This establishes the necessity. The proof of the theorem is completed.

REMARK. Let Q be a quiver of type $\mathbb{A}_\infty$ without infinite paths. We should point out that our results will enable us to construct all the cluster-tilting subcategories of $\mathscr{C}(Q)$. Indeed, using Proposition 4.2.24 and Corollary 4.2.25, we are able to construct all the section-generators in $\Gamma_{\mathscr{C}(Q)}$ which are locally finite or almost contained in some ray or some co-ray. Given a section-generator $\mathcal{S}$ in $\Gamma_{\mathscr{C}(Q)}$, as indicated in Define 5.3.20, we shall apply Theorem 5.2.9 to construct all possible addends Θ to $\mathcal{S}$ in $\Gamma_{\mathscr{C}(Q)}$.

To conclude this section, we shall give a complete description of the cluster-tilting subcategories of $\mathscr{C}(Q)$. A rigid set $\mathcal{T}$ of objects in $\Gamma_{\mathscr{C}(Q)}$ is called *locally maximal* in $\Gamma_{\mathscr{C}(Q)}$ if $|\mathcal{T} \cap \mathcal{W}_X| = \ell(X)$, for every $X \in \mathcal{T}$.

6.1.9 THEOREM. *Let $\mathscr{C}(Q)$ be the cluster category associated with a quiver Q of type $\mathbb{A}_\infty$ without infinite paths, and let $\mathscr{T}$ be a strictly additive subcategory of $\mathscr{C}(Q)$. Then $\mathscr{T}$ is cluster-tilting in $\mathscr{C}(Q)$ if and only if* ind$\mathscr{T}$ *is locally maximal rigid satisfying one the following conditions.*

(1) *$\mathcal{T}$ contains an infinite chain which is locally finite in $\Gamma_{\mathscr{C}(Q)}$.*

(2) *$\mathcal{T}$ contains infinitely many objects of some ray R_i^+ and infinitely many objects of the co-ray R_{i+2}^- in $\Gamma_{\mathscr{C}(Q)}$.*

Proof. Let $\mathscr{T}$ be a strictly additive subcategory of $\mathscr{C}(Q)$. Observe that ind$\mathscr{T}$ is a set of objects in $\Gamma_{\mathscr{C}(Q)}$, as well as, a set of objects in $\mathcal{C}_Q$.

Assume that $\mathscr{T}$ is cluster-tilting. Then ind$\mathscr{T}$ verifies Theorem 6.1.8. Since every section-generator of $\Gamma_{\mathscr{C}(Q)}$ is a chain by Proposition 4.2.24(2), ind$\mathscr{T}$ verifies Statement (1) or (2) stated in the theorem. Considering ind$\mathscr{T}$ as a set of objects in $\mathcal{C}_Q$, by Proposition 5.1.7, ind$\mathscr{T}$ is maximal τ_D-rigid in $\mathcal{C}_Q$. By Lemma 5.3.6, ind$\mathscr{T}$ is locally maximal in $\mathcal{C}_Q$. Thus, as a set of objects in $\Gamma_{\mathscr{C}(Q)}$, ind$\mathscr{T}$ is locally maximal in $\Gamma_{\mathscr{C}(Q)}$. The necessity is established.

Conversely, assume first that ind$\mathscr{T}$ is locally maximal verifying Statement (1) stated in the theorem. Considering ind$\mathscr{T}$ as a set of objects in $\mathcal{C}_Q$, by Theorem 5.3.17(1), ind$\mathscr{T}$ is maximal τ_D-rigid in $\mathcal{C}_Q$. Therefore, ind$\mathscr{T}$ is of one of the three forms stated in Theorem 5.3.25. Since ind$\mathscr{T}$ contains an infinite chain in $\mathcal{C}_Q$ which is locally finite, by Lemma 5.3.16, it is locally finite. Hence, ind$\mathscr{T}$ is of the form stated in Statement (1) in Theorem 5.3.25. Therefore, as objects of $\Gamma_{\mathscr{C}(Q)}$, ind$\mathscr{T}$ is of the form stated in Statement (1) in Theorem 6.1.8. Thus, $\mathscr{T}$ is cluster-tilting.

Assume secondly that ind$\mathscr{T}$ is locally maximal verifying Statement (2) stated in the theorem. Similarly, considering ind$\mathscr{T}$ as a set of objects in $\mathcal{C}_Q$, by Theorem 5.3.17(2), ind$\mathscr{T}$ is maximal τ_{D}-rigid in $\mathcal{C}_Q$. Since ind$\mathscr{T}$ contain infinitely many objects of some ray R_i^+ and the co-ray R_{i+2}^- in $\mathcal{C}_Q$, it is not locally finite. By Lemma 5.3.16, ind$\mathscr{T}$ contains no infinite chain which is locally finite in $\mathcal{C}_Q$. Hence, ind$\mathscr{T}$ verifies Statement (2) or (3) in Theorem 5.3.25. Since i is unique by Lemma 5.3.4, ind$\mathscr{T}$ verifies Statement (2) in Theorem 5.3.25. Therefore, as objects of $\Gamma_{\mathscr{C}(Q)}$, ind$\mathscr{T}$ verifies Statement (2) in Theorem 6.1.8. Thus, $\mathscr{T}$ is cluster-tilting. The sufficiency is established. The proof of the theorem is completed.

Chapter 7

Cluster-tilting subcategories of a cluster category of type $\mathbb{A}_\infty^\infty$

The main objective of this chapter is to provide a way to construct all the cluster-tilting subcategories of a cluster category of type $\mathbb{A}_\infty^\infty$. Throughout this chapter, Q stands for a quiver of type $\mathbb{A}_\infty^\infty$ without infinite paths.

7.1 The coordinate systems for the fundamental domain

The objective of this section is to introduce the coordinate systems for the fundamental domain $\mathscr{F}(Q)$. Recall that the skeleton $\mathscr{D}^b(Q)$ of $D^b(\text{rep}(Q))$ chosen in Section 3.3 is an Auslander-Reiten category, whose Auslander-Reiten translation will be simply written as τ. The fundamental domain $\mathscr{F}(Q)$ for the cluster category $\mathscr{C}(Q)$ consists of three standard components of $\varGamma_{\mathscr{D}^b(Q)}$, namely, the connecting component $\mathcal{C}_Q$ and two orthogonal regular components $\mathcal{R}$ and $\mathcal{L}$. Each of these three components will be equipped with a coordinate system, which will be used throughout this chapter.

Let us start with a coordinate system for the connecting component $\mathcal{C}_Q$. Recall that $\mathcal{C}_Q$ is of shape $\mathbb{Z}\mathbb{A}_\infty^\infty$. As in Section 4.3, we fix two double infinite sectional paths L_0 and R_0, whose intersection contains exactly one object. Write $L_i = \tau^i L_0$ and $R_j = \tau^j R_0$, for $i, j \in \mathbb{Z}$. For each $X \in \mathcal{C}_Q$, there exists a unique pair (i_X, j_X) of integers such that $\{X\} = L_{i_X} \cap R_{j_X}$. For convenience, we shall write $X = (i_X, j_X)$. By Lemma 4.3.7, $(\mathcal{C}_Q, \preceq)$ is partially ordered in such a way that $X \preceq Y$ if $i_X \le i_Y$ and $j_X \ge j_Y$.

Next, observe that $\mathcal{L}$ and $\mathcal{R}$ are of shape $\mathbb{Z}\mathbb{A}_\infty$. In order to fix a coordinate system for each of them, we need to reformulate a result stated in [48, (4.1)]. For this purpose, recall from Definition 4.2.26 that, for a quasi-simple object S in $\mathcal{L}$ or in $\mathcal{R}$, the infinite co-wing $\mathcal{W}(S)$ is the full subquiver generated by the object X for which there exist paths $N \rightsquigarrow X \rightsquigarrow M$, where M belongs to the ray starting with S and N belongs to the co-ray ending with S.

7.1.1 Lemma. *Let L_i and R_j with $i, j \in \mathbb{Z}$ be the previously fixed double infinite sectional paths in $\mathcal{C}_Q$. There is a unique quasi-simple object $S_L \in \mathcal{L}$ and a unique quasi-simple object $T_R \in \mathcal{R}$ such that the following statements hold.*

(1) *If $X \in \mathcal{C}_Q$ and $i \in \mathbb{Z}$, then $X \in L_i$ if and only if $\mathrm{Hom}_{\mathscr{D}^b(Q)}(X, \tau^i S_L) \neq 0$; and in this case, for each $N \in \mathcal{L}$, one has $\mathrm{Hom}_{\mathscr{D}^b(Q)}(X, N) \neq 0$ if and only if $N \in \mathcal{W}(\tau^i S_L)$.*

(2) *If $Y \in \mathcal{C}_Q$ and $j \in \mathbb{Z}$, then $Y \in R_j$ if and only if $\mathrm{Hom}_{\mathscr{D}^b(Q)}(Y, \tau^j T_R) \neq 0$; and in this case, for each $M \in \mathcal{R}$, one has $\mathrm{Hom}_{\mathscr{D}^b(Q)}(Y, M) \neq 0$ if and only if $M \in \mathcal{W}(\tau^j T_R)$.*

Proof. We shall prove only Statement (1). Let $X \in \mathcal{C}_Q$ and $N \in \mathcal{L}$. We claim that $\mathrm{Hom}_{\mathscr{C}(Q)}(X, N) \cong \mathrm{Hom}_{\mathscr{D}^b(Q)}(X, N)$. By Lemma 3.3.3,

$$\mathrm{Hom}_{\mathscr{C}(Q)}(X, N) \cong \mathrm{Hom}_{\mathscr{D}^b(Q)}(X, N) \oplus D\mathrm{Hom}_{\mathscr{D}^b(Q)}(N, \tau^2 X).$$

By the property of $\mathcal{C}_Q$, there exists an integer $n \ge 0$ such that $\tau^{-n+2}X$ is a representation. That is, $\tau^{-n+2}X$ lies in the preprojective component $\mathcal{P}$ of $\Gamma_{\mathrm{rep}(Q)}$. Since $\tau^{-n}N \in \mathcal{L}$ is a representation, we have

$$\begin{aligned} \mathrm{Hom}_{\mathscr{D}^b(Q)}(N, \tau^2 X) &\cong \mathrm{Hom}_{\mathscr{D}^b(Q)}(\tau^{-n}N, \tau^{-n+2}X) \\ &\cong \mathrm{Hom}_{\mathrm{rep}(Q)}(\tau^{-n}N, \tau^{-n+2}X) \\ &= 0. \end{aligned}$$

The last equality holds since $\mathrm{Hom}_{\mathrm{rep}(Q)}(\mathcal{L}, \mathcal{P}) = 0$. This establishes our claim. Compare the L_i and R_i chosen above with those chosen in Section 4 in [48].

Since $\pi : \mathscr{F}(Q) \to \Gamma_{\mathscr{C}(Q)}$ is a translation-quiver-isomorphism acting identically on underlying quiver, our statement follows immediately from Lemma 4.1 in [48]. The proof of the lemma is completed.

Now, we are ready to fix a coordinate system for $\mathcal{L}$. Set $S_0 = \tau^- S_L \in \mathcal{L}$. Let L_i^+ with $i \in \mathbb{Z}$ stand for the ray in $\mathcal{L}$ starting with $\tau^i S_0$ and L_j^- with $j \in \mathbb{Z}$ for the co-ray in $\mathcal{L}$ ending with $\tau^j S_0$. For each $M \in \mathcal{L}$, by Lemma 4.2.1, there is a unique pair (i_M, j_M) of integers, with $i_M \ge j_M$, such that $\{M\} = L^+_{i_M} \cap L^-_{j_M}$. For simplicity, we write $M = (i_M, j_M)$. Given an integer n, write $\mathcal{L}^+_{<n} = \bigcup_{i<n} L_i^+$ and $\mathcal{L}^-_{>n} = \bigcup_{j>n} L_j^-$. By Lemma 4.2.16(2), if $m > n+1$, then $\mathcal{L}^+_{<m} \cap \mathcal{L}^-_{>n}$ is a wing in $\mathcal{L}$ with a wing vertex $Z = (m-1, n+1)$.

To fix a coordinate system for $\mathcal{R}$, we set $T_0 = \tau^- T_R$ and denote by R_i^+ with $i \in \mathbb{Z}$ the ray in $\mathcal{R}$ starting with $\tau^i T_0$ and by R_j^- with $j \in \mathbb{Z}$ the co-ray in $\mathcal{R}$ ending with $\tau^j T_0$. For each $N \in \mathcal{R}$, by Lemma 4.2.1, there is a unique pair (i_N, j_N) of integers, with $i_N \ge j_N$, such that $\{N\} = R^+_{i_N} \cap R^-_{j_N}$. For simplicity, we write $N = (i_N, j_N)$. Given an integer n, write $\mathcal{R}^+_{<n} = \bigcup_{i<n} R_i^+$ and $\mathcal{R}^-_{>n} = \bigcup_{j>n} R_j^-$. By Lemma 4.2.16(2), if $m > n+1$, then $\mathcal{R}^+_{<m} \cap \mathcal{R}^-_{>n}$ is a wing in $\mathcal{R}$ with a wing vertex $Z = (m-1, n+1)$.

Let X, Y be objects in $\mathcal{C}_Q$ with $X \prec Y$ connected by a path p. By Lemma 4.3.9, p is a sectional path. Thus, by Lemma 4.3.4, if $s(p) = X$, then $i_X = i_Y$ and $j_X > j_Y$; if $t(p) = X$, then $i_X < i_Y$ and $j_X = j_Y$. The following definition is important in our later investigation.

7.1.2 Definition. Let X, Y be two objects in $\mathcal{C}_Q$ with $X \prec Y$ and connected by a path p. Define

$$\mathcal{W}_{X,Y} = \begin{cases} \mathcal{R}^+_{<j_X} \cap \mathcal{R}^-_{>j_Y}, & \text{if } s(p) = X; \\ \mathcal{L}^+_{<i_Y} \cap \mathcal{L}^-_{>i_X}, & \text{if } t(p) = X. \end{cases}$$

7.1.3 Lemma. *Let $X, Y \in \mathcal{C}_Q$, with $X \prec Y$, be connected by a path p.*

(1) *If $l(p) = 1$, then $\mathcal{W}_{X,Y} = \emptyset$.*

(2) *If* $l(p) > 1$ *and* $s(p) = X$*, then* $\mathcal{W}_{X,Y} = \mathcal{W}_Z$*, where* $Z \in \mathcal{R}$ *with* $(i_{_Z}, j_{_Z}) = (j_{_X} - 1, j_{_Y} + 1)$.

(3) *If* $l(p) > 1$ *and* $t(p) = X$*, then* $\mathcal{W}_{X,Y} = \mathcal{W}_Z$*, where* $Z \in \mathcal{L}$ *with* $(i_{_Z}, j_{_Z}) = (i_{_Y} - 1, i_{_X} + 1)$.

Proof. We shall consider only the case where $t(p) = X$. By definition, $\mathcal{W}_{X,Y}$ lies in $\mathcal{R}$. If $l(p) = 1$, then $(i_{_X}, j_{_X}) = (i_{_Y}, j_{_Y} + 1)$ by Lemma 4.3.2(2). Since $j_{_X} = j_{_Y} + 1$, by Lemma 4.2.16(1), $\mathcal{W}_{X,Y} = \emptyset$. If $l(p) > 1$ then, by Lemma 4.3.3, $i_{_X} = i_{_Y}$ and $j_{_X} > j_{_Y} + 1$. By Lemma 4.2.16(2), $\mathcal{W}_{X,Y} = \mathcal{W}_Z$ with $(i_{_Z}, j_{_Z}) = (j_{_X} - 1, j_{_Y} + 1)$. The proof of the lemma is completed.

7.2 Maximal τ-rigid sets in the fundamental domain

The main objective of this section is to give a method to construct all the maximal τ-rigid sets in the fundamental domain $\mathscr{F}(Q)$. For this purpose, we shall make a frequent use of the double infinite sectional paths L_i and R_j which determine the coordinate system for $\mathcal{C}_Q$. Recall, moreover, that the quasi-simple objects in $\mathcal{L}$ are $S_i = \tau^{i-1} S_L$ with $i \in \mathbb{Z}$ and those in $\mathcal{R}$ are $T_j = \tau^{j-1} T_R$ with $j \in \mathbb{Z}$, where S_L and T_R are as stated in Lemma 7.1.1.

Recall that an object X in $\mathcal{L}$ (respectively, $\mathcal{R}$) determines a wing $\mathcal{W}_X$ in $\mathcal{L}$ (respectively, $\mathcal{R}$) formed by the objects M with $j_{_X} \le j_{_M} \le i_{_M} \le i_{_X}$. Moreover, let $m, n \in \mathbb{Z}$ with $m \le n$, we shall denote by $[m, n]$ the interval of the integers i with $m \le i \le n$. The following is a criterion for two objects from different components forming a τ-rigid pair.

7.2.1 Lemma. *Let* $X \in \mathcal{C}_Q$*,* $M \in \mathcal{L}$*, and* $N \in \mathcal{R}$*.*

(1) (X, M) *is a* τ*-rigid pair if and only if* $S_{i_X} \notin \mathcal{W}_M$ *if and only if* $i_{_X} \notin [j_{_M}, i_{_M}]$.

(2) (X, N) *is a* τ*-rigid pair if and only if* $T_{j_X} \notin \mathcal{W}_N$ *if and only if* $j_{_X} \notin [j_{_N}, i_{_N}]$.

Proof. We shall prove only Statement (1). Since τ is an automorphism of $\mathscr{D}^b(Q)$, in view of Theorem 3.2.2(2), $\mathrm{Hom}_{\mathscr{D}^b(Q)}(M, \tau X) = 0$. Write $i_{_X} = i$, that is,

$X \in L_i$. By Lemma 7.1.1, $\mathrm{Hom}_{\mathscr{D}^b(Q)}(X, \tau M) \neq 0$ if and only if $\tau M \in \mathcal{W}(\tau^i S_L)$ if and only if $M \in \mathcal{W}(\tau^{i-1} S_L) = \mathcal{W}(S_i)$. Therefore, (X, M) is τ-rigid if and only if $M \notin \mathcal{W}(S_i)$. In view of Lemma 4.2.27, the latter is equivalent to $S_i \notin \mathcal{W}_M$. Then our statement follows from Lemma 4.2.7 and Lemma 4.2.1. The proof of the lemma is completed.

The following statement follows immediately from Lemma 7.2.1.

7.2.2 COROLLARY. *Let $X \in \mathcal{C}_Q$, $M \in \mathcal{L}$, and $N \in \mathcal{R}$.*

(1) *If (X, M) is a τ-rigid pair, then (Y, M) is a τ-rigid pair, for any Y with $i_{_Y} = i_{_X}$.*

(2) *If (X, N) is a τ-rigid pair, then (Z, N) is a τ-rigid pair, for any Z with $j_{_Z} = j_{_X}$.*

We start with maximal τ-rigid sets in $\mathscr{F}(Q)$ which contain no objects of the connecting components $\mathcal{C}_Q$.

7.2.3 THEOREM. *Let $\mathcal{T}$ be a set of objects in $\mathscr{F}(Q)$ with $\mathcal{T} \cap \mathcal{C}_Q = \emptyset$. Then $\mathcal{T}$ is maximal τ-rigid in $\mathscr{F}(Q)$ if and only if the following statements are satisfied.*

(1) *$\mathcal{T} \cap \mathcal{L}$ is maximal τ-rigid in $\mathcal{L}$.*

(2) *$\mathcal{T} \cap \mathcal{R}$ is maximal τ-rigid in $\mathcal{R}$.*

(3) *$\mathcal{T} \cap \mathcal{L}$ is dense in $\mathcal{L}$ or $\mathcal{T} \cap \mathcal{R}$ is dense in $\mathcal{R}$.*

Proof. Suppose that $\mathcal{T}$ is maximal τ-rigid in $\mathscr{F}(Q)$. Since $\mathcal{T}$ has no object in $\mathcal{C}_Q$ and $\mathcal{L}, \mathcal{R}$ are orthogonal, $\mathcal{T} \cap \mathcal{L}$ is maximal τ-rigid in $\mathcal{L}$ and $\mathcal{T} \cap \mathcal{R}$ is maximal τ-rigid in $\mathcal{R}$. If Statement (3) is not true, then there is a quasi-simple object S in $\mathcal{L}$ which is not covered by any object in $\mathcal{T} \cap \mathcal{L}$ and a quasi-simple object T in $\mathcal{R}$ which is not covered by any object in $\mathcal{T} \cap \mathcal{R}$. Consider the object $X \in \mathcal{C}_Q$ with $(i_{_X}, j_{_X}) = (i_S, i_T)$. By the definition of the coordinate systems, we have $S_{i_X} = S$ and $T_{j_X} = T$. Let $M \in \mathcal{T} \cap \mathcal{L}$. Since S is not covered by M, that is, $S_{i_X} \notin \mathcal{W}_M$. By Lemma 7.2.1(1), (X, M) is τ-rigid in $\mathscr{F}(Q)$. Therefore, $(\mathcal{T} \cap \mathcal{L}) \cup \{X\}$ is τ-rigid. Similarly, $(\mathcal{T} \cap \mathcal{R}) \cup \{X\}$ is τ-rigid. Hence, $\mathcal{T} \cup \{X\}$ is τ-rigid, which is a contradiction to the τ-rigid maximality of $\mathcal{T}$. Thus, Statement (3) holds. The necessity is established.

Conversely, we assume that $\mathcal{T}\cap\mathcal{L}$ is densely maximal τ-rigid in $\mathcal{L}$, and $\mathcal{T}\cap\mathcal{R}$ is maximal τ-rigid in $\mathcal{R}$. Since $\mathcal{L}$ and $\mathcal{R}$ are orthogonal, $\mathcal{T}$ is τ-rigid. Let $X \in \mathcal{C}_Q$. By the density of $\mathcal{T} \cap \mathcal{L}$ in $\mathcal{L}$, the quasi-simple object $S_{i_X} \in \mathcal{L}$ is covered by some object $M \in \mathcal{T} \cap \mathcal{L}$, that is, $S_{i_X} \in \mathcal{W}_M$. By Lemma 7.2.1(1), (M, X) is not a τ-rigid pair. Therefore, $\mathcal{T}$ is maximal τ-rigid in $\mathscr{F}(Q)$. The proof of the proposition is completed.

For the rest of this section, we shall focus on the τ-rigid sets in $\mathscr{F}(Q)$ which contain some objects in $\mathcal{C}_Q$. Recall that $\mathcal{C}_Q$ is a poset. The following statement decribes the τ-rigid sets in $\mathcal{C}_Q$.

7.2.4 Lemma. *A set of objects in $\mathcal{C}_Q$ is τ-rigid if and only if it is a chain.*

Proof. Since $\mathcal{C}_Q$ is a standard component of $\Gamma_{D^b(\mathrm{rep}(Q))}$ of shape $\mathbb{Z}\mathbb{A}_\infty^\infty$, by Lemma 5.4.1, a pair of objects in $\mathcal{C}_Q$ is τ-rigid if and only if they are comparable. Immediately, our statement holds. The proof of the lemma is completed.

7.2.5 Lemma. *Let $\mathcal{S}$ be a chain in $\mathcal{C}_Q$.*

(1) *If $M \in \mathcal{L}$, then $\mathcal{S} \cup \{M\}$ is τ-rigid if and only if $i_X \notin [j_M, i_M]$ for every $X \in \mathcal{S}$; and in this case, $\mathcal{S} \cup \{L\}$ is τ-rigid for every $L \in \mathcal{W}_M$.*

(2) *If $N \in \mathcal{R}$, then $\mathcal{S} \cup \{N\}$ is τ-rigid if and only if $j_X \notin [j_N, i_N]$ for every $X \in \mathcal{S}$; and in this case, $\mathcal{S} \cup \{L\}$ is τ-rigid for every $L \in \mathcal{W}_N$.*

Proof. We shall prove only Statement (1). Let $M \in \mathcal{L}$. Since $\mathcal{S}$ is τ-rigid, the first part of the statement follows immediately from Lemma 7.2.1(1).

Now, assume that $\mathcal{S}\cup\{M\}$ is τ-rigid. Let $L \in \mathcal{W}_M$. Then $j_M \le j_L \le i_L \le i_M$. In particular, $i_X \notin [j_L, i_L]$ for every $X \in \mathcal{S}$. This in turn implies that $\mathcal{S} \cup \{L\}$ is τ-rigid. The proof of the lemma is completed.

Recall that a τ-rigid set $\mathcal{T}$ of objects in $\mathcal{L}$ (respectively, $\mathcal{R}$) is said to be locally maximal if $\mathcal{T} \cap \mathcal{W}_X$ is a maximal τ-rigid set in $\mathcal{W}_X$, for each $X \in \mathcal{T}$.

7.2.6 Corollary. *Let $\mathcal{T}$ be a maximal τ-rigid set in $\mathscr{F}(Q)$.*

(1) $\mathcal{T} \cap \mathcal{L} = \emptyset$ *if and only if* $\mathcal{T} \cap L_i \neq \emptyset$ *for every* $i \in \mathbb{Z}$; *and otherwise,* $\mathcal{T} \cap \mathcal{L}$ *is locally maximal in* $\mathcal{L}$.

(2) $\mathcal{T} \cap \mathcal{R} = \emptyset$ *if and only if* $\mathcal{T} \cap R_j \neq \emptyset$ *for every* $j \in \mathbb{Z}$; *and otherwise,* $\mathcal{T} \cap \mathcal{R}$ *is locally maximal in* $\mathcal{R}$.

Proof. We shall prove only Statement (1). Assume that $\mathcal{T} \cap L_i \neq \emptyset$ for every $i \in \mathbb{Z}$. That is, for any integer i, there is an object $X \in \mathcal{T} \cap \mathcal{C}_Q$ such that $i_{_X} = i$. Then, by Lemma 7.2.5(1), $\mathcal{T} \cup \{M\}$ is not τ-rigid, for any object $M \in \mathcal{L}$. Therefore, $\mathcal{T} \cap \mathcal{L} = \emptyset$.

Assume, conversely, that $\mathcal{T} \cap \mathcal{L} = \emptyset$ but $\mathcal{T} \cap L_i = \emptyset$ for some integer i. That is, for any $X \in \mathcal{T} \cap \mathcal{C}_Q$, we have $i_{_X} \neq i$. Since $\mathcal{T} \cap \mathcal{C}_Q$ is a chain by Lemma 7.2.4, applying Lemma 7.2.5(1) to the quasi-simple object $S_i \in \mathcal{L}$, we conclude that $(\mathcal{T} \cap \mathcal{C}_Q) \cup \{S_i\}$ is τ-rigid. Moreover, since $\mathcal{R}$ and $\mathcal{L}$ are orthogonal, $(\mathcal{T} \cap \mathcal{R}) \cup \{S_i\}$ is τ-rigid, and consequently, $\mathcal{T} \cup \{S_i\}$ is τ-rigid. This contradiction to the τ-rigid maximality of $\mathcal{T}$ establishes the first part of Statement (1).

Next, assume that $\mathcal{T} \cap \mathcal{L} \neq \emptyset$. Fix $M \in \mathcal{T} \cap \mathcal{L}$. We claim that $\mathcal{T} \cap \mathcal{W}_M$ is maximal τ-rigid in $\mathcal{W}_M$. Indeed, assume that $N \in \mathcal{W}_M$ is such that $(\mathcal{T} \cap \mathcal{W}_M) \cup \{N\}$ is τ-rigid. Let $X \in \mathcal{T} \cap \mathcal{L}$. If $X \in \mathcal{W}_M$, then (N, X) is trivially τ-rigid. Otherwise, since (M, X) is τ-rigid, we deduce from Lemma 5.3.6 that (N, X) is τ-rigid. Hence, $(\mathcal{T} \cap \mathcal{L}) \cup \{N\}$ is τ-rigid. Moreover, since $(\mathcal{T} \cap \mathcal{C}_Q) \cup \{M\}$ is τ-rigid, by Lemma 7.2.5(1), $(\mathcal{T} \cap \mathcal{C}_Q) \cup \{N\}$ is τ-rigid. Since $(\mathcal{T} \cap \mathcal{R}) \cup \{N\}$ is τ-rigid, we conclude that $\mathcal{T} \cup \{N\}$ is τ-rigid. By the τ-rigid maximality, $N \in \mathcal{T}$, that is, $N \in \mathcal{T} \cap \mathcal{W}_M$. This establishes our claim. The proof of the corollary is completed.

The following statement describes when a maximal τ-rigid set in $\mathcal{C}_Q$ is also a maximal τ-rigid set in $\mathscr{F}(Q)$.

7.2.7 Proposition. *A set* $\mathcal{S}$ *of objects in* $\mathcal{C}_Q$ *is a maximal* τ*-rigid set in* $\mathscr{F}(Q)$ *if and only if it is the vertex set of a section in* $\mathcal{C}_Q$ *containing no infinite path.*

Proof. Let $\mathcal{S}$ be a set of objects in $\mathcal{C}_Q$. Assume that $\mathcal{S}$ is a maximal τ-rigid set in $\mathscr{F}(Q)$. In view of Corollary 7.2.6, $\mathcal{S}$ passes through every L_i and every R_j for $i, j \in \mathbb{Z}$. Observe that $\mathcal{S}$ is maximal τ-rigid in $\mathcal{C}_Q$. By Theorem 5.4.3 and Lemma 4.3.20, there is a section Σ in $\mathcal{C}_Q$ containing no infinite path such that

$\mathcal{S} = \Sigma_0$. Conversely, assume that there is a section Σ of $\mathcal{C}_Q$ containing no infinite path such that $\mathcal{S} = \Sigma_0$. By Theorem 5.4.3 , $\mathcal{S}$ is maximal τ-rigid in $\mathcal{C}_Q$; and by Lemma 4.3.20, $\mathcal{S}$ passes every L_i and every R_j for $i, j \in \mathbb{Z}$. That is, $\mathcal{S} \cap L_i \neq \emptyset$ and $\mathcal{S} \cap R_j \neq \emptyset$ for all $i, j \in \mathbb{Z}$. By Lemma 7.2.5, $\mathcal{S} \cup \{M\}$ is not τ-rigid, for any object $M \in \mathcal{L} \cup \mathcal{R}$. Therefore, $\mathcal{S}$ is maximal τ-rigid in $\mathscr{F}(Q)$. The proof of the proposition is completed.

Recall now that a chain $\mathcal{S}$ in $\mathcal{C}_Q$ is called a sectional chain if, every two objects of $\mathcal{S}$, with one being a minimal cover of the other, are connected by a path in $\mathcal{C}_Q$; and if $X, Y \in \mathcal{C}_Q$ with $X \prec Y$ are connected by a path, then they determine a (possibly empty) wing $\mathcal{W}_{X,Y}$ in $\mathcal{L}$ or in $\mathcal{R}$ as shown in Definition 7.1.2.

7.2.8 Lemma. *Let $\mathcal{S}$ be a sectional chain in $\mathcal{C}_Q$, and let $X, Y \in \mathcal{S}$ with Y being a minimal cover of X.*

1. *If $Z \in \mathcal{W}_{X,Y}$, then $\mathcal{S} \cup \{Z\}$ is τ-rigid.*
2. *If $M, N \in \mathcal{S}$ with N being a minimal cover of M such that $N \preceq X$, then (L, Z) is a τ-rigid pair, for any $L \in \mathcal{W}_{M,N}$ and $Z \in \mathcal{W}_{X,Y}$.*

Proof. We shall only consider the case where $\emptyset \neq \mathcal{W}_{X,Y} \subseteq \mathcal{L}$. By Lemma 7.1.3(3), $\mathcal{W}_{X,Y} = \mathcal{W}_U$, where $U \in \mathcal{L}$ with $(i_U, j_U) = (i_Y - 1, i_X + 1)$.

First, we claim that $\mathcal{S} \cup \{U\}$ is τ-rigid. By Lemma 7.2.5(1), it suffices to show that $\mathcal{S}$ contains no object lying in L_i, for every i with $j_U \leq i \leq i_U$. If this was not the case, then there exists some $V \in \mathcal{S}$ such that $i_V = i$ with $j_U \leq i \leq i_U$. In particular, $V \notin \{X, Y\}$. Since $\mathcal{S}$ is a chain, we obtain $V \prec X$ or $V \succ Y$. In the first case, $i = i_V \leq i_X = j_U - 1$; and in the second case, $i = i_V \geq i_Y = i_U + 1$, a contradiction. Our claim is true. Then, by Lemma 7.2.5(1), $\mathcal{S} \cup \{Z\}$ is τ-rigid for every $Z \in \mathcal{W}_U$. Statement (1) is established.

For proving Statement (2), let $M, N \in \mathcal{S}$ with N being a minimal cover of M such that $N \preceq X$. In particular, $i_N \leq i_X$. Assume that $L \in \mathcal{W}_{M,N}$ and $Z \in \mathcal{W}_{X,Y}$. If $\mathcal{W}_{M,N} \subseteq \mathcal{R}$, since $\mathcal{R}, \mathcal{L}$ are orthogonal, (L, Z) trivially is a τ-rigid pair. Assume now that $\mathcal{W}_{M,N} \subseteq \mathcal{L}$. By Lemma 7.1.3(3), $\mathcal{W}_{M,N} = \mathcal{W}_V$, where $V = (i_N - 1, i_M + 1)$. Observe that $i_V = i_N - 1$ and $j_U = i_X + 1$. Since $i_N \leq i_X$, we obtain $i_V \leq j_U + 2$. That is, $\mathcal{W}_V$ and $\mathcal{W}_U$ are separable by Lemma 4.2.8. Thus, for any $L \in \mathcal{W}_{M,N} = \mathcal{W}_V$ and $Z \in \mathcal{W}_{X,Y} = \mathcal{W}_U$, we have $\mathcal{W}_L, \mathcal{W}_Z$ are separable and hence, (L, Z) is a τ-rigid pair by Lemma 5.3.1. The proof of the lemma is completed.

In Proposition 7.2.3, we have characterized the maximal τ-rigid sets of $\mathscr{F}(Q)$ without objects of $\mathcal{C}_Q$; and in Proposition 7.2.7, we have characterized the maximal τ-rigid sets of $\mathscr{F}(Q)$ containing only objects of $\mathcal{C}_Q$. In the following, we shall study general maximal τ-rigid sets of $\mathscr{F}(Q)$.

Given a chain $\mathcal{S}$ in $\mathcal{C}_Q$, recall that the coordinates of the objects of $\mathcal{S}$ determine two integer sets $I_{\mathcal{S}} = \{i_X \mid X \in \mathcal{S}\}$ and $J_{\mathcal{S}} = \{j_X \mid X \in \mathcal{S}\}$. Moreover, given $n \in \mathbb{Z}$, we have $\mathcal{L}^+_{<n} = \bigcup_{i<n} L^+_i$ and $\mathcal{L}^-_{>n} = \bigcup_{j>n} L^-_j$, which are subquivers of $\mathcal{L}$. Similarly, $\mathcal{R}^+_{<n} = \bigcup_{i<n} R^+_i$ and $\mathcal{R}^-_{>n} = \bigcup_{j>n} R^-_j$, which are subquivers of $\mathcal{R}$. The following statement collects some properties of maximal τ-rigid sets in $\mathscr{F}(Q)$, which contains some objects of $\mathcal{C}_Q$.

7.2.9 PROPOSITION. *Let $\mathcal{T}$ be a maximal τ-rigid set in $\mathscr{F}(Q)$ with $\mathcal{T} \cap \mathcal{C}_Q \neq \emptyset$.*

(1) *$\mathcal{T} \cap \mathcal{C}_Q$ is a sectional chain in $\mathcal{C}_Q$.*

(2) *If $\mathcal{W}_{X,Y} \neq \emptyset$ for some $X, Y \in \mathcal{T} \cap \mathcal{C}_Q$ with Y a minimal cover of X, then $\mathcal{T} \cap \mathcal{W}_{X,Y}$ is maximal τ-rigid in $\mathcal{W}_{X,Y}$.*

(3) *If $I_{\mathcal{T}\cap\mathcal{C}_Q}$ has a minimal (respectively, maximal) element i_0, then $\mathcal{T} \cap \mathcal{L}^+_{<i_0}$ (respectively, $\mathcal{T} \cap \mathcal{L}^-_{>i_0}$) is maximal τ-rigid in $\mathcal{L}^+_{<i_0}$ (respectively, $\mathcal{L}^-_{>i_0}$).*

(4) *If $J_{\mathcal{T}\cap\mathcal{C}_Q}$ has a minimal (respectively, maximal) element j_0, then $\mathcal{T} \cap \mathcal{R}^+_{<j_0}$ (respectively, $\mathcal{T} \cap \mathcal{R}^-_{>j_0}$) is maximal τ-rigid in $\mathcal{R}^+_{<j_0}$ (respectively, $\mathcal{R}^-_{>j_0}$).*

Proof. Being τ-rigid, by Lemma 7.2.4, $\mathcal{T} \cap \mathcal{C}_Q$ is a chain in $\mathcal{C}_Q$. Now, assume that $X, Y \in \mathcal{T} \cap \mathcal{C}_Q$, where Y is a minimal cover of X. We need to show that X, Y are connected by a path in $\mathcal{C}_Q$. Since $X \preceq Y$, we have $i_X \le i_Y$ and $j_X \ge j_Y$. If $i_X = i_Y$ or $j_X = j_Y$, we deduce from Lemma 4.3.1 that X, Y lie on L_{i_X} or R_{j_X}. Otherwise, we have $i_X < i_Y$ and $j_X > j_Y$. Consider the object $Z \in \mathcal{C}_Q$ with $(i_Z, j_Z) = (i_X, j_Y)$. Then, $X \prec Z \prec Y$. Since Y is the minimal cover of X in $\mathcal{T} \cap \mathcal{C}_Q$, we see that $Z \notin \mathcal{T}$. We claim that $\mathcal{T} \cup \{Z\}$ is τ-rigid. Since $(\mathcal{T} \cap \mathcal{C}_Q) \cup \{Z\}$ is a chain, by Lemma 7.2.4, it is τ-rigid. Now let $M \in \mathcal{T} \cap \mathcal{L}$. Since (X, M) is τ-rigid with $i_Z = i_X$, by Corollary 7.2.2(1), (Z, M) is τ-rigid. This shows that $(\mathcal{T}\cap\mathcal{L})\cup\{Z\}$ is τ-rigid. Similarly, $(\mathcal{T}\cap\mathcal{R})\cup\{Z\}$ is τ-rigid. This establishes our claim, which contradicts the τ-rigid maximality of $\mathcal{T}$. Statement (1) is established.

For proving Statement (2), let $X, Y\in\mathcal{T}\cap\mathcal{C}_Q$ with Y a minimal cover of X such that $\mathcal{W}_{X,Y}\neq\emptyset$. In view of Lemma 7.1.3(3), we may assume that $\mathcal{W}_{X,Y}=\mathcal{W}_Z$, where $Z\in\mathcal{L}$ with $(i_{_Z}, j_{_Z})=(i_{_Y}-1, i_{_X}+1)$. Since $\mathcal{T}\cap\mathcal{L}$ is locally maximal in $\mathcal{L}$ by Lemma 7.2.6, it suffices to show that $Z\in\mathcal{T}$. Indeed, by Lemma 7.2.8(1), we see that $(\mathcal{T}\cap\mathcal{C}_Q)\cup\{Z\}$ is τ-rigid. Since $\mathcal{L}$ and $\mathcal{R}$ are orthogonal, $(\mathcal{T}\cap\mathcal{R})\cup\{Z\}$ is τ-rigid. Finally, take an object $M\in\mathcal{T}\cap\mathcal{L}$. Since (X, M) and (Y, M) are τ-rigid pairs, by Lemma 7.2.1(1), $i_{_X}, i_{_Y}\notin[j_{_M}, i_{_M}]$. Since $i_{_X}\le i_{_Y}$, we conclude that $i_{_X}<j_{_M}\le i_{_M}<i_{_Y}$, or $i_{_X}>i_{_M}$, or $i_{_Y}<j_{_M}$. In the first case, $j_{_Z}=i_{_X}+1\le j_{_M}\le i_{_M}\le i_{_Y}-1=i_{_Z}$. Then, $\mathcal{W}_M\subseteq\mathcal{W}_Z$, and hence, (M, Z) is τ-rigid by Lemma 5.3.1. In the second case, $j_{_Z}=i_{_X}+1\ge i_{_M}+2$; and in the third case, $i_{_Z}=i_{_Y}-1\le j_{_M}-2$. In both cases, $\mathcal{W}_M, \mathcal{W}_Z$ are separable; and by Lemma 5.3.1, (M, Z) is τ-rigid. This shows that $(\mathcal{T}\cap\mathcal{L})\cup\{Z\}$ is τ-rigid. Consequently, $\mathcal{T}\cup\{Z\}$ is τ-rigid. By the τ-rigid maximality of $\mathcal{T}$, we obtain $Z\in\mathcal{T}$. Statement (2) is established.

For proving Statement (3), we shall consider only the case where $I_{\mathcal{T}\cap\mathcal{C}_Q}$ has a minimal element i_0. Let $N\in\mathcal{L}^+_{<i_0}$ be such that $(\mathcal{T}\cap\mathcal{L}^+_{<i_0})\cup\{N\}$ is τ-rigid. In particular, $i_{_N}<i_0$. We claim that $N\in\mathcal{T}$, or equivalently, $\mathcal{T}\cup\{N\}$ is τ-rigid. Indeed, since $\mathcal{L}$ and $\mathcal{R}$ are orthogonal, $(\mathcal{T}\cap\mathcal{R})\cup\{N\}$ is τ-rigid. Next, let $X\in\mathcal{T}\cap\mathcal{C}_Q$. Since $i_{_X}\in I_{\mathcal{T}\cap\mathcal{C}_Q}$, by the minimality of i_0, we have $i_{_X}\ge i_0>i_{_N}\ge j_{_N}$, and in particular, $i_{_X}\notin[j_{_N}, i_{_N}]$. By Lemma 7.2.5(1), $(\mathcal{T}\cap\mathcal{C}_Q)\cup\{N\}$ is τ-rigid.

Finally, since $i_0\in I_{\mathcal{T}\cap\mathcal{C}_Q}$, we have $i_{_X}=i_0$ for some $X\in\mathcal{T}\cap\mathcal{C}_Q$. Take an object $M\in\mathcal{T}\cap\mathcal{L}$. Since (X, M) is τ-rigid, by Lemma 7.2.1, $i_0\notin[j_{_M}, i_{_M}]$. Then, $i_{_M}<i_0$ or $j_{_M}>i_0$. That is, either $M\in\mathcal{L}^+_{<i_0}$ or $M\in\mathcal{L}^-_{>i_0}$. In the first case, (M, N) is τ-rigid by our assumption. In the second case, by Corollary 5.3.3(2), (M, N) is τ-rigid. This shows that $(\mathcal{T}\cap\mathcal{L})\cup\{N\}$ is τ-rigid. This establishes our claim. By the τ-rigid maximality of $\mathcal{T}$, we have $N\in\mathcal{T}$, and hence, $N\in\mathcal{T}\cap\mathcal{L}^+_{<i_0}$. Therefore, $\mathcal{T}\cap\mathcal{L}^+_{<i_0}$ is maximal τ-rigid in $\mathcal{L}^+_{<i_0}$. By a similar discussion, Statement (4) holds. The proof of the lemma is completed.

The following statement collects of some properties of a maximal τ-rigid set $\mathcal{T}$ in $\mathscr{F}(Q)$ with $\mathcal{T}\cap\mathcal{C}_Q$ a double infinite chain.

7.2.10 Lemma. *Let $\mathcal{T}$ be a maximal τ-rigid set in $\mathscr{F}(Q)$ such that $\mathcal{T} \cap \mathcal{C}_Q$ is a double infinite chain.*

(1) *If $\mathcal{T} \cap L_i$ has a minimal (respectively, maximal) element , for some $i \in \mathbb{Z}$, then $\mathcal{T} \cap L^+_{i-1}$ (respectively, $\mathcal{T} \cap L^-_{i+1}$) is finite.*

(2) *If $\mathcal{T} \cap R_i$ has a minimal (respectively, maximal) element , for some $i \in \mathbb{Z}$, then $\mathcal{T} \cap R^-_{i+1}$ (respectively, $\mathcal{T} \cap R^+_{i-1}$) is finite.*

(3) *If $\mathcal{T} \cap L_i$ and $\mathcal{T} \cap R_j$ are both finite, for all $i, j \in \mathbb{Z}$, then $I_{\mathcal{T}\cap\mathcal{C}_Q}$ has neither minimal nor maximal element and $J_{\mathcal{T}\cap\mathcal{C}_Q}$ has neither minimal nor maximal element.*

Proof. We shall only prove Statements (1) and (3). Assume that $\mathcal{T} \cap L_i$ has a minimal element X, for some $i \in \mathbb{Z}$. Obviously, $\mathcal{T} \cap L_i \subseteq \mathcal{T} \cap \mathcal{C}_Q$. Being a double infinite chain, $\mathcal{T} \cap \mathcal{C}_Q$ contains an object M such that $M \prec X$. That is, $i_{_M} \le i_{_X} = i$. Since X is the minimal element in $\mathcal{T} \cap L_i$, we have $i_{_M} \le i - 1$. Assume that $\mathcal{T} \cap L^+_{i-1}$ is infinite. Then it contains an object Z with $i_{_Z} = i - 1$ and $j_{_Z} < i_{_M}$. Thus, $i_{_M} \in [j_{_Z}, i_{_Z}]$. By Lemma 7.2.1, (M, Z) is not a τ-rigid pair, a contradiction. Hence, $\mathcal{T} \cap L^+_{i-1}$ is finite. Similarly, if $\mathcal{T} \cap L_i$ has a maximal element, then $\mathcal{T} \cap L^-_{i+1}$ is finite. This establishes Statement (1).

For proving Statement (3), assume that $\mathcal{T} \cap L_i$ and $\mathcal{T} \cap R_j$ are both finite, for all $i, j \in \mathbb{Z}$. We shall consider only the case that $I_{\mathcal{T}\cap\mathcal{C}_Q}$ has a minimal element i_0. Since $\mathcal{T} \cap L_{i_0}$ is finite, it has a minimal element N. Now let $X \in \mathcal{T} \cap \mathcal{C}_Q$ such that $X \preceq N$. Then $i_{_X} \le i_{_N} = i_0$. By the minimality of i_0, we see that $i_{_X} = i_0$. That is, $X \in L_{i_0}$. By the minimality of N, we see that $X = N$. Thus, N is a minimal element in $\mathcal{T} \cap \mathcal{C}_Q$ which is a contradicts our assumption that $\mathcal{T} \cap \mathcal{C}_Q$ is a double infinite chain. Thus, Statement (3) is established. The proof of the lemma is completed.

In view of Proposition 7.2.9(1), given a maximal τ-rigid set $\mathcal{T}$ in $\mathscr{F}(Q)$, if $\mathcal{T} \cap \mathcal{C}_Q \neq \emptyset$, then it is a sectional chain in $\mathcal{C}_Q$. Next we shall extend a sectional chain in $\mathcal{C}_Q$ to a maximal τ-rigid set in $\mathscr{F}(Q)$. We start with the following definition.

7.2.11 DEFINITION. Let $\mathcal{S} = \{X_n\}_{n\in I}$ be a sectional chain in $\mathcal{C}_Q$, where I is some interval of $\mathbb{Z}$ such that $X_n \prec X_{n+1}$ for every non-maximal integer $n \in I$. Let Θ_n be a maximal τ-rigid set in $\mathcal{W}_{X_n,X_{n+1}}$, for every non-maximal $n \in I$. By Lemma 7.2.8(2), the set

$$\Theta = \bigcup_{n,n+1\in I} \Theta_n$$

is τ-rigid. Moreover, by Lemma 7.2.8(1), $\mathcal{S} \cup \Theta$ is τ-rigid. Therefore, we shall call Θ an *addend* to $\mathcal{S}$ in $\mathscr{F}(Q)$.

Observe that $\mathcal{S} \cup \Theta$ is not necessarily a maximal τ-rigid set in $\mathscr{F}(Q)$, as shown below.

7.2.12 EXAMPLE. Let

$$\mathcal{S} : \cdots \prec X_{n-1} \prec X_n \prec X_{n+1} \prec \cdots$$

be a chain in $\mathcal{C}_Q$ contained in the double infinite section path R_0 such that $(i_{X_n}, j_{X_n}) = (2n, 0)$ for $n \in \mathbb{Z}$. Obviously, $\mathcal{S}$ is a sectional chain. By definition, $\mathcal{W}_{X_n,X_{n+1}} = W_Z$, where $Z = (2n+1, 2n+1)$ is the quasi-simple object S_{2n+1} in $\mathcal{L}$, for every $n \in \mathbb{Z}$. That is, the addend Θ to $\mathcal{S}$ is the set

$$\{\cdots, S_{-1}, S_1, S_3, \cdots, S_{2n-1} \cdots\}.$$

Since $\mathcal{S} \cup \Theta \cup \{T_1\}$ is τ-rigid, where T_1 is the quasi-simple object in $\mathcal{R}$, we see that $\mathcal{S} \cup \Theta$ is not maximal τ-rigid in $\mathscr{F}(Q)$.

The following result shows some properties of an addend to a sectional chain in $\mathscr{F}(Q)$.

7.2.13 LEMMA. *Let $\mathcal{S}$ be a sectional chain in $\mathcal{C}_Q$, and let Θ be an addend to $\mathcal{S}$ in $\mathscr{F}(Q)$.*

(1) *If $I_{\mathcal{S}}$ has a smallest (respectively, largest) integer i_0, then $\Theta \cap \mathcal{L}$ is contained in $\mathcal{L}^-_{>i_0}$ (repectively, $\mathcal{L}^+_{<i_0}$).*

(2) *If $J_{\mathcal{S}}$ has a smallest (respectively, largest) integer j_0, then $\Theta \cap \mathcal{R}$ is contained in $\mathcal{R}^-_{>j_0}$ (repectively, $\mathcal{R}^+_{<j_0}$).*

Proof. We shall only prove Statement (1) in case $I_{\mathcal{S}}$ has a smallest integer i_0. Take an object $M \in \Theta \cap \mathcal{L}$. By definition, $M \in \mathcal{W}_{X,Y}$ for some $X, Y \in \mathcal{S}$ with Y a minimal cover of X. By Lemma 7.1.3(3), $\mathcal{W}_{X,Y} = \mathcal{W}_Z$, where $Z \in \mathcal{L}$ with $(i_{_Z}, j_{_Z}) = (i_{_Y} - 1, i_{_X} + 1)$. Since $M \in \mathcal{W}_Z$ and $i_{_X} \in I_{\mathcal{S}}$, we have $j_{_M} \geq j_{_Z} = i_{_X} + 1 > i_0$. That is, $M \in \mathcal{L}^-_{>i_0}$. This shows that $\Theta \cap \mathcal{L} \subseteq \mathcal{L}^-_{>i_0}$. The proof of the lemma is completed.

Let $\mathcal{S}$ be a sectional chain in $\mathcal{C}_Q$, and let Θ be an addend to $\mathcal{S}$ in $\mathscr{F}(Q)$. The following lemma shows how to complete $\mathcal{S} \cup \Theta$ to a maximal τ-rigid set in $\mathscr{F}(Q)$.

7.2.14 Lemma. *Let $\mathcal{S}$ be a sectional chain in $\mathcal{C}_Q$, and let Θ be an addend to $\mathcal{S}$ in $\mathscr{F}(Q)$. If $M \in \mathscr{F}(Q) \backslash (\mathcal{S} \cup \Theta)$, then $\mathcal{S} \cup \Theta \cup \{M\}$ is τ-rigid if and only if one of the following statements holds.*

(1) *$M \in \mathcal{C}_Q$ is either a lower bound or an upper bound for $\mathcal{S}$.*

(2) *$M \in \mathcal{L}$ is such that either $i_{_M} < i$ for all $i \in I_{\mathcal{S}}$ or $j_{_M} > i$ for all $i \in I_{\mathcal{S}}$.*

(3) *$M \in \mathcal{R}$ is such that either $i_{_M} < i$ for all $j \in J_{\mathcal{S}}$ or $j_{_M} > j$, for all $j \in J_{\mathcal{S}}$.*

Proof. Fix an object $M \in \mathscr{F}(Q) \backslash (\mathcal{S} \cup \Theta)$. Suppose first that $\mathcal{S} \cup \Theta \cup \{M\}$ is τ-rigid. Firstly, assume that $M \in \mathcal{C}_Q$. Being τ-rigid, $\mathcal{S} \cup \{M\}$ is a chain by Lemma 7.2.4. We shall need to show that Statement (1) holds. Indeed, suppose on the contrary that M is neither a lower bound nor an upper bound for $\mathcal{S}$. Since $\mathcal{S}$ is interval-finite, there are $X, Y \in \mathcal{S}$ with Y being the minimal cover of X such that $X \prec M \prec Y$. By the assumption, X, Y are connected by a path p. Consider first the case that $t(p) = X$. By Lemma 4.3.9, p is the unique sectional path in $\mathcal{C}_Q$ from Y to X. Then, by Lemma 4.3.4, $i_{_X} < i_{_Y}$ and $j_{_X} = j_{_Y}$. By the definition of the partial order in $\mathcal{C}_Q$, we have $i_{_X} < i_{_M} < i_{_Y}$ and $j_{_X} = j_{_M} = j_{_Y}$. Therefore, M lies on p by Lemma 4.3.4, In particular, $l(p) > 1$. By Lemma 7.1.3(3), $\mathcal{W}_{X,Y} = \mathcal{W}_Z$, where $Z \in \mathcal{L}$ with $(i_Z, j_{_Z}) = (i_{_Y} - 1, i_{_X} + 1)$. In view of Lemma 5.1.4 and Lemma 5.2.6(2), Z lies in every maximal τ-rigid set in $\mathcal{W}_Z$. In view of the definition of Θ, we see that $Z \in \Theta$. Since $j_{_Z} = i_{_X} + 1 \leq i_{_M} \leq i_{_Y} - 1 = i_{_Z}$, by Lemma 7.2.1, (M, Z) is not τ-rigid, a contradiction to $\Theta \cup \{M\}$ being τ-rigid. Similarly, we will obtain a contradiction if $s(p) = X$. This establishes Statement (1).

Secondly, assume that $M \in \mathcal{L}$. We need to show that Statement (2) holds. Suppose on the contrary that there exist some $i_0, i_1 \in I_{\mathcal{S}}$ such that $i_{_M} \geq i_0$ and $j_{_M} \leq i_1$. Since $\mathcal{S} \cup \{M\}$ is τ-rigid, by Lemma 7.2.5, i does not lie in the interval $[j_{_M}, i_{_M}]$, for every $i \in I_{\mathcal{S}}$. In particular, $i_0, i_1 \notin [j_{_M}, i_{_M}]$, and therefore, $i_0 < j_{_M} \leq i_{_M} < i_1$. Moreover, we may assume that i_0 is the largest integer in $I_{\mathcal{S}}$ such that $i_0 < j_{_M}$ and i_1 is the smallest integer in $I_{\mathcal{S}}$ such that $i_{_M} < i_1$. Then, i_1 is the minimal cover of i_0 in $I_{\mathcal{S}}$. By Lemma 4.3.16(1), there exist $X, Y \in \mathcal{S}$ with Y being a minimal cover of X such that $i_{_X} = i_0$ and $i_{_Y} = i_1$. Let p be a path in $\mathcal{C}_Q$ between X and Y. Since X, Y are comparable, Lemma 4.3.9, p is sectional. Since $i_{_X} < i_{_Y}$, by Lemma 4.3.4(2), $s(p) = Y$. Therefore, $\mathcal{W}_{X,Y}$ is contained in $\mathcal{L}$. By Lemma 7.1.3(3), $\mathcal{W}_{X,Y} = \mathcal{W}_Z$ with $(i_{_Z}, j_{_Z}) = (i_1 - 1, i_0 + 1)$. Since $i_0 + 1 \leq j_{_M} \leq i_{_M} \leq i_1 - 1$, we have $M \in \mathcal{W}_{X,Y}$. By definition, Θ contains a maximal τ-rigid set $\Theta_{X,Y}$ in $\mathcal{W}_{X,Y}$. In particular, $\Theta_{X,Y} \cup \{M\}$ is τ-rigid. By the τ-rigid maximality of $\Theta_{X,Y}$, we have $M \in \Theta_{X,Y}$. In particular, $M \in \Theta$. This is a contradiction. Hence, Statement (2) holds. Similarly, if $M \in \mathcal{R}$, then Statement (3) holds. The necessity is established.

Next, we shall prove the sufficiency, that is, each of the three statements stated in the lemma implies that $\mathcal{S} \cup \Theta \cup \{M\}$ is τ-rigid. Assume first that Statement (1) is true. Then $M \in \mathcal{C}_Q$, which is either a lower bound or an upper bound for $\mathcal{S}$. We shall consider only the case where the first situation occurs. In view of Lemma 4.3.10, $\mathcal{S}$ has a minimal element X. By the definition of the partial order in $\mathcal{C}_Q$, we deduce that $i_{_X}$ is the smallest integer in $I_{\mathcal{S}}$ and $j_{_X}$ is the largest integer in $J_{\mathcal{S}}$. Since $M \prec X$, we have $i_{_M} \leq i_{_X}$ and $j_{_M} \geq j_{_X}$. Take $N \in \Theta$. Assume that $N \in \mathcal{L}$. By Lemma 7.2.13(1), $N \in \mathcal{L}^-_{>i_X}$. Observe that $i_{_N} \geq j_{_N} > i_{_X}$. That is, $i_{_M} \notin [j_{_N}, i_{_N}]$. By Lemma 7.2.1(1), (M, N) is τ-rigid. In case $N \in \mathcal{R}$, we similarly have (M, N) is τ-rigid. This shows that $\Theta \cup \{M\}$ is τ-rigid. Furthermore, since M is lower bound of $\mathcal{S}$, the set $\mathcal{S} \cup \{M\}$ is a chain in $\mathcal{C}_Q$. By Lemma 7.2.4, $\mathcal{S} \cup \{M\}$ is τ-rigid. This proves that $\mathcal{S} \cup \Theta \cup \{M\}$ is τ-rigid.

Assume now that Statement (2) holds. Then $M \in \mathcal{L}$ such that either $i_{_M} < i$ for all $i \in I_{\mathcal{S}}$ or $j_{_M} > i$ for all $i \in I_{\mathcal{S}}$. We shall consider only the case where the first situation occurs. In particular, $[j_{_M}, i_{_M}] \cap I_{\mathcal{S}} = \emptyset$ and $I_{\mathcal{S}}$ has a smallest integer i_0. That is, $\mathcal{S} \cap L_i = \emptyset$ for every $i \in [j_{_M}, i_{_M}]$. By Lemma 7.2.5(1), $\mathcal{S} \cup \{M\}$ is τ-rigid. Next, take $N \in \Theta$. If $N \in \mathcal{R}$, since $\mathcal{L}, \mathcal{R}$ are orthogonal, then (M, N) is τ-rigid. Otherwise, $N \in \mathcal{L}$. Since $I_{\mathcal{S}}$ has minimal element i_0, by

Lemma 7.2.13(1), $\mathbb{N} \in \mathcal{L}^-_{>i_0}$. Since $i_{_M} < i_0$, we have $M \in \mathcal{L}^+_{<i_0}$. By Corollary 5.3.3(2), (M, N) is τ-rigid. This shows that $\mathcal{S} \cup \Theta \cup \{M\}$ is τ-rigid. Finally, in case Statement (3) holds, we may show in a similar fashion that $\mathcal{S} \cup \Theta \cup \{M\}$ is τ-rigid. The sufficiency is established. The proof of the lemma is completed.

Recall that a set of object in $\mathcal{C}_Q$ is a section-generator of $\mathcal{C}_Q$ if its convex hull in $\mathcal{C}_Q$ is a section in $\mathcal{C}_Q$.

7.2.15 Proposition. *Let $\mathcal{S}$ be a sectional chain in $\mathcal{C}_Q$, and let Θ be an addend to $\mathcal{S}$ in $\mathscr{F}(Q)$. Then $\mathcal{S} \cup \Theta$ is a maximal τ-rigid set in $\mathscr{F}(Q)$ if and only if $\mathcal{S}$ is a section-generator of $\mathcal{C}_Q$ such that its convex hull has no infinite path.*

Proof. Let Σ be the convex hull of $\mathcal{S}$ in $\mathcal{C}_Q$. Assume that $\mathcal{S} \cup \Theta$ is a maximal τ-rigid set in $\mathscr{F}(Q)$. We claim that neither of $I_\mathcal{S}$ and $J_\mathcal{S}$ has an upper or lower bound. Otherwise, assume first that $I_\mathcal{S}$ has a smallest integer i_0. Choose some object $M \in \mathcal{L}$ with $i_{_M} \le i_0$. In particular, $j_{_M} \le i_0$. By Lemma 7.2.13(1), $M \notin \Theta$. By Lemma 7.2.14(2), $\mathcal{S} \cup \Theta \cup \{M\}$ is τ-rigid. This contradicts the τ-rigid maximality of $\mathcal{S} \cup \Theta$. Thus, $I_\mathcal{S}$ has no lower bound. Similarly, we could show that $I_\mathcal{S}$ has no upper bound. In a similar fashion, we can show that $J_\mathcal{S}$ has neither an upper bound nor a lower bound. By Proposition 4.3.24, Σ is a section of $\mathcal{C}_Q$ having no infinite path. In particular, $\mathcal{S}$ is a section-generator of $\mathcal{C}_Q$.

Now assume that $\mathcal{S}$ is a section-generator of $\mathcal{C}_Q$ and Σ is a section in $\mathcal{C}_Q$ containing no infinite path. By Proposition 4.3.23, $\mathcal{S}$ is a double infinite chain. Moreover, by Proposition 4.3.24, neither of $I_\mathcal{S}$ and $J_\mathcal{S}$ has an upper or lower bound. Suppose that $\mathcal{S} \cup \Theta$ is not maximal τ-rigid in $\mathscr{F}(Q)$. Then, there exists an object $M \in \mathscr{F}(Q)\backslash(\mathcal{S} \cup \Theta)$ such that $\mathcal{S} \cup \Theta \cup \{M\}$ is τ-rigid. If $M \in \mathcal{C}_Q$, then Lemma 7.2.14(1), M is a lower bound or an upper bound of $\mathcal{S}$, contradiction to the fact that $\mathcal{S}$ is double infinite. If $M \in \mathcal{L}$ then, by Lemma 7.2.14(2), $i_{_M}$ is a lower bound or an upper bound of $I_\mathcal{S}$, contradiction. If $M \in \mathcal{R}$, then, by Lemma 7.2.14(3), $j_{_M}$ is a lower bound or an upper bound of $J_\mathcal{S}$, a contradiction again. The proof of the proposition is completed.

The following definitions will be used in our main result of this section.

7.2.16 Definition. Let $\mathcal{S}$ be a chain in $\mathcal{C}_Q$.

(1) Let $\mathcal{L}^+_{<I_\mathcal{S}} = \mathcal{L}^+_{<i_0}$ in case $I_\mathcal{S}$ has a smallest integer i_0; and otherwise, the empty set. Moreover, let $\mathcal{L}^-_{>I_\mathcal{S}} = \mathcal{L}^-_{>i_1}$ in case $I_\mathcal{S}$ has a largest integer i_1; and otherwise, the empty set.

(2) Let $\mathcal{R}^+_{<J_\mathcal{S}} = \mathcal{R}^+_{<j_0}$ in case $J_\mathcal{S}$ has a smallest integer j_0; and otherwise, the empty set. Moreover, let $\mathcal{R}^-_{>J_\mathcal{S}} = \mathcal{R}^-_{>j_1}$ in case $J_\mathcal{S}$ has a largest integer j_1; and otherwise, the empty set.

Fix an integer i. Let Φ be a set of objects in $\mathcal{L}^+_{<i}$ (respectively, $\mathcal{L}^-_{>i}$). Recall that Φ is dense in $\mathcal{L}^+_{<i}$ (respectively, $\mathcal{L}^-_{>i}$) if, for any $M \in \mathcal{L}^+_{<i}$ (respectively, $\mathcal{L}^-_{>i}$), there exists $N \in \Phi$ such that $\mathcal{W}_M \subseteq \mathcal{W}_N$. We have the same fashion for $\mathcal{R}^+_{<i}$ and $\mathcal{R}^-_{>i}$.

The following is the main result of this chapter, which gives a description of all the maximal τ-rigid sets in $\mathscr{F}(Q)$ containing some objects in $\mathcal{C}_Q$.

7.2.17 Theorem. *Let $\mathcal{T}$ be a set of objects of $\mathscr{F}(Q)$. Then $\mathcal{T}$ is a maximal τ-rigid set in $\mathscr{F}(Q)$ with $\mathcal{T} \cap \mathcal{C}_Q \neq \emptyset$ if and only if there is a sectional chain $\mathcal{S}$ in $\mathcal{C}_Q$ such that*

$$\mathcal{T} = \mathcal{S} \cup \Theta \cup \Phi_\mathcal{L}^{<I_\mathcal{S}} \cup \Phi_\mathcal{L}^{>I_\mathcal{S}} \cup \Phi_\mathcal{R}^{<J_\mathcal{S}} \cup \Phi_\mathcal{R}^{>J_\mathcal{S}},$$

where

(1) *Θ is an addend to $\mathcal{S}$ in $\mathscr{F}(Q)$;*

(2) *$\Phi_\mathcal{L}^{<I_\mathcal{S}}$ is a maximal τ-rigid set in $\mathcal{L}^+_{<I_\mathcal{S}}$, which is dense in $\mathcal{L}^+_{<I_\mathcal{S}}$ in case $\mathcal{S}$ has a minimal element;*

(3) *$\Phi_\mathcal{L}^{>I_\mathcal{S}}$ is a maximal τ-rigid set in $\mathcal{L}^-_{>I_\mathcal{S}}$, which is dense in $\mathcal{L}^-_{>I_\mathcal{S}}$ in case $\mathcal{S}$ has a maximal element;*

(4) *$\Phi_\mathcal{R}^{<J_\mathcal{S}}$ is a maximal τ-rigid set in $\mathcal{R}^+_{<J_\mathcal{S}}$, which is dense in $\mathcal{R}^+_{<J_\mathcal{S}}$ in case $\mathcal{S}$ has a maximal element;*

(5) *$\Phi_\mathcal{R}^{>J_\mathcal{S}}$ is a maximal τ-rigid set in $\mathcal{R}^-_{>J_\mathcal{S}}$, which is dense in $\mathcal{R}^-_{>J_\mathcal{S}}$ in case $\mathcal{S}$ has a minimal element.*

Proof. Assume first that $\mathcal{S}$ is a sectional chain in $\mathcal{C}_Q$ such that

$$\mathcal{T} = \mathcal{S} \cup \Theta \cup \Phi_{\mathcal{L}}^{<I_\mathcal{S}} \cup \Phi_{\mathcal{L}}^{>I_\mathcal{S}} \cup \Phi_{\mathcal{R}}^{<J_\mathcal{S}} \cup \Phi_{\mathcal{R}}^{>J_\mathcal{S}}$$

as stated in the theorem. Consider an object $M \in \Phi_{\mathcal{L}}^{<I_\mathcal{S}}$. Then, $I_\mathcal{S}$ has a minimal element i_0 with $\mathcal{L}^+_{<I_\mathcal{S}} = \mathcal{L}^+_{<i_0}$. By Lemma 7.2.13(1), $\Theta \cap \mathcal{L} \subseteq \mathcal{L}^-_{>i_0}$. In particular, $M \notin \Theta \cap \mathcal{L}$, and hence, $M \notin \mathcal{S} \cup \Theta$. Since $i_{_M} < i_0 \le i$ for all $i \in I_\mathcal{S}$, by Lemma 7.2.14(2), $\mathcal{S} \cup \Theta \cup \{M\}$ is τ-rigid. That is, (M, N) is τ-rigid, for any $N \in \mathcal{S} \cup \Theta$. Now, assume that $N \in \Phi_{\mathcal{L}}^{>I_\mathcal{S}}$. Then, $I_\mathcal{S}$ has a a maximal element i_1 with $\mathcal{L}^-_{>I_\mathcal{S}} = \mathcal{L}^-_{>i_1}$. Since $i_0 \le i_1$, we have $N \in \mathcal{L}^-_{>i_1} \subseteq \mathcal{L}^-_{>i_0}$. Since $M \in \Phi_{\mathcal{L}}^{<I_\mathcal{S}} \subseteq \mathcal{L}^+_{<i_0}$, by Corollary 5.3.3(2), (M, N) is τ-rigid. Since $\mathcal{L}, \mathcal{R}$ are orthogonal, (M, N) is τ-rigid, for every $N \in \Phi_{\mathcal{R}}^{>J_\mathcal{S}} \cup \Phi_{\mathcal{R}}^{<J_\mathcal{S}}$. Since $\Phi_{\mathcal{L}}^{<I_\mathcal{S}}$ is τ-rigid, we have shown that (M, N) is τ-rigid, for every $N \in \mathcal{T}$. If $M \in \Phi_{\mathcal{L}}^{<I_\mathcal{S}} \cup \Phi_{\mathcal{R}}^{>J_\mathcal{S}} \cup \Phi_{\mathcal{R}}^{<J_\mathcal{S}}$, using a similar argument, we may show that (M, N) is τ-rigid, for every N in $\mathcal{T}$. This shows that $\mathcal{T}$ is τ-rigid in $\mathscr{F}(Q)$.

For proving the maximal τ-rigidity of $\mathcal{T}$ in $\mathscr{F}(Q)$, let $M \in \mathscr{F}(Q)$ be such that $\mathcal{T} \cup \{M\}$ is τ-rigid. We shall show that $M \in \mathcal{T}$. Suppose that $M \notin \mathcal{S} \cup \Theta$. In view of Lemma 7.2.14, we need to consider three possibilities. Assume first that $M \in \mathcal{C}_Q$, which is either a lower or upper bound of $\mathcal{S}$. We shall consider only the case where M is a lower bound of $\mathcal{S}$. In particular, $\mathcal{S}$ has a minimal element X. Then, $i_{_X}$ is the minimal element $I_\mathcal{S}$ and $j_{_X}$ is the maximal element in $J_\mathcal{S}$. By Statement (2), $\Phi_{\mathcal{L}}^{<I_\mathcal{S}}$ is dense in $\mathcal{L}^+_{<I_\mathcal{S}} = \mathcal{L}^+_{<i_X}$, and by Statement (5), $\Phi_{\mathcal{R}}^{>J_\mathcal{S}}$ is dense in $\mathcal{R}^-_{>I_\mathcal{S}} = \mathcal{R}^-_{>j_X}$. Since $M \prec X$, we have $i_{_M} < i_{_X}$ or $j_{_M} > j_{_X}$. Assume first that $i_{_M} < i_{_X}$. Consider the quasi-simple object $S \in \mathcal{L}$ with $(i_{_S}, j_{_S}) = (i_{_M}, i_{_M})$. Since $i_{_S} = i_{_M} < i_{_X}$, we have $S \in \mathcal{L}^+_{<i_X}$. Since $\Phi_{\mathcal{L}}^{<I_\mathcal{S}}$ is dense in $\mathcal{L}^+_{<i_X}$, there exists some $N \in \Phi_{\mathcal{L}}^{<I_\mathcal{S}}$ such that $S \in \mathcal{W}_N$. Then, $i_{_M} = i_{_S} \in [j_{_N}, i_{_N}]$. By Lemma 7.2.1(1), (M, N) is not τ-rigid, a contradiction. In case $j_{_M} > j_{_X}$, a dual argument will yield a contradiction.

Assume now that the second case stated in Lemma 7.2.14 occurs, that is, $M \in \mathcal{L}$ such that $i_{_M} < i$ for all $i \in I_\mathcal{S}$ or $j_{_M} > i$ for all $i \in I_\mathcal{S}$. We shall consider only the first case. Then, $I_\mathcal{S}$ has a minimal integer i_0. Since $i_{_M} < i_0$, we have $M \in \mathcal{L}^+_{<i_0} = \mathcal{L}^+_{<I_\mathcal{S}}$. Since $\Phi_{\mathcal{L}}^{<I_\mathcal{S}}$ is maximal τ-rigid in $\mathcal{L}^+_{<I_\mathcal{S}}$ and $\Phi_{\mathcal{L}}^{<I_\mathcal{S}} \cup \{M\}$ is τ-rigid by our assumption, we obtain $M \in \Phi_{\mathcal{L}}^{<I_\mathcal{S}}$. Similarly, if the third case stated in Lemma 7.2.14 occurs, then $M \in \Phi_{\mathcal{R}}^{>J_\mathcal{S}}$. This establishes the sufficiency.

Conversely, assume that $\mathcal{T}$ is a maximal τ-rigid set in $\mathscr{F}(Q)$ with $\mathcal{T} \cap \mathcal{C}_Q \neq \emptyset$.

By Lemma 7.2.9(1), $\mathcal{S} = \mathcal{T} \cap \mathcal{C}_Q$ is a sectional chain in $\mathcal{C}_Q$. We may write $\mathcal{S} = \{X_n\}_{n\in\mathcal{I}}$, where $\mathcal{I}$ is an interval of $\mathbb{Z}$, such that $X_n \prec X_{n+1}$, for any non-maximal $n \in \mathcal{I}$. Given a non-maximal integer $n \in \mathcal{I}$, by Lemma 7.2.9(2), $\Theta_{X_n,X_{n+1}} = \mathcal{T} \cap \mathcal{W}_{X_n,X_{n+1}}$ is a maximal τ-rigid set in $\mathcal{W}_{X_n,X_{n+1}}$. By definition, $\Theta = \cup_{n,n+1\in\mathcal{I}} \Theta_{X_n,X_{n+1}}$ is an addend to $\mathcal{S}$ in $\mathscr{F}(Q)$. In view of Definition 7.2.16 and Lemma 7.2.9(3), we see that $\Phi_{\mathcal{L}}^{<I_\mathcal{S}} = \mathcal{T} \cap \mathcal{L}^+_{<I_\mathcal{S}}$ is a maximal τ-rigid set in $\mathcal{L}^+_{<I_\mathcal{S}}$. We shall verify the second part of Statement (2). Assume that $\mathcal{S}$ has a minimal element X. Then, $\mathcal{L}^+_{<I_\mathcal{S}} = \mathcal{L}^+_{<i_X}$ with i_X the minimal element in $I_\mathcal{S}$. By Proposition 5.3.9(1), $\Phi_{\mathcal{L}}^{<I_\mathcal{S}}$ contains infinitly many objects of the ray $L^+_{i_0}$ for some $i_0 < i_X$. Suppose on the contrary that $\Phi_{\mathcal{L}}^{<I_\mathcal{S}}$ is not dense in $\mathcal{L}^+_{<i_X}$. Then, $i_0 < i_X - 1$ by Lemma 4.2.20. Consider the quasi-simple object $S \in \mathcal{L}$ with $(i_S, j_S) = (i_0+1, i_0+1)$. We claim that S has no cover in $\mathcal{T}\cap\mathcal{L}$. Indeed, let $N \in \mathcal{L}$ be such that $S \prec N$. Then $j_N \le j_S = i_0 + 1 = i_S \le i_N$. That is, $N \notin \mathcal{L}^+_{<i_0+1}$ and $N \notin \mathcal{L}^-_{>i_0+1}$. By Lemma 5.3.4(1), $\Phi_{\mathcal{L}}^{<I_\mathcal{S}} \cup \{N\}$ is not τ-rigid. Since $\Phi_{\mathcal{L}}^{<I_\mathcal{S}} \subseteq \mathcal{T}$, we have $\mathcal{T} \cup \{N\}$ is not τ-rigid. In particular, $N \notin \mathcal{T}$, that is, $N \notin \mathcal{T} \cap \mathcal{L}$. Thus, our claim is true. Therefore, $i_0 + 1 \notin [j_M, i_M]$ for every $M \in \mathcal{T}\cap\mathcal{L}$. Consider now $Z \in \mathcal{C}_Q$ with $(i_Z, j_Z) = (i_0 + 1, j_X)$. Since $i_Z = i_0 + 1 < i_X$ and $j_Z = j_X$, we have $Z \prec X$. Since X is the minimal element in $\mathcal{S}$, we see that $Z \notin \mathcal{S}$, and hence, $Z \notin \mathcal{T}$. Observing that $\mathcal{S}\cup\{Z\}$ is a chain, by Lemma 7.2.14(1), $\mathcal{S}\cup\{Z\}$ is τ-rigid. Since $i_Z = i_0+1 \notin [j_M, i_M]$ for every $M \in \mathcal{T}\cap\mathcal{L}$, we deduce from Lemma 7.2.1(1) that $(\mathcal{T} \cap \mathcal{L}) \cup \{Z\}$ is τ-rigid. Furthermore, since $(\mathcal{T} \cap \mathcal{R}) \cup \{X\}$ is τ-rigid and $j_X = j_Z$, we conclude from by Corollary 7.2.2(2) that $(\mathcal{T} \cap \mathcal{R}) \cup \{Z\}$ is τ-rigid. As a consequence, $\mathcal{T} \cup \{Z\}$ is τ-rigid. Since $Z \notin \mathcal{T}$, we obtain a contradiction to the maximality of $\mathcal{T}$. Thus, Statement (2) holds.

Similarly, $\Phi_{\mathcal{L}}^{>I_\mathcal{S}} = \mathcal{T} \cap \mathcal{L}^-_{>I_\mathcal{S}}$ verifies Statement (3), $\Phi_{\mathcal{R}}^{<J_\mathcal{S}} = \mathcal{T} \cap \mathcal{R}^+_{<J_\mathcal{S}}$ verifies Statement (4), and $\Phi_{\mathcal{L}}^{>I_\mathcal{S}} = \mathcal{T} \cap \mathcal{R}^-_{<J_\mathcal{S}}$ verifies Statement (5). Thus,

$$\mathcal{T}' = \mathcal{S} \cup \Theta \cup \Phi_{\mathcal{L}}^{<I_\mathcal{S}} \cup \Phi_{\mathcal{L}}^{>I_\mathcal{S}} \cup \Phi_{\mathcal{R}}^{<J_\mathcal{S}} \cup \Phi_{\mathcal{R}}^{>J_\mathcal{S}}$$

is a maximal τ-rigid set in $\mathscr{F}(Q)$. Since $\mathcal{T}' \subseteq \mathcal{T}$, we have $\mathcal{T}' = \mathcal{T}$. The proof of the theorem is completed.

7.2.18 Remark. To conclude this section, we should point out that our results enable us to construct all the maximal τ-rigid sets in $\mathscr{F}(Q)$. Indeed, applying Theorem 5.3.25 enables us to construct all (densely) maximal τ-rigid sets in each of the regular components $\mathcal{L}$ and $\mathcal{R}$. Thus, Theorem 7.2.3 tells us how to find all the maximal τ-rigid sets in $\mathscr{F}(Q)$ containing no objects of $\mathcal{C}_Q$. Next, applying Theorem 7.2.17, we shall be able to construct all the maximal τ-rigid sets in $\mathscr{F}(Q)$ containing some objects in the connecting component $\mathcal{C}_Q$. Indeed, using Proposition 4.3.22, we are able to construct all the sectional chains in $\mathcal{C}_Q$. Given a sectional chain $\mathcal{S}$ in $\mathcal{C}_Q$, as indicated in Define 7.2.11, we shall apply Theorem 5.2.9 to construct all possible addends Θ to $\mathcal{S}$ in $\mathscr{F}(Q)$. Moreover, using Theorem 5.3.27 and 5.3.28, we are able to construct all the (densely) maximal τ-rigid sets $\Phi_{\mathcal{L}}^{<I_{\mathcal{S}}}$, $\Phi_{\mathcal{L}}^{>I_{\mathcal{S}}}$, $\Phi_{\mathcal{R}}^{>J_{\mathcal{S}}}$ and $\Phi_{\mathcal{R}}^{<J_{\mathcal{S}}}$ in $\mathcal{L}^+_{<I_{\mathcal{S}}}$, $\mathcal{L}^-_{>I_{\mathcal{S}}}$, $\mathcal{R}^+_{<J_{\mathcal{S}}}$ and $\mathcal{R}^-_{>J_{\mathcal{S}}}$, respectively.

7.3 Cluster-tilting subcategories of a cluster category of type $\mathbb{A}_\infty^\infty$

The objective of this section is to give a method to construct all the cluster-tilting subcategories of a cluster category $\mathscr{C}(Q)$ of type $\mathbb{A}_\infty^\infty$.

In view of Theorem 7.2.17, we are able to characterize and construct all the maximal rigid subcategories of $\mathscr{C}(Q)$. Indeed, Liu and Paquette have given a geometric criterion for a maximal rigid subcategory of $\mathscr{C}(Q)$ to be cluster-tilting; see [48]. Next, combining these results, we shall provide a method to construct all the cluster-tilting subcategories of $\mathscr{C}(Q)$.

For this purpose, we shall recall some geometric notions and terminology from [48]. Denote by $\mathfrak{B}_\infty$ the infinite strip in the plane of the points (x, y) with $0 \le y \le 1$. The points $\mathfrak{l}_i = (i, 1), i \in \mathbb{Z}$, are called the *upper marked points*; and $\mathfrak{r}_i = (-i, 0), i \in \mathbb{Z}$, the *lower marked points*. An upper or lower marked point will be simply called a marked point.

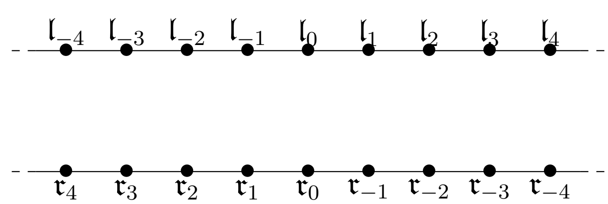


Figure 7.1: An infinite strip with marked points.

A *simple curve* in $\mathfrak{B}_\infty$ is a curve which does not cross itself and joins two (maybe identical) marked points called *endpoints*. A simple curve is called *internal* if it intersects the boundary of $\mathfrak{B}_\infty$ only at the endpoints. Two distinct simple curves in $\mathfrak{B}_\infty$ are said to *cross* if they have a common point which is not an endpoint of any of the curves. Let $\mathfrak{p}, \mathfrak{q}$ be two distinct marked points. There is an isotopy class of internal simple curves in $\mathfrak{B}_\infty$ connecting $\mathfrak{p}$ and $\mathfrak{q}$, which is called a *segment* of endpoints $\mathfrak{p}, \mathfrak{q}$; denoted by $[\mathfrak{p}, \mathfrak{q}]$ or $[\mathfrak{q}, \mathfrak{p}]$. A segment α in $\mathfrak{B}_\infty$ is called a *connecting arc* if $\alpha = [\mathfrak{l}_i, \mathfrak{r}_j]$ with $i, j \in \mathbb{Z}$; an *upper arc* if $\alpha = [\mathfrak{l}_i, \mathfrak{l}_j]$ with $j - i \geq 2$, and a *lower arc* if $\alpha = [\mathfrak{r}_p, \mathfrak{r}_q]$ with $p - q \geq 2$. A segment is called an *arc* in $\mathfrak{B}_\infty$ if it is a connecting, upper or lower arc. We shall denote by $\mathrm{arc}(\mathfrak{B}_\infty)$ all the arcs in $\mathfrak{B}_\infty$. One says that two arcs α, β *cross* if every curve in α crosses each of the curves in β.

Remark. It is easy to see that two upper arcs $[\mathfrak{l}_i, \mathfrak{l}_j]$ with $j - i \geq 2$ and $[\mathfrak{l}_p, \mathfrak{l}_q]$ with $q - p \geq 2$ cross if $j > q > i > p$ or $q > j > p > i$. Similarly, two lower arcs $[\mathfrak{r}_i, \mathfrak{r}_j]$ with $i - j \geq 2$ and $[\mathfrak{r}_p, \mathfrak{r}_q]$ with $p - q \geq 2$ cross if $i > p > j > q$ or $p > i > q > j$. Moreover, two connecting arcs $[\mathfrak{l}_i, \mathfrak{r}_j]$ and $[\mathfrak{l}_p, \mathfrak{r}_q]$ cross if $i < p$ and $j < q$ or $p < i$ and $q < j$. Furthermore, an upper arc $[\mathfrak{l}_i, \mathfrak{l}_j]$ with $j - i \geq 2$ and a connecting arc $[\mathfrak{l}_p, \mathfrak{r}_q]$ cross if $j > p > i$. Finally, a lower arc $[\mathfrak{r}_i, \mathfrak{r}_j]$ with $i - j \geq 2$ and a connecting arc $[\mathfrak{l}_p, \mathfrak{r}_q]$ cross if $i > q > j$. An upper arc and a lower arc never cross.

7.3.1 Definition. A maximal set of pairwise non-crossing arcs of $\mathfrak{B}_\infty$ is called a *triangulation*.

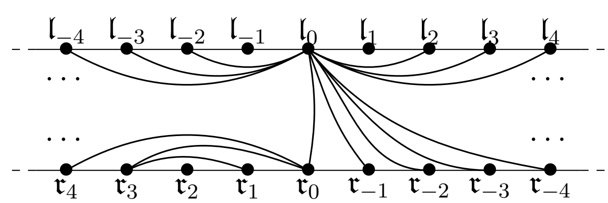


Figure 7.2: A triangulation $\mathbb{T}$ of $\mathfrak{B}_\infty$.

Let $\mathbb{T}$ be a triangulation of $\mathfrak{B}_\infty$. A marked point in $\mathfrak{B}_\infty$ is called $\mathbb{T}$-*bounded* if it is an endpoint of at most finitely many arcs in $\mathbb{T}$. An upper marked point $\mathfrak{l}_p$ is called

(1) a *left* $\mathbb{T}$-*fountain base* provided that the set $\{j \in \mathbb{Z} \mid [\mathfrak{l}_p, \mathfrak{r}_j] \in \mathbb{T}\}$ has a minimal integer but no maximal integer, and there exist infinitely integers $i < p-1$ such that $[\mathfrak{l}_i, \mathfrak{l}_p] \in \mathbb{T}$ but at most finitely many integers $j > p+1$ such that $[\mathfrak{l}_p, \mathfrak{l}_j] \in \mathbb{T}$;

(2) a *right* $\mathbb{T}$-*fountain base* provided that the set $\{j \in \mathbb{Z} \mid [\mathfrak{l}_p, \mathfrak{r}_j] \in \mathbb{T}\}$ has a maximal integer but no minimal integer, and there exist infinitely integers $i > p+1$ such that $[\mathfrak{l}_p, \mathfrak{l}_i] \in \mathbb{T}$ but at most finitely many integers $j < p-1$ such that $[\mathfrak{l}_j, \mathfrak{l}_p] \in \mathbb{T}$;

Similarly, a lower marked point $\mathfrak{r}_p$ is called

(1) a *left* $\mathbb{T}$-*fountain base* if provided that the set $\{j \in \mathbb{Z} \mid [\mathfrak{l}_j, \mathfrak{r}_p] \in \mathbb{T}\}$ has a minimal integer but no maximal integer, and there exist infinitely integers $i > p+1$ such that $[\mathfrak{r}_i, \mathfrak{r}_p] \in \mathbb{T}$ but at most finitely many integers $j < p-1$ such that $[\mathfrak{r}_p, \mathfrak{r}_j] \in \mathbb{T}$;

(2) a *right* $\mathbb{T}$-*fountain base* provided that the set $\{j \in \mathbb{Z} \mid [\mathfrak{l}_p, \mathfrak{r}_j] \in \mathbb{T}\}$ has a maximal integer but no minimal integer, and there exist infinitely integers $i < p-1$ such that $[\mathfrak{r}_p, \mathfrak{r}_i] \in \mathbb{T}$ but at most finitely many integers $j > p+1$ such that $[\mathfrak{r}_j, \mathfrak{r}_p] \in \mathbb{T}$;

Moreover, an upper marked point in $\mathfrak{B}_\infty$ is called a *full* $\mathbb{T}$-*fountain base* provided that the set $\{j \in \mathbb{Z} \mid [\mathfrak{l}_p, \mathfrak{r}_j] \in \mathbb{T}\}$ has neither minimal nor maximal integer, and there exist infinitely integers $i < p-1$ such that $[\mathfrak{l}_i, \mathfrak{l}_p] \in \mathbb{T}$ and infinitely many integers $j > p+1$ such that $[\mathfrak{l}_p, \mathfrak{l}_j] \in \mathbb{T}$; a lower marked point $\mathfrak{r}_p$ is called a *full* $\mathbb{T}$-*fountain base* provided that the set the set $\{j \in \mathbb{Z} \mid [\mathfrak{l}_j, \mathfrak{r}_p] \in \mathbb{T}\}$ has neither minimal nor maximal integer, and there exist infinitely integers $i > p+1$ such that $[\mathfrak{r}_i, \mathfrak{r}_p] \in \mathbb{T}$ and infinitely many integers $j < p-1$ such that $[\mathfrak{r}_p, \mathfrak{r}_j] \in \mathbb{T}$.

For brevity, a left, right or full $\mathbb{T}$-fountain base will be simply called a $\mathbb{T}$-fountain base. It is easy to see that the above definitions are equivalent to those given in [48, Section 3]. In Figure 7.2, the upper marked point $\mathfrak{l}_0$ is neither $\mathbb{T}$-bounded nor a $\mathbb{T}$-fountain base. In Figure 7.3 below, the upper marked point $\mathfrak{l}_0$ is a full $\mathbb{T}$-fountain base.

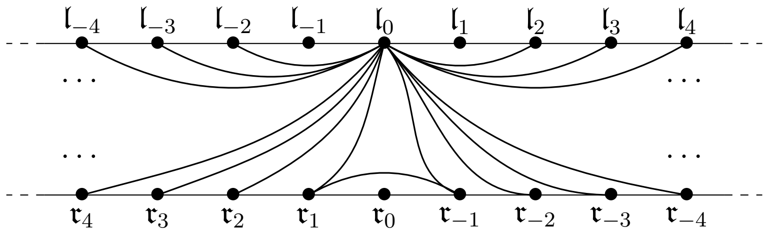


Figure 7.3: A triangulation $\mathbb{T}$ of $\mathfrak{B}_\infty$ with a $\mathbb{T}$-fountain base.

Recall that by Q we denote a quiver of type $\mathbb{A}_\infty^\infty$ having no infinite path. The skeleton $\mathscr{D}^b(Q)$ of the derived category $D^b(\text{rep}(Q))$ chosen in Section 3.3 is an Auslander-Reiten category, whose Auslander-Reiten quiver and Auslander-Reiten translation are denoted by $\varGamma_{\mathscr{D}^b(Q)}$ and τ. Moreover, $\varGamma_{\mathscr{D}^b(Q)}$ contains three connected components: the connecting component $\mathcal{C}_Q$ of shape $\mathbb{Z}\mathbb{A}_\infty^\infty$, and two regular components $\mathcal{L}$ and $\mathcal{R}$ which are of shape $\mathbb{Z}\mathbb{A}_\infty$. On the other hand, the cluster category $\mathscr{C}(Q)$ is also an Auslander-Reiten category, whose Auslander-Reiten quiver and Auslander-Reiten translation are denoted by $\varGamma_{\mathscr{C}(Q)}$ and $\tau_{\mathscr{C}}$. Since $\mathcal{C}_Q$, $\mathcal{L}$ and $\mathcal{R}$ form a fundamental domain $\mathscr{F}(Q)$ for $\mathscr{C}(Q)$, the canonical functor $\pi : \mathscr{D}^b(Q) \to \mathscr{C}(Q)$ induces a translation-quiver-isomorphism $\pi : \mathscr{F}(Q) \to \varGamma_{\mathscr{C}(Q)}$, which acts identically on the objects. Moreover, π restricted to $\mathcal{C}_Q$, $\mathcal{L}$ or $\mathcal{R}$ is a translation-quiver-isomorphism, respectively. Thus, $\varGamma_{\mathscr{C}(Q)}$ consists of three connected components, namely, $\mathcal{C}_{\mathscr{C}} = \pi(\mathcal{C}_Q)$ which is of shape $\mathbb{Z}\mathbb{A}_\infty^\infty$, and $\mathcal{L}_{\mathscr{C}} = \pi(\mathcal{L})$ and $\mathcal{R}_{\mathscr{C}} = \pi(\mathcal{R})$ which are of shape $\mathbb{Z}\mathbb{A}_\infty$.

As in Section 7.1, we shall give a coordinate system for each of the three components of $\varGamma_{\mathscr{C}(Q)}$ in such a way that the three coordinate systems are related to each other. To start with, we fix two double infinite sectional path L_0 and R_0 in $\mathcal{C}_{\mathscr{C}}$, whose intersection contains a unique object U_0. Then, for any $i, j \in \mathbb{Z}$, write $L_i = \tau_{\mathscr{C}}^i L^0$ and $R_j = \tau_{\mathscr{C}}^j R_0$. Thus, given an object $X \in \mathcal{C}_{\mathscr{C}}$, there is a unique pair (i_X, j_X) of integers such that $L_{i_X} \cap R_{j_X} = \{X\}$. For convenience, we shall write $X = (i_X, j_X)$. This yields a coordinate system for $\mathcal{C}_{\mathscr{C}}$, which coincides with the coordinate system $\mathcal{C}_Q$.

Now, we shall give a coordinate system for each of $\mathcal{L}_{\mathscr{C}}$ and $\mathcal{R}_{\mathscr{C}}$. Consider U_0 as an object in $\varGamma_{D^b(\text{rep}(Q))}$, there is a unique quasi-simple object $S_L \in \mathcal{L}_{\mathscr{C}}$ such that $\text{Hom}_{\mathscr{D}^b(Q)}(U_0, S_L) \neq 0$. The quasi-simple objects in $\mathcal{L}_{\mathscr{C}}$ are $S_i = \tau_{\mathscr{C}}^{i-1} S_L$, with $i \in \mathbb{Z}$. The ray starting with S_i is denoted by L_i^+, and the co-ray ending with S_i is denoted by L_i^-. Given an object $M \in \mathcal{L}_{\mathscr{C}}$, by Lemma 4.2.1, there is a unique pair of integers $(i_{_M}, j_{_M})$ with $i_{_M} \geq j_{_M}$, such that $M = L_{i_M}^+ \cap L_{j_M}^-$. We shall write $M = (i_{_M}, j_{_M})$. This yields a coordinate system for $\mathcal{L}_{\mathscr{C}}$, which coincides with the coordinate system for $\mathcal{L}$.

Similarly, $\mathrm{Hom}_{\mathscr{D}^b(Q)}(U_0, T_R) \neq 0$ for a unique quasi-simple object $T_R \in \mathcal{R}_{\mathscr{C}}$. The quasi-simple objects in $\mathcal{R}_{\mathscr{C}}$ are T_i with $T_i = \tau_{\mathscr{C}}^{i-1} T_R$, with $i \in \mathbb{Z}$. The ray starting with T_i is denoted by R_i^+, and the co-ray ending with T_i is denoted by R_i^-. Given an object $N \in \mathcal{R}_{\mathscr{C}}$, by Lemma 4.2.1, there is a unique pair of integers $(i_{_N}, j_{_N})$ with $i_{_N} \geq j_{_N}$, such that $N = R_{i_N}^+ \cap R_{j_N}^-$. We shall write $N = (i_{_N}, j_{_N})$. This yields a coordinate system for $\mathcal{R}_{\mathscr{C}}$, which coincides with the coordinate system for $\mathcal{R}$.

The following result is a reformation of some facts stated in [48, Section 4].

7.3.2 Lemma. *There is a bijection* $\Psi : (\varGamma_{\mathscr{C}(Q)})_0 \to \mathrm{arc}(\mathfrak{B}_\infty)$, *defined by*

$$\Psi(X) = \begin{cases} [\mathfrak{l}_{i_X}, \mathfrak{r}_{j_X}], & \text{if } X \in \mathcal{C}_{\mathscr{C}}; \\ [\mathfrak{l}_{j_X - 1}, \mathfrak{l}_{i_X + 1}], & \text{if } X \in \mathcal{L}_{\mathscr{C}}; \\ [\mathfrak{r}_{i_X + 1}, \mathfrak{r}_{j_X - 1}], & \text{if } X \in \mathcal{R}_{\mathscr{C}}. \end{cases}$$

Given a strictly additive subcategory $\mathscr{T}$ of $\mathscr{C}(Q)$, denote by $\mathrm{ind}\mathscr{T}$ the set of objects of $\varGamma_{\mathscr{C}(Q)}$ which lie in $\mathscr{T}$. Moreover, denote by $\mathrm{arc}(\mathscr{T})$ the set of arcs $\Psi(X)$ with $X \in \mathrm{ind}\mathscr{T}$.

7.3.3 Theorem ([48]). *Let Q be a quiver of type $\mathbb{A}_\infty^\infty$ having no infinite path. A strictly additive subcategory $\mathscr{T}$ of $\mathscr{C}(Q)$ is weakly cluster-tilting if and only if* $\mathrm{arc}(\mathscr{T})$ *is a triangulation of $\mathfrak{B}_\infty$.*

The following statement gives a condition for a weakly cluster-tilting subcategory of $\mathscr{C}(Q)$ being cluster-tilting.

7.3.4 Theorem ([48]). *Let Q be a quiver of type $\mathbb{A}_\infty^\infty$ having no infinite path. A strictly additive subcategory $\mathscr{T}$ of $\mathscr{C}(Q)$ is cluster-tilting if and only if* $\mathrm{arc}(\mathscr{T})$ *is a triangulation of $\mathfrak{B}_\infty$ with infinitely many connecting arcs, and every marked point in $\mathfrak{B}_\infty$ is either an* $\mathrm{arc}(\mathscr{T})$*-fountain base or* $\mathrm{arc}(\mathscr{T})$*-bounded.*

Given a weakly cluster-tilting subcategory, the following statement gives a combinatorial interpretation of the $\mathrm{arc}(\mathscr{T})$-bounded marked points and the left (right) $\mathrm{arc}(\mathscr{T})$-fountain bases in $\mathfrak{B}_\infty$.

7.3.5 LEMMA. *Let $\mathscr{T}$ be a weakly cluster-tilting subcategory of $\mathscr{C}(Q)$. Consider an upper marked point $\mathfrak{l}_i$ and a lower marked point $\mathfrak{r}_j$ in $\mathfrak{B}_\infty$ with $i, j \in \mathbb{Z}$.*

(1) *The upper marked point $\mathfrak{l}_i$ is $\mathrm{arc}(\mathscr{T})$-bounded if and only if $L_i \cap \mathrm{ind}\mathscr{T}$, $L^+_{i-1} \cap \mathrm{ind}\mathscr{T}$ and $L^-_{i+1} \cap \mathrm{ind}\mathscr{T}$ are finite.*

(2) *The lower marked point $\mathfrak{r}_j$ is $\mathrm{arc}(\mathscr{T})$-bounded if and only if $R_j \cap \mathrm{ind}\mathscr{T}$, $R^+_{j-1} \cap \mathrm{ind}\mathscr{T}$ and $R^-_{j+1} \cap \mathrm{ind}\mathscr{T}$ are finite.*

Proof. Consider the bijection $\Psi : (\varGamma_{\mathscr{C}(Q)})_0 \to \mathrm{arc}(\mathfrak{B}_\infty)$, stated in Lemma 7.3.2. We obtain

$$\Phi(L_i \cap \mathrm{ind}\mathscr{T}) = \{[\mathfrak{l}_i, \mathfrak{r}_p] \in \mathrm{arc}(\mathscr{T})\},$$
$$\Phi(L^+_{i-1} \cap \mathrm{ind}\mathscr{T}) = \{[\mathfrak{l}_p, \mathfrak{l}_i] \in \mathrm{arc}(\mathscr{T}) \mid p < i-1\},$$
$$\Phi(L^-_{i+1} \cap \mathrm{ind}\mathscr{T}) = \{[\mathfrak{l}_i, \mathfrak{l}_q] \in \mathrm{arc}(\mathscr{T}) \mid q > i+1\}.$$

By definition, $\mathfrak{l}_i$ is $\mathrm{arc}(\mathscr{T})$-bounded if and only if the sets $\{[\mathfrak{l}_i, \mathfrak{r}_j] \in \mathrm{arc}(\mathscr{T})\}$, $\{[\mathfrak{l}_p, \mathfrak{l}_i] \in \mathrm{arc}(\mathscr{T}) \mid p < i-1\}$ and $\{[\mathfrak{l}_i, \mathfrak{l}_q] \in \mathrm{arc}(\mathscr{T}) \mid q > i+1\}$ are all finite. This is evidently equivalent to that $R_i \cap \mathrm{ind}\mathscr{T}$, $\mathcal{L}^+_{i-1} \cap \mathrm{ind}\mathscr{T}$ and $\mathcal{L}^-_{i+1} \cap \mathrm{ind}\mathscr{T}$ are finite. This establishes Statement (1). Statement(2) is similar to show. The proof of the lemma is completed.

Given a weakly cluster-tilting subcategory, the following statement gives a combinatorial interpretation of the left (right) $\mathrm{arc}(\mathscr{T})$-fountain bases in $\mathfrak{B}_\infty$.

7.3.6 LEMMA. *Let $\mathscr{T}$ be a weakly cluster-tilting subcategory of $\mathscr{C}(Q)$. Consider an upper marked point $\mathfrak{l}_i$ and a lower marked point $\mathfrak{r}_j$ in $\mathfrak{B}_\infty$ with $i, j \in \mathbb{Z}$.*

(1) *The upper marked point $\mathfrak{l}_i$ is a left $\mathrm{arc}(\mathscr{T})$-fountain base if and only if $L_i \cap \mathrm{ind}\mathscr{T}$ is a chain having a maximal element but no minimal element, and $L^+_{i-1} \cap \mathrm{ind}\mathscr{T}$ is infinite but $L^-_{i+1} \cap \mathrm{ind}\mathscr{T}$ is finite.*

(2) *The upper marked point $\mathfrak{l}_i$ is a right $\mathrm{arc}(\mathscr{T})$-fountain base if and only if $L_i \cap \mathrm{ind}\mathscr{T}$ is a chain having a minimal element but no maximal element, and $L^-_{i+1} \cap \mathrm{ind}\mathscr{T}$ is infinite but $L^+_{i-1} \cap \mathrm{ind}\mathscr{T}$ is finite.*

(3) *The lower marked point* $\mathfrak{r}_j$ *in* $\mathfrak{B}_\infty$ *is a left* $\mathrm{arc}(\mathscr{T})$*-fountain base if and only if* $R_j \cap \mathrm{ind}\mathscr{T}$ *is a chain having a maximal element but no minimal element, and* $R^-_{j+1} \cap \mathrm{ind}\mathscr{T}$ *is infinite but* $R^+_{j-1} \cap \mathrm{ind}\mathscr{T}$ *is finite.*

(4) *The lower marked point* $\mathfrak{r}_j$ *in* $\mathfrak{B}_\infty$ *is a right* $\mathrm{arc}(\mathscr{T})$*-fountain base if and only if* $R_j \cap \mathrm{ind}\mathscr{T}$ *is a chain having a maximal element but no minimal element, and* $R^+_{j-1} \cap \mathrm{ind}\mathscr{T}$ *is infinite but* $R^-_{j+1} \cap \mathrm{ind}\mathscr{T}$ *is finite.*

Proof. Considering the bijection $\Psi : (\Gamma_{\mathscr{C}(Q)})_0 \to \mathrm{arc}(\mathfrak{B}_\infty)$, stated in Lemma 7.3.2, we obtain

$$\Phi(L_i \cap \mathrm{ind}\mathscr{T}) = \{[\mathfrak{l}_i, \mathfrak{r}_p] \in \mathrm{arc}(\mathscr{T})\},$$

$$\Phi(L^+_{i-1} \cap \mathrm{ind}\mathscr{T}) = \{[\mathfrak{l}_p, \mathfrak{l}_i] \in \mathrm{arc}(\mathscr{T}) \mid p < i-1\},$$

$$\Phi(L^-_{i+1} \cap \mathrm{ind}\mathscr{T}) = \{[\mathfrak{l}_i, \mathfrak{l}_q] \in \mathrm{arc}(\mathscr{T}) \mid q > i+1\}.$$

By definition, $\mathfrak{l}_i$ is a left $\mathrm{arc}(\mathscr{T})$-fountain base if and only if the following conditions hold.

(a) $\{p \in \mathbb{Z} \mid [\mathfrak{l}_i, \mathfrak{r}_p] \in \mathrm{arc}(\mathscr{T})\}$ has a minimal integer but no maximal one.

(b) There exist infinitely integers $p < i-1$ such that $[\mathfrak{l}_p, \mathfrak{l}_i] \in \mathrm{arc}(\mathscr{T})$.

(c) There exist at most finitely many integers $p > i+1$ such that $[\mathfrak{l}_i, \mathfrak{l}_p] \in \mathrm{arc}(\mathscr{T})$.

Since L_i is a chain by Proposition 4.3.19(2), $L_i \cap \mathrm{ind}\mathscr{T}$ is a chain with respect to the partial order over $\mathcal{C}_{\mathscr{C}}$. Condition (a) is equivalent to the fact that $L_i \cap \mathrm{ind}\mathscr{T}$ has a maximal element but no minimal element. Condition (b) is equivalent to the fact that $L^+_{i-1} \cap \mathrm{ind}\mathscr{T}$ is infinite, while Condition (c) is equivalent to the fact that $L^+_{i+1} \cap \mathrm{ind}\mathscr{T}$ is finite. This establishes Statement (1). The other statements can be verified similarly. The proof of the lemma is completed.

Given a weakly cluster-tilting subcategory $\mathscr{T}$, the following statement gives a combinatorial interpretation of the full $\mathrm{arc}(\mathscr{T})$-fountain bases in $\mathfrak{B}_\infty$. We shall omit the proof, since it is similar to the ones of the two preceding lemmas.

7.3.7 Lemma. *Let* $\mathscr{T}$ *be a weakly cluster-tilting subcategory of* $\mathscr{C}(Q)$*. Consider an upper marked point* $\mathfrak{l}_i$ *and a lower marked point* $\mathfrak{r}_j$ *in* $\mathfrak{B}_\infty$ *with* $i, j \in \mathbb{Z}$*.*

(1) *The upper marked point* $\mathfrak{l}_i$ *is a full* $\operatorname{arc}(\mathscr{T})$*-fountain base if and only if* $L_i \cap \operatorname{ind}\mathscr{T}$ *is a double infinite chain, while* $L^+_{i-1} \cap \operatorname{ind}\mathscr{T}$ *and* $L^-_{i+1} \cap \operatorname{ind}\mathscr{T}$ *are infinite.*

(2) *The lower marked point* $\mathfrak{r}_j$ *is a full* $\operatorname{arc}(\mathscr{T})$*-fountain base if and only if* $R_j \cap \operatorname{ind}\mathscr{T}$ *is a double infinite chain, while* $R^+_{j-1} \cap \operatorname{ind}\mathscr{T}$ *and* $R^-_{j+1} \cap \operatorname{ind}\mathscr{T}$ *are infinite.*

Recall that the restricted map $\pi : \mathcal{L} \to \mathcal{L}_{\mathscr{C}}$ is also an isomorphism of translation quivers acting identically on the underlying quiver. Thus, $\mathcal{L}$ and $\mathcal{L}_{\mathscr{C}}$ may share the same coordinate system as defined in Section 2.2. Let $\mathcal{S}$ be a section-generator of $\mathcal{L}_{\mathscr{C}}$. Thus, $\mathcal{S}$ is also a section-generator of $\mathcal{L}$. Since $\mathcal{L}$ is a standard component of $\mathit{\Gamma}_{\mathscr{D}^b(Q)}$ and is of shape $\mathbb{Z}\mathbb{A}_\infty$, we see that $\mathcal{S}$ admits an addend Θ of $\mathcal{S}$ in $\mathcal{L}$ as defined in Definition 5.3.20. Considering Θ as a set of objects in $\mathcal{L}_{\mathscr{C}(Q)}$, we shall call it an *addend* to $\mathcal{S}$ in $\mathcal{L}_{\mathscr{C}}$. We may have the same discussion for $\mathcal{R}_{\mathscr{C}}$.

Given $i \in \mathbb{Z}$, write

$$\mathcal{L}^+_{<i} = \bigcup\nolimits_{m<i} L^+_m \text{ and } \mathcal{L}^-_{>i} = \bigcup\nolimits_{m>i} L^-_m,$$

which are convex subquivers of $\mathcal{L}_{\mathscr{C}}$. Similarly, write

$$\mathcal{R}^+_{<i} = \bigcup\nolimits_{m<i} R^+_m \text{ and } \mathcal{R}^-_{>i} = \bigcup\nolimits_{m>i} R^-_m,$$

which are convex subquivers of $\mathcal{R}_{\mathscr{C}}$.

The following statement shall enable us to construct all the densely maximal rigid sets in $\mathcal{L}^+_{<n}$ or $\mathcal{L}^-_{>n}$, for any $n \in \mathbb{Z}$.

7.3.8 PROPOSITION. *Let n be an integer.*

(1) *A set* $\mathcal{T}$ *of objects in* $\mathcal{L}^+_{<n}$ *is densely maximal rigid in* $\mathcal{L}^+_{<n}$ *if and only if there is a section-generator* $\mathcal{S}$ *of* $\mathcal{L}_{\mathscr{C}}$ *which is almost contained in* L^+_{n-1} *such that* $\mathcal{T} = \mathcal{S} \cup \Theta$*, where* Θ *is an addend to* $\mathcal{S}$ *in* $\mathcal{L}_{\mathscr{C}}$*.*

(2) *A set* $\mathcal{T}$ *of objects in* $\mathcal{L}^-_{>n}$ *is densely maximal rigid in* $\mathcal{L}^-_{>n}$ *if and only if there is a section-generator* $\mathcal{S}$ *of* $\mathcal{L}_{\mathscr{C}}$ *which is almost contained in* L^-_{n+1} *such that* $\mathcal{T} = \mathcal{S} \cup \Theta$*, where* Θ *is an addend to* $\mathcal{S}$ *in* $\mathcal{L}_{\mathscr{C}}$*.*

Proof. Let $\mathcal{T}$ be a set of objects in $\mathcal{L}_{\mathscr{C}}$. Assume that there is a section-generator $\mathcal{S}$ of $\mathcal{L}_{\mathscr{C}}$ which is almost contained in L^+_{n-1} such that $\mathcal{T} = \mathcal{S} \cup \Theta$, where Θ is an addend to $\mathcal{S}$ in $\mathcal{L}_{\mathscr{C}}$. Consider $\mathcal{T}$ as a set of objects of $\mathcal{L}$. Then $\mathcal{S}$ is a section-generator of $\mathcal{L}$, which is almost contained in the ray $\pi^-(L^+_{n-1})$ in $\mathcal{L}$ and Θ is an addend to $\mathcal{S}$ in $\mathcal{L}$. Since $\mathcal{L}$ is of shape $\mathbb{Z}\mathbb{A}_\infty$, by Proposition 5.3.27, $\mathcal{T}$ is densely maximal τ-rigid in $\pi^-(\mathcal{L}^+_{<n})$. Considering $\mathcal{T}$ as objects of $\mathcal{L}_{\mathscr{C}}$, by Lemma 5.1.6, $\mathcal{T}$ is densely maximal rigid in $\mathcal{L}^+_{<n}$.

Conversely, assume that $\mathcal{T}$ is a densely maximal rigid set in $\mathcal{L}^+_{<n}$. Now consider $\mathcal{T}$ as a set of objects in $\mathscr{F}(Q)$. In particular, $\mathcal{T}$ is contained in $\pi^-(\mathcal{L}^+_{<n})$ a subquiver of $\mathcal{L}$. By Lemma 5.1.6, $\mathcal{T}$ is a densely maximal τ-rigid set in $\pi^-(\mathcal{L}^+_{<n})$. By Proposition 5.3.27, there is a section-generator $\mathcal{S}$ of $\mathcal{L}$ which is almost contained in a ray L^+_{n-1} in $\mathcal{L}$ such that $\mathcal{T} = \mathcal{S} \cup \Theta$, where Θ is an addend to $\mathcal{S}$ in $\mathcal{L}$. Consider $\mathcal{T}$ as a set of objects in $\mathcal{L}_{\mathscr{C}}$. In particular, $\mathcal{S}$ is a section-generator of $\mathcal{L}_{\mathscr{C}}$ which is almost contained in the ray L^+_{n-1} in $\mathcal{L}_{\mathscr{C}}$, and Θ an addend to $\mathcal{S}$ in $\mathcal{L}_{\mathscr{C}}$. The proof of the proposition is completed.

The following statement shall enable us to construct all the densely maximal rigid sets in $\mathcal{R}^+_{<n}$ or $\mathcal{R}^-_{>n}$, for any $n \in \mathbb{Z}$. We shall omit the proof, since it is similar to that of Proposition 7.3.8.

7.3.9 Proposition. *Let n be an integer.*

(1) *A set $\mathcal{T}$ of objects in $\mathcal{R}^+_{<n}$ is densely maximal rigid if and only if there is a section-generator $\mathcal{S}$ in $\mathcal{R}_{\mathscr{C}}$, which is almost contained in R^+_{n-1}, such that $\mathcal{T} = \mathcal{S} \cup \Theta$, where Θ is an addend to $\mathcal{S}$ in $\mathcal{R}_{\mathscr{C}}$.*

(2) *A set $\mathcal{T}$ of objects in $\mathcal{R}^-_{>n}$ is densely maximal rigid if and only if there is a section-generator $\mathcal{S}$ in $\mathcal{R}_{\mathscr{C}}$, which is almost contained in R^-_{n+1}, such that $\mathcal{T} = \mathcal{S} \cup \Theta$, where Θ is an addend to $\mathcal{S}$ in $\mathcal{R}_{\mathscr{C}}$.*

Recall that $(\mathcal{C}_{\mathscr{C}}, \preceq)$ is a partially ordered set as defined in Section 4.3. Let $\mathcal{S}$ be a chain in $\mathcal{C}_{\mathscr{C}}$. Writing $I_{\mathcal{S}} = \{i \in \mathbb{Z} \mid \mathcal{S} \cap L_i \neq \emptyset\}$, we have $\mathcal{L}^+_{<I_{\mathcal{S}}} = \mathcal{L}^+_{<i_0}$ in case $I_{\mathcal{S}}$ has a smallest integer i_0; and the empty set otherwise; and $\mathcal{L}^-_{>I_{\mathcal{S}}} = \mathcal{L}^-_{>i_1}$ in case $I_{\mathcal{S}}$ has a largest integer i_1; and the empty set otherwise. Similarly, writing $J_{\mathcal{S}} = \{j \in \mathbb{Z} \mid \mathcal{S} \cap R_j \neq \emptyset\}$, we have $\mathcal{R}^+_{<J_{\mathcal{S}}} = \mathcal{R}^+_{<j_0}$ in case $J_{\mathcal{S}}$ has a smallest

integer j_0; and the empty set otherwise; and $\mathcal{R}^-_{>J_\mathcal{S}} = \mathcal{R}^-_{>j_1}$ in case $J_\mathcal{S}$ has a largest integer j_1; and the empty set otherwise.

The following is a necessary condition for a weakly cluster-tilting subcategory of $\mathscr{C}(Q)$ to be a cluster-tilting subcategory.

7.3.10 LEMMA. *If $\mathscr{T}$ is a cluster-tilting subcategory of $\mathscr{C}(Q)$, then $\mathcal{C}_\mathscr{C} \cap \mathrm{ind}\mathscr{T}$ is a section-generator of $\mathcal{C}_\mathscr{C}$.*

Proof. Assume that $\mathscr{T}$ is a cluster-tilting subcategory of $\mathscr{C}(Q)$. By Theorem 7.3.4(2), $\mathrm{arc}(\mathscr{T})$ contains infinitely many connecting arcs. By Lemma 7.3.2, $\mathrm{ind}\mathscr{T} \cap \mathcal{C}_\mathscr{C}$ is infinite. Since $\Gamma_{\mathscr{C}(Q)}$ and $\mathscr{F}(Q)$ have the same set of objects, $\mathrm{ind}\mathscr{T} \cap \mathcal{C}_Q$ is infinite.

Since $\mathscr{T}$ is a maximal rigid subcategory in $\mathscr{C}(Q)$, by Proposition 5.1.7, $\mathrm{ind}\mathscr{T}$ is a maximal τ-rigid set in $\mathscr{F}(Q)$. By Theorem 7.2.17, there is a sectional chain $\mathcal{S}$ in $\mathcal{C}_Q$ such that

$$\mathrm{ind}\mathscr{T} = \mathcal{S} \cup \Theta \cup \Phi_\mathcal{L}^{<I_\mathcal{S}} \cup \Phi_\mathcal{L}^{>I_\mathcal{S}} \cup \Phi_\mathcal{R}^{<J_\mathcal{S}} \cup \Phi_\mathcal{R}^{>J_\mathcal{S}},$$

where $\Theta, \Phi_\mathcal{L}^{<I_\mathcal{S}}, \Phi_\mathcal{L}^{>I_\mathcal{S}}, \Phi_\mathcal{R}^{<J_\mathcal{S}}$ and $\Phi_\mathcal{R}^{>J_\mathcal{S}}$ are as stated in Theorem 7.2.17. Obviously, $\mathcal{S} = \mathcal{C}_Q \cap \mathrm{ind}\mathscr{T} = \mathcal{C}_\mathscr{C} \cap \mathrm{ind}\mathscr{T}$.

To show that $\mathcal{C}_\mathscr{C} \cap \mathrm{ind}\mathscr{T}$ is a section-generator of $\mathcal{C}_\mathscr{C}$, by Proposition 4.3.23, it remains to show that $\mathcal{S}$ has neither maximal nor minimal elements. Assume on the contrary that $\mathcal{S}$ has a minimal element X. In this case, i_X is the minimal element in $I_\mathcal{S}$, and hence, $\mathcal{L}^+_{<I_\mathcal{S}} = \mathcal{L}^+_{<i_X}$. By Theorem 7.2.17(2), $\Phi_\mathcal{L}^{<I_\mathcal{S}}$ is dense in $\mathcal{L}^+_{<i_X}$, and by Lemma 4.2.20, $\Phi_\mathcal{L}^{<I_\mathcal{S}}$ contains infinitely many objects of the ray $L^+_{i_X-1}$ in $\mathcal{L}$. In particular, $L^+_{i_X-1} \cap \mathrm{ind}\mathscr{T}$ is infinite. By Lemma 7.3.5(1), the upper marked point $\mathfrak{l}_{i_X}$ is not $\mathrm{arc}(\mathscr{T})$-bounded; and by Lemma 7.3.6(3), $\mathfrak{l}_{i_X}$ is not a right $\mathrm{arc}(\mathscr{T})$-fountain base. On the other hand, since $\mathcal{S}$ is an interval-finite chain with a minimal element, its subchain $\mathcal{S} \cap L_{i_X} = \mathrm{ind}\mathscr{T} \cap L_{i_X}$ has a minimal element. By Lemma 7.3.6(1), $\mathfrak{l}_{i_X}$ is not a left $\mathrm{arc}(\mathscr{T})$-fountain base; and by Lemma 7.3.7(1), $\mathfrak{l}_{i_X}$ is not a full $\mathrm{arc}(\mathscr{T})$-fountain base. By Theorem 7.3.4, $\mathscr{T}$ is not cluster-tilting, a contradiction. Hence, $\mathcal{S}$ has no minimal element. Similarly, $\mathcal{S}$ has no maximal element. The proof of the lemma is completed.

Recall that $\pi : \mathscr{F}(Q) \to \Gamma_{\mathscr{C}(Q)}$ is an isomorphism of translation quivers acting identically on the underlying quiver. Let $\mathcal{S}$ be a section-generator of $\mathcal{C}_\mathscr{C}$. Observe that $\mathcal{S}$ is also a section-generator of $\mathcal{C}_Q$. Let Θ be an addend to $\mathcal{S}$ in $\mathscr{F}(Q)$ as defined in Definition 7.2.11. Considering Θ as a set of objects in $\Gamma_{\mathscr{C}(Q)}$, we shall call it an *addend* to $\mathcal{S}$ in $\Gamma_{\mathscr{C}(Q)}$.

The following statement is one of our main results, which enables us to construct all the cluster-tilting subcategories of a cluster category of type $\mathbb{A}_\infty^\infty$. We should mention that, for convenience, the empty set is considered as a dense subset of itself.

7.3.11 THEOREM. *Let $\mathscr{C}(Q)$ be the cluster category associated with a quiver Q of type $\mathbb{A}_\infty^\infty$ without infinite paths. A strictly additive subcategory $\mathscr{T}$ of $\mathscr{C}(Q)$ is cluster-tilting if and only if there is a section-generator $\mathcal{S}$ of $\mathcal{C}_{\mathscr{C}}$ such that*

$$\mathrm{ind}\mathscr{T} = \mathcal{S} \cup \Theta \cup \Phi_{\mathcal{L}}^{<I_\mathcal{S}} \cup \Phi_{\mathcal{L}}^{>I_\mathcal{S}} \cup \Phi_{\mathcal{R}}^{<J_\mathcal{S}} \cup \Phi_{\mathcal{R}}^{>J_\mathcal{S}},$$

where

(1) *Θ is an addend to $\mathcal{S}$ in $\Gamma_{\mathscr{C}(Q)}$;*

(2) *$\Phi_{\mathcal{L}}^{<I_\mathcal{S}}$ is a densely maximal rigid set in $\mathcal{L}^+_{<I_\mathcal{S}}$; while $\Phi_{\mathcal{L}}^{>I_\mathcal{S}}$ is a densely maximal rigid set in $\mathcal{L}^-_{>I_\mathcal{S}}$;*

(3) *$\Phi_{\mathcal{R}}^{<J_\mathcal{S}}$ is a densely maximal rigid set in $\mathcal{R}^+_{<J_\mathcal{S}}$, while $\Phi_{\mathcal{R}}^{>J_\mathcal{S}}$ is a densely maximal rigid set in $\mathcal{R}^-_{>J_\mathcal{S}}$.*

Proof. Let $\mathscr{T}$ be a strictly additive subcategory $\mathscr{T}$ of $\mathscr{C}(Q)$. Suppose that there is a section-generator $\mathcal{S}$ of $\mathcal{C}_{\mathscr{C}}$ such that

$$\mathrm{ind}\mathscr{T} = \mathcal{S} \cup \Theta \cup \Phi_{\mathcal{L}}^{<I_\mathcal{S}} \cup \Phi_{\mathcal{L}}^{>I_\mathcal{S}} \cup \Phi_{\mathcal{R}}^{<J_\mathcal{S}} \cup \Phi_{\mathcal{R}}^{>J_\mathcal{S}},$$

where $\Theta, \Phi_{\mathcal{L}}^{<I_\mathcal{S}}, \Phi_{\mathcal{L}}^{>I_\mathcal{S}}, \Phi_{\mathcal{R}}^{<J_\mathcal{S}}$ and $\Phi_{\mathcal{R}}^{>J_\mathcal{S}}$ are as in Statements (1), (2) and (3) stated in the theorem. By Proposition 4.3.23, $\mathcal{S}$ is a double infinite chain.

Now, we consider $\mathrm{ind}\mathscr{T}$ as a set of objects of $\mathscr{F}(Q)$. Then $\mathcal{S}$ is a section-generator of $\mathcal{C}_Q$ and Θ is an addend to $\mathcal{S}$ in $\mathscr{F}(Q)$. In view of Lemma 5.1.6, $\Phi_{\mathcal{L}}^{<I_\mathcal{S}}$, $\Phi_{\mathcal{L}}^{>I_\mathcal{S}}, \Phi_{\mathcal{R}}^{<J_\mathcal{S}}$ and $\Phi_{\mathcal{R}}^{>J_\mathcal{S}}$ verify respectively the Statements (2), (3), (4) and (5) stated in Theorem 7.2.17. Therefore, $\mathrm{ind}\mathscr{T}$ is a maximal τ-rigid set in $\mathscr{F}(Q)$. By Proposition 5.1.7, $\mathscr{T}$ is a weakly cluster-tilting subcategory of $\mathscr{C}(Q)$. By

Theorem 7.3.3, $\operatorname{arc}(\mathscr{T})$ is a triangulation of $\mathfrak{B}_\infty$. We shall show that $\operatorname{arc}(\mathscr{T})$ verifies the conditions stated in Theorem 7.3.4.

Fix an integer $i \in \mathbb{Z}$. Consider first the case where $L_i \cap \operatorname{ind}\mathscr{T}$ is finite. Observe that $\operatorname{ind}\mathscr{T}$ is a maximal τ-rigid set in $\mathscr{F}(Q)$ and $\operatorname{ind}\mathscr{T} \cap \mathcal{C}_Q = \mathcal{S}$ is a double infinite chain. As a finite chain, $L_i \cap \operatorname{ind}\mathscr{T}$ has a minimal element as well a maximal element. By Lemma 7.2.10(1), $L^+_{i-1} \cap \operatorname{ind}\mathscr{T}$ and $L^-_{i+1} \cap \operatorname{ind}\mathscr{T}$ are both finite. By Lemma 7.3.5(1), $\mathfrak{l}_i$ is $\operatorname{arc}(\mathscr{T})$-bounded.

Suppose now that $\operatorname{ind}\mathscr{T} \cap L_i$ is infinite. First, assume that $\operatorname{ind}\mathscr{T} \cap L_i$ is a double infinite chain. That is, $\mathcal{S} \cap L_i = \operatorname{ind}\mathscr{T} \cap L_i$ is a double infinite chain. By Lemma 4.3.14(1), $I_\mathcal{S} = \{i\}$. That is, i is the minimal element in $I_\mathcal{S}$. In particular, $\mathcal{L}^+_{<I_\mathcal{S}} = \mathcal{L}^+_{<i} \neq \emptyset$. Being dense in $\mathcal{L}^+_{<i}$, by Lemma 4.2.20, $\Phi^{<I_\mathcal{S}}_\mathcal{L} \cap L^+_{i-1}$ is infinite, and hence $L^+_{i-1} \cap \operatorname{ind}\mathscr{T}$ is infinite. Since i is the maximal element in $I_\mathcal{S}$, we similarly show that $L^-_{i+1} \cap \operatorname{ind}\mathscr{T}$ is infinite. Therefore, by Lemma 7.3.7(1), $\mathfrak{l}_i$ is a full fountain.

Secondly, assume that $L_i \cap \operatorname{ind}\mathscr{T}$ has a minimal element. That is, $\mathcal{S} \cap L_i = L_i \cap \operatorname{ind}\mathscr{T}$ has a minimal element. By Lemma 4.3.14(3), i is the largest integer in $I_\mathcal{S}$. Thus, $\mathcal{L}^-_{>I_\mathcal{S}} = \mathcal{L}^-_{>i} \neq \emptyset$. Since $\Phi^{>I_\mathcal{S}}_\mathcal{L}$ is dense in $\mathcal{L}^-_{>i}$, by Lemma 4.2.20, $\Phi^{>I_\mathcal{S}}_\mathcal{L} \cap L^-_{i+1}$ is infinite. Hence, $\operatorname{ind}\mathscr{T} \cap L^-_{i+1}$ is infinite. On the other hand, observe that $\operatorname{ind}\mathscr{T}$ is a maximal τ-rigid set in $\mathscr{F}(Q)$ and $\mathcal{S} = \operatorname{ind}\mathscr{T} \cap \mathcal{C}_Q$ is a double infinite chain. Since $L_i \cap \operatorname{ind}\mathscr{T}$ has a minimal element, by Lemma 7.2.10(1), $L^+_{i-1} \cap \operatorname{ind}\mathscr{T}$ is finite. Therefore, by Lemma 7.3.6(2), $\mathfrak{l}_i$ is a right $\operatorname{arc}(\mathscr{T})$-fountain base. Finally, assume that $L_i \cap \operatorname{ind}\mathscr{T}$ has a maximal element. Then it is similar to show that $\mathfrak{l}_i$ is a left $\operatorname{arc}(\mathscr{T})$-fountain base. This shows that every upper marked point is either $\operatorname{arc}(\mathscr{T})$-bounded or an $\operatorname{arc}(\mathscr{T})$-fountain base. Similarly, we can show that every lower marked point is either $\operatorname{arc}(\mathscr{T})$-bounded or an $\operatorname{arc}(\mathscr{T})$-fountain base. By Theorem 7.3.4, $\mathscr{T}$ is cluster-tilting in $\mathscr{C}(Q)$.

Conversely, assume that $\mathscr{T}$ is a cluster-tilting subcategory of $\mathscr{C}(Q)$. By Theorem 7.3.3, $\operatorname{arc}(\mathscr{T})$ is a triangulation of $\mathfrak{B}_\infty$. Consider $\operatorname{ind}\mathscr{T}$ as a set of objects of $\mathscr{F}(Q)$. By Proposition 5.1.7, $\operatorname{ind}\mathscr{T}$ is maximal τ-rigid in $\mathscr{F}(Q)$. By Theorem 7.2.17, there is a sectional chain $\mathcal{S}$ in $\mathcal{C}_Q$ such that

$$\operatorname{ind}\mathscr{T} = \mathcal{S} \cup \Theta \cup \Phi^{<I_\mathcal{S}}_\mathcal{L} \cup \Phi^{>I_\mathcal{S}}_\mathcal{L} \cup \Phi^{<J_\mathcal{S}}_\mathcal{R} \cup \Phi^{>J_\mathcal{S}}_\mathcal{R},$$

where $\Theta, \Phi^{<I_\mathcal{S}}_\mathcal{L}, \Phi^{>I_\mathcal{S}}_\mathcal{L}, \Phi^{<J_\mathcal{S}}_\mathcal{R}$ and $\Phi^{>J_\mathcal{S}}_\mathcal{R}$ satisfy the conditions stated in Theorem 7.2.17. Consider $\operatorname{ind}\mathscr{T}$ as a set of objects in $\Gamma_{\mathscr{C}(Q)}$. Since $\mathcal{S} = \mathcal{C}_\mathscr{C} \cap \operatorname{ind}\mathscr{T}$, by

Lemma 7.3.10, $\mathcal{S}$ is a section-generator of $\mathcal{C}_{\mathscr{C}}$. By our definition, Θ is an addend to $\mathcal{S}$ in $\Gamma_{\mathscr{C}(Q)}$. Moreover, in view of Lemma 5.1.6, $\Phi_{\mathcal{L}}^{<I_{\mathcal{S}}}$ is a maximal rigid set in $\mathcal{L}^{+}_{<I_{\mathcal{S}}}$. We claim that $\Phi_{\mathcal{L}}^{<I_{\mathcal{S}}}$ is dense in $\mathcal{L}^{+}_{<I_{\mathcal{S}}}$. Indeed, we may assume that $\mathcal{L}^{+}_{<I_{\mathcal{S}}} \neq \emptyset$. Then, $\mathcal{L}^{+}_{<I_{\mathcal{S}}} = \mathcal{L}^{+}_{<i}$, where i is a minimal element in $I_{\mathcal{S}}$. Since $\mathcal{S}$ is a double infinite sectional chain by Proposition 4.3.23, we deduce from Lemma 4.3.15(1) that $\mathcal{S} \cap L_i$ is a chain having no minimal element. Then, $L_i \cap \operatorname{ind}\mathscr{T} = \mathcal{S} \cap L_i$ is infinite. Hence, by Lemma 7.3.5(1), $\mathfrak{l}_i$ is not arc($\mathscr{T}$)-bounded. By Theorem 7.3.4 and Lemma 7.3.6(2), $\mathfrak{l}_i$ is either a left arc($\mathscr{T}$)-fountain base or a full arc($\mathscr{T}$)-fountain base. By Lemmas 7.3.5(3) and 7.3.7(1), $L^{+}_{i-1} \cap \operatorname{ind}\mathscr{T}$ is infinite. Since $\Phi_{\mathcal{L}}^{<I_{\mathcal{S}}}$ is maximal rigid in $\mathcal{L}^{+}_{<I_{\mathcal{S}}}$ and $L^{+}_{i-1} \subseteq \mathcal{L}^{+}_{<I_{\mathcal{S}}}$, we have $L^{+}_{i-1} \cap \operatorname{ind}\mathscr{T} \subseteq \Phi_{\mathcal{L}}^{<I_{\mathcal{S}}}$. That is, $\Phi_{\mathcal{L}}^{<I_{\mathcal{S}}}$ contains infinitely many objects of L^{+}_{i-1}. By Proposition 4.2.20, $\Phi_{\mathcal{L}}^{<I_{\mathcal{S}}}$ is dense in $\mathcal{L}^{+}_{<I_{\mathcal{S}}}$. Similarly, we can show that $\Phi_{\mathcal{L}}^{>I_{\mathcal{S}}}$ is a densely maximal rigid set in $\mathcal{L}^{-}_{>I_{\mathcal{S}}}$. This shows that Statement (2) holds. In a similar fashion, we may show that Statement (3) holds. The proof of the theorem is completed.

Remark. Let Q be a quiver of type $\mathbb{A}_{\infty}^{\infty}$ without infinite paths. We point out that our results enable us to construct all the cluster-tilting subcategories of the cluster category $\mathscr{C}(Q)$. Indeed, by Proposition 4.3.23, we are able to obtain all the section-generators of $\mathcal{C}_{\mathscr{C}}$. Given a section-generator $\mathcal{S}$, as indicated in Definition 7.2.11, by applying Theorem 5.2.9, we are enable to obtain all possible addends Θ to $\mathcal{S}$ in $\Gamma_{\mathscr{C}(Q)}$. Finally, applying Proposition 7.3.8 and 7.3.9 enables us to construct all the densely maximal rigid sets in $\mathcal{L}^{+}_{<I_{\mathcal{S}}}$, $\mathcal{L}^{-}_{>I_{\mathcal{S}}}$, $\mathcal{R}^{+}_{<J_{\mathcal{S}}}$, and $\mathcal{R}^{-}_{>J_{\mathcal{S}}}$, respectively.

Conclusion

Let Q be a locally finite quiver without infinite path. The orbit category $\mathscr{C}(Q)$ is a cluster category in the sense that its cluster-tilting subcategories form a cluster structure. Particularly, it is true if Q is of infinite Dynkin type, that is, Q is of type $\mathbb{A}_\infty$, $\mathbb{A}_\infty^\infty$ or $\mathbb{D}_\infty$. In this thesis, we provide an effective method to construct all the maximal rigid subcategories and cluster-tilting subcategories of $\mathscr{C}(Q)$ in case Q is of type $\mathbb{A}_\infty$ or $\mathbb{A}_\infty^\infty$.

Now let Q be a quiver of type $\mathbb{D}_\infty$ without infinite paths. Observe that $\mathscr{C}(Q)$ shares several properties with the cluster categories of types $\mathbb{A}_\infty$ and $\mathbb{A}_\infty^\infty$. For instance, a strictly additive subcategory of $\mathscr{C}(Q)$ is cluster-tilting if and only if it is maximal rigid and functorially finite; see [48, (2.11)]. Moreover, the Auslander-Reiten quiver $\Gamma_{\mathscr{C}(Q)}$ of $\mathscr{C}(Q)$ consists of a connecting component of shape $\mathbb{Z}\mathbb{D}_\infty$ and one regular component of shape $\mathbb{Z}\mathbb{A}_\infty$; see [48, (2.9)]. These inspire us to attempt to address the following problems by applying the techniques introduced in this thesis.

(1) Construct the maximal rigid subcategories of $\mathscr{C}(Q)$.

(2) Give a criterion for a maximal rigid subcategory of $\mathscr{C}(Q)$ to be cluster-tilting.

(3) Construct all the cluster-tilting subcategories of $\mathscr{C}(Q)$.

Bibliography

[1] T. Adachi, O. Iyama, I. Reiten, "τ-tilting theory," Compositio Mathematica, 150 (2014) 415 - 452.

[2] C. Amiot "On generalized cluster categories", Representations of Algebras and Related Topics, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, (2011) 1 - 53.

[3] I. Assem, "Algèbres et Modules", (Masson, Paris, 1997).

[4] I. Assem and D. Happel, "Generalized tilted algebras of type $\mathbb{A}_n$", Comm. Algebra 9 (1981) 2101 - 2125.

[5] I. Assem, D. Simson, A.Skowroński, "Elements of the Representation Theory of Associative Algebras", London Mathematical Society Student Texts 65 (Cambridge University Press, Cambridge, 2006).

[6] M. Auslander, "Functors and morphisms determined by objects", Lecture Notes in Pure Appl. Math. 37 (Dekker, New York, 1978) 1 - 244.

[7] M. Auslander, M. I. Platzeck, and I. Reiten, " Coxeter functors without diagrams", Trans. Amer. Math. Soc. 250 (1979) 1 - 46.

[8] M. Auslander and I. Reiten, "Representation theory of artin algebras III", Comm. Algebra 3 (1975) 239 - 294.

[9] M. Auslander and I. Reiten, "Representation theory of artin algebras IV", Comm. Algebra 5 (1977) 443 - 518.

[10] M. Auslander, I. Reiten and S. O. Smalø, “Representation Theory of Artin Algebras”, Cambridge Studies in Advanced Mathematics 36 (Cambridge University Press, Cambridge, 1995).

[11] M. Auslander, S. O. Smalo, “Almost split sequences in subcategories”, J. Algebra 69 (1981) 426 - 454.

[12] R. Bautista, “Irreducible morphisms and the radical of a category”, Ann.Inst. Math. Univ. Nac. Auto. Mex. 22 (1982) 83 - 135.

[13] R. Bautista and M. Kleiner, “Almost split sequences for relatively projective modules”, J. Algebra 135 (1990) 19 - 56.

[14] R. Bautista and S. Liu, “Covering theory for linear categories with application to derived categories”, J. Algebra 406 (2014) 173 - 225.

[15] R. Bautista, S. Liu and C. Paquette, “Representation theory of strongly locally finite quivers”, Proc. London Math. Soc. 106 (2013) 97 - 162.

[16] A. Berenstein, S. Fomin, and A. Zelevinsky, “Cluster algebras III: Upper bounds and double Bruhat cells”, Duke Math. J. 126 1 (2005) 1 - 52.

[17] I. N. Bernstein, I. M. Gelfand, and V. A. Ponomarev, “Coxeter functors and Gabriel’s theorem”, Usp. Mat. Nauk 28 (1973) 19 - 33; English translation: Russian Math. Surveys 28 (1973) 17 - 32.

[18] K. Bongartz, “Tilted algebras”, Proceedings of ICRA III, Springer Lecture Notes 903 (Springer, Heidelberg, 1981) 26 - 38.

[19] S. Brenner and M. C. R. Butler, “Generalization of the Bernstein-Gelfand-Ponomarev reflection functors”, Lecture Notes in Mathematics 839 (Springer-Verlag, 1980) 103 - 169.

[20] A. B. Buan, R. Marsh, M. Reineke, I. Reiten and G. Todorov, “Tilting theory and cluster combinatorics”, Adv. Math. 204 (2006) 572 - 618.

[21] A. B. Buan, O. Iyama, I. Reiten and J. Scott, "Cluster structures for 2-Calabi-Yau categories and unipotent groups", Compos. Math. 145 (2009) 1035 - 1079.

[22] P. Caldero, F. Chapoton, R. Schiffler " Quivers with relations arising from clusters (A_n case)", Trans. Amer. Math. Soc. 358 3 (2006) 1347 - 1364.

[23] R. Dehy, B. Keller, " On the combinatorics of rigid objects in 2-Calabi-Yau categories", Internat. Math. Res. Notices 2008 (2008).

[24] S. Fomin and A. Zelevinsky, "Cluster algebras. I. Foundations," J. Amer. Math. Soc., 15 (2) (2002) 497 - 529.

[25] S. Fomin and A. Zelevinsky, "Cluster algebras II: Finite type classification", Invent. Math.154 (2003) 63 - 121.

[26] P. Gabriel, "Unzerlegbare Darstellungen I", Manuscripta Math. 6 (1972) 71 -103.

[27] P. Gabriel, "Auslander-Reiten sequences and representation-finite algebras", Lecture Notes in Math. 831 (Springer, Berlin, 1980) 1 - 71.

[28] P. Gabriel and A. V. Roiter, "Representations of Finite-Dimensional Algebras", Algebra VIII, Encyclopedia Math. Sci. 73 (Springer, Berlin, 1992).

[29] J.E. Grabowski and S. Gratz, "Cluster algebras of infinite rank", J. London Math. Soc. 89 (2014) 337 - 363.

[30] L. Hügel, D. Happel, H. Krause, "Handbook of tilting theory", London Mathematical Society Lecture Note Series 332. (Cambridge University Press, Cambridge, 2007).

[31] D. Happel, "Triangulated Categories in the Representation Theory of Finite Dimensional Algebras", London Mathematical Society Lectures Note Series 119 (Cambridge University Press, Cambridge, 1988).

[32] D. Happel and CM. Ringel, “Tilted algebras”, Trans. Amer. Math. Soc. 274 (1982) 399 - 443.

[33] D. Happel and L. Unger, “ Almost complete tilting modules”, Proc. Amer. Math. Soc. 107 (1989) 603 - 610.

[34] T. Holm and P. Jørgensen, “On a cluster category of infinite Dynkin type, and the relation to triangulations of the infinity-gon”, Math. Z. 270 (2012) 277 - 295.

[35] O. Iyama, I. Reiten, “Introduction to τ-tilting theory”, Proc. Natl. Acad. Sci. 111(27) (2014), 9704 - 9711.

[36] P. Jørgensen, “Auslander-Reiten theory over topological spaces”, Comment. Math. Helv. 79 (2004) 160-182.

[37] V. G. Kac, “Infinite root systems, representations of graphs and invariant theory”, Invent. Math. 56 (1980) 57 - 92.

[38] B. Keller, “Cluster algebras, quiver representations and triangulated categories”, London Mathematical Society Lecture Note Series. (Cambridge University Press, 2010) 76 - 160.

[39] B. Keller, “On triangulated orbit categories”, Doc. Math. 10 (2005) 551 - 581.

[40] G. M. Kelly, “On the radical of a category”, J. Austral. Math. Soc. 4 (1964) 299 - 307.

[41] O. Kerner, “Tilting wild algebras”, J. London Math. Soc. (2) 39 (1989) 29 - 47.

[42] D. König, “Über eine Schlussweise aus dem Endlichen ins Unendliche”, Acta Sci. Math. (Szeged) 3 (1927) 121 - 130.

[43] H. Krause, “Krull-Schmidt Categories and Projective Covers”, Expo. Math. 33 (2015) 535 - 549.

[44] W. Krull, “Über verallgemeinerte endliche Abelsche Gruppen”, Math. Z. 23 (1925) 161 - 196.

[45] S. Liu, "Tilted algebras and generalized standard Auslander-Reiten components", Arch. Math. 61 (1993) 12 - 19.

[46] S. Liu, "Auslander-Reiten theory in a Krull-Schmidt category", São Paulo J. Math. Sci. 4 (2010) 425 - 472.

[47] S. Liu and C. Paquette, "Standard components of a Krull-Schmidt category", Proc. Amer. Math. Soc. 143 (2015) 45 - 59.

[48] S. Liu and C. Paquette, "Cluster categories of type A double infinity and triangulations of the infinite strip", Math. Z. 286 (2017) 197 - 222.

[49] D. Milicic, "Lectures on Derived Categories," preprint, available at www.math.utah.edu/ milicic/Eprints/dercat.pdf

[50] R. Marsh, M, Reineke, A. Zelevinsky, "Generalized associahedra via quiver representations", Trans. Amer. Math. Soc. 355 (2003) 4171 - 4186.

[51] Y. Palu, "Grothendieck group and generalized mutation rule for 2-Calabi-Yau triangulated categories", J. Pure Appl. Algebra 213 (2009) 1438 - 1449.

[52] R. Remak, "Über die Zerlegung der endlichen Gruppen in direkte unzerlegbare Faktoren", J. reine angew. Math. 139 (1911), 293 - 308.

[53] I. Reiten and M. Van den Bergh, "Noetherian hereditary abelian categories satisfying Serre duality", J. Amer. Math. Soc. 15 (2002) 295 - 366.

[54] C. M. Ringel, "Finite dimensional hereditary algebras of wild representation type", Math. Z. 161 (1978) 235 - 255.

[55] C. M. Ringel, "Tame Algebras and Integral Quadratic Forms", Lecture Notes in Mathematics 1099 (Springer-Verlag, Berlin, 1984).

[56] J. Rotman, "An Introduction to Homological Algebra", Universitext (Springer New York, 2008).

[57] O. Schmidt, “Sur les produits directs”, Bull. Soc. Math. France 41 (1913) 161 - 164.

[58] A. Skowronski, “Regular Auslander-Reiten components containing directing modules”, Proc. Amer. Math. Soc. 120(1) (1994) 19 - 26.

[59] J. Stovicek, A. Roosmalen, “2-Calabi-Yau categories with a directed cluster-tilting subcategory”, arXiv:1611.03836.

[60] J. Verdier, “Des Catégories dérivées des Catégories Abéliennes”, Asterisque. 239 (1996).

[61] C.A. Weibel, “An Introduction to Homological Algebra”, Cambridge Studies in Advanced Mathematics 38 (Cambridge University Press, 1995).

[62] Y. Yang, “On Cluster Categories of Type $\mathbb{D}_\infty$”, J. Algebra 478 (2017) 12 - 37.

[63] Y. Zhou, B. Zhu, “Maximal rigid subcategories in 2-Calabi-Yau triangulated categories,” J. Algebra 348 1 (2011) 49 - 60.