La Théorie des Représentations des Algèbres graduées et des Algèbres de Koszul

The Representation Theory of Graded Algebras and Koszul Algebras

par

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Abstract

In this thesis, we give a novel proof of Auslander-Reiten formulas and describe a new connection between Koszul theory and representation theory.

Let Λ be a graded algebra defined by a locally finite quiver with relations. We construct a graded Nakayama functor for Λ, essential for our new proof of Auslander-Reiten formulas and the existence of almost split triangles in their graded derived categories. In case Λ is quadratic, we first give a combinatorial description for the local Koszul complexes and the quadratic dual $\Lambda^!$.

As applications, we obtain a new class of Koszul algebras and prove the Extension Conjecture for finite dimensional Koszul algebras with a left noetherian Koszul dual. Then, we construct a left and a right Koszul functor for Λ, which respectively induce a 2-real-parameter family of left and right derived Koszul functors from categories derived from graded Λ-modules into those derived from graded $\Lambda^!$-modules. A left or right derived Koszul functor for Λ is paired with a unique right or left Koszul functor for $\Lambda^!$, respectively. In case Λ is Koszul, any two paired derived Koszul functors are mutually quasi-inverse, and the Koszul duality of Beilinson, Ginzburg and Soergel is one of these pairs. If Λ and $\Lambda^!$ are locally bounded on opposite sides, then the Koszul functors induce two equivalences of bounded derived categories: one for finitely piece-supported graded modules, and one for finite dimensional graded modules.

Finally, if Λ and $\Lambda^!$ are locally bounded, then the bounded derived category of finite dimensional graded Λ-modules has almost split triangles, and the Auslander-Reiten translations and the Serre functors are composite functors of the derived Koszul functors.

Sommaire

Dans cette thèse, nous donnons une nouvelle preuve des formules d'Auslander-Reiten et décrivons une nouvelle connexion entre la théorie de Koszul et la théorie des représentations.

Soit Λ une algèbre graduée définie par un carquois localement fini avec des relations. Nous construisons un foncteur de Nakayama gradué pour Λ, essentiel pour notre nouvelle preuve des formules d'Auslander-Reiten et l'existence de triangles presque scindés dans leurs catégories dérivées graduées. Dans le cas où Λ est quadratique, nous donnons d'abord une description combinatoire des complexes locaux de Koszul et du dual quadratique $\Lambda^!$.

Comme applications, nous obtenons une nouvelle classe d'algèbres de Koszul et prouvons la conjecture d'extension pour les algèbres de Koszul de dimension finie avec un dual de Koszul noethérien gauche. Ensuite, nous construisons un foncteur Koszul à gauche et à droite pour Λ, qui induisent respectivement une famille paramétrée par 2 paramètres réels de foncteurs Koszul dérivés à gauche et à droite des catégories dérivées des Λ-module gradués vers celles dérivées de $\Lambda^!$-modules gradués. Un foncteur Koszul dérivé à gauche ou à droite pour Λ est associé à un foncteur Koszul unique à droite ou à gauche pour $\Lambda^!$, respectivement. Dans le cas où Λ est Koszul, deux foncteurs Koszul dérivés appariés sont mutuellement quasi-inverses, et la dualité de Koszul de Beilinson, Ginzburg et Soergel est l'une de ces paires. Si Λ et $\Lambda^!$ sont localement délimités sur des côtés opposés, alors les foncteurs de Koszul induisent deux équivalences de catégories dérivées limitées : une pour les modules gradués à support par pièces finies et une pour les modules gradués de dimension finie.

Enfin, si Λ et $\Lambda^!$ sont localement bornés, alors la catégorie dérivée bornée des modules Λ gradués de dimension finie a des triangles presque scindés, et les translations d'Auslander-Reiten et les foncteurs de Serre sont des foncteurs composés des foncteurs Koszul dérivés.

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Contents

Introduction

The history of Koszul theory traces back to Cartan and Eilenberg's computing the cohomology groups of a Lie algebra using the Koszul resolution; see [18, Section 8.7]. It is connected to many branches of mathematics such as algebraic topology; see [21, 54], algebraic geometry; see [13], Hopf algebras and Lie theory; see [13, 47, 48]. Beilinson, Ginzburg and Soergel described the Koszul duality between a Koszul algebra Λ and its Koszul dual $\Lambda^!$ at the level of derived categories; see [13]. More precisely, they constructed a pair of mutually quasi-inverse equivalences between a category derived from complexes of graded Λ-modules and a category derived from complexes of graded $\Lambda^!$-modules. In case Λ is finite dimensional and $\Lambda^!$ is left noetherian, they obtained equivalences of the bounded derived categories of finitely generated graded modules. Later, the Koszul duality has been generalized to positively graded Koszul categories in [48]; see also [45]. Moreover, finite dimensional Koszul algebras have been studied extensively by many representation theorists from other aspects; see [22, 24, 42, 43, 44, 46].

The main objective of this thesis is to study Koszul algebras defined by a locally finite quiver with relations from a novel viewpoint of connecting the Koszul theory with the representation theory. Our contribution is twofold. As to Koszul theory, not only our Koszul algebras have infinitely many graded simple modules up to grading shifts, but we shall also extend Beilinson, Ginzburg and Soergel's Koszul duality to a 2-real-parameter family of equivalence pairs. Moreover, under a hypothesis weaker than theirs, we shall obtain two equivalences of bounded derived categories, one for finitely piece-supported graded modules and one for finite dimensional graded modules. Moreover, in contrast to their technique of spectral sequences, ours is elementary.

As to the representation theory, we shall extract some homological properties of Koszul algebras and study almost split triangles in their graded derived categories. Indeed, our combinatorial description of the local Koszul complexes and the quadratic dual enables us not only to obtain a new class of Koszul algebras, but also to describe the linear projective resolution and the colinear injective coresolution, if they exist, of a graded simple module in terms of subspaces of the quadratic dual. This description allows us to establish the Extension Conjecture for finite dimensional Koszul algebra with a left noetherian Koszul dual. More generally, we use the Koszul functors to describe explicitly a graded projective resolution and a graded injective coresolution for any finite dimensional graded module over a Koszul algebra. This could be applied to study more homological properties, for instance, to prove or disprove the Generalized Nakayama Conjecture; see [6] and the Finitistic Dimensional Conjecture, of finite dimensional Koszul algebras. Finally, our graded Nakayama functor provides an indispensable tool for studying almost split sequences in graded module categories and almost split triangles in graded derived categories in general. Using the graded Nakayama functor, we give a new proof of graded Aslander-Reiten formulas. An immediate consequence is to establish the existence of almost split sequences in graded module categories. In the locally bounded Koszul case, we shall establish the existence of almost split triangles in the bounded derived category of finite dimensional graded modules, and describe the Auslander-Reiten translations and the Serre functors in terms of the derived Koszul functors. This may stimulate future work on the graded Auslander-Reiten components of a hereditary or radical squared zero algebra, as is done under the non-graded setting; see [10, 11].

Now, we outline the content section by section. Let Λ be a graded algebra defined by a locally finite quiver with relations. It is important to note that our algebras do not necessarily have an identity. We denote by $\mathrm{GMod}\Lambda$ the category of unitary graded left Λ-modules, and by $\mathrm{GMod}^b\Lambda$, $\mathrm{gmod}\Lambda$ and $\mathrm{gmod}^b\Lambda$ its subcategories of finitely piece-supported modules, of piecewise finite dimensional modules and of finite dimensional modules, respectively. If $\mathcal{A}$ is an additive category, then $C(\mathcal{A})$, $K(\mathcal{A})$, $D(\mathcal{A})$ and $D^b(\mathcal{A})$ stand for the category of complexes, the homotopy category, the derived category and the bounded derived category of $\mathcal{A}$, respectively.

We shall lay down the foundation of this thesis in Chapters 1 and 2. For this purpose, we collect some basic notions and preliminary results related to k-linear categories and k-linear algebras.

In Chapter 3, we shall investigate the graded module categories over a graded algebra given by a quiver with relations. In Section 3, we shall construct a contravariant functor $\mathfrak{D} : \mathrm{GMod}\Lambda \to \mathrm{GMod}\Lambda^{\mathrm{o}}$, which restricts to a duality $\mathfrak{D} : \mathrm{gmod}\Lambda \to \mathrm{gmod}\Lambda^{\mathrm{o}}$. In Sections 4 and 5, we shall provide explicit descriptions of the morphisms in $\mathrm{GProj}\Lambda$ and $\mathrm{GInj}\Lambda$; see, (3.4.3) and (3.4.6); and study the graded radical and graded socle of graded modules, especially to generalize some classical results under the non-graded setting; compare [1] and [2]. Moreover, we show how to construct the graded projective cover of a finitely generated module and the graded injective envelope of a finitely cogenerated module, respectively; see (3.8.5). For this purpose, we shall first describe the finitely generated graded modules and the finitely cogenerated graded modules; see (3.6.6); and, superfluous graded epimorphisms and essential graded monomorphisms; see (3.7.1). In the final section, we shall introduce several Hom-finite Krull-Schmidt k-subcategories of GModΛ.

In Chapter 4, we shall construct a Nakayama functor for GModΛ; see (4.1.3), which ensures the existence of almost split triangles in $D(\text{GMod}\Lambda)$ for bounded complexes of finitely generated projective modules and for those of finitely cogenerated injective modules; see (4.3.2). More importantly, using the Nakayama functor, we provide a new method to prove the Auslander-Reiten formulas; see (4.2.10). The classical approach to this well known formula consists of the following two identifications; see [4, (I.3.4)] and the corollary to [44, (1.6.3)]. First, the covariant stable Hom functor given by a finitely presented module is identified with the Tor^1 functor given by its transpose; see [4, (I.3.2)] and [44, (1.6.3)]. Secondly, the dual of the Tor^1 functor given by a module is identified by the adjunction isomorphism with the contravariant Ext^1 functor given by its dual; see [4, (I.3.3)], [18, (VI.5.1)] and [44, (1.6.1)]. Our approach is to apply the Nakayama functor; see (4.1.3), which does not pass through the Tor^1 functor and does not involve the tensor product over the algebra or the adjunction isomorphism. As a consequence, we shall study the existence of almost split sequences in GModΛ. Furthermore, if Λ is locally left noetherian, then an indecomposable complex of finitely generated graded Λ-modules is the ending term of an almost split triangle in $D^b(\text{gmod}\Lambda)$ if and only if it has a finite graded projective resolution; see (4.3.4).

In Chapter 5, we shall give a combinatorial description of the local Koszul complexes and the quadratic dual $\Lambda^!$ for a quadratic algebra Λ; see (5.2.3). This allows us to obtain a new class of Koszul algebras; see (5.5.2) and to describe the linear projective resolution and the colinear injective co-resolution of a graded simple module, if they exist, in terms of subspaces of $\Lambda^!$; see (5.4.1) and (5.4.2). Using this description, we shall show that Λ is Koszul if and only if $\Lambda^!$ or Λ^{op} is Koszul if and only if every graded simple module has a colinear injective coresolution; see (5.4.3) and compare [13, (2.2.1), (2.9.1)], and establish the Extension Conjecture for finite dimensional Koszul algebras with a noetherian Koszul dual;

see (5.6.4).

In Chapter 6, we shall describe our generalized Koszul duality. In case Λ is quadratic, we construct a left and a right Koszul functor from $\mathrm{GMod}\Lambda$ into $C(\mathrm{GMod}\Lambda^!)$; see (6.1.1), which extend respectively to a left and a right complex Koszul functor from $C(\mathrm{Mod}\Lambda)$ into $C(\mathrm{Mod}\Lambda^!)$. The latter induce respectively a 2-real-parameter family of left and right derived Koszul functors from categories derived from subcategories of $C(\mathrm{Mod}\Lambda)$ into those derived from subcategories of $C(\mathrm{Mod}\Lambda^!)$ so that a left or right derived Koszul functor for Λ is paired with a unique right or left derived Koszul functor for $\Lambda^!$, respectively; see (6.3.3). They also induce a left and a right bounded derived Koszul functors from $D^b(\mathrm{GMod}^b\Lambda)$ into $D^b(\mathrm{GMod}\Lambda^!)$; see (6.3.4). In case Λ is Koszul, by composing the Koszul functors and the complex Koszul functors, we obtain a graded projective resolution and a graded injective co-resolution of a graded Λ-module M in terms of subspaces of $\Lambda^!$ and M; see (6.4.2) and (6.4.3). This is essential for us to show that the derived Koszul functors in any pair are mutually quasi-inverse; see (6.4.6), including the Koszul duality stated in [13, (2.12.1)]. If Λ and $\Lambda^!$ are locally bounded on opposite sides, then the bounded derived Koszul functors restrict to two equivalences $D^b(\mathrm{GMod}^b\Lambda) \cong D^b(\mathrm{GMod}^b\Lambda^!)$ and $D^b(\mathrm{gmod}^b\Lambda) \cong D^b(\mathrm{gmod}^b\Lambda^!)$; see (6.4.7). This strengthens the result in [13, (2.12.6)]. As applications, in case Λ is quadratic, the images of a complex in $C^b(\mathrm{gmod}^b\Lambda^!)$ under the two bounded derived Koszul functors fit into an almost split triangle in $D^b(\mathrm{gmod}\Lambda)$ if they are indecomposable; see (6.5.1). In case Λ is Koszul, $D^b(\mathrm{gmod}^b\Lambda)$ is a full triangulated subcategory of $D^b(\mathrm{gmod}\Lambda)$, and every indecomposable object in $D^b(\mathrm{gmod}^b\Lambda)$ is the ending term of an almost split triangle in $D^b(\mathrm{gmod}\Lambda)$ if and only if $\Lambda^!$ is locally right bounded; see (6.5.2). If Λ and $\Lambda^!$ are locally bounded, then $D^b(\mathrm{gmod}^b\Lambda)$ has almost split triangles, and the Auslander-Reiten translations and the Serre functors are composites of derived Koszul functors; see (6.5.4).

Chapter 1

Categories

Throughout this thesis, k denotes a commutative field. In this chapter, we assemble some basic notions and results concerning k-linear categories, with a specific focus on additive k-categories, abelian k-categories, exact k-categories, triangulated k-categories, derived k-categories, and double complex k-categories. This enables us to provide the necessary foundational knowledge for subsequent in-depth research endeavours.

1.1 k-linear categories

Given any category, we shall compose the morphisms from the right to the left. A full subcategory of a category is called **strictly** if it is closed under isomorphisms. A k-**linear category** (or simply a k-**category**) is a category in which the morphism sets are k-vector spaces and the composition of morphisms is k-bilinear. All functors between k-categories are assumed to be additive.

Throughout this section, $\mathcal{A}$ denotes a k-category. One says that $\mathcal{A}$ is **Hom-finite** if its morphism spaces are finite dimensional over k. An object X in $\mathcal{A}$ is called a **zero object** if $\mathrm{id}_X = 0$. Let $f : X \to Y$ be a morphism in $\mathcal{A}$. One says that f is a **monomorphism** provided that $f \circ g = 0$ only if $g = 0$ and an **epimorphism** provided that $g \circ f = 0$ only if $g = 0$. A morphism $q : U \to X$ is called a **kernel** of f provided that $f \circ q = 0$, and for any morphism $g : V \to X$ with $f \circ g = 0$, there exists a unique morphism $h : V \to U$ such that $g = q \circ h$. On defines a **cokernel** of f dually. It is evident that a kernel of a morphism is a monomorphism and a cokernel of a morphism is an epimorphism. Moreover, f is called a **section** or a **retraction** if there exists a morphism $g : Y \to X$ such that $g \circ f = \mathrm{id}_X$ or $f \circ g = \mathrm{id}_Y$, respectively. It is evident that a section is a monomorphism while a retraction is an epimorphism.

1.1.1 **Definition**. Let $\mathcal{A}$ be a k-category. A **coproduct** or **direct sum** of a family of objects $\{X_\sigma\}_{\sigma\in\Sigma}$ in $\mathcal{A}$ is an object X with a family of morphisms $\{q_\sigma : X_\sigma \to X\}_{\sigma\in\Sigma}$, called the **canonical injections**, satisfying the following universal property: for any object Y with a family of morphisms $\{f_\sigma : X_\sigma \to Y\}_{\sigma\in\Sigma}$ in $\mathcal{A}$, there exists a unique morphism $g : X \to Y$ such that the diagram

$$\begin{array}{ccc} X_\sigma & \xrightarrow{q_\sigma} & X \\ {\scriptstyle f_\sigma}\downarrow & \swarrow{\scriptstyle g} & \\ Y & & \end{array}$$

commutes for all $\sigma \in \Sigma$. In this case, one writes $X = \coprod_{\sigma\in\Sigma} X_\sigma$ or $X = \oplus_{\sigma\in\Sigma} X_\sigma$.

Dually, we have the following notion.

1.1.2 **Definition**. Let $\mathcal{A}$ be a k-category. A **product** of a family of objects $\{X_\sigma\}_{\sigma\in\Sigma}$ in $\mathcal{A}$ is an object X with a family of morphisms $\{p_\sigma : X \to X_\sigma\}_{\sigma\in\Sigma}$, called the **canonical projections**, satisfying the following universal property: for any object Y with a family of morphisms $f_\sigma : Y \to X_\sigma$ in $\mathcal{A}$, there exists a unique morphism $g : Y \to X$ such that the diagram

$$\begin{array}{ccc} X_\sigma & \xleftarrow{p_\sigma} & X \\ {\scriptstyle f_\sigma}\uparrow & \nearrow{\scriptstyle g} & \\ Y & & \end{array}$$

commutes for all $\sigma \in \Sigma$. In this case, one write $X = \prod_{\sigma\in\Sigma} X_\sigma$.

Remark. It is evident that the canonical injections for a direct sum are monomorphisms and the canonical projections for a product are epimorphisms.

The following statement is well known.

1.1.3 **Proposition**. *Let $\mathcal{A}$ be a k-category. If $X, X_1, \ldots, X_n$ are objects in $\mathcal{A}$, then $X = X_1 \oplus \cdots \oplus X_n$ if and only if there exist morphisms $q_i : X_i \to X$ and $p_i : X \to X_i$ such that $\mathrm{id}_X = \sum_{i=1}^n q_i \circ p_i$ and*

$$p_i \circ q_i = \begin{cases} \mathrm{id}_{X_i} & \text{if } i = j; \\ 0 & \text{if } i \neq j, \end{cases}$$

for all $1 \le i, j \le n$.

Remark. In the above situation, X is the coproduct and product of $X_1, \ldots, X_n$ with canonical injections q_i and canonical projections p_i. Moreover, the $e_i = q_i \circ p_i$ with $1 \le i \le n$ are pairwise orthogonal idempotents in $\mathrm{End}_{\mathcal{A}}(X)$.

Let X be an object in $\mathcal{A}$. An idempotent $e \in \mathrm{End}_{\mathcal{A}}(X)$ is said to **split** if there exist morphisms $p : X \to Y$ and $q : Y \to X$ such that $e = q \circ p$ and $p \circ q = \mathrm{id}_Y$. The following statement is well known. For the reader's convenience, we shall include a short proof.

1.1.4 **Lemma**. *Let $\mathcal{A}$ be a k-category with an object X. Then, $X = X_1 \oplus \cdots \oplus X_n$ if and only if $\mathrm{id}_X = e_1 + \cdots + e_n$, where the e_i are pairwise orthogonal split non-zero idempotents in $\mathrm{End}_{\mathcal{A}}(X)$; and in this case, $e_i = q_i \circ p_i$, where q_i and p_i are the canonical injections and canonical projections of the direct sum respectively.*

Proof. The sufficiency follows from Proposition 1.1.3, we only need to prove the necessity. Suppose that $e_1, \ldots, e_n$ are pairwise orthogonal split non-zero idempotents in $\mathrm{End}_{\mathcal{A}}(X)$ such that $e_1 + \cdots + e_n = \mathrm{id}_X$. For every $1 \le i \le n$, since e_i splits, there exist morphisms $p_i : X \to X_i$ and $q_i : X_i \to X$ such that $q_i \circ p_i = e_i$ and $p_i \circ q_i = \mathrm{id}_{X_i}$. For any $1 \le i \le n$, since $e_i \ne 0$, we see that $\mathrm{id}_{X_i} \ne 0$. That is, the X_i are all non-zero objects. Now, $\sum_{i=1}^n q_i \circ p_i = \sum_{i=1}^n e_i = \mathrm{id}_X$, and since $e_1, \ldots, e_n$ are pairwise orthogonal, $p_i \circ q_j = 0$ for all $1 \le i, j \le n$ with $i \ne j$. By Proposition 1.1.3, $X = X_1 \oplus \cdots \oplus X_n$. The proof of the lemma is completed.

1.2 Additive k-categories

An **additive** k-category is a k-category, which has a zero object and finite direct sums. Throughout this section, $\mathcal{A}$ denotes an additive k-category. Let X be a non-zero object in $\mathcal{A}$. An object Y is called a **direct summand** of X if $X \cong Y \oplus Z$ for some object Z in $\mathcal{A}$. And a direct sum decomposition $X = Y \oplus Z$ is called **proper** if Y and Z are non-zero. One says that X is **indecomposable** provided that it is non-zero and admits no proper decomposition. If $\mathrm{End}_{\mathcal{A}}(X)$ is a local algebra, then it has no proper idempotent, and by Proposition 1.1.3, X is indecomposable. In this case, we call X **strongly indecomposable**.

1.2.1 **Definition**. A non-zero additive k-category $\mathcal{A}$ is called **Krull-Schmdit** if every non-zero object in $\mathcal{A}$ is a direct sum of finitely many strongly indecomposable objects.

To characterize Krull-Schmidt k-categories, we recall that a k-algebra Σ with an identity is **semiperfect** if it has a complete orthogonal set $\{e_1, \ldots, e_n\}$ of idempotents such that the $e_i \Sigma e_i$ with $1 \le i \le n$ are local rings; see [1, page 304], and also, [30, (4.1)].

The following statement is well known; see, for example, [39, (1.1)].

1.2.2 **Theorem**. *Let $\mathcal{A}$ be an additive k-category with a non-zero object X. Then, $X = \oplus_i^n X_i$ with X_i strongly indecomposable if and only if $\mathrm{End}_{\mathcal{A}}(X)$ is semiperfect with all idempotents split in $\mathcal{A}$. In this case, the direct sum decomposition for X is unique up to isomorphism and permutation, and every non-zero direct summand of X is a direct sum of objects of a subfamily of $\{X_1, \ldots, X_n\}$.*

As an immediate consequence, we have the following statement.

1.2.3 **Corollary**. *Let $\mathcal{A}$ be a Hom-finite additive k-category. Then $\mathcal{A}$ is Krull-Schmidt if and only if all idempotent endomorphisms in $\mathcal{A}$ split.*

Proof. It is well known that every finite dimensional k-algebra with an identity is semi-perfect. Now, the statement follows immediately from Theorem 1.2.2. The proof of the corollary is completed.

Let $\mathcal{A}$ be an additive k-category. A **two-sided ideal** (or simply, ideal) $\mathcal{I}$ of $\mathcal{A}$ consists of subspaces $\mathcal{I}(X,Y)$ of $\mathrm{Hom}_{\mathcal{A}}(X,Y)$ with $X, Y \in \mathcal{A}$ such that $h \circ f \circ g \in \mathcal{I}(U,V)$ for all $f \in \mathcal{I}(X,Y)$, $g \in \mathrm{Hom}_{\mathcal{A}}(U,X)$ and $h \in \mathrm{Hom}_{\mathcal{A}}(Y,V)$. The following easy result will be used later; see, for example, [39, 1.2].

1.2.4 **Lemma**. *Let $\mathcal{A}$ be an additive k-category and $\mathcal{I}$ be a proper ideal of $\mathcal{A}$. If $\mathcal{A}$ is Krull-Stchmidt, then so is $\mathcal{A}/\mathcal{I}$.*

1.2.5 **Definition**. Let $\mathcal{A}$ be an additive k-category. A morphism $f : X \to Y$ in $\mathcal{A}$ is called

(1) **right minimal** provided that every morphism $g : X \to X$ such that $f \circ g = f$ is an automorphism.

(2) **right almost split** provided that f is not a retraction and every non-retraction morphism $u : U \to Y$ factors through f.

(3) **minimal right almost split** provided that it is right minimal and right almost split.

Dually, one has the following notions.

1.2.6 **Definition**. Let $\mathcal{A}$ be an additive k-category. A morphism $f : X \to Y$ in $\mathcal{A}$ is called

(1) **left minimal** provided that every morphism $h : Y \to Y$ such that $h \circ f = f$ is an automorphism.

(2) **left almost split** provided that f is not a section and every non-section morphism $v : X \to V$ factors through f.

(3) **minimal left almost split** provided that it is left minimal and left almost split.

The following statement is due to Auslander and Reiten; see [7, (2.3)].

1.2.7 **Proposition**. *Let $\mathcal{A}$ be an additive k-category. If $f : X \to Y$ is a left* (*respectively, right*) *almost split morphism in $\mathcal{A}$, then X* (*respectively, Y*) *is strongly indecomposable.*

Finally, let $\mathcal{A}$ be Hom-finite. An additive functor $\mathbb{S} : \mathcal{A} \to \mathcal{A}$ is called a **left** (respectively, **right**) **Serre functor** if there exist binatural k-linear isomorphisms $\mathrm{Hom}_{\mathcal{A}}(X, Y) \cong D\mathrm{Hom}_{\mathcal{A}}(\mathbb{S}Y, X)$ (respectively, $\mathrm{Hom}_{\mathcal{A}}(X, Y) \cong D\mathrm{Hom}_{\mathcal{A}}(Y, \mathbb{S}X)$), for all $X, Y \in \mathcal{A}$; see [55, (I.1)]. Moreover, a left (respectively, right) Serre functor F is called a **left** (respectively, **right**) **Serre equivalence** if it is an equivalence of categories.

1.3 Abelian k-categories

An additive k-category is called **abelian** if every morphism has a kernel and a cokernel, while every monomorphism is a kernel of some morphism and every epimorphism is a cokernel of some morphism; see [61, (1.2.2)]. Throughout this section, let $\mathfrak{A}$ denote an abelian k-category. We start with some sufficient conditions for an additive k-subcategory of $\mathfrak{A}$ to be Krull-Schmidt. The following statement is well known. For the convenience of the reader, we will provide a brief proof.

1.3.1 **Lemma**. *Let $\mathfrak{A}$ be an abelian k-category. Then idempotent endomorphisms in $\mathfrak{A}$ split.*

Proof. Let $e : X \to X$ be an idempotent endomorphism in $\mathfrak{A}$. Since $\mathfrak{A}$ is abelian, the endomorphism $\mathrm{id}_X - e : X \to X$ has a kernel $f : Y \to X$. In particular, $(\mathrm{id}_X - e) \circ f = 0$, and hence, $f = e \circ f$. On the other hand, since $(\mathrm{id}_X - e) \circ e = 0$, there exists a morphism $g : X \to Y$ such that $e = f \circ g$. Therefore, $f \circ (g \circ f) = (f \circ g) \circ f = e \circ f = f$. Since f is a monomorphism, $g \circ f = \mathrm{id}_Y$. By definition, e splits in $\mathfrak{A}$. The proof of the lemma is completed.

We shall use frequently the following statement.

1.3.2 **Proposition**. *Let $\mathfrak{A}$ be an abelian k-category, and let $\mathcal{A}$ be a strictly full additive k-subcategory of $\mathfrak{A}$. If $\mathcal{A}$ is Hom-finite, then $\mathcal{A}$ is Krull-Schmidt if and only if it is closed under direct summands.*

Proof. Let $\mathcal{A}$ be Hom-finite. Suppose first that $\mathcal{A}$ is Krull-Schmidt. Consider a non-zero object $X \in \mathcal{A}$. Then, $X = \oplus_i^n X_i$, where X_i are strongly indecomposable objects in $\mathcal{A}$. Let Y be a non-zero direct summand of X. By Theorem 1.2.2, Y is a direct sum of objects of a subfamily of $\{X_1, \ldots, X_n\}$. In particular, $Y \in \mathcal{A}$.

Conversely, suppose that $\mathcal{A}$ is closed under direct summands. Let $e : X \to X$ be a proper idempotent endomorphism. Then, $\mathrm{id}_X = e + (\mathrm{id}_X - e)$, where e and $\mathrm{id}_X - e$ are orthogonal idempotents in $\mathrm{End}_{\mathcal{A}}(X)$. By Lemma 1.3.1, e and $\mathrm{id}_X - e$ split in $\mathfrak{A}$. And by Lemma 1.1.4, $X = Y \oplus Z$ such that $e = q \circ p$, where $q : Y \to X$ is the canonical injection and canonical projection in $\mathfrak{A}$. In particular, Y is a direct summand of X, and hence, $Y \in \mathcal{A}$ by the assumption. Therefore, e splits in $\mathcal{A}$. By Corollary 1.2.3, $\mathcal{A}$ is Krull-Schmidt. The proof of the proposition is completed.

Let $f : X \to Y$ be a morphism in $\mathfrak{A}$. Recall that the **image** $\mathrm{Im}(f)$ of f is the kernel of its cokernel. The following statement follows from the definition of an abelian category.

1.3.3 **Lemma**. *Let $\mathfrak{A}$ be an abelian k-category. Every morphism $f : X \to Y$ in $\mathfrak{A}$ admits a canonical factorization $f = p \circ q$, where $p : X \to \mathrm{Im}(f)$ is an epimorphism $p : X \to \mathrm{Im}(f)$ and $q : \mathrm{Im}(f) \to Y$ is a monomorphism.*

Remark. In the sequel, the morphisms p, q in Lemma 1.3.3 will be called respectively the **canonical epimorphism** and the **canonical monomorphism** associated with f.

1.3.4 **Definition**. Let $\mathfrak{A}$ be an abelian k-category.

(1) An epimorphism $f : X \to Y$ in $\mathfrak{A}$ is called **superfluous** provided, for any morphism $g : U \to X$, that the composite $f \circ g$ is an epimorphism only if g is an epimorphism.

(2) A monomorphism $f : X \to Y$ in $\mathfrak{A}$ is called **essential** provided, for any morphism $h : Y \to V$, that the composite $h \circ f$ is a monomorphism only if h is a monomorphism.

An object P in $\mathfrak{A}$ is called **projective** provided, for any epimorphism $f : X \to Y$ and any morphism $g : P \to Y$, that there exists a morphism $h : P \to X$ such that $f \circ h = g$. Dually, an object I in $\mathfrak{A}$ is called **injective** provided, for any monomorphism $f : X \to Y$ and any morphism $g : X \to I$, that there exists a morphism $h : Y \to I$ such that $h \circ f = g$.

1.3.5 **Definition**. Let $\mathfrak{A}$ be an abelian k-category. Given an object X in $\mathfrak{A}$,

(1) a **projective cover** of X is a superfluous epimorphism $f : P \to X$ with P projective;

(2) an **injective envelope** of X is an essential monomorphism $g : X \to I$ with I injective.

Example. If P is a projective object in $\mathfrak{A}$, then $\mathrm{id}_P : P \to P$ is a projective cover of P. Dually, if I is an injective object in $\mathfrak{A}$, then $\mathrm{id}_I : I \to I$ is an injective envelope of I.

The projective covers and the injective envelope can be characterized in terms of minimal morphisms as follows; see [30, (3.4)] and its dual.

1.3.6 **Lemma**. *Let $\mathfrak{A}$ be an abelian k-category with an object X.*

(1) *An epimorphism $f : P \to X$ in $\mathfrak{A}$ with P projective is a projective cover if and only if it is right minimal; and in this case, f is unique up to isomorphism.*

(2) *A monomorphism $g : X \to I$ in $\mathfrak{A}$ with I injective is an injective envelope if and only if it is left minimal; and in this case, g is unique up to isomorphism.*

More generally, we have the following well known result. For the reader's convenience, we shall include a short proof.

1.3.7 **Lemma**. *Let $\mathfrak{A}$ be an abelian k-category with an object X.*

(1) *A morphism $f : P \to X$ in $\mathfrak{A}$ with P projective is right minimal if and only if the canonical epimorphism $f' : P \to \mathrm{Im}(f)$ is a projective cover of $\mathrm{Im}(f)$.*

(2) *A morphism $g : X \to I$ in $\mathfrak{A}$ with I injective is left minimal if and only if the canonical monomorphism $g' : \mathrm{Im}(g) \to I$ is an injective envelope of $\mathrm{Im}(g)$.*

Proof. We shall only prove Statement (1), since the proof of Statement (2) is dual. Let $f : P \to X$ be a morphism with P projective. Write $f = h \circ f'$, where $f' : P \to \mathrm{Im}(f)$ is the canonical epimorphism and $h : \mathrm{Im}(f) \to M$ is the canonical monomorphism. By Lemma 1.3.6, it amounts to show that f is right minimal if and only if so is f'.

Suppose that f is right minimal. Let $u : P \to P$ be such that $f' = f' \circ u$. Then, $f = h \circ f' = h \circ (f' \circ u) = f \circ u$. Since f is right minimal, u is an automorphism. Thus, f' is right minimal. Conversely, assume that f' is right minimal. Let $v : P \to P$ be such that $f = f \circ v$. Then, $h \circ f' = f = f \circ v = h \circ f' \circ v$. Since h is a monomorphism, $f' = f' \circ v$, and hence, v is an automorphism. So, f is right minimal. The proof of the lemma is completed.

In order to introduce exact sequences in $\mathfrak{A}$, we shall need the following easy statement.

1.3.8 **Lemma**. *Let $\mathfrak{A}$ be an abelian k-category. Consider a morphism $f : X \to Y$ with $q : \mathrm{Im}(f) \to Y$ the kernel of the cokernel of f, and a morphism $g : Y \to Z$ with $j : \mathrm{Ker}(g) \to Y$ the kernel of g. If $g \circ f = 0$, then there exists a canonical monomorphism $i : \mathrm{Im}(f) \to \mathrm{Ker}(g)$ such that $q = j \circ i$.*

Proof. Let $c : Y \to \mathrm{Coker}(f)$ be the cokernel of f. Suppose that $g \circ f = 0$. Then, there exists a unique morphism $u : \mathrm{Coker}(f) \to Z$ such that $u \circ c = g$. By the assumption, $q : \mathrm{Im}(f) \to Y$ is the kernel of c and $j : \mathrm{Ker}(g) \to Y$ is a kernel of g. Since $g \circ q = u \circ c \circ q = 0$, there exist a unique morphism $i : \mathrm{Im}(f) \to \mathrm{Ker}(g)$ such that $j \circ i = q$. Since q is a monomorphism, so is i. The proof of the lemma is completed.

Now, a finite or infinite sequence of at least two morphisms

$$\cdots \longrightarrow X_{n-1} \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} X_{n+1} \longrightarrow \cdots$$

in $\mathfrak{A}$ is called **exact** provided that there is an isomorphism $\mathrm{Im}(f_{n-1}) \cong \mathrm{Ker}(f_n)$ for each n.

1.3.9 **Definition**. Let $\mathfrak{A}$ be an abelian k-category with an object X.

(1) A **projective** n**-presentation** of X is an exact sequence

$$P^{-n} \xrightarrow{d^{-n}} P^{1-n} \longrightarrow \cdots \longrightarrow P^{-1} \xrightarrow{d^{-1}} P^0 \xrightarrow{d^0} X \longrightarrow 0$$

with the P^{-i} projective, which is called **minimal** if d^{-i} is right minimal, that is, it induces a projective cover $d^{-i} : P^{-i} \to \mathrm{Im}(d^{-i})$, for $i = 0, \ldots, n$.

(2) An **injective** n**-copresentation** of X is an exact sequence

$$0 \longrightarrow X \xrightarrow{d^0} I^0 \xrightarrow{d^1} I^1 \longrightarrow \cdots \longrightarrow I^n$$

with the I^i injective, which is called **minimal** if d^i is left minimal, that is, it induces an injective envelope $d^i : \mathrm{Im}(d^i) \to I^i$, for $i = 0, \ldots, n$.

In particular, a projective 1-presentation of X is called a **projective presentation**, and an injective 1-copresentation of X is called an **injective copresentation**. Observe that a minimal projective 0-presentation of X is simply a projective cover of X, while a minimal injective 0-copresentation of X is an injective envelope of X, More generally, one has the following notions.

1.3.10 **Definition**. Let $\mathfrak{A}$ be an abelian k-category with an object X.

(1) A **projective resolution** of X in $\mathfrak{A}$ is a semi-infinite exact sequence

$$\cdots \longrightarrow P^{-n} \xrightarrow{d^{-n}} P^{1-n} \longrightarrow \cdots \longrightarrow P^{-1} \xrightarrow{d^{-1}} P^{0} \xrightarrow{d^{0}} X \longrightarrow 0$$

with the P^{-n} projective, which is called **minimal** if $d^{-n} \neq 0$ is right minimal for every $n \geq 0$. Moreover, given such a projective resolution, the double infinite sequence

$$\cdots \longrightarrow P^{-n} \xrightarrow{d^{-n}} P^{1-n} \longrightarrow \cdots \longrightarrow P^{-1} \xrightarrow{d^{-1}} P^{0} \longrightarrow 0 \longrightarrow \cdots$$

will be called a **truncated projective resolution** of M.

(2) An **injective coresolution** of X in $\mathfrak{A}$ is a semi-infinite exact sequence

$$0 \longrightarrow X \xrightarrow{d^{0}} I^{0} \xrightarrow{d^{1}} I^{1} \longrightarrow \cdots \longrightarrow I^{n} \xrightarrow{d^{n}} I^{n+1} \longrightarrow \cdots$$

with the I^n injective, which is called **minimal** if $d^n \neq 0$ is left minimal for every $n \geq 0$. Moreover, given such an injective coresolution, the double infinite sequence

$$\cdots \longrightarrow 0 \longrightarrow I^{0} \xrightarrow{d^{1}} I^{1} \longrightarrow \cdots \longrightarrow I^{n} \xrightarrow{d^{n}} I^{n+1} \longrightarrow \cdots$$

will be called a **truncated injective coresolution** of X.

One says that $\mathfrak{A}$ has **enough projective objects** if every object X in $\mathfrak{A}$ admits an epimorphism $f : P \to X$ with P projective; and **enough injective objects** if every object Y in $\mathfrak{A}$ admits a monomorphism $g : Y \to I$ with I injective; see, for example [61, (2.2 and 2.3)]. The following statement is evident.

1.3.11 **Lemma**. *Let $\mathfrak{A}$ be an abelian k-category.*

(1) *If $\mathfrak{A}$ has enough projective objects, then every object in $\mathfrak{A}$ has a projective resolution.*

(2) *If $\mathfrak{A}$ has enough injective objects, then every object in $\mathfrak{A}$ has a injective coresolution.*

1.4 Extension groups

Throughout this section, let $\mathfrak{A}$ be an abelian k-category. Our objective is to introduce extension groups in $\mathfrak{A}$. For this purpose, we start with the following evident fact.

1.4.1 **Lemma**. *Let $\mathfrak{A}$ be an abelian k-category. A sequence*

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

in $\mathfrak{A}$ is exact if and only if f is a kernel of g, and g is a cokernel of f.

An exact sequence as stated in Lemma 1.4.1 will be called a **short exact sequence**. The following statement is well known; see [57, (4.1.7)].

1.4.2 **Proposition**. *Let $\mathfrak{A}$ be an abelian k-category. Given a short exact sequence $0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$ in $\mathfrak{A}$, the following statements are equivalent.*

(1) *The morphism f is a section.*

(2) *The morphism g is a retraction.*

(3) *There is an isomorphism of exact sequences*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & 0 \\ & & \| & & \downarrow{\cong} & & \| & & \\ 0 & \longrightarrow & X & \xrightarrow{\binom{\mathrm{id}_X}{0}} & X \oplus Z & \xrightarrow{(0,\mathrm{id}_Z)} & Z & \longrightarrow & 0. \end{array}$$

Remark. A short exact sequence $0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$ in $\mathfrak{A}$ is said to **split** if any of the equivalent statements in Proposition 1.4.2 holds true.

Now, we are ready to define the extension groups; see [42], [51], [57] and [61]. Fix two objects X, Y in $\mathfrak{A}$. A short exact sequence

$$\xi : 0 \longrightarrow Y \xrightarrow{f} E \xrightarrow{g} X \longrightarrow 0$$

is called an **extension** of Y by X. One says that it is **equivalent to** another extension $\xi' : 0 \longrightarrow Y \xrightarrow{f'} E' \xrightarrow{g'} X \longrightarrow 0$ if there exists a commutative diagram

$$\begin{array}{ccccccccc} \xi : 0 & \longrightarrow & Y & \xrightarrow{f} & E & \xrightarrow{g} & X & \longrightarrow & 0 \\ & & \| & & \downarrow u & & \| & & \\ \xi' : 0 & \longrightarrow & Y & \xrightarrow{f'} & E' & \xrightarrow{g'} & X & \longrightarrow & 0. \end{array}$$

In this case, u is an isomorphism. Thus, this yields clearly an equivalence relation on the extensions of Y by X. We shall write $[\xi]$ for the equivalence class of ξ. Moreover, one denotes by $\mathrm{Ext}^1_{\mathfrak{A}}(X, Y)$ the set of all equivalence classes of extensions of Y by X. Given a morphism $v : X' \to X$ in $\mathfrak{A}$, it is clear that we have a map

$$\mathrm{Ext}^1_{\mathfrak{A}}(v, Y) : \mathrm{Ext}^1_{\mathfrak{A}}(X, Y) \to \mathrm{Ext}^1_{\mathfrak{A}}(X', Y); [\xi] \mapsto [\xi{\cdot}v]$$

given by a pull-back diagram

$$\begin{array}{ccccccccc} \xi{\cdot}v : 0 & \longrightarrow & Y & \longrightarrow & E' & \longrightarrow & X' & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow v & & \\ \xi : 0 & \longrightarrow & Y & \longrightarrow & E & \longrightarrow & X & \longrightarrow & 0. \end{array}$$

Dually, given a morphism $u : Y \to Y'$ in $\mathfrak{A}$, we have a map

$$\mathrm{Ext}^1_{\mathfrak{A}}(X, u) : \mathrm{Ext}^1_{\mathfrak{A}}(X, Y) \to \mathrm{Ext}^1_{\mathfrak{A}}(X, Y'); [\xi] \mapsto [u{\cdot}\xi]$$

given by a push-out diagram

$$\begin{array}{ccccccccc} \xi : 0 & \longrightarrow & Y & \longrightarrow & E & \longrightarrow & X & \longrightarrow & 0 \\ & & \downarrow u & & \downarrow & & \| & & \\ u{\cdot}\xi : 0 & \longrightarrow & Y' & \longrightarrow & E' & \longrightarrow & X & \longrightarrow & 0. \end{array}$$

Now, we are ready to recall the **Baer sum** of extensions of Y by X; see, for example, [61, (3.4.4)]. Given two short exact sequences

$$\xi_1 : 0 \longrightarrow Y \longrightarrow E_1 \longrightarrow X \longrightarrow 0$$

$$\xi_2 : 0 \longrightarrow Y \longrightarrow E_2 \longrightarrow X \longrightarrow 0$$

in $\mathfrak{A}$, we consider their direct sum

$$\xi_1 \oplus \xi_2 : \qquad 0 \longrightarrow Y \oplus Y \longrightarrow E_1 \oplus E_2 \longrightarrow X \oplus X \longrightarrow 0$$

and two morphisms $\nabla = (\mathrm{id}_Y, \mathrm{id}_Y) : Y \oplus Y \to Y$ and

$$\triangle = \begin{pmatrix} \mathrm{id}_X \\ \mathrm{id}_X \end{pmatrix} : X \to X \oplus X$$

in $\mathfrak{A}$. Then, one sets

$$[\xi_1] + [\xi_2] = [\nabla \cdot (\xi_1 \oplus \xi_2) \cdot \triangle] \in \mathrm{Ext}^1_{\mathfrak{A}}(X, Y),$$

which is illustrated as follows:

$$\begin{array}{rccccccccc}
\xi_1 \oplus \xi_2 : & 0 & \longrightarrow & Y \oplus Y & \longrightarrow & E_1 \oplus E_2 & \longrightarrow & X \oplus X & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle \nabla} & & \downarrow & & \Vert & & \\
\nabla \cdot (\xi_1 \oplus \xi_2) : & 0 & \longrightarrow & Y & \longrightarrow & E' & \longrightarrow & X \oplus X & \longrightarrow & 0 \\
 & & & \Vert & & \uparrow & & \uparrow{\scriptstyle \triangle} & & \\
\nabla \cdot (\xi_1 \oplus \xi_2) \cdot \triangle : & 0 & \longrightarrow & Y & \longrightarrow & E & \longrightarrow & X & \longrightarrow & 0.
\end{array}$$

It is well known that this makes $\mathrm{Ext}^1_{\mathfrak{A}}(X, Y)$ into an abelian group, whose zero is the equivalence class of the split short exact sequence

$$0 \longrightarrow Y \xrightarrow{\binom{\mathrm{id}_Y}{0}} Y \oplus X \xrightarrow{(0, \mathrm{id}_X)} X \longrightarrow 0.$$

Indeed, in view of Lemmas 1.3 and 1.4 in [51, (VII)], we have the following well known result.

1.4.3 **Proposition**. *Let $\mathfrak{A}$ be an abelian k-category. If X, Y are objects in $\mathfrak{A}$, then $\mathrm{Ext}^1_{\mathfrak{A}}(X, Y)$ is an $\mathrm{End}_{\mathfrak{A}}(X)$-$\mathrm{End}_{\mathfrak{A}}(Y)$-bimodule such that*

$$u \cdot ([\xi] \cdot v) = [u \cdot (\xi \cdot v)],$$

for all $[\xi] \in \mathrm{Ext}^1_{\mathfrak{A}}(X, Y)$, $v \in \mathrm{End}_{\mathfrak{A}}(X)$ and $u \in \mathrm{End}_{\mathfrak{A}}(Y)$.

Remark. In particular, $\mathrm{Ext}^1_{\mathfrak{A}}(X, Y)$ is a k-vector space such that

$$\lambda \cdot [\xi] = [(\lambda \cdot \mathrm{id}_Y) \cdot \xi] = [\xi \cdot (\lambda \cdot \mathrm{id}_X)],$$

for all $\lambda \in k$ and $[\xi] \in \mathrm{Ext}^1_{\mathfrak{A}}(X, Y)$.

We shall need the following notion.

1.4.4 **Definition**. Let $\mathfrak{A}$ be an abelian k-category. A full additive subcategory $\mathcal{A}$ of $\mathfrak{A}$ is called **extension-closed** provided, for any short exact sequence $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ in $\mathfrak{A}$, that $Y \in \mathcal{A}$ whenever $X, Z \in \mathcal{A}$.

More generally, one can define the higher degree Yoneda extension groups $\mathrm{Ext}^n_{\mathfrak{A}}(X, Y)$ in $\mathfrak{A}$ for all $n > 1$; see, for example, [51, (VII.3)] and [61, (3.4.6)]. An n**-fold extension** of X by Y is an exact sequence

$$\mu : 0 \longrightarrow Y \xrightarrow{f_n} U_n \xrightarrow{f_{n-1}} \cdots \longrightarrow U_1 \xrightarrow{f_0} X \longrightarrow 0$$

in $\mathfrak{A}$. Fix μ be a n-fold extension of X by Y. We obtain the following short exact sequences from μ:

$$\xi_i : 0 \longrightarrow V_i \longrightarrow U_i \longrightarrow V_{i-1} \longrightarrow 0,$$

where $V_0 = X$, $V_n = Y$ and $V_i = \mathrm{Ker}(f_{i-1})$ for all $i = 1, \ldots, n-1$. In this case, we shall write $\mu = \xi_n \xi_{n-1} \cdots \xi_2 \xi_1$. Given two morphisms $u : Y \to Y'$ and $v : X' \to X$ in $\mathfrak{A}$, we define $u \cdot \mu = (u \cdot \xi_n)\xi_{n-1} \cdots \xi_2 \xi_1$ and $\mu \cdot v = \xi_n \xi_{n-1} \cdots \xi_2(\xi_1 \cdot v)$. Moreover, given two short exact sequences

$$\xi : 0 \longrightarrow Y \longrightarrow E \longrightarrow X \longrightarrow 0$$

$$\xi' : 0 \longrightarrow Y \longrightarrow E' \longrightarrow X \longrightarrow 0$$

in $\mathfrak{A}$ such that $(\xi \cdot v)\xi'$ is defined, we shall call a morphism $(\xi \cdot v)\xi' \to \xi \cdot (v\xi')$ is a **switch**. It is well known; see [51, (VII.3.1)] that there exists an equivalence relation on these n-fold extensions of X by Y such that two n-fold extensions $\mu = \xi_n \xi_{n-1} \cdots \xi_2 \xi_1$ and $\mu' = \xi'_n \xi'_{n-1} \cdots \xi'_2 \xi'_1$ are **equivalent** if μ can obtain from μ' by a finite number of switches. The equivalence class of μ will be denoted again by $[\mu]$. And one writes $\mathrm{Ext}^n_{\mathfrak{A}}(X, Y)$ for the set of equivalence classes of n-fold extensions of X by Y. Given two n-fold extensions μ and μ', one defines

$$[\mu] + [\mu'] = [\nabla \cdot (\mu \oplus \mu') \cdot \triangle],$$

where $\nabla \cdot (\mu \oplus \mu') \cdot \triangle$ is the n-fold extension

$$0 \to Y \to V \to U_{n-1} \oplus U'_{n-1} \to \cdots \to U_2 \oplus U'_2 \to W \to X \to 0$$

with V the push-out of $f_n \oplus f'_n : Y \oplus Y \to U_n \oplus U'_n$ and $\nabla : Y \oplus Y \to Y$, and W the pull-back of $f_0 \oplus f'_0 : U_0 \oplus U'_0 \to X \oplus X$ and $\triangle$: $X \to X \oplus X$. Equipped with this addition, $\mathrm{Ext}^n_{\mathfrak{A}}(X,Y)$ is an abelian group, whose zero is the equivalent class of the n-fold extension

$$0 \longrightarrow Y \xrightarrow{\mathrm{id}_Y} Y \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow X \xrightarrow{\mathrm{id}_X} X \longrightarrow 0.$$

As in the degree one case, $\mathrm{Ext}^n_{\mathfrak{A}}(X,Y)$ is an $\mathrm{End}_{\mathfrak{A}}(X)$-$\mathrm{End}_{\mathfrak{A}}(Y)$-bimodule; see [51, (VII.3.2)], and in particular, it is a k-vector space, for every $n > 1$.

The following statement is well known; see, [51, (VII.6.3)] and its dual, which is useful for calculating Yoneda extension groups.

1.4.5 **Lemma**. *Let $\mathfrak{A}$ be an abelian k-category with objects X, Y.*

(1) *If $\mathfrak{A}$ has an exact sequence*

$$0 \longrightarrow L^{-n} \xrightarrow{q} P^{-(n-1)} \longrightarrow \cdots \longrightarrow P^{-1} \longrightarrow P^0 \longrightarrow X \longrightarrow 0$$

with $n > 0$ and P^{-s} projective for $0 \le s < n$, then

$$\mathrm{Ext}^n_{\mathfrak{A}}(X,Y) \cong \mathrm{Hom}_{\mathfrak{A}}(L^{-n},Y)/\mathrm{ImHom}_{\mathfrak{A}}(q,Y)$$

and $\mathrm{Ext}^{n+i}_{\mathfrak{A}}(X,Y) \cong \mathrm{Ext}^i_{\mathfrak{A}}(L^{-n},Y)$ *for $i > 0$.*

(2) *If $\mathfrak{A}$ has an exact sequence*

$$0 \longrightarrow Y \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots \longrightarrow I^{n-1} \xrightarrow{p} L^n \longrightarrow 0$$

with $n > 0$ and I^s injective for $0 \le s < n$, then

$$\mathrm{Ext}^n_{\mathfrak{A}}(X,Y) \cong \mathrm{Hom}_{\mathfrak{A}}(X,L^n)/\mathrm{ImHom}_{\mathfrak{A}}(X,p)$$

and $\mathrm{Ext}^{n+i}_{\mathfrak{A}}(X,Y) \cong \mathrm{Ext}^i_{\mathfrak{A}}(X,L^n)$ *for $i > 0$.*

Remark. Lemma 1.4.5 is particularly useful for constructing finitely generated projective resolutions or finitely cogenerated injective coresolutions in concrete abelian categories.

In case $\mathfrak{A}$ has enough projective objects or enough injective objects, the following well known statement; see, for example, [51, (VII.7)] provides an alternative interpretation of Yoneda extension groups.

1.4.6 **Proposition**. *Let $\mathfrak{A}$ be an abelian k-category with objects X, Y.*

(1) *Given a projective resolution*

$$\cdots \longrightarrow P^{-n-1} \xrightarrow{d^{-n-1}} P^{-n} \xrightarrow{d^{-n}} P^{-n+1} \longrightarrow \cdots \longrightarrow P^0 \xrightarrow{d^0} X \longrightarrow 0$$

in $\mathfrak{A}$, for any $n > 0$, we have

$$\mathrm{Ext}^n_{\mathfrak{A}}(X, Y) \cong \mathrm{Ker}(\mathrm{Hom}_{\mathfrak{A}}(d^{-n-1}, Y))/\mathrm{Im}(\mathrm{Hom}_{\mathfrak{A}}(d^{-n}, Y)).$$

(2) *Given an injective coresolution*

$$0 \longrightarrow Y \xrightarrow{d^0} I^0 \longrightarrow \cdots \longrightarrow I^{n-1} \xrightarrow{d^{n-1}} I^n \xrightarrow{d^n} I^{n+1} \longrightarrow \cdots$$

in $\mathfrak{A}$, for any $n > 0$, we have

$$\mathrm{Ext}^n_{\mathfrak{A}}(X, Y) \cong \mathrm{Ker}(\mathrm{Hom}_{\mathfrak{A}}(X, d^n))/\mathrm{Im}(\mathrm{Hom}_{\mathfrak{A}}(X, d^{n-1})).$$

We conclude this section with the following well-known notion; see, for example, [10, page 5].

1.4.7 **Definition**. An abelian k-category $\mathfrak{A}$ is called **hereditary** provided that $\mathrm{Ext}^2_{\mathfrak{A}}(X, Y) = 0$ for all $X, Y \in \mathfrak{A}$.

The following statement is probably well known. However, we could not find it explicitly in the existing literature.

1.4.8 **Proposition**. *Let $\mathfrak{A}$ be an abelian category with enough projective objects. Then $\mathfrak{A}$ is hereditary if and only if the subobjects of a projective object are projective.*

Proof. The sufficiency follows immediately from Proposition 1.4.6(1). Suppose that $\mathfrak{A}$ is hereditary. Let P be a projective object in $\mathfrak{A}$ with a subobject M. Suppose that M is not projective. Since $\mathfrak{A}$ has enough projective objects, we have a non-split short exact sequence

$$0 \longrightarrow N \longrightarrow P' \xrightarrow{f} M \longrightarrow 0$$

in $\mathfrak{A}$ with P' projective. In particular, $\mathrm{Ext}^1_{\mathfrak{A}}(M,N) \neq 0$. Setting $g = j \circ f$, where $j : M \to P$ is the inclusion map, we obtain an exact sequence

$$0 \longrightarrow N \longrightarrow P' \xrightarrow{\;g\;} P \longrightarrow P/M \longrightarrow 0.$$

Since N has a projective resolution by the hypothesis, we deduce from Proposition 1.4.6(1) that

$$\mathrm{Ext}^2_{\mathfrak{A}}(P/M, N) \cong \mathrm{Ext}^1_{\mathfrak{A}}(M,N) \neq 0,$$

a contradiction. The proof of the proposition is completed.

1.5 Exact k-categories

Throughout this section let $\mathcal{A}$ be an **exact k-category**, that is an extension-closed additive full subcategory, closed under direct summands, of an abelian k-category $\mathfrak{A}$; see [39, Section 2]. We first recall the notion of stable categories of $\mathcal{A}$; see [35, 38], which is related to the existence of almost split sequences. Given $X, Y \in \mathcal{A}$, we shall write $\mathrm{Ext}^1_{\mathcal{A}}(X,Y) = \mathrm{Ext}^1_{\mathfrak{A}}(X,Y)$. A morphism $f : M \to N$ in $\mathcal{A}$ is called **projectively trivial** provided that, given any commutative diagram with exact rows

$$\begin{array}{lccccccccc}
\eta \cdot f: & 0 & \longrightarrow & X & \longrightarrow & Z & \longrightarrow & M & \longrightarrow & 0 \\
 & & & \| & & \downarrow & & \downarrow{\scriptstyle f} & & \\
\eta: & 0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & N & \longrightarrow & 0
\end{array}$$

in $\mathcal{A}$, the upper row splits; and **injectively trivial** provided that, given any commutative diagram with exact rows

$$\begin{array}{lccccccccc}
\delta: & 0 & \longrightarrow & M & \longrightarrow & Y & \longrightarrow & X & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle f} & & \downarrow & & \| & & \\
f \cdot \delta: & 0 & \longrightarrow & N & \longrightarrow & Z & \longrightarrow & X & \longrightarrow & 0
\end{array}$$

in $\mathcal{A}$, the lower row splits. An object $M \in \mathcal{A}$ is called **Ext-projective** if id_M is projectively trivial, or equivalently, $\mathrm{Ext}^1_{\mathcal{A}}(M,X) = 0$ for all $X \in \mathcal{A}$. Dually, an object $N \in \mathcal{A}$ is called **Ext-injective** if id_N is injectively trivial, or equivalently, $\mathrm{Ext}^1_{\mathcal{A}}(Y,N) = 0$ for all $Y \in \mathcal{A}$; see [38, page 9].

It is easy to see that the projectively trivial morphisms in $\mathcal{A}$ generate an ideal $\mathcal{P}(\mathcal{A})$, and the injectively trivial morphisms in $\mathcal{A}$ generate an ideal $\mathcal{I}(\mathcal{A})$. The quotient categories $\underline{\mathcal{A}} = \mathcal{A}/\mathcal{P}(\mathcal{A})$ and $\overline{\mathcal{A}} = \mathcal{A}/\mathcal{I}(\mathcal{A})$ are called the **projectively stable category** and the **injectively stable category** of $\mathcal{A}$, respectively. Given $X, Y \in \mathcal{A}$, we shall denote by $\mathcal{P}(X,Y)$ and $\mathcal{I}(X,Y))$ the k-vector subspace of $\mathrm{Hom}_{\mathcal{A}}(X,Y)$ of projectively trivial morphisms and that of injectively trivial morphisms, respectively, and we put

$$\underline{\mathrm{Hom}}_{\mathcal{A}}(X,Y) = \mathrm{Hom}_{\mathcal{A}}(X,Y)/\mathcal{P}(X,Y)$$

and

$$\overline{\mathrm{Hom}}_{\mathcal{A}}(X,Y) = \mathrm{Hom}_{\mathcal{A}}(X,Y)/\mathcal{I}(X,Y).$$

The following statement shows that for an abelian category with enough projective objects and enough injective objects, the stable categories as defined above coincide with the classical ones.

1.5.1 **Lemma**. *Let $\mathfrak{A}$ be an abelian k-category, having enough projective objects and enough injective objects.*

(1) *An object in $\mathfrak{A}$ is Ext-projective* (*respectively, Ext-injective*) *if and only if it is projective* (*respectively, injective*).

(2) *A morphism in $\mathfrak{A}$ is projectively* (*respectively, injectively*) *trivial if and only if it factors through a projective* (*respectively, injective*) *object.*

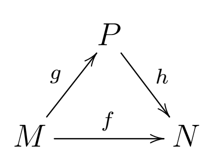
Proof. Since $\mathfrak{A}$ has enough projective objects and enough injective objects, Statement (1) is evident. For Statement (2), we shall only prove the first part, since the second part is dual. Let $f : M \to N$ be a morphism in $\mathfrak{A}$. Suppose that f is projectively trivial. By the hypothesis, there exists an exact sequence

$$\eta : \qquad 0 \longrightarrow K \longrightarrow P \xrightarrow{\ u\ } N \longrightarrow 0$$

in $\mathfrak{A}$ with P being projective. Consider the pull-back diagram

$$\begin{array}{rcccccccc}
\eta \cdot f : 0 & \longrightarrow & K & \longrightarrow & E & \longrightarrow & M & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow{\scriptstyle f} & & \\
\eta : 0 & \longrightarrow & K & \longrightarrow & P & \xrightarrow{\ g\ } & N & \longrightarrow & 0
\end{array}$$

in $\mathfrak{A}$. Since the upper row splits by hypothesis, f factors through u. In particular, it factors through P. Conversely, assume that there exists a commutative diagram



commutes in $\mathfrak{A}$ with P being projective. Consider a commutative diagram with exact rows

$$\begin{array}{lccccccccc}
\eta \cdot f: & 0 & \longrightarrow & X & \xrightarrow{u} & Z & \longrightarrow & M & \longrightarrow & 0\\
 & & & \| & & \downarrow & & \downarrow{\scriptstyle f} & & \\
\eta: & 0 & \longrightarrow & X & \longrightarrow & Y & \xrightarrow{v} & N & \longrightarrow & 0
\end{array}$$

in $\mathfrak{A}$. Since P is projective, $h = v \circ h'$ for some $h' : P \to v$, and consequently, $f = h \circ g = v \circ (h' \circ g)$. Now, it is easy to verify that u is a section. That is, the upper row in the above commutative diagram splits. The proof of the lemma is completed.

1.5.2 **Definition**. Let $\mathcal{A}$ be an exact k-category. A short exact sequence

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

in $\mathcal{A}$ is called an **almost split sequence** if f is minimal left almost split and g is minimal right almost split. In this case, we call X the **starting term** and Z the **ending term**, and write $X = \tau Z$ and $Z = \tau^{-} X$.

The almost split sequences in an exact category are characterized as follows; see [7, (2.14)], and also, [12, (7.9)].

1.5.3 **Theorem**. *Let $\mathcal{A}$ be an exact k-category with a short exact sequence*

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0.$$

The the following statements are equivalent.

(1) *The given sequence is almost split.*

(2) *The endomorphism ring of Z is local and f is left almost split.*

(3) *The endomorphism ring of X is local and g is right almost split.*

We say that $\mathcal{A}$ **has almost split sequences on the right** if every strongly indecomposable not Ext-projective object in $\mathcal{A}$ is the ending term of an almost split sequence; and in this case, τ is called the **right Auslander-Reiten translation** and that $\mathcal{A}$ **has almost split sequences on the left** if every strongly indecomposable not Ext-injective object in $\mathcal{A}$ is the starting term of an almost split sequence; and in this case, τ^{-} is called the **left Auslander-Reiten translation**. Moreover, one says that $\mathcal{A}$ **has almost split sequences** if it has almost split sequences on the right and on the left; see [39, page 5].

1.6 Triangulated k-categories

Throughout this section, let $\mathcal{T}$ be a **triangulated k-category** with translation functor [1]. That is, $\mathcal{T}$ is an additive k-category and [1] is an automorphism of $\mathcal{T}$, together with a class of sextuples

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1],$$

satisfying the properties (TR 1), (TR 2), (TR 3) and (TR 4); see[49, (II.1.1)]. In this case, a sextuple as stated above is called an **exact triangle**. An additive functor $F : \mathcal{T} \to \mathcal{T}'$ between triangulated k-categories is called **triangle-exact** if it commutes with the translation functors and sends exact triangles to exact triangles. The following notion is due to Happel; see [27, (4.1)].

1.6.1 **Definition**. Let $\mathcal{T}$ be a triangulated k-category. An exact triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{\delta} X[1]$$

in $\mathcal{T}$ is called an **almost split triangle** if f is minimal left almost split and g minimal right almost split. In this case, we call X the **starting term** and Z the **ending term**, and write $X = \tau Z$ and $Z = \tau^{-}X$.

One says that $\mathcal{T}$ **has almost split triangles on the right** (respectively, **left**) if every strongly indecomposable object in $\mathcal{T}$ is the ending (respectively, starting) term of an almost split triangle; and in this case, τ is called the **right** (respectively, **left**) **Auslander-Reiten translation**. And one says that $\mathcal{T}$ **has almost split triangles** if it has almost split triangles on the right and on the left. The following statement is due to Reiten and Van den Bergh; see [55, (I.2.3)].

1.6.2 **Theorem**. *Let $\mathcal{T}$ be a Hom-finite Krull-Schmidt triangulated k-category.*

(1) *$\mathcal{T}$ has almost split triangles on the right if and only if there exists a right Serre functor $\mathbb{S} : \mathcal{T} \to \mathcal{T}$; and in this case, $\tau X = \mathbb{S}(X)[-1]$ for any indecomposable object $X \in \mathcal{T}$.*

(2) *$\mathcal{T}$ has almost split triangles on the left if and only if there exists a left Serre functor $\mathbb{S} : \mathcal{T} \to \mathcal{T}$; and in this case, $\tau^{-}X = \mathbb{S}(X)[1]$ for any indecomposable object $X \in \mathcal{T}$.*

(3) *$\mathcal{T}$ has almost split triangles if and only if it admits a right Serre equivalence if and only if it admits a left Serre equivalence.*

1.7 Derived k-categories

Throughout this section, by an additive k-category we mean a strictly full additive subcategory of some abelian k-category. Let $\mathcal{A}$ stand for a strictly full additive k-subcategory of an abelian k-category $\mathfrak{A}$. A **complex** $(X^\bullet, d_X^\bullet)$, or simply $X^\bullet$, over $\mathcal{A}$ is a double infinite sequence

$$\cdots \longrightarrow X^{n-1} \xrightarrow{d_X^{n-1}} X^n \xrightarrow{d_X^n} X^{n+1} \longrightarrow \cdots, \quad n \in \mathbb{Z}$$

of morphisms in $\mathcal{A}$ such that $d_X^{n+1} \circ d_X^n = 0$ for all $n \in \mathbb{Z}$. The object X^n (May be equal to 0) is called the **component of degree** n and the morphism d_X^n is called the **differential of degree** n, of $X^\bullet$. A morphism $f^\bullet : X^\bullet \to Y^\bullet$ of complexes over $\mathcal{A}$ is a family of morphisms $f^n : X^n \to Y^n$ in $\mathcal{A}$ such that $f^{n+1} \circ d_X^n = d_Y^{n+1} \circ f^n$ for all $n \in \mathbb{Z}$. With these morphisms, the complexes over $\mathcal{A}$ form an additive k-category $C(\mathcal{A})$, called the **category of complexes** of $\mathcal{A}$. It is evident that $C(\mathfrak{A})$ is an abelian k-category. A complex $X^\bullet$ in $C(\mathcal{A})$ is called **bounded below** (respectively, **bounded above**) if there exists $n_0 \in \mathbb{Z}$ such that $X^n = 0$ for $n < n_0$ (respectively, for $n > n_0$); **bounded** if it is bounded below and bounded above; and **stalk complex concentrated in degree** s if $X^n = 0$ for any $n \neq s$. We shall denote by $C^b(\mathcal{A})$, $C^+(\mathcal{A})$ and $C^-(\mathcal{A})$ the full subcategories of $C(\mathcal{A})$ of bounded complexes, of bounded-below complexes and of bounded-above complexes, respectively.

Now, we shall define the cohomology functors of $\mathcal{A}$. First, we shall fix some notation for the differentials of a complex in the following statement, which follows immediately from Lemma 1.3.8.

1.7.1 **Lemma**. *Let $\mathcal{A}$ be an additive k-category, and let $X^\bullet$ be a complex in $C(\mathcal{A})$. Given $n \in \mathbb{Z}$, since $d_X^n \circ d_X^{n-1} = 0$, we have a commutative diagram*

$$\begin{array}{ccc} X^{n-1} & \xrightarrow{d_X^{n-1}} & X^n \\ {\scriptstyle p_X^{n-1}}\downarrow & \nearrow{\scriptstyle q_X^{n-1}} & \uparrow{\scriptstyle j_X^{n-1}} \\ \mathrm{Im}(d_X^{n-1}) & \xrightarrow{i_X^{n-1}} & \mathrm{Ker}(d_X^n), \end{array}$$

where p_X^{n-1} is the canonical epimorphism, q_X^{n-1} is the cokernel of d_X^{n-1}, and j_X^{n-1} is the kernel of d_X^n, while i_X^n is the canonical monomorphism as stated in Lemma 1.3.8(2).

The following statement is well known. However, we are not able to find any rigorous categorical proof.

1.7.2 **Lemma**. *Let $\mathcal{A}$ be an additive k-category, and let $f^\bullet : X^\bullet \to Y^\bullet$ be a morphism in $C(\mathcal{A})$. For any $n \in \mathbb{Z}$, there exists a commutative diagram*

$$\begin{array}{ccccc}
\mathrm{Im}(d_X^{n-1}) & \xrightarrow{i_X^{n-1}} & \mathrm{Ker}(d_X^n) & \xrightarrow{j_X^{n-1}} & X^n \\
{\scriptstyle \tilde{f}^n}\downarrow & & \downarrow{\scriptstyle \bar{f}^n} & & \downarrow{\scriptstyle f^n} \\
\mathrm{Im}(d_Y^{n-1}) & \xrightarrow{i_Y^{n-1}} & \mathrm{Ker}(d_Y^n) & \xrightarrow{j_Y^{n-1}} & Y^n
\end{array}$$

in $\mathcal{A}$, where the horizontal morphisms are the canonical morphisms as stated in Lemma 1.7.1.

Proof. Recall that j_X^{n-1} and j_Y^{n-1} are kernels of d_X^n and d_Y^n, respectively. Since $d_Y^n \circ f^n \circ j_X^{n-1} = f^{n+1} \circ d_X^n \circ j_X^{n-1} = 0$, we have $\bar{f}^n : \mathrm{Ker}(d_X^n) \to \mathrm{Ker}(d_Y^n)$ such that $f^n \circ j_X^{n-1} = j_Y^{n-1} \circ \bar{f}^n$. In view of Lemma 1.7.1, we obtain a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
 & & X^{n-1} & \xrightarrow{d_X^{n-1}} & X^n & \xrightarrow{c_X^{n-1}} & \mathrm{Coker}(d_X^{n-1}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle p_X^{n-1}} & & \| & & \| & & \\
0 & \longrightarrow & \mathrm{Im}(d_X^{n-1}) & \xrightarrow{q_X^{n-1}} & X^n & \xrightarrow{c_X^{n-1}} & \mathrm{Coker}(d_X^{n-1}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle i_X^{n-1}} & & \| & & & & \\
0 & \longrightarrow & \mathrm{Ker}(d_X^n) & \xrightarrow{j_X^{n-1}} & X^n & \xrightarrow{d_X^n} & X^{n+1} & & \\
 & & \downarrow{\scriptstyle \bar{f}^n} & & \downarrow{\scriptstyle f^n} & & \downarrow{\scriptstyle f^{n+1}} & & \\
0 & \longrightarrow & \mathrm{Ker}(d_Y^n) & \xrightarrow{j_Y^{n-1}} & Y^n & \xrightarrow{d_Y^n} & Y^{n+1} & & \\
 & & \uparrow{\scriptstyle i_Y^{n-1}} & & \| & & & & \\
0 & \longrightarrow & \mathrm{Im}(d_Y^{n-1}) & \xrightarrow{q_Y^{n-1}} & Y^n & \xrightarrow{c_Y^{n-1}} & \mathrm{Coker}(d_Y^{n-1}) & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle p_Y^{n-1}} & & \| & & \| & & \\
 & & Y^{n-1} & \xrightarrow{d_Y^{n-1}} & Y^n & \xrightarrow{c_Y^{n-1}} & \mathrm{Coker}(d_Y^{n-1}) & \longrightarrow & 0.
\end{array}$$

Now, $c_Y^{n-1} \circ f^n \circ d_X^{n-1} = c_Y^{n-1} \circ d_Y^{n-1} \circ f^{n-1} = 0$. Recall that p_X^{n-1} is an epimorphism. Thus, $c_Y^{n-1} \circ f^n \circ q_X^{n-1} = 0$, and hence, there exists $\tilde{f}^n : \mathrm{Im}(d_X^{n-1}) \to \mathrm{Im}(d_Y^{n-1})$ such that $f^n \circ q_X^{n-1} = q_Y^{n-1} \circ \tilde{f}^n$. This yields

$$\begin{aligned} j_Y^{n-1} \circ \bar{f}^n \circ i_X^{n-1} &= f^n \circ j_X^{n-1} \circ i_X^{n-1} \\ &= f^n \circ q_X^{n-1} \\ &= q_Y^{n-1} \circ \tilde{f}^{n-1} \\ &= j_Y^{n-1} \circ i_Y^{n-1} \circ \tilde{f}^{n-1}. \end{aligned}$$

Since j_Y^{n-1} is a monomorphism, $\bar{f}^n \circ i_X^{n-1} = i_Y^{n-1} \circ \tilde{f}^{n-1}$. The proof of the lemma is completed.

Let $X^\bullet$ be a complex in $C(\mathcal{A})$. Considering the canonical monomorphism i_X^{n-1} : $\mathrm{Im}(d_X^{n-1}) \to \mathrm{Ker}(d_X^n)$ as stated in Lemma 1.7.1, we define the n**-th cohomology group** of $X^\bullet$ to be the object

$$H^n(X^\bullet) = \mathrm{Ker}(d_X^n)/\mathrm{Im}(d_X^{n-1}) \in \mathfrak{A}.$$

Let $f^\bullet : X^\bullet \to Y^\bullet$ be a morphism in $C(\mathcal{A})$. For any $n \in \mathbb{Z}$, in view of Lemma 1.7.2, we obtain a unique morphism $H^n(f^\bullet) : H^n(X^\bullet) \to H^n(Y^\bullet)$ such that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Im}(d_X^{n-1}) & \xrightarrow{i_X^{n-1}} & \mathrm{Ker}(d_X^n) & \longrightarrow & H^n(X^\bullet) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \tilde{f}^n} & & \downarrow{\scriptstyle \bar{f}^n} & & \downarrow{\scriptstyle H^n(f^\bullet)} & & \\ 0 & \longrightarrow & \mathrm{Im}(d_Y^{n-1}) & \xrightarrow{i_Y^{n-1}} & \mathrm{Ker}(d_Y^{n-1}) & \longrightarrow & H^n(Y^\bullet) & \longrightarrow & 0 \end{array}$$

is a commutative diagram with exact rows. This yields an additive functor

$$H^n : C(\mathcal{A}) \to \mathfrak{A}; X^\bullet \mapsto H^n(X^\bullet); f^\bullet \mapsto H^n(f^\bullet),$$

called the n**-th cohomology functor** of $C(\mathcal{A})$, for every integer n. One says that a complex $X^\bullet$ is **acyclic** if $H^n(X^\bullet) = 0$ for all $n \in \mathbb{Z}$, and a morphism $f^\bullet$ is a **quasi-isomorphism** if $H^n(f^\bullet)$ is an isomorphism, for every $n \in \mathbb{Z}$.

Next, we prepare to introduce the homotopy categories. Let $X^\bullet$ be a complex in $C(\mathcal{A})$. Given an integer s, the s-**shift** of $X^\bullet$ is a complex $(X^\bullet[s], d^\bullet_{X^\bullet[s]})$ defined by $(X^\bullet[s])^n = X^{n+s}$ and $d^n_{X^\bullet[s]} = (-1)^s d_X^{n+s}$ for all $n \in \mathbb{Z}$. Moreover, the s-**shift** of a morphism $f^\bullet : X^\bullet \to Y^\bullet$ is the morphism $f^\bullet[s] : X^\bullet[s] \to Y^\bullet[s]$, defined by $(f^\bullet[s])^n = f^{n+s} : X^{n+s} \to Y^{n+s}$, for all $n \in \mathbb{Z}$. In particular, we have an automorphism

$$[1] : C(\mathcal{A}) \to C(\mathcal{A}); X^\bullet \to X^\bullet[1]; f^\bullet \mapsto f^\bullet[1],$$

called the **translation functor**, of $C(\mathcal{A})$. Moreover, the **twist complex** $\mathfrak{t}(X^\bullet)$ of $X^\bullet$ is the complex $(M^\bullet, d_M^\bullet)$ defined by $M^n = X^n$ and $d_M^n = -d_X^n$; see [10]. This induces an automorphism $\mathfrak{t}$ of $C(\mathcal{A})$, called the **twist functor** of $\mathcal{C}(\mathcal{A})$. Given a morphism $f^\bullet : X^\bullet \to Y^\bullet$ in $C(\mathcal{A})$, its **mapping cone** is the complex $C_{f^\bullet}$ defined by $C_{f^\bullet}^n = X^{n+1} \oplus Y^n$ and

$$d_{C_{f^\bullet}}^n = \begin{pmatrix} -d_X^{n+1} & 0 \\ f^{n+1} & d_Y^n \end{pmatrix},$$

for all $n \in \mathbb{Z}$. Clearly, we have two complex morphisms $i_{f^\bullet} : Y^\bullet \to C_{f^\bullet}$ and $p_{f^\bullet} : C_{f^\bullet} \to X^\bullet[1]$ defined by

$$i_{f^\bullet}^n = \begin{pmatrix} 0 \\ \mathrm{id}_{Y^n} \end{pmatrix} \text{ and } p_{f^\bullet}^n = (\mathrm{id}_{X^{n+1}}, 0),$$

for all $n \in \mathbb{Z}$. This yields a sextuple $X^\bullet \xrightarrow{f^\bullet} Y^\bullet \xrightarrow{i_{f^\bullet}} C_{f^\bullet} \xrightarrow{p_{f^\bullet}} X^\bullet[1]$ in $\mathcal{A}$, which is called the **standard triangle** attached to $f^\bullet$.

A morphism $f^\bullet : X^\bullet \to Y^\bullet$ in $C(\mathcal{A})$ is called **null-homotopic** if there exist morphisms $h^n : X^n \to Y^{n-1}$ in $\mathcal{A}$ such that $f^n = d_Y^{n-1} \circ h^n + h^{n+1} \circ d_X^n$ for all $n \in \mathbb{Z}$. And two morphisms $f^\bullet, g^\bullet : X^\bullet \to Y^\bullet$ are called **homotopic** if $f^\bullet - g^\bullet$ is null-homotopic. The following statement is well known; see, for example, [49, (III.1.4.1)]. Here, we provide a rigorous categorical proof.

1.7.3 **Lemma**. *Let $\mathcal{A}$ be an additive k-subcategory of an abelian k-category $\mathfrak{A}$. Consider a morphism $f^\bullet : X^\bullet \to Y^\bullet$ be in $C(\mathcal{A})$. If $f^\bullet$ is null-homotopic, then $H^n(f^\bullet) = 0$, for every $n \in \mathbb{Z}$.*

Proof. Let $h^n : X^n \to Y^{n-1}$ be morphisms such that $f^n = d_Y^{n-1} \circ h^n + h^{n+1} \circ d_X^n$ for all $n \in \mathbb{Z}$. As stated in the proof of Lemma 1.7.2, we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Ker}(d_X^n) & \xrightarrow{j_X^{n-1}} & X^n & \xrightarrow{d_X^n} & X^{n+1} & & \\
 & & \downarrow{\scriptstyle \bar f^n} & & \downarrow{\scriptstyle f^n} & & \downarrow{\scriptstyle f^{n+1}} & & \\
0 & \longrightarrow & \mathrm{Ker}(d_Y^n) & \xrightarrow{j_Y^{n-1}} & Y^n & \xrightarrow{d_Y^n} & Y^{n+1} & & \\
 & & \uparrow{\scriptstyle i_Y^{n-1}} & & \| & & & & \\
0 & \longrightarrow & \mathrm{Im}(d_Y^{n-1}) & \xrightarrow{q_Y^{n-1}} & Y^n & \xrightarrow{c_Y^{n-1}} & \mathrm{Coker}(d_Y^{n-1}) & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle p_Y^{n-1}} & & \| & & \| & & \\
 & & Y^{n-1} & \xrightarrow{d_Y^{n-1}} & Y^n & \xrightarrow{c_Y^{n-1}} & \mathrm{Coker}(d_Y^{n-1}) & \longrightarrow & 0
\end{array}$$

in $\mathfrak{A}$. Thus,

$$\begin{aligned} j_Y^{n-1} \circ \bar{f}^n &= f^n \circ j_X^{n-1} \\ &= (d_Y^{n-1} \circ h^n + h^{n+1} \circ d_X^n) \circ j_X^{n-1} \\ &= d_Y^{n-1} \circ h^n \circ j_X^{n-1} \\ &= j_Y^{n-1} \circ i_Y^{n-1} \circ p_Y^{n-1} \circ h^n \circ j_X^{n-1}. \end{aligned}$$

Since j_Y^{n-1} is a monomorphism, $\bar{f}^n = i_Y^{n-1} \circ p_Y^{n-1} \circ h^n \circ j_X^{n-1}$. On the other hand, by definition, we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Im}(d_X^{n-1}) & \xrightarrow{i_X^{n-1}} & \mathrm{Ker}(d_X^n) & \xrightarrow{u_X^n} & H^n(X^\bullet) & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \tilde{f}^n} & & \Big\downarrow{\scriptstyle \bar{f}^n} & & \Big\downarrow{\scriptstyle H^n(f^\bullet)} & & \\ 0 & \longrightarrow & \mathrm{Im}(d_Y^{n-1}) & \xrightarrow{i_Y^{n-1}} & \mathrm{Ker} d_Y^{n-1} & \xrightarrow{u_Y^n} & H^n(Y^\bullet) & \longrightarrow & 0 \end{array}$$

in $\mathfrak{A}$. Thus, $u_Y^n \circ \bar{f}^n = u_Y^n \circ i_Y^{n-1} \circ p_Y^{n-1} \circ h^n \circ j_X^{n-1} = 0$, and consequently, $H^n(f^\bullet) \circ u_X^n = 0$. Since u_X^n is an epimorphism, $H^n(f^\bullet) = 0$. The proof of the lemma is completed.

Consider $C^*(\mathcal{A})$, where $* \in \{\emptyset, -, +, b\}$. Observing that the null-homotopic morphisms in $C^*(\mathcal{A})$ form an ideal, one defines the **homotopy category** $K^*(\mathcal{A})$ to be the quotient category of $C^*(\mathcal{A})$ modulo the null-homotopic morphisms. In particular, $K(\mathcal{A})$ is called the **homotopy category** of $\mathcal{A}$. We denote by $P_{\mathcal{A}}^* : C^*(\mathcal{A}) \to K^*(\mathcal{A})$ the canonical projection functor. Given a morphism $f^\bullet : X^\bullet \to Y^\bullet$ in $C^*(\mathcal{A})$, we obtain a morphism $\bar{f}^\bullet = P_{\mathcal{A}}^*(f^\bullet) : X^\bullet \to Y^\bullet$ in $K^*(\mathcal{A})$. We call the sextuple

$$X^\bullet \xrightarrow{\bar{f}^\bullet} Y^\bullet \xrightarrow{\bar{i}_{f^\bullet}} C_{f^\bullet} \xrightarrow{\bar{p}_{f^\bullet}} X^\bullet[1]$$

the **standard triangle** in $K^*(\mathcal{A})$ attached to $\bar{f}^\bullet$, and $C_{f^\bullet}$ the **mapping cone** of $\bar{f}^\bullet$. It is well known; see [49, (III.2.1.1)] that $K^*(\mathcal{A})$ is a triangulated category, whose translation functor is the automorphism induced from the shift functor of $C(\mathcal{A})$ and the exact triangles are the sextuples isomorphic to standard triangles in $K^*(\mathcal{A})$.

Consider two homotopic morphisms $f^\bullet, g^\bullet : X^\bullet \to Y^\bullet$ in $C^*(\mathcal{A})$. In view of Lemma 1.7.3, we see that $H^n(f^\bullet) = H^n(g^\bullet)$, for all $n \in \mathbb{Z}$. In particular, $f^\bullet$ is a quasi-isomorphism if and only if so is $g^\bullet$. Thus, one calls $\bar{f}^\bullet$ a **quasi-isomorphism** in $K^*(\mathcal{A})$ if $f^\bullet$ is a quasi-isomorphism in $C^*(\mathcal{A})$. We shall need the following well known result; see, for example, [49, (III.3.1.1)].

1.7.4 **Lemma**. *Let $\mathcal{A}$ be an additive k-category. Then, a morphism in $K^*(\mathcal{A})$ with $* \in \{\emptyset, +, -, b\}$ is a quasi-isomorphism if and only if its mapping cone is acyclic.*

Finally, we shall introduce derived categories. Fix $* \in \{\emptyset, -, +, b\}$. It is well known; see [49, (III.3.1.2)] that the class of quasi-isomorphisms in $K^*(\mathcal{A})$ is a localizing class compatible with the triangulation; see, for definition, [49, Chapter I, Section 1.3] and [49, Chapter II, Section 1.6]. One defines the **derived category** $D^*(\mathcal{A})$ to be the localization of $K^*(\mathcal{A})$ with respect to quasi-isomorphisms, which is a triangulated k-category with translation functor and exact triangles induced from those of $K^*(\mathcal{A})$; see [49, (II.1.6.1)]. We shall denote by $L^*_{\mathcal{A}} : K^*(\mathcal{A}) \to D^*(\mathcal{A})$ the localization functor. In particular, $D(\mathcal{A})$ and $D^b(\mathcal{A})$ are called the **derived category** and the **bounded derived category** of $\mathcal{A}$, respectively. Clearly, we have a canonical additive functor

$$j^*_{\mathcal{A}} : \mathcal{A} \to D^*(\mathcal{A}); X \mapsto X[0]; f \mapsto f[0].$$

The following statement is well known; see, for example, [49, Chapter III, (3.4.3), (3.4.4), (3.4.5), (3.4.7)].

1.7.5 **Proposition**. *Let $\mathfrak{A}$ be an abelian k-category. Given $* \in \{\emptyset, -, +, b\}$, the following statements hold.*

(1) *The canonical functor $j^*_{\mathcal{A}} : \mathfrak{A} \to D^*(\mathfrak{A})$ is fully faithful.*

(2) *There exists a fully faithful triangle-exact functor $i^* : D^*(\mathfrak{A}) \to D(\mathfrak{A})$.*

Remark. In the sequel, we shall regard $D^*(\mathfrak{A})$ as a full triangulated subcategory of $D(\mathfrak{A})$ for $* \in \{-, +, b\}$.

More generally, we have the following statement.

1.7.6 **Proposition**. *Let $\mathscr{A}$ be a full additive subcategory of $C(\mathfrak{A})$, where $\mathfrak{A}$ is an abelian k-category. If $\mathscr{A}$ is closed under shifts, then*

(1) *the quotient category $\mathcal{K}(\mathscr{A})$ of $\mathscr{A}$ modulo the null-homotopic morphisms is a full triangulated subcategory of $K(\mathfrak{A})$;*

(2) *the localization $\mathcal{D}(\mathscr{A})$ of $\mathcal{K}(\mathscr{A})$ with respect to quasi-isomorphisms is a triangulated category.*

Proof. We sketch a proof of this statement. It is evident that the null-homotopic morphisms in $\mathscr{A}$ form an ideal of $\mathscr{A}$. Thus, the quotient category $\mathcal{K}(\mathscr{A})$ of $\mathscr{A}$ modulo the null-homotopic morphisms is a full additive subcategory of $K(\mathfrak{A})$. Assume that $\mathscr{A}$ is closed under shifts. Then, $\mathscr{A}$ is closed under mapping cones. Therefore, $\mathcal{K}(\mathscr{A})$ is closed under the translation functor of $K(\mathfrak{A})$ and under mapping cones. Hence, $\mathcal{K}(\mathscr{A})$ is a full triangulated subcategory of $K(\mathfrak{A})$; see [49, Chapter II, Section 1.7]. Moreover, in view of the proof of Proposition 3.1.2 in [49, Chapter III], we see that the quasi-isomorphisms in $\mathcal{K}(\mathscr{A})$ form a localizing class compatible with the triangulation. Therefore, $\mathcal{D}(\mathscr{A})$ is a triangulated category; see [49, (II.1.6.1)]. The proof of the proposition is completed.

Remark. In the sequel, we shall say that a full additive subcategory $\mathscr{A}$ of $C(\mathfrak{A})$ is **derivable** if it is closed under shifts, and in this case, $\mathcal{D}(\mathscr{A})$ is called the **category derived from** $\mathscr{A}$. Some sufficient conditions for $\mathcal{D}(\mathscr{A})$ to be a full triangulated subcategory of $D(\mathfrak{A})$ can be found in [49, (II.1.7.1), (II.1.7.2)].

Let $X^\bullet$ be a complex in $C(\mathfrak{A})$. A complex of projective objects $P^\bullet$ is called a **projective resolution** of $X^\bullet$ if there exists a quasi-isomorphism $f^\bullet : P^\bullet \to X^\bullet$ in $C(\mathfrak{A})$, which is **finite** if $P^\bullet$ is a bounded complex. Dually, a complex $I^\bullet$ of injective objects is called an **injective coresolution** of $X^\bullet$ if there exists a quasi-isomorphism $g^\bullet : X^\bullet \to I^\bullet$ in $C(\mathfrak{A})$, which is **finite** if $I^\bullet$ is a bounded complex. The following statement is evident.

1.7.7 **Lemma**. *Let $\mathfrak{A}$ be an abelian k-category. Consider a complex $X^\bullet$ in $C^*(\mathfrak{A})$ with $* \in \{\emptyset, +, -, b\}$.*

(1) *If $X^\bullet$ admits a projective resolution $P^\bullet$ in $C^*(\mathfrak{A})$, then $X^\bullet \cong P^\bullet$ in $D^*(\mathfrak{A})$.*

(2) *If $X^\bullet$ admits an injective coresolution $I^\bullet$ in $C^*(\mathfrak{A})$, then $X^\bullet \cong I^\bullet$ in $D^*(\mathfrak{A})$.*

The following statement; see [61, (10.4.7)] says that it is easy to compute the morphisms in the derived category starting (respectively, ending) in a complex with a bounded above projective resolution (respectively, bounded below injective so-resolution).

1.7.8 **Lemma**. *Let $\mathfrak{A}$ be an abelian k-category. Let $P^\bullet \in C^-(\mathfrak{A})$ be a complex of projective objects and $I^\bullet \in C^+(\mathfrak{A})$ a complex of injective objects. Given any* *complex $X^\bullet$ over $\mathfrak{A}$, the localization functor $L : K(\mathfrak{A}) \to D(\mathfrak{A})$ induces two isomorphisms*

$$L_{P^\bullet, X^\bullet} : \mathrm{Hom}_{K(\mathfrak{A})}(P^\bullet, X^\bullet) \to \mathrm{Hom}_{D(\mathfrak{A})}(P^\bullet, X^\bullet)$$

and

$$L_{X^\bullet, I^\bullet} : \mathrm{Hom}_{K(\mathfrak{A})}(X^\bullet, I^\bullet) \to \mathrm{Hom}_{D(\mathfrak{A})}(X^\bullet, I^\bullet).$$

We shall need the following result.

1.7.9 **Lemma**. *Let $\mathfrak{A}$ be an abelian k-category.*

(1) *If $\mathcal{P}$ is a full additive subcategory of projective objects of $\mathfrak{A}$, then $K^b(\mathcal{P})$ can be regarded as a full triangulated subcategory of $D^b(\mathfrak{A})$.*

(2) *If $\mathcal{I}$ is a full additive subcategory of injective objects of $\mathfrak{A}$, then $K^b(\mathcal{I})$ can be regarded as a full triangulated subcategory of $D^b(\mathfrak{A})$.*

Proof. We shall only prove Statement (1). It is evident that $K^b(\mathcal{P})$ is a full triangulated subcategory of $K^b(\mathfrak{A})$. Restricting the localization functor $L^b_{\mathfrak{A}} : K^b(\mathfrak{A}) \to D^b(\mathfrak{A})$, we obtain a triangle-exact functor $j : K^b(\mathcal{P}) \to D^b(\mathfrak{A})$. Let $P^\bullet, U^\bullet \in K^b(\mathcal{P})$. Since $K^b(\mathcal{P})$ is a full subcategory of $K(\mathcal{P})$, we deduce from Lemma 1.7.8 and Proposition 1.7.5(2) that

$$\mathrm{Hom}_{K^b(\mathcal{P})}(P^\bullet, U^\bullet) = \mathrm{Hom}_{K(\mathfrak{A})}(P^\bullet, U^\bullet) \cong \mathrm{Hom}_{D(\mathfrak{A})}(P^\bullet, U^\bullet) \cong \mathrm{Hom}_{D^b(\mathfrak{A})}(P^\bullet, U^\bullet).$$

The proof of the lemma is completed.

Remark. We see from Proposition 1.7.5(2) that $K^b(\mathcal{P})$ and $K^b(\mathcal{I})$ can also be regarded as a full triangulated subcategories of $D(\mathfrak{A})$.

To conclude this section, we shall study when an additive functor between categories of complexes induces a triangle-exact functor between derived categories. The following statement is well known; see [49, (II.1.6.2)]. For the convenience of the reader, we will provide a brief proof.

1.7.10 **Proposition**. *Let $\mathcal{A}$ and $\mathcal{B}$ be additive k-categories. Consider an additive functor $\mathfrak{F} : C^*(\mathcal{A}) \to C^*(\mathcal{B})$, where $* \in \{\emptyset, -, +, b\}$, such that*

(1) $\mathfrak{F} \circ [1] \cong [1] \circ \mathfrak{F}$;

(2) $\mathfrak{F}$ *sends acyclic complexes to acyclic complexes.*

(3) $\mathfrak{F}$ *sends the cone of a morphism to the cone of its image*;

(4) $\mathfrak{F}$ *sends null-homotopic morphisms to null-homotopic morphisms*;

Then, $\mathfrak{F}$ *induces a commutative diagram of additive functors*

$$\begin{array}{ccccc} C^*(\mathcal{A}) & \xrightarrow{P^*_{\mathcal{A}}} & K^*(\mathcal{A}) & \xrightarrow{L^*_{\mathcal{A}}} & D^*(\mathcal{A}) \\ \downarrow{\scriptstyle \mathfrak{F}} & & \downarrow{\scriptstyle \overline{\mathfrak{F}}} & & \downarrow{\scriptstyle \tilde{\mathfrak{F}}} \\ C^*(\mathcal{B}) & \xrightarrow{P^*_{\mathcal{B}}} & K^*(\mathcal{B}) & \xrightarrow{L^*_{\mathcal{B}}} & D^*(\mathcal{B}), \end{array}$$

where $\overline{\mathfrak{F}}$ *and* $\tilde{\mathfrak{F}}$ *are triangle-exact..*

Proof. By Statement (4), $\mathfrak{F}$ induces an additive functor $\overline{\mathfrak{F}} : K^*(\mathcal{A}) \to K^*(\mathcal{B})$ such that $\overline{\mathfrak{F}} \circ P^*_{\mathcal{A}} = P^*_{\mathcal{B}} \circ \mathfrak{F}$. By Statements (1) and (3), $\overline{\mathfrak{F}}$ is triangle-exact. Let $\bar{f}^\bullet : X^\bullet \to Y^\bullet$ in $K^*(\mathcal{A})$ be a quasi-isomorphism. Then, there is a standard triangle $X^\bullet \xrightarrow{\bar{f}^\bullet} Y^\bullet \xrightarrow{\bar{i}_{f^\bullet}} C_{f^\bullet} \xrightarrow{\bar{p}_{f^\bullet}} X^\bullet[1]$ in $K^*(\mathcal{A})$, where $C_{f^\bullet}$ is acyclic by Lemma 1.7.4. By Statements (1) and (3), we have a standard triangle

$$\overline{\mathfrak{F}}(X^\bullet) \xrightarrow{\overline{\mathfrak{F}}(\bar{f}^\bullet)} \overline{\mathfrak{F}}(Y^\bullet) \xrightarrow{\overline{\mathfrak{F}}(\bar{i}_{f^\bullet})} \overline{\mathfrak{F}}(C_{f^\bullet}) \xrightarrow{\overline{\mathfrak{F}}(\bar{p}_{f^\bullet})} \overline{\mathfrak{F}}(X^\bullet)[1]$$

in $K^*(\mathfrak{B})$, where $\overline{\mathfrak{F}}(C_{f^\bullet})$ is acyclic by Statement (2). Again by Lemma 1.7.4, $\overline{\mathfrak{F}}(\bar{f}^\bullet)$ is a quasi-isomorphism. This shows that $\overline{\mathfrak{F}}$ sends quasi-isomorphisms to quasi-isomorphisms. By the universal property of localization, there exists a unique triangle-exact functor $\tilde{\mathfrak{F}} : D^*(\mathcal{A}) \to D^*(\mathcal{B})$ such that $\tilde{\mathfrak{F}} \circ L^*_{\mathcal{A}} = L^*_{\mathcal{B}} \circ \overline{\mathfrak{F}}$. The proof of the proposition is completed.

Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between additive k-categories. Then, it induces an additive functor $F^C : C^*(\mathcal{A}) \to C^*(\mathcal{B})$ as follows. Given a complex $X^\bullet$ in $C^*(\mathcal{A})$, we have a complex $F^C(X^\bullet) \in C^*(\mathcal{B})$ defined by $F^C(X^\bullet)^n = F(X^n)$ and $d^n_{F^C(X^\bullet)} = F(d^n_X) : F(X^n) \to F(X^{n+1})$, for all $n \in \mathbb{Z}$. And given a morphism $f^\bullet : X^\bullet \to Y^\bullet$ in $C^*(\mathcal{A})$, we have a morphism $F^C(f^\bullet) : F^C(X^\bullet) \to F^C(Y^\bullet)$ in $C^*(\mathcal{B})$ defined by $F^C(f^\bullet)^n = F(f^n) : F(X^n) \to F(Y^n)$ for all $n \in \mathbb{Z}$. The following statement is well known; see, for example, [49, (V.1.1.1)] and [49, (V.1.2.2)]. For the sake of the reader's convenience, we shall include a short proof.

1.7.11 **Proposition**. *Let $F : \mathfrak{A} \to \mathfrak{B}$ be an exact functor between abelian k-categories. Given $* \in \{\emptyset, -, +, b\}$, there exists a commutative diagram of functors*

$$\begin{array}{ccccc} C^*(\mathfrak{A}) & \xrightarrow{P^*_{\mathfrak{A}}} & K^*(\mathfrak{A}) & \xrightarrow{L^*_{\mathfrak{A}}} & D^*(\mathfrak{A}) \\ \downarrow{\scriptstyle F^C} & & \downarrow{\scriptstyle F^K} & & \downarrow{\scriptstyle F^D} \\ C^*(\mathfrak{B}) & \xrightarrow{P^*_{\mathfrak{B}}} & K^*(\mathfrak{B}) & \xrightarrow{L^*_{\mathfrak{B}}} & D^*(\mathfrak{B}), \end{array}$$

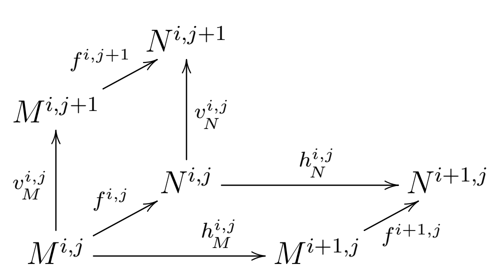
where F^K and F^D are triangle-exact.

Proof. It is evident that F^C has the properties in Statements (1) and (4) of Proposition 1.7.10. As shown in Section 1.1 in [49, Chapter V], F^C sends the cone of a morphism in $C^*(\mathfrak{A})$ to the cone of its image. Finally, since F is exact, F^C sends acyclic complexes in $C^*(\mathfrak{A})$ to acyclic complexes. Now, the proposition follows from Proposition 1.7.10. The proof of the proposition is completed.

1.8 Double complexes

In order to provide a tool for constructing our generalized Koszul duality, we shall recall the double categories of complexes as defined in [16, section 4]. An additive k-category is called **concrete** if the objects are equipped with a k-vector space structure, which is compatible with the composition of morphisms. Throughout this section, $\mathcal{A}, \mathcal{B}, \mathcal{C}$ stand for full additive subcategories of concrete abelian k-categories.

Let $(M^{\bullet\bullet}, v_M^{\bullet\bullet}, h_M^{\bullet\bullet})$ be a double complex over $\mathcal{A}$, where $v_M^{\bullet\bullet}$ is the vertical differential and $h_M^{\bullet\bullet}$ is the horizontal one. Given $i, j \in \mathbb{Z}$, the complexes $(M^{i,\bullet}, v_M^{i,\bullet})$ and $(M^{\bullet,j}, h_M^{\bullet,j})$ are called the **i-th column** and the **j-th row** of $M^{\bullet\bullet}$, respectively. A double complex morphism $f^{\bullet\bullet} : M^{\bullet\bullet} \to N^{\bullet\bullet}$ consists of morphisms $f^{i,j} : M^{i,j} \to N^{i,j}$ in $\mathcal{A}$ with $i, j \in \mathbb{Z}$ making



commute for all $i, j \in \mathbb{Z}$, that is, $f^{i,\bullet} : M^{i,\bullet} \to N^{i,\bullet}$ and $f^{\bullet,j} : M^{\bullet,j} \to N^{\bullet,j}$ are complex morphisms, called the i-**column** and the j-**th row** of $f^{\bullet\bullet}$ respectively, for all $i, j \in \mathbb{Z}$. In this way, the double complexes over $\mathcal{A}$ form an additive k-category written as $DC(\mathcal{A})$. In case $\mathcal{A}$ has countable direct sums, we shall define a functor $\mathbb{T} : DC(\mathcal{A}) \to C(\mathcal{A})$ as follows. Given $M^{\bullet\bullet} \in DC(\mathcal{A})$, one defines its **total complex** $\mathbb{T}(M^{\bullet\bullet}) \in C(\mathcal{A})$ by setting $\mathbb{T}(M^{\bullet\bullet})^n = \oplus_{i\in\mathbb{Z}} M^{i,n-i}$ and

$$d^n_{\mathbb{T}(M^{\bullet\bullet})} = (d^n_{\mathbb{T}(M^{\bullet\bullet})}(j,i))_{(j,i)\in\mathbb{Z}\times\mathbb{Z}} : \oplus_{i\in\mathbb{Z}} M^{i,n-i} \to \oplus_{j\in\mathbb{Z}} M^{j,n+1-j},$$

where $d^n_{\mathbb{T}(M^{\bullet\bullet})}(j,i) : M^{i,n-i} \to M^{j,n+1-j}$ is defined such that $d^n_{\mathbb{T}(M^{\bullet\bullet})}(i,i) = v^{i,n-i}_{_M}$; $d^n_{\mathbb{T}(M^{\bullet\bullet})}(i+1,i) = h^{i,n-i}_{_M}$ and $d^n_{\mathbb{T}(M^{\bullet\bullet})}(j,i) = 0$ if $j \notin i$ or $i+1$. Given a morphism $f^{\bullet\bullet} : M^{\bullet\bullet} \to N^{\bullet\bullet}$, we define its **total morphism** $\mathbb{T}(f^{\bullet\bullet}) : \mathbb{T}(M^{\bullet\bullet}) \to \mathbb{T}(N^{\bullet\bullet})$ by setting

$$\mathbb{T}(f^{\bullet\bullet})^n = (\mathbb{T}\,(f^{\bullet\bullet})^n(j,i))_{(j,i)\in\mathbb{Z}\times\mathbb{Z}} : \oplus_{i\in\mathbb{Z}} M^{i,n-i} \to \oplus_{j\in\mathbb{Z}} N^{j,n-j}$$

for $n \in \mathbb{Z}$, where $\mathbb{T}(f^{\bullet\bullet})^n(j,i) : M^{i,n-i} \to N^{j,n-j}$ is given by $\mathbb{T}(f^{\bullet\bullet})^n(i,i) = f^{i,n-i}$ and $\mathbb{T}(f^{\bullet\bullet})^n(j,i) = 0$ for all $j \neq i$.

We shall study when the total complex of a double complex is acyclic. For this purpose, we need some terminology. Let $M^{\bullet\bullet} \in DC(\mathcal{A})$. Given $n \in \mathbb{Z}$, the n-**diagonal** of $M^{\bullet\bullet}$ consists of $M^{i,n-i}$ with $i \in \mathbb{Z}$. We shall say that $M^{\bullet\bullet}$ is n-**diagonally bounded** (respectively, **bounded-above**, **bounded-below**) if $M^{i,n-i} = 0$ for all but finitely many (respectively, positive, negative) integers i. Moreover, $M^{\bullet\bullet}$ is called **diagonally bounded** (respectively, **bounded-above**, **bounded-below**) if it is n-diagonally bounded (respectively, bounded-above, bounded-below) for every $n \in \mathbb{Z}$.

The following two statements; see [16, (4.2), (4.3)] tell us when the total complex of a double complex is acyclic.

1.8.1 **Lemma**. *Let $\mathcal{A}$ be a concrete additive category with countable direct sums. If $M^{\bullet\bullet} \in DC(\mathcal{A})$ and $n \in \mathbb{Z}$, then $\mathrm{H}^n(\mathbb{T}(M^{\bullet\bullet})) = 0$ in case*

(1) *$M^{\bullet\bullet}$ is n-diagonally bounded-below with $\mathrm{H}^{n-j}(M^{\bullet,j}) = 0$ for all $j \in \mathbb{Z}$; or*

(2) *$M^{\bullet\bullet}$ is n-diagonally bounded-above with $\mathrm{H}^{n-i}(M^{i,\bullet}) = 0$ for all $i \in \mathbb{Z}$.*

As a consequence, one obtains the Acyclic Assembly Lemma; see [61, (2.7.3)].

1.8.2 **Proposition**. *Let $\mathcal{A}$ be a concrete additive category with countable direct sums. If $M^{\bullet\bullet} \in DC(\mathcal{A})$, then $\mathbb{T}(M^{\bullet\bullet})$ is acyclic in case $M^{\bullet\bullet}$ is diagonally bounded-below with acyclic rows or diagonally bounded-above with acyclic columns.*

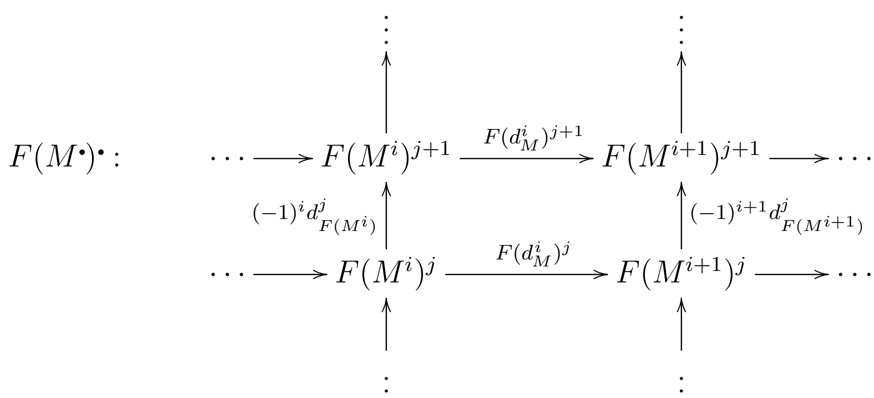
The following statement; see [16, (4.6)] tells us when the total morphism of a double complex morphism is a quasi-isomorphism.

1.8.3 **Lemma**. *Let $\mathcal{A}$ be a concrete additive category with countable direct sums. Consider a morphism $f^{\bullet\bullet} : M^{\bullet\bullet} \to N^{\bullet\bullet}$ in $DC(\mathcal{A})$ such that $f^{i,\bullet} : M^{i,\bullet} \to N^{i,\bullet}$ is a quasi-isomorphism, for every $i \in \mathbb{Z}$. If $M^{\bullet\bullet}$ and $N^{\bullet\bullet}$ are diagonally bounded-above, then $\mathbb{T}(f^{\bullet\bullet})$ is a quasi-isomorphism.*

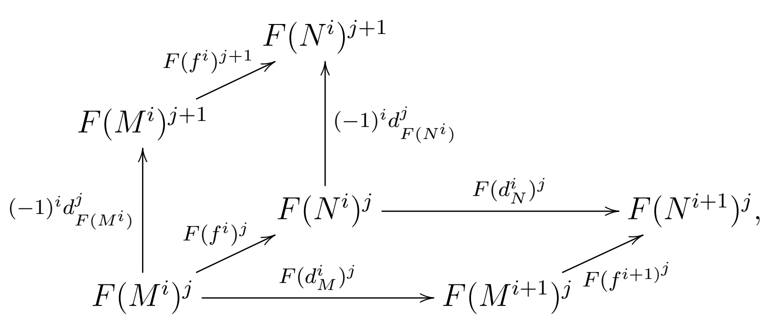
Next, we recall a technique to extend a functor from an additive category into a category of complexes to the category of complexes. Consider an additive functor

$$F : \mathcal{A} \to C(\mathcal{B}) : M \to F(M)^{\bullet}; f \mapsto F(f)^{\bullet},$$

where $\mathcal{B}$ has countable direct sums. In order to extend F to $C(\mathcal{A})$, one first constructs a functor $F^{DC} : C(\mathcal{A}) \to DC(\mathcal{B})$ as follows. Given an object $M^{\bullet} \in C(\mathcal{A})$, applying F component-wise yields a double complex



whose i-th column is $\mathfrak{t}^i(F(M^i)^{\bullet})$, the i-th twist of the complex $F(M^i)^{\bullet}$. Then, one puts $F^{DC}(M^{\bullet}) = F(M^{\bullet})^{\bullet}$. Given a morphism $f^{\bullet} : M^{\bullet} \to N^{\bullet}$ in $C(\mathcal{A})$, applying F component-wise yields a commutative diagram



for $i, j \in \mathbb{Z}$. So, $F(f^\bullet)^\bullet = (F(f^i)^j)_{i,j\in\mathbb{Z}} : F(M^\bullet)^\bullet \to F(N^\bullet)^\bullet$ is a morphism in $DC(\mathcal{B})$. Then, one puts $F^{DC}(f^\bullet) = F(f^\bullet)^\bullet$.

The following statement is quoted from [16, (4.8)], which collects some basic properties of the extended functor F^C and is a general version of the result stated in [10, (3.7)].

1.8.4 **Proposition**. *Let $\mathcal{A}$ and $\mathcal{B}$ be concrete additive categories such that $\mathcal{B}$ has countable direct sums. Then, every additive functor $F : \mathcal{A} \to C(\mathcal{B})$ extends to a functor $F^C = \mathbb{T} \circ F^{DC} : C(\mathcal{A}) \to C(\mathcal{B})$ with the following properties.*

(1) *If M is an object in $\mathcal{A}$, then $F^C(M) = F(M)$.*

(2) *If $M^\bullet$ is a complex in $C(\mathcal{A})$, then $F^C(M^\bullet[1]) = F^C(M^\bullet)[1]$.*

(3) *If $f^\bullet$ is a morphism in $C(\mathcal{A})$, then $F^C(C_{f^\bullet}) = C_{F^C(f^\bullet)}$; and in case $f^\bullet$ is null-homotopic, $F^C(f^\bullet)$ is null-homotopic.*

We show that the extension of functors preserves the exactness and the faithfulness.

1.8.5 **Proposition**. *Consider an additive functor $F : \mathcal{A} \to C(\mathcal{B})$, where $\mathcal{A}$ and $\mathcal{B}$ are concrete additive categories such that $\mathcal{B}$ has countable direct sums. If F is exact or faithful, then F^C is exact or faithful respectively.*

Proof. Assume that F is exact. Let $0 \longrightarrow X^\bullet \longrightarrow Y^\bullet \longrightarrow Z^\bullet \longrightarrow 0$ be a short exact sequence in $C(\mathfrak{A})$. Then, we have short exact sequences

$$0 \longrightarrow F(X^i)^{n-i} \longrightarrow F(Y^i)^{n-i} \longrightarrow F(Z^i)^{n-i} \longrightarrow 0,$$

for all $i, n \in \mathbb{Z}$. This yields a short exact sequence

$$0 \longrightarrow F^C(X^\bullet)^n \longrightarrow F^C(Y^\bullet)^n \longrightarrow F^C(Z^\bullet)^n \longrightarrow 0,$$

for every $n \in \mathbb{Z}$. Thus, this gives rise to a short exact sequence

$$0 \longrightarrow F^C(X^\bullet) \longrightarrow F^C(Y^\bullet) \longrightarrow F^C(Z^\bullet) \longrightarrow 0.$$

That is, F^C is an exact functor.

Suppose that F is faithful. Consider a morphism $f^\bullet : X^\bullet \to Y^\bullet$ in $C(\mathcal{A})$. By definition, $F^C(f^\bullet) = \mathbb{T}(F(f^\bullet)^\bullet)$, where $F(f^\bullet)^\bullet$ is the double complex morphism given by $F(f^i)^j : F(X^i)^j \to F(Y^i)^j$, for $i, j \in \mathbb{Z}$. Thus, for any $n \in \mathbb{Z}$, we have

$$F^C(f^\bullet)^n = \mathbb{T}(F(f^\bullet)^\bullet)^n = \oplus_{i\in\mathbb{Z}} F(f^i)^{n-i}.$$

Since F is faithful, we see that

$$\begin{aligned} F^C(f^\bullet) = 0 \;\; &\Rightarrow \;\; \oplus_{i\in\mathbb{Z}} F(f^i)^{n-i} = 0, \text{ for all } n \in \mathbb{Z} \\ &\Rightarrow \;\; F(f^i)^{n-i} = 0, \text{ for all } i, n \in \mathbb{Z} \\ &\Rightarrow \;\; F(f^i)^j = 0, \text{ for all } i, j \in \mathbb{Z} \\ &\Rightarrow \;\; F(f^i)^\bullet = 0, \text{ for all } i \in \mathbb{Z} \\ &\Rightarrow \;\; f^i = 0, \text{ for all } i \in \mathbb{Z} \\ &\Rightarrow \;\; f^\bullet = 0. \end{aligned}$$

So, F^C is faithful. The proof of the proposition is completed.

We quote the following important statement from [16, (4.10)], which says that the extension of functors is compatible with the composition of functors.

1.8.6 **Proposition**. *Let $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ be concrete additive categories with $\mathcal{B}$ and $\mathcal{C}$ having countable direct sums. If $F : \mathcal{A} \to C(\mathcal{B})$ and $G : \mathcal{B} \to C(\mathcal{C})$ are additive functors, then $(G^C \circ F)^C = G^C \circ F^C$.*

In view of Proposition 1.8.2, we see that the total complex of a double complex is not necessarily acyclic even if the double complex has exact rows or exact columns. Therefore, the extended functor F^C does not send all acyclic complexes to acyclic ones, and hence, it does not descend to the derived category $D(\mathcal{A})$ as stated in Proposition 1.7.10. We quote the following statement from[16, (4.9)], which says that F^C descends to categories derived from some suitable derivable subcategories of $C(\mathcal{A})$.

1.8.7 **Theorem**. *Let $\mathcal{A}, \mathcal{B}$ be concrete additive categories such that $\mathcal{B}$ has countable direct sums. Consider an additive functor $F : \mathcal{A} \to C(\mathcal{B})$, sending acyclic complexes to acyclic ones. Suppose that F^C sends a derivable subcategory $\mathscr{A}$ of $C(\mathcal{A})$ into a derivable subcategory $\mathscr{B}$ of $C(\mathcal{B})$.*

(1) *If F^{DC} sends complexes in $\mathscr{A}$ to diagonally bounded-below double complexes, then F^C sends acyclic complexes in $\mathscr{A}$ to acyclic ones.*

(2) *If F^C sends acyclic complexes in $\mathscr{A}$ to acyclic ones, then it induces a commutative diagram of functors*

$$\begin{array}{ccccc} \mathscr{A} & \longrightarrow & \mathcal{K}(\mathscr{A}) & \longrightarrow & \mathcal{D}(\mathscr{A}) \\ {\scriptstyle F^C}\downarrow & & \downarrow{\scriptstyle F^K} & & \downarrow{\scriptstyle F^D} \\ \mathscr{B} & \longrightarrow & \mathcal{K}(\mathscr{B}) & \longrightarrow & \mathcal{D}(\mathscr{B}), \end{array}$$

where F^K and F^D are triangle-exact.

We conclude this section by quoting the following statement from [16, (4.11)], which says that functorial morphisms between functors can also be extended.

1.8.8 **Lemma**. *Let $\mathcal{A}, \mathcal{B}$ be concrete additive categories such that $\mathcal{B}$ has countable direct sums. Let $F, G : \mathcal{A} \to C(\mathcal{B})$ be additive functors. Then every functorial morphism $\eta = (\eta^{\bullet}_{M})_{M \in \mathcal{A}} : F \to G$ extends to a functorial morphism $\eta^C = (\eta^C_{M^\bullet})_{M^\bullet \in C(\mathcal{A})} : F^C \to G^C$, where $\eta^C_{M^\bullet} = \mathbb{T}(\eta^{\bullet}_{M^\bullet}) : F^C(M^\bullet) \to G^C(M^\bullet)$ with $\eta^{\bullet}_{M^\bullet} : F(M^\bullet)^\bullet \to G(M^\bullet)^\bullet$ given by $\eta^j_{M^i} : F(M^i)^j \to G(M^i)^j$ with $i, j \in \mathbb{Z}$;*

Chapter 2

Algebras and modules

The main objective of this chapter is to recall some background and collect some preliminary results. The terminology and notation introduced in this section will be used throughout the thesis. Let k denotes a commutative field. All tensor products will be over k unless explicitly stated otherwise. An algebra is always over k and does not necessarily have an identity unless explicitly stated otherwise, and an ideal of an algebra is always a two-sided ideal.

2.1 Linear algebras

The content of this section is taken from [16, Section 1]. Given a set $\mathcal{S}$, the k-vector space spanned by $\mathcal{S}$ will be written as $k\mathcal{S}$. The category of all k-vector spaces and that of finite dimensional k-vector spaces will be denoted by $\mathrm{Mod}k$ and $\mathrm{mod}k$, respectively. We shall make a frequent use of the exact functor $D = \mathrm{Hom}_k(-, k) : \mathrm{Mod}k \to \mathrm{Mod}k$, which restricts to a duality $D : \mathrm{mod}k \to \mathrm{mod}k$. The following result is important.

2.1.1 **Lemma**. *Given $U, V; M, N \in \mathrm{Mod}k$, we obtain a k-linear map*

$$\rho : \mathrm{Hom}_k(U, V) \otimes \mathrm{Hom}_k(M, N) \to \mathrm{Hom}_k(U \otimes M, V \otimes N); f \otimes g \mapsto \rho(f \otimes g),$$

where $\rho(f \otimes g)(u \otimes m) = f(u) \otimes g(m)$ for $u \in U$ and $m \in M$, which is natural in all variables. Moreover, ρ is an isomorphism in case U, V are finite dimensional, or else, M, N are finite dimensional.

Proof. The first part of the lemma is evident. For the second part, we shall consider only the case where U, V are finite dimensional. Let $\{u_1, \ldots, u_s\}$ be a basis of U and $\{v_1, \ldots, v_t\}$ a basis of V. Consider the k-linear maps $q_i : M \to U \otimes M; m \mapsto u_i \otimes m$, and $p_j : V \otimes N \to N; \sum_{l=1}^t v_l \otimes n_l \mapsto n_j$, and $f_{ij} : U \to V; \sum_{l=1}^s \lambda_l u_l \mapsto \lambda_i v_j$, for $i = 1, \ldots, s$ and $j = 1, \ldots, t$.

Let $w \in \mathrm{Hom}_k(U, V) \otimes \mathrm{Hom}_k(M, N)$ such that $\rho(w) = 0$. Since the f_{ij} form a basis of $\mathrm{Hom}_k(U, V)$, Then, we may write $w = \sum_{i=1}^s \sum_{j=1}^t f_{ij} \otimes g_{ij}$, with $g_{ij} \in \mathrm{Hom}_k(M, N)$. Given any $m \in M$, we obtain $\sum_{j=1}^t v_j \otimes g_{lj}(m) = \sum_{i=1}^s \sum_{j=1}^t f_{ij}(u_l) \otimes g_{ij}(m) = \rho(w)(u_l \otimes m) = 0$ for $l = 1, \ldots, s$. Therefore, $g_{lj}(m) = 0$, and hence, $g_{lj} = 0$, for $l = 1, \ldots, s$; $j = 1, \ldots, t$. That is, $w = 0$.

Let $h \in \mathrm{Hom}_k(U \otimes M, V \otimes N)$. Consider $g_{ij} = p_j \circ h \circ q_i \in \mathrm{Hom}_k(M, N)$. Given $w = \sum_{l=1}^s u_l \otimes m_l \in U \otimes M$, we may write $h(u_l \otimes m_l) = \sum_{j=1}^t v_j \otimes n_{lj}$, for some $n_{lj} \in N$. Then, $g_{lj}(m_l) = p_j(h(u_l \otimes m_l)) = n_{lj}$. Now,

$$\begin{aligned}
\varphi\left(\sum_{i=1}^s \sum_{j=j}^t f_{ij} \otimes g_{ij}\right)(w) &= \sum_{i,j,l} \varphi(f_{ij} \otimes g_{ij})(u_l \otimes m_l) \\
&= \sum_{1 \le i,l \le s} \sum_{1 \le j \le t} f_{ij}(u_l) \otimes g_{ij}(m_l) \\
&= \sum_{l=1}^s \sum_{j=1}^t (v_j \otimes n_{lj}) \\
&= \sum_{l=1}^s h(u_l \otimes m_l) \\
&= h(w).
\end{aligned}$$

Thus, $\varphi(\sum_{i=1}^s \sum_{j=j}^t f_{ij} \otimes g_{ij}) = h$. The proof of the lemma is completed.

Remark. We shall identify $f \otimes g$ with $\rho(f \otimes g)$ in case $U, V \in \mathrm{mod}k$ or $M, N \in \mathrm{mod}k$.

Observing that $V \otimes k \cong V \cong \mathrm{Hom}_k(k, V)$ for any $V \in \mathrm{Mod}k$, we obtain the following immediate consequence of Lemma 2.1.1.

2.1.2 **Corollary**. *Given $U \in \mathrm{Mod}k$ and $V \in \mathrm{Mod}k$, we obtain*

(1) *a binatural k-linear isomorphism*

$$\sigma : D(U) \otimes V \to \mathrm{Hom}_k(U, V); f \otimes v \mapsto \sigma(f \otimes v),$$

where $\sigma(f \otimes v)(u) = f(u)v$ for $u \in U$ and $v \in V$, which is an isomorphism in case U or V is finite dimensional.

(2) *a binatural k-linear isomorphism*

$$\rho : D(V) \otimes D(U) \to D(V \otimes U); f \otimes g \mapsto \rho(f \otimes g),$$

where $\rho(f \otimes g)(v \otimes u) = f(v)g(u)$ for $u \in U$ and $v \in V$, which is an isomorphism in case U or V is finite dimensional.

The following statement will be needed for our later investigation.

2.1.3 **Lemma**. *Given morphisms $f : U \to M$ and $g : N \to V$ in* mod$k$*, we obtain a commutative diagram with vertical isomorphisms as follows*:

$$\begin{array}{ccc} U \otimes D(V) & \xrightarrow{f \otimes D(g)} & M \otimes D(N) \\ {\scriptstyle \theta_{U,V}} \downarrow & & \downarrow {\scriptstyle \theta_{M,N}} \\ D(V \otimes D(U)) & \xrightarrow{D(g \otimes D(f))} & D(N \otimes D(M)). \end{array}$$

We conclude this section with the following easy statement.

2.1.4 **Lemma**. *Let V be a k-vector space, and let U be a subspace of V. If $v \in V \backslash U$, then there exist a subspace W of V containing U such that $V = W \oplus kv$.*

Proof. Assume that $v \in V \backslash U$. Let $\mathcal{U}$ be a k-basis of U. Being linearly independent, $\mathcal{U} \cup \{v\}$ extend to a k-basis $\mathcal{W} \cup \{v\}$ of V. Letting W be the subspace generated by $\mathcal{W}$, we see that $U \subseteq W$ and $V = W \oplus kv$. The proof of the lemma is completed.

2.2 Quivers and algebras

In this section, we shall fix some notions and terminology for quivers, which will be used throughout this thesis. Let $Q = (Q_0, Q_1)$ be a quiver, where Q_0 is a set of vertices and Q_1 is a set of arrows between the vertices. Given an arrow $\alpha : x \to y$ in Q_1, we call x the **starting point** and y the **ending point** of α; and write $s(\alpha) = x$ and $e(\alpha) = y$. One says that Q is **finite** if both Q_0 and Q_1 are finite sets; **locally finite**, that is, for any $x \in Q_0$, the set of arrows α with $s(\alpha) = x$ or $e(\alpha) = x$ is finite; **gradable** if $Q_0 = \cup_{i \in \mathbb{Z}} Q_0^i$ such that every arrow is of the form $x \to y$, where $x \in Q^i$, $y \in Q^{i+1}$ and $i \in \mathbb{Z}$. A **path** ρ of **length** $n \geq 1$ in Q is a sequence

$$\rho : \quad x_0 \xrightarrow{\alpha_1} x_1 \xrightarrow{\alpha_2} x_2 \longrightarrow \cdots \longrightarrow x_{n-1} \xrightarrow{\alpha_n} x_n$$

where $\alpha_i \in Q_1$ for all $1 \leq i \leq n$ such that $e(\alpha_i) = s(\alpha_{i+1})$ for $1 \leq i < n-1$. Such a path will be denoted by $\rho = \alpha_n \cdots \alpha_1$. In this case, we call α_1 the **initial arrow** and α_n the **terminal arrow** of ρ. Moreover, we associate with each vertex $x \in Q_0$ a **trivial path** ε_x, which is of length 0. A path of length $n \geq 1$ is said to be a **oriented cycle** whenever its starting point and ending point coincide. In particular, an oriented cycle of length 1 is called a **loop**.

Fix an integer $n \geq 0$ and some vertices x, y of Q. We shall denote by Q_n the set of paths of length n and by $Q(x, y)$ the set of paths from x to y. Moreover, we shall write $Q_n(x, y)$, $Q_{\leq n}(x, y)$, and $Q_{\geq n}(x, y)$ for the subsets of $Q(x, y)$ of paths of length n, of length $\leq n$, and of length $\geq n$, respectively. Further, we put $Q_n(x, -) = \cup_{z \in Q_0} Q_n(x, z)$ and $Q_n(-, x) = \cup_{z \in Q_0} Q_n(z, x)$. Finally, we define $Q_{\leq n}(x, -) = \cup_{z \in Q_0} Q_{\leq n}(x, z)$ and $Q_{\leq n}(-, x) = \cup_{z \in Q_0} Q_{\leq n}(z, x)$, and similarly, $Q_{\geq n}(x, -) = \cup_{z \in Q_0} Q_{\geq n}(x, z)$ and $Q_{\geq n}(-, x) = \cup_{z \in Q_0} Q_{\geq n}(z, x)$. For convenience, we shall put $Q_s(x, y) = \emptyset$ for an integer $s < 0$.

The **opposite** quiver of Q is a quiver Q^{o} defined in such a way that $(Q^{\mathrm{o}})_0 = Q_0$ and $(Q^{\mathrm{o}})_1 = \{\alpha^{\mathrm{o}} : y \to x \mid \alpha : x \to y \in Q_1\}$. A non-trivial path $\rho = \alpha_n \cdots \alpha_1$ in $Q(x, y)$, where $\alpha_i \in Q_1$, corresponds to a non-trivial path $\rho^{\mathrm{o}} = \alpha_1^{\mathrm{o}} \cdots \alpha_n^{\mathrm{o}}$ in $Q^{\mathrm{o}}(y, x)$. However, the trivial path in Q at a vertex x will be identified with the trivial path in Q^{o} at x.

2.3 Algebras given by a quiver with relations

In this thesis, an algebra does not necessarily have an identity, and an ideal in an algebra is always a two-sided ideal unless stated otherwise. In this section, we shall fix some notions and terminology for algebras defined by a quiver with relations.

2.3.1 **Definition**. Let Q be a locally finite quiver. The **path algebra** kQ of Q is the k-vector space having the set of all paths in Q as a basis. The product of two paths $\alpha_n \cdots \alpha_1$ and $\beta_m \cdots \beta_1$ is defined by

$$(\beta_m \cdots \beta_1)(\alpha_n \cdots \alpha_1) = \begin{cases} \beta_m \cdots \beta_1 \alpha_n \cdots \alpha_1, & \text{if } e(\alpha_n) = s(\beta_1); \\ 0, & \text{otherwise.} \end{cases}$$

This multiplication is then extended by k-bilinearity to all elements in kQ.

The **opposite algebra** of kQ is the path algebra kQ° of the opposite quiver Q^{o}. Given $\omega = \sum_{i=1}^{s} \lambda_i \rho_i \in kQ$, where $\lambda_i \in k$ and ρ_i are paths, we shall write

$$\omega^{\circ} = \sum_{i=1}^{s} \lambda_i \rho_i^{\circ} \in kQ^{\circ}.$$

Then, we have an algebra anti-isomorphism $kQ \to kQ^{\circ}$ given by $\omega \mapsto \omega^{\circ}$ for $\omega \in kQ$.

Consider the path algebra kQ. An element $\rho \in kQ$ is called **quadratic** if $\rho \in Q_2$ and **homogeneous** if $\rho \in kQ_n$ for some $n \geq 1$. Moreover, an element $\rho \in kQ(x,y)$ with $x, y \in Q_0$ is called **monomial** if $\rho = 0$ or $\rho = \lambda p$, where $\lambda \in k$ and p a path in Q, and **polynomial** otherwise.

Let R be an ideal of kQ. We write $R_n = R \cap kQ_n$ for $n \geq 0$, $R(x,y) = R \cap kQ(x,y)$ for $x, y \in Q_0$, and $R_n(x,y) = R \cap kQ_n(x,y)$. Finally, we put $R(x,-) = \cup_{z \in Q_0} R(x,z)$ and $R(-,y) = \cup_{z \in Q_0} R(z,y)$. We shall say that R is **quadratic** or **homogeneous** if R is generated by a set of quadratic elements or by a set of homogeneous elements in kQ, respectively. Moreover, R is called a **relation ideal** if it is contained in $(kQ^+)^2$, where kQ^+ is the ideal of kQ generated by Q_1.

Consider now a quotient algebra $\Lambda = kQ/R$, where R is a relation ideal of kQ. Given $x, y \in Q_0$. An element $\rho = \sum_{i=1}^{s} \lambda_i p_i \in R(x,y)$ is called a **relation** for Λ if the λ_i are non-zero scalars in k and the p_i are pairwise distinct paths in $Q(x,y)$ such that $\sum_{i \in \Sigma} \lambda_i p_i \notin R$ for any $\emptyset \neq \Sigma \subset \{1, \ldots, s\}$. In this case, the $\lambda_i p_i$ are called the **summands** of the relation ρ. In the sequel, we shall say that Λ is the algebra defined by Q with relations in R. Moreover, we call Λ a **quadratic** or **graded algebra** if R is a quadratic or homogeneous ideal, respectively.

Let us fix some notations for $\Lambda = kQ/R$, which will be used for the rest of the thesis. Write $\bar{\gamma} = \gamma + R \in \Lambda$ for $\gamma \in kQ$, and $e_x = \bar{\varepsilon}_x$ for $x \in Q_0$. Then, $\{e_x \mid x \in Q_0\}$ is a complete set of orthogonal idempotents in Λ. The opposite algebra of Λ is given by $\Lambda^{\mathrm{o}} = kQ^{\mathrm{o}}/R^{\mathrm{o}}$, where $R^{\mathrm{o}} = \{\rho^{\mathrm{o}} \mid \rho \in R\}$. We shall write $\bar{\gamma}^{\mathrm{o}} = \gamma^{\mathrm{o}} + R^{\mathrm{o}}$ for $\gamma \in kQ$, but $e_x = \varepsilon_x + R^{\mathrm{o}}$ for $x \in Q_0$. In this way, we have an algebra anti-isomorphism $\Lambda \to \Lambda^{\mathrm{o}}$ given by $\bar{\gamma} \to \bar{\gamma}^{\mathrm{o}}$ for $\bar{\gamma} \in \Lambda$.

A left Λ-module M is called **unitary** if $M = \sum_{x \in Q_0} e_x M$. In this case, we shall write $M(x) = e_x M$, called the x-**component**, for all $x \in Q_0$. We shall denote by $\mathrm{Mod}\Lambda$ the category of all unitary left Λ-modules, and by $\mathrm{mod}^b\Lambda$ the full subcategory of $\mathrm{Mod}\Lambda$ of finite dimensional modules.

2.4 Locally noetherian algebras

Throughout this section, let $\Lambda = kQ/R$, where Q is a locally finite quiver and R is a relation ideal. A left or right unitary Λ-module is called **noetherian** if all of its submodules are finitely generated. We shall say that Λ is **locally left noetherian** if Λe_a is noetherian for every $a \in Q_0$; and **locally right noetherian** if $e_a\Lambda$ is noetherian for every $a \in Q_0$. As examples of locally noetherian algebras, we introduce the following class of algebras.

2.4.1 **Definition**. Let $\Lambda = kQ/R$, where Q is a locally finite quiver and R is a relation ideal. We say that Λ is **multi-serial** provided, for any $\alpha \in Q_1$, that there exists at most one arrow β such that $\beta\alpha \notin R$ and at most one arrow γ such that $\alpha\gamma \notin R$.

Remark. It is evident that string algebras and special biserial algebras are multi-serial; see [17, 59].

2.4.2 **Proposition**. *Let $\Lambda = kQ/R$ be a multi-serial algebra, where Q is a locally finite quiver and R is a relation ideal. Then Λ is locally left and right noetherian.*

Proof. Fix $a \in Q_0$. Given $\alpha \in Q_1(a, -)$, we claim that the left Λ-module $\Lambda\alpha$ is noetherian. For this purpose, we may assume that $\Lambda\alpha$ is infinite dimensional. Since Λ is multi-serial, Q contains an infinite path

$$a = a_0 \xrightarrow{\alpha_1} a_1 \longrightarrow \cdots \longrightarrow a_{i-1} \xrightarrow{\alpha_i} a_i \longrightarrow \cdots$$

with $\alpha_1 = \alpha$ such that $\Lambda\alpha$ has a k-basis $\{u_1, u_2, \ldots, u_i, \ldots, \}$, where $u_i = \bar{\alpha}_i \cdots \bar{\alpha}_1$. Thus, every non-zero element $u \in \Lambda\alpha$ is uniquely written as $u = \sum_{i=1}^{n} \lambda_i u_i$, where $\lambda_i \in k$ with $\lambda_n \neq 0$, and we write $\deg(u) = n$. Given non-zero elements $u, v \in \Lambda\alpha$, it is not hard to to see that $v = qu + w$, where $q \in \Lambda$ and $w \in \Lambda\alpha$ such that $w = 0$ or $\deg(w) < \deg(u)$. Using this fact, we deduce that if L is a non-zero left Λ-submodule of $\Lambda\alpha$, then $L = \Lambda u$, where $u \in \Lambda\alpha$ with $\deg(u)$ being minimal. This establishes our claim. Since Q is locally finite, $J_a = \sum_{\alpha \in Q_1(a,-)} \Lambda\alpha$ is noetherian. Since $\Lambda e_a/J_a$ is one-dimensional, we conclude that Λe_a is noetherian. Similarly, we can show that $e_a\Lambda$ is a noetherian right Λ-module. The proof of the proposition is completed.

Finally, we say that Λ is **locally left bounded** if Λe_a is finite dimensional for any $a \in Q_0$; **locally right bounded** if $e_a\Lambda$ is finite dimensional for any $a \in Q_0$, and **locally bounded** if Λ is locally right and locally left bounded; compare [14, (2.1)]. It is evident that a locally left or right bounded algebra is locally left or right noetherian respectively.

2.5 Local trace function

Throughout this section, we let $\Lambda = kQ/R$, where Q is a locally finite quiver and R is a relation ideal. In case Λ is finite dimensional, Lenzing's trace function; see [34] is localized to a local trace function in [29] for finite dimensional modules, in order to establish the **Strong No Loop Conjecture**, that is, Q has no loop at a vertex a if the corresponding simple module S_a is of finite projective dimension. In this section, we shall further extend the notion of local trace function and reformulate the main result in [29] under our most general setting.

We start with some notations and terminology. Given $x \in Q_0$, we shall write $P_x = \Lambda e_x$, which is clearly a projective module in $\mathrm{Mod}\Lambda$. We denote by $[\Lambda, \Lambda]$ the **commutator group** of Λ, that is the k-vector subspace of Λ generated by the elements $uv - vu$ with $u, v \in \Lambda$. And we write $\mathrm{HH}_0(\Lambda) = \Lambda/[\Lambda, \Lambda]$, called the **0-th Hochschild homology group** of Λ. Given $a \in Q_0$, we put $\Lambda_a = \Lambda/(\sum_{x \in Q_0 \backslash \{a\}} \Lambda e_x \Lambda)$. Then, the canonical algebra projection $\Lambda \to \Lambda_a$ induces a k-linear map $H_a : \mathrm{HH}_0(\Lambda) \to \mathrm{HH}_0(\Lambda_a)$. By definition, we see that $H_a(uv + [\Lambda, \Lambda]) = H_a(vu + [\Lambda, \Lambda])$, for all $u, v \in \Lambda$.

An **index set** for Q is a set Ω equipped with a map $\pi : \Omega \to Q_0$. In this case, we shall write $e_i = e_{\pi(i)}$ and $P_i = P_{\pi(i)}$ for $i \in \Omega$. And for $a \in Q_0$, we shall write $\Omega_a = \pi^{-1}(a)$ and say that Ω is a-**finite** if Ω_a is finite.

Fix $a \in Q_0$ and an a-finite index set Ω. We write $P^{(\Omega)} = \oplus_{i \in \Omega} P_i$ with canonical injections $q_i : P_i \to P^{(\Omega)}$ and canonical projections $p_i : P^{(\Omega)} \to P_i$. Consider $\varphi \in \mathrm{End}_\Lambda(P^{(\Omega)})$. We may write $\varphi = (u_{ij})_{(i,j) \in \Omega \times \Omega}$, where $u_{ij} = (p_j \varphi q_i)(e_i) \in e_i \Lambda e_j$. Observe that each row of the matrix $(u_{ij})_{(i,j) \in \Omega \times \Omega}$ has at most finitely many non-zero entries. Since $H_a(u_{ii} + [\Lambda, \Lambda]) = 0$ for all $i \in \Omega \backslash \Omega_a$, we may define the e_a-**trace** $\mathrm{tr}_a(\varphi)$ of φ by setting

$$\mathrm{tr}_a(\varphi) := \textstyle\sum_{i \in \Omega} H_a(u_{ii} + [\Lambda, \Lambda]) = H_a(\sum_{i \in \Omega_a} u_{ii} + [\Lambda, \Lambda]) \in \mathrm{HH}_0(\Lambda_a).$$

In particular, $\mathrm{tr}_a(\varphi) = 0$ if Ω_a is empty.

2.5.1 **Lemma**. *Let $\Lambda = kQ/R$, where Q is a locally finite quiver and R is a relation ideal. Fix a vertex a of Q.*

(1) *Let $\varphi_u : \Lambda e_a \to \Lambda e_a$ be the right multiplication by some $u \in e_a \Lambda e_a$. Then $\mathrm{tr}_a(\varphi_u) = \tilde{u} + [\Lambda_a, \Lambda_a]$, where $\tilde{u} = u + (\sum_{x \in Q_0 \setminus a} \Lambda e_x \Lambda) \in \Lambda_a$.*

(2) *Let $\varphi : P^{(\Omega)} \to P^{(\Theta)}$ and $\psi : P^{(\Theta)} \to P^{(\Omega)}$ be Λ-linear morphisms, where Ω and Θ are a-finite index sets for some $a \in Q_0$. Then $\mathrm{tr}_a(\varphi\psi) = \mathrm{tr}_a(\psi\varphi)$.*

Proof. Statement (1) follows immediately from the definition of the e_a-trace. For proving Statement (2), we consider the canonical injections $q_i : P_i \to P^{(\Omega)}$ and $q'_s : P_s \to P^{(\Theta)}$ and the canonical projections $p_i : P^{(\Omega)} \to P_i$ and $p'_s : P^{(\Theta)} \to P_s$, for all $i \in \Omega$ and $s \in \Theta$. Then, we may write $\varphi = (u_{is})_{(i,s) \in \Omega \times \Theta}$, where $u_{is} = (p'_s \varphi q_i)(e_i) \in e_i \Lambda e_s$, and $\psi = (v_{sj})_{(s,j) \in \Theta \times \Omega}$, where $v_{sj} = (p_j \psi q'_s)(e_s) \in e_s \Lambda e_j$. Now, $\psi\varphi = (w_{ij})_{(i,j) \in \Omega \times \Omega}$, where

$$\begin{aligned} w_{ij} = (p_j(\psi\varphi)q_i)(e_i) &= (p_j\psi)(\textstyle\sum_{s\in\Theta}(q'_s((p'_s\varphi q_i)(e_i)) \\ &= (p_j\psi)(\textstyle\sum_{s\in\Theta} q'_s(u_{is})) \\ &= \textstyle\sum_{s\in\Theta}(p_j\psi q'_s)(u_{is}\cdot e_s) \\ &= \textstyle\sum_{s\in\Theta} u_{is}\cdot(p_j\psi q'_s)(e_s) \\ &= \textstyle\sum_{s\in\Theta} u_{is}v_{sj}. \end{aligned}$$

Similarly, $\varphi\psi = (w'_{st})_{(s,t)\in\Theta\times\Theta}$, where $w'_{st} = \sum_{i\in\Omega} v_{si}u_{it}$. By definition, we have

$$\begin{aligned} \mathrm{tr}_a(\psi\varphi) &= \textstyle\sum_{i\in\Omega} H_a(w_{ii} + [\Lambda,\Lambda]) \\ &= \textstyle\sum_{i\in\Omega} H_a(\sum_{s\in\Theta} u_{is}v_{si} + [\Lambda,\Lambda]) \\ &= \textstyle\sum_{i\in\Omega}\sum_{s\in\Theta} H_a(u_{is}v_{si} + [\Lambda,\Lambda]) \end{aligned}$$

and

$$\begin{aligned} \mathrm{tr}_a(\varphi\psi) &= \textstyle\sum_{s\in\Theta} H_a(w'_{ss} + [\Lambda,\Lambda]) \\ &= \textstyle\sum_{s\in\Theta} H_a(\sum_{i\in\Omega} v_{si}u_{is} + [\Lambda,\Lambda]) \\ &= \textstyle\sum_{i\in\Omega}\sum_{s\in\Theta} H_a(u_{is}v_{si} + [\Lambda,\Lambda]). \end{aligned}$$

So, $\mathrm{tr}_a(\varphi\phi) = \mathrm{tr}_a(\phi\varphi)$. The proof of the lemma is completed.

Let $a \in Q_0$. We shall denote by $\mathrm{Proj}(\Lambda, a)$ the full additive subcategory of $\mathrm{Mod}\Lambda$ generated by the projective modules isomorphic to some projective Λ-module $P^{(\Omega)}$, where Ω is an a-finite index set. Consider $\varphi \in \mathrm{End}_\Lambda(P)$, where

$P \in \mathrm{Proj}(\Lambda, a)$. Choosing a Λ-linear isomorphism $\omega : P \to P^{(\Omega)}$, where Ω is an a-finite index set, we define the e_a-**trace** $\mathrm{tr}_a(\varphi)$ of φ by setting

$$\mathrm{tr}_a(\varphi) := \mathrm{tr}_a(\omega\varphi\omega^{-1})$$

which is well-defined as shown below.

2.5.2 **Lemma**. *Let $\Lambda = kQ/R$, where Q is a locally finite quiver and R is a relation ideal. Consider a projective module $P \in \mathrm{Proj}(\Lambda, a)$. Then, the e_a-trace $\mathrm{tr}_a(\varphi)$ is well-defined, for every $\varphi \in \mathrm{End}_\Lambda(P)$.*

Proof. Assume that $\omega : P \to P^{(\Omega)}$ and $\theta : P \to P^{(\Theta)}$ are Λ-linear isomorphisms, where Ω and Θ are a-finite index sets for some $a \in Q_0$. Given $\varphi \in \mathrm{End}_\Lambda(P)$, by Lemma 2.5.1(2), we have

$$\mathrm{tr}_a(\omega\varphi\omega^{-1}) = \mathrm{tr}_a((\omega\varphi\theta^{-1})(\theta\omega^{-1})) = \mathrm{tr}_a((\theta\omega^{-1})(\omega\varphi\theta^{-1})) = \mathrm{tr}_a(\theta\varphi\theta^{-1}).$$

The proof of the lemma is completed.

The following statement collects some basic properties of our local trace function; compare [29, (1.1)].

2.5.3 **Proposition**. *Let $\Lambda = kQ/R$, where Q is a locally finite quiver and R is a relation ideal. Let P and P' be projective modules in $\mathrm{Proj}(\Lambda, a)$ for some $a \in Q_0$.*

(1) *If $\varphi, \varphi' \in \mathrm{End}_\Lambda(P)$, then $\mathrm{tr}_a(\varphi + \varphi') = \mathrm{tr}_a(\varphi) + \mathrm{tr}_a(\varphi')$.*

(2) *If $\varphi = \begin{pmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{pmatrix} : P \oplus P' \to P \oplus P'$ is a Λ-linear morphism, then*

$$\mathrm{tr}_a(\varphi) = \mathrm{tr}_a(\varphi_{11}) + \mathrm{tr}_a(\varphi_{22}).$$

(3) *If $\varphi : P \to P'$ and $\phi : P' \to P$ are Λ-linear morphisms, then*

$$\mathrm{tr}_a(\varphi\phi) = \mathrm{tr}_a(\phi\varphi).$$

(4) *If $\phi : P \to P'$ is an isomorphism and $\varphi \in \mathrm{End}_\Lambda(P)$, then*

$$\mathrm{tr}_a(\phi\varphi\phi^{-1}) = \mathrm{tr}_a(\varphi).$$

Proof. By the assumption, we have Λ-linear isomorphisms $\omega : P \to P^{(\Omega)}$ and $\theta : P' \to P^{(\Theta)}$, where Ω and Θ are a-finite index sets for Q.

(1) Let $\varphi, \varphi' \in \mathrm{End}_\Lambda(P)$. Assume first that $P = P^{(\Omega)}$. Write $\varphi = (u_{ij})_{(i,j)\in\Omega\times\Omega}$ and $\varphi' = (v_{ij})_{(i,j)\in\Omega\times\Omega}$, where $u_{ij}, v_{ij} \in e_i\Lambda e_j$. Then, $\varphi + \varphi' = (u_{ij} + v_{ij})_{(i,j)\in\Omega\times\Omega}$. By definition, we have

$$\begin{aligned}\mathrm{tr}_a(\varphi) + \mathrm{tr}_a(\varphi') &= \textstyle\sum_{i\in\Omega} H_a(u_{ii} + [\Lambda,\Lambda]) + \sum_{i\in\Omega} H_a(v_{ii} + [\Lambda,\Lambda]) \\ &= \textstyle\sum_{i\in\Omega} H_a((u_{ii} + v_{ii}) + [\Lambda,\Lambda]) \\ &= \mathrm{tr}_a(\varphi + \varphi').\end{aligned}$$

In general, we have $\omega\varphi\omega^{-1}, \omega\varphi'\omega^{-1} \in \mathrm{End}_\Lambda(P^{(\Omega)})$ such that $\omega(\varphi + \varphi')\omega^{-1} = \omega\varphi\omega^{-1} + \omega\varphi'\omega^{-1}$. By definition, we have

$$\mathrm{tr}_a(\varphi + \varphi') = \mathrm{tr}_a(\omega(\varphi + \varphi')\omega^{-1}) = \mathrm{tr}_a(\omega\varphi\omega^{-1}) + \mathrm{tr}_a(\omega\varphi'\omega^{-1}) = \mathrm{tr}_a(\varphi) + \mathrm{tr}_a(\varphi').$$

(2) Consider a Λ-linear morphism

$$\varphi = \begin{pmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{pmatrix} : P \oplus P' \to P \oplus P'.$$

Suppose first that $P = P^{(\Omega)}$ and $P' = P^{(\Theta)}$. Then, we may write

$$\varphi = \begin{pmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{pmatrix} = \begin{pmatrix} (u_{ij})_{(i,j)\in\Omega\times\Omega} & (u_{it})_{(i,t)\in\Omega\times\Theta} \\ (u_{sj})_{(s,j)\in\Theta\times\Omega} & (u_{st})_{(s,t)\in\Theta\times\Theta} \end{pmatrix},$$

where $u_{ij} \in e_i\Lambda e_j$, $u_{it} \in e_i\Lambda e_t$, $u_{sj} \in e_s\Lambda e_j$ and $u_{st} \in e_s\Lambda e_t$. By definition,

$$\mathrm{tr}_a(\varphi) = \textstyle\sum_{i\in\Omega_a} H_a(u_{ii} + [\Lambda,\Lambda]) + \sum_{s\in\Theta_a} H_a(u_{ss} + [\Lambda,\Lambda]) = \mathrm{tr}_a(\varphi_{11}) + \mathrm{tr}_a(\varphi_{22}).$$

In general, we have a Λ-linear isomorphism

$$\rho = \begin{pmatrix} \omega & 0 \\ 0 & \theta \end{pmatrix} : P \oplus P' \to P^{(\Omega)} \oplus P^{(\Theta)}$$

such that

$$\rho\varphi\rho^{-1} = \begin{pmatrix} \omega\varphi_{11}\omega^{-1} & \omega\varphi_{12}\theta^{-1} \\ \theta\varphi_{21}\omega^{-1} & \theta\varphi_{22}\theta^{-1} \end{pmatrix} : P^{(\Omega)} \oplus P^{(\Theta)} \to P^{(\Omega)} \oplus P^{(\Theta)}.$$

By definition, we have

$$\mathrm{tr}_a(\varphi) = \mathrm{tr}_a(\rho\varphi\rho^{-1}) = \mathrm{tr}_a(\omega\varphi_{11}\omega^{-1}) + \mathrm{tr}_a(\theta\varphi_{22}\theta^{-1}) = \mathrm{tr}_a(\varphi_{11}) + \mathrm{tr}_a(\varphi_{22}).$$

(3) Given Λ-linear morphisms $\varphi : P \to P'$ and $\phi : P' \to P$, we obtain Λ-linear morphisms $\omega\phi\theta^{-1} : P^{(\Theta)} \to P^{(\Omega)}$ and $\theta\phi\omega^{-1} : P^{(\Omega)} \to P^{(\Theta)}$. By Lemma 2.5.1(2),

$$\begin{aligned}
\mathrm{tr}_a(\varphi\phi) &= \mathrm{tr}_a(\theta(\varphi\phi)\theta^{-1}) \\
&= \mathrm{tr}_a((\theta\varphi\omega^{-1})(\omega\phi\theta^{-1})) \\
&= \mathrm{tr}_a((\omega\phi\theta^{-1})(\theta\varphi\omega^{-1})) \\
&= \mathrm{tr}_a(\omega\phi\varphi\omega^{-1}) \\
&= \mathrm{tr}_a(\phi\varphi).
\end{aligned}$$

(4) Let $\varphi \in \mathrm{End}_\Lambda(P)$ and let $\phi : P' \to P$ be a Λ-linear isomorphism. By Statement (3), we have $\mathrm{tr}_a(\phi\varphi\phi^{-1}) = \mathrm{tr}_a((\varphi\phi^{-1})\phi) = \mathrm{tr}_a(\varphi)$. The proof of the proposition is completed.

Fix $a \in Q_0$. Given M a module in $\mathrm{Mod}\Lambda$, a projective resolution

$$\cdots \longrightarrow P^{-n} \xrightarrow{d^{-n}} P^{1-n} \cdots \longrightarrow P^0 \xrightarrow{d^0} M \longrightarrow 0$$

is called called e_a-**bounded** if $P^{-n} \cong P^{(\Omega_n)}$ for $n \geq 0$, where the Ω_n are a-finite index sets for Q such that $(\Omega_n)_a$ is empty for all but finitely many $n \geq 0$. In this case, every $\varphi \in \mathrm{End}_\Lambda(M)$ induces a commutative diagram

$$\begin{array}{ccccccccccc}
\cdots \longrightarrow & P^{-n} & \xrightarrow{d^{-n}} & P^{1-n} & \longrightarrow \cdots \longrightarrow & P^0 & \xrightarrow{d^0} & M & \longrightarrow & 0 \\
& \downarrow{\scriptstyle \varphi^{-n}} & & \downarrow{\scriptstyle \varphi^{1-n}} & & \downarrow{\scriptstyle \varphi^{0}} & & \downarrow{\scriptstyle \varphi} & & \\
\cdots \longrightarrow & P^{-n} & \xrightarrow{d^{-n}} & P^{1-n} & \longrightarrow \cdots \longrightarrow & P^0 & \xrightarrow{d^0} & M & \longrightarrow & 0,
\end{array}$$

and we define the e_a-**trace** $\mathrm{tr}_a(\varphi)$ of φ by setting

$$\mathrm{tr}_a(\varphi) := \textstyle\sum_{n=0}^{\infty}(-1)^n \mathrm{tr}_a(\varphi^{-n}) \in \mathrm{HH}_0(\Lambda_a).$$

In view of Proposition 2.5.3, we may establish the following important statement; see, for details, [29, (1.3), (1.4)].

2.5.4 **Proposition**. *Let $\Lambda = kQ/R$, where Q is a locally finite quiver and R is a relation ideal.*

(1) *If $M \in \mathrm{Mod}\Lambda$ admits an e_a-bounded projective resolution, then $\mathrm{tr}_a(\varphi)$ is well defined for every $\varphi \in \mathrm{End}_\Lambda(M)$.*

(2) *Consider a commutative diagram with exact rows*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L & \xrightarrow{f} & M & \xrightarrow{g} & N & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \varphi_L} & & \downarrow{\scriptstyle \varphi_M} & & \downarrow{\scriptstyle \varphi_N} & & \\
0 & \longrightarrow & L & \xrightarrow{f} & M & \xrightarrow{g} & N & \longrightarrow & 0
\end{array}$$

in $\mathrm{Mod}\Lambda$. *If* L *and* N *admit* e_a*-bounded projective resolutions, then so does* M *and* $\mathrm{tr}_a(\varphi_M) = \mathrm{tr}_a(\varphi_L) + \mathrm{tr}_a(\varphi_N)$.

Chapter 3

Graded algebras and graded modules

The objective of this chapter is to investigate graded module categories over a graded algebra given by a quiver with relations, especially to generalize some classical results under the non-graded setting; compare [1] and [2].

Let $\Lambda = kQ/R$ be a graded algebra, where Q is locally finite and R is a homogeneous relation ideal of kQ. Then, Λ is a positively graded algebra with $\Lambda = \oplus_{i\geq 0}\Lambda_i$, where $\Lambda_i = \{\bar{\gamma} \mid \gamma \in kQ_i\}$.

To study graded module categories over Λ, we begin by constructing a crucial tool for our investigation: a contravariant functor $\mathfrak{D}$: GModΛ $\rightarrow$ GMod$\Lambda^{\rm o}$, which restricts to a duality $\mathfrak{D}$: gmodΛ $\rightarrow$ gmod$\Lambda^{\rm o}$. In Section 4, we provide descriptions of the morphisms in GProjΛ and GInjΛ, while in Section 5, we delve into the graded radical and graded socle in GModΛ. In order to study the graded projective covers and the graded injective envelopes, we explicitly describe the finitely generated graded modules, the finitely cogenerated graded modules, superfluous graded epimorphisms and essential graded monomorphisms. In the final section, we shall introduce several Hom-finite Krull-Schmidt k-subcategories of GModΛ.

3.1 General positively graded algebras

The purpose of this section is to recall the notions of general positively graded algebras and graded modules from [52, 53]. It is important to note that our algebras do not necessarily have an identity.

3.1.1 **Definition**. A k-algebra A is called **positively graded** if there is a family of k-vector subpaces $\{A_i\}_{i\geq 0}$ of A such that

(1) $A = \oplus_{i\geq 0} A_i$ as a k-vector space, called the **grading** of A;

(2) $A_i A_j \subseteq A_{i+j}$, for all $i, j \geq 0$.

Remark. Let A be a positively graded algebra. Then A_0 is a subalgebra of A, which does not necessarily have an identity.

For the rest of this section, A stands for a positively graded k-algebra.

3.1.2 **Definition**. Let A be a positively graded k-algebra. A left A-module M is called **graded** provided that, for every $i \in \mathbb{Z}$, there is a k-vector subspace M_i, call the i-th **homogeneous component**, of M such that

(1) $M = \oplus_{i\in\mathbb{Z}} M_i$ as a k-vector space, called an A-**grading**;

(2) $A_i M_j \subseteq M_{i+j}$, for all $i \geq 0$ and $j \in \mathbb{Z}$.

Remark. Clearly, A is a graded left A-module with A-grading $A = \oplus_{i\geq 0} A_i$, which is written as ${}_A A$.

Let $M = \oplus_{i\in\mathbb{Z}} M_i$ be a graded left A-module. Given $m \in M$, we shall always write $m = \sum_{i\in\mathbb{Z}} m_i$ with $m_i \in M_i$ and finitely many nonzero m_i. An element $m_i \in M_i$ with $i \in \mathbb{Z}$ is called **homogeneous of degree** i. An A-submodule L of M is said to be **graded** if $L = \sum_{i\in\mathbb{Z}} (M_i \cap L)$, and in this case, L is a graded left A-module with A-grading $L = \oplus_{i\in\mathbb{Z}} L_i$, where $L_i = M_i \cap L$. In particular, a graded submodule of ${}_A A$ is called a **graded left ideal** of A. The following statement collects some well known properties of graded submodules of a graded module. For the convenience of the reader, we include a proof.

3.1.3 **Lemma**. *Let A be a positively graded k-algebra, and let $M = \oplus_{i\in\mathbb{Z}} M_i$ be a graded left A-module.*

(1) *An A-submodule N of M is graded if and only if, given $m = \sum_{i\in\mathbb{Z}} m_i \in N$ with $m_i \in M_i$, we have $m_i \in N$ for all $i \in \mathbb{Z}$.*

(2) *If L and N are graded submodules of M, then $L + N$ and $L \cap N$ are graded submodules of M.*

(3) *If I is a graded left ideal of A and m is a homogeneous element of M, then Im is a graded submodule of M.*

(4) *If L is a graded submodule of M, then the quotient M/L is a graded left A-module with A-grading $M/L = \oplus_{i\in\mathbb{Z}}(M/L)_i$, where $(M/L)_i = (M_i + L)/L$.*

Proof. (1) Let N be an A-submodule of M. Suppose first that N is a graded. Let $m = \sum_{i\in\mathbb{Z}} m_i \in N$, where $m_i \in M_i$. By definition, we can also write $m = \sum_{i\in\mathbb{Z}} m_i'$, where $m_i' \in M_i \cap N \subseteq M_i$. Thus, $m_i = m_i' \in N$, for all $i \in \mathbb{Z}$.

Conversely, assume that this condition is satisfied. Given $m \in N$, we may write $m = \sum_{i\in\mathbb{Z}} m_i$, where $m_i \in M_i$. By the condition, $m_i \in N$, and hence, $m_i \in M_i \cap N$. This shows that $N = \sum_{i\in\mathbb{Z}}(M_i \cap N)$.

(2) Let L and N be graded submodules of M. Then $L+N$ is an A-submodule of M. Consider $x \in L + N$. We may assume that $x = m + m'$ with $m \in L$ and $m' \in N$. Then $m = \sum_{i\in\mathbb{Z}} m_i$ and $m' = \sum_{i\in\mathbb{Z}} m_i'$, where $m_i \in M_i \cap L$ and $m_i' \in M_i \cap N$. Then, $x = \sum_{i\in\mathbb{Z}}(m_i + m_i')$, where $m_i + m_i' \in M_i \cap (L + N)$. Therefore, $L + N = \sum_{i\in\mathbb{Z}} M_i \cap (L + N)$.

(3) Assume that I is a graded left ideal of A. Then, $I = \oplus_{j\in\mathbb{Z}} I_j$ where $I_j = A_j \cap I$. Consider $m \in M_s$ with $s \in \mathbb{Z}$. Clearly, Im is an A-submodule of M. Let $um \in Im$, where $u \in I$. Writing $u = \sum_{j\in\mathbb{Z}} u_j$ with $u_j \in I_j$, we have $x = \sum_{j\in\mathbb{Z}}(u_j m) = \sum_{i\in\mathbb{Z}} u_{i-s}m$, where $u_{i-s}m \in M_i \cap (Im)$. This shows that $Im = \sum_{i\in Z} M_i \cap (Im)$.

(4) Let L be a graded submodule of M. Observe that $M/L = \sum_{i\in\mathbb{Z}}(M_i+L)/L$. By definition, $A_j \cdot (M_i + L)/L = (A_j M_i + L)/L \subseteq (M_{i+j} + L)/L$, for all $i, j \in \mathbb{Z}$. Let $\sum_{i\in\mathbb{Z}}(m_i + L) = \bar{0}$, where $m_i \in M_i$ such that $m_i + L = \bar{0}$ for all but finitely many $i \in \mathbb{Z}$. Without loss of generality, we may assume that $m_i = 0$ for all but finitely many $i \in \mathbb{Z}$. Then, $m = \sum_{i\in\mathbb{Z}} m_i \in L$. By Statement (1), $m_i \in L$ for all $i \in \mathbb{Z}$. Thus, $M/L = \oplus_{i\in\mathbb{Z}}(M_i + L)/L$. The proof of the lemma is completed.

As an immediate consequence of Lemma 3.1.3, we obtain the following statement.

3.1.4 **Corollary**. *Let A be a positively graded k-algebra, and let M be a graded left A-module. If $m_1, \ldots, m_r \in M$ are homogeneous, then $Am_1 + \cdots + Am_r$ is a graded submodule of M.*

Proof. Assume that m_1 is homogeneous of degree i. Then, $Am_1 = \oplus_{j\geq 0} A_j m_1$, which is clearly graded submodule of M. By Lemma 3.1.3(2), $Am_1 + \cdots + Am_r$ is a graded submodule of M. The proof of the corollary is completed.

Let M be a graded left A-module. A graded submodule L of M is said to be **graded essential** in M if $M \neq 0$ and $L \cap N \neq 0$, for any non-zero graded submodule N of M; and **graded superfluous** in M if L is a proper graded submodule of M and M is the only graded submodule of M such that $L + M = M$. The following statement is evident.

3.1.5 **Lemma**. *Let A be a positively graded k-algebra, and let M be a graded left A-module.*

(1) *A graded submodule of M is graded superfluous if and only if it is contained in a graded superfluous submodule of M.*

(2) *A graded submodule of M is graded essential if and only if it contains a graded essential submodule of M.*

Let M be a graded left A-module. One says that M is **graded simple** if it is non-zero with exactly two graded submodules 0 and M; **graded semisimple** if it is a direct sum of graded simple modules. Moreover, a graded submodule L of M is called **graded maximal** if there exists no graded submodule N of M with $L \subsetneqq N \subsetneqq M$. The following statement is well known.

3.1.6 **Lemma**. *Let A be a positively graded k-algebra, and let M be a graded left A-module. A graded submodule L of M is graded maximal if and only if M/L is graded simple.*

The following definitions will play an essential role in our later study of graded modules.

3.1.7 **Definition**. Let A be a positively graded k-algebra, and let M be a graded left A-module.

(1) The **graded socle** soc$M$ of M is defined to be the sum of all graded simple submodules of M in case M has graded simple submodules; and otherwise, soc$M = 0$.

(2) The **graded radical** rad$M$ of M is defined to be the intersection of all maximal graded submodules of M if M has maximal graded submodules; and otherwise, rad$M = M$.

(3) The **graded top** of M is defined to be the graded quotient module $\text{top}M = M/\text{rad}M$.

Let M, N be graded left A-modules. An A-linear morphism $f : M \to N$ is called **graded** if $f(M_i) \subseteq N_i$ for all $i \in \mathbb{Z}$. We shall write $f_i : M_i \to N_i$, where $i \in \mathbb{Z}$, for the maps obtained by restricting f. The following statement is well known.

3.1.8 **Proposition**. *Let A be a positively graded k-algebra, and let $f : M \to N$ be a graded A-linear morphism of graded left A-modules.*

(1) $\text{Im}f = \oplus_{i\in\mathbb{Z}}(\text{Im}f)_i$ *is a graded submodule of N, where* $(\text{Im}f)_i = \text{Im}(f_i)$.

(2) $\text{Ker}f = \oplus_{i\in\mathbb{Z}}(\text{Ker}f)_i$ *is a graded submodule of M, where* $(\text{Ker}f)_i = \text{Ker}(f_i)$.

(3) *There is a one-one correspondence $L \mapsto L/\text{Ker}(f)$ from the class of graded submodules L of M with $\text{Ker}(f) \subseteq L \subseteq M$ onto the class of graded submodules of* $\text{Im}(f)$.

Let $M = \oplus_{i\in\mathbb{Z}}M_i$ be a graded left A-module. Given $s \in \mathbb{Z}$, we define the **grading** s**-shift** $M\langle s\rangle$ of M by $M\langle s\rangle_i = M_{i+s}$ for all $i \in \mathbb{Z}$. And for a graded A-linear morphism $f : M \to N$, we define the **grading** s**-shift** $f\langle s\rangle : M\langle s\rangle \to N\langle s\rangle$ of f by $f\langle s\rangle_i = f_{i+s}$, for all $i \in \mathbb{Z}$. The following statement is evident.

3.1.9 **Lemma**. *Let A be a positively graded k-algebra. If $f : M \to N$ is a graded A-linear morphism of graded left A-modules, then* $\text{Ker}(f\langle s\rangle) = (\text{Ker}f)\langle s\rangle$ *and* $\text{Im}(f\langle s\rangle) = (\text{Im}f)\langle s\rangle$, *for all $s \in \mathbb{Z}$.*

Let $M = \oplus_{i\in\mathbb{Z}}M_i$ be a graded left A-module. Given a k-vector space V, it is clear that $M \otimes V = \oplus_{i\in\mathbb{Z}}M_i \otimes V$ is a graded left A-module such that $a(m \otimes v) = (am) \otimes v$, for all $a \in A$, $m \in M$ and $v \in V$. The following statement is evident.

3.1.10 **Lemma**. *Let A be a positively graded k-algebra. Given a graded left A-module M and a k-vector space V, we have $(M \otimes V)\langle s\rangle = M\langle s\rangle \otimes V$, for all $s \in \mathbb{Z}$.*

3.2 Positively graded algebras given by a quiver with relations

Throughout the rest of this chapter, we shall concentrate on graded algebras given by a quiver with relations. For this purpose, we shall always assume that $\Lambda = kQ/R$ is a graded algebra, where Q is a locally finite quiver and R is a homogeneous relation ideal of kQ. It is important to note that Λ does not necessarily have an identity. The terminology, the notations and the results stated in this section will be used frequently in the sequel.

A graded left Λ-module $M = \oplus_{i\in\mathbb{Z}} M_i$ is called **unitary** if $M = \sum_{x\in Q_0} e_x M$. Let M be such a unitary graded left Λ-module. Since the e_x in Λ with $x \in Q_0$ are pairwise orthogonal, it is easy to see that $M = \oplus_{x\in Q_0} M(x)$, where $M(x) = e_x M$, called the x-**point component** of M. As a consequence, $M = \oplus_{i\in\mathbb{Z};x\in Q_0} M_i(x)$, where $M_i(x) = e_x M_i$, called the (i,x)-**piece** of M. Given $m \in M$, by writing $m = \sum_{(i,x)\in\mathbb{Z}\times Q_0} m_{i,x}$ with $m_{i,x} \in M_i(x)$, we shall assume implicitly that $m_{i,x} = 0$ for all but finitely pairs $(i,x) \in \mathbb{Z} \times Q_0$. An element $m \in M$ will be called **pure** if $m \in M_i(x)$ for some $(i,x) \in \mathbb{Z} \times Q_0$.

The following easy statement is useful.

3.2.1 **Lemma**. *Let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver. Consider a graded left $\Lambda-$module M. If N is a Λ-submodule of M, then the following statements are equivalent*:

(1) *N is a graded Λ-submodule of M;*

(2) *$N = \sum_{(i,x)\in\mathbb{Z}\times Q_0} N_{i,x}$, where $N_{i,x}$ is a k-vector subspace of $M_i(x)$;*

(3) *for any $m = \sum_{(i,x)\in\mathbb{Z}\times Q_0} m_{i,x} \in N$ with $m_{i,x} \in M_i(x)$, we have $m_{i,x} \in N$ for all $(i,x) \in \mathbb{Z} \times Q_0$.*

Proof. Let N be a Λ-submodule of M. First, suppose that Statement (1) holds. Given $m \in N$, we may write $m = \sum_{i\in\mathbb{Z}} m_i$, where $m_i \in M_i$. By Lemma 3.1.3(1), $m_i \in N$ for all $i \in \mathbb{Z}$. Since M is unitary, we have $m_i = \sum_{x\in Q_0} e_x m_i$ with $e_x m_i \in M_i(x) \cap N$ for all $(i,x) \in \mathbb{Z} \times Q_0$. Thus, $N = \sum_{(i,x)\in\mathbb{Z}\times Q_0} (M_i(x) \cap N)$. In particular, Statement (2) holds.

Now, suppose that Statement (3) holds. Given any $m \in N$, we may write $m = \sum_{(i,x)\in\mathbb{Z}\times Q_0} m_{i,x}$ where $m_{i,x} \in M_i(x)$. By Statement (3), $m_{i,x} \in N$, and hence,

$m_{i,x} \in M_i(x) \cap N$ for all $(i,x) \in \mathbb{Z} \times Q_0$. This shows that $N = \sum_{(i,x)\in\mathbb{Z}\times Q_0}(M_i \cap N)$. In particular, Statement (2) holds.

Finally, suppose that Statement (2) holds, that is, $N = \sum_{(i,x)\in\mathbb{Z}\times Q_0} N_{i,x}$, where $N_{i,x}$ is a k-vector subspace of $M_i(x)$. Then, any element $m \in N$ can be written as $m = \sum_{(i,x)\in\mathbb{Z}\times Q_0} n_{i,x}$ with $n_{i,x} \in N_{i,x} \subseteq M_i(x) \subseteq M_i$. Assume that $m = \sum_{(i,x)\in\mathbb{Z}\times Q_0} m_{i,x}$ with $m_{i,x} \in M_i(x)$. Then, $m_{i,x} = n_{i,x} \in N$ for all $(i,x) \in \mathbb{Z} \times Q_0$. Therefore, Statement (3) holds. Moreover, assume that $m = \sum_{\in\mathbb{Z}} m_i$, where $m_i \in M_i$. Observing that $\sum_{x\in Q_0} n_{i,x} \in M_i$, we get $m_i = \sum_{x\in Q_0} n_{i,x} \in N$, for all $i \in \mathbb{Z}$. By Lemma 3.1.3(1), N is a graded submodule of M. That is, Statement (1) holds. The proof of the lemma is completed.

Let $f : M \to N$ be a graded Λ-linear morphism between unitary graded left Λ-modules. We shall write $f_{i,x} : M_i(x) \to N_i(x)$, where $i \in \mathbb{Z}$ and $x \in Q_0$, for the maps obtained by restricting f. Observe that such a graded Λ-linear morphism f is uniquely determined by a family of k-linear maps $f_{i,x} : M_i(x) \to N_i(x)$ with $(i,x) \in \mathbb{Z} \times Q_0$ such that $uf_{i,x}(m) = f_{i+j,y}(um)$, for all $u \in e_y\Lambda_j e_x$ and $m \in M_i(x)$.

Clearly, the unitary graded left Λ-modules together with the graded Λ-linear morphisms form an abelian k-category, which will be denoted by $\mathrm{GMod}\Lambda$. Given modules $M, N \in \mathrm{GMod}\Lambda$, we shall write $\mathrm{GHom}_\Lambda(M,N)$ for the k-vector space of graded Λ-linear morphisms $f : M \to N$. In particular, $\mathrm{GEnd}_\Lambda(M) = \mathrm{GHom}_\Lambda(M,M)$. The following statement is evident.

3.2.2 **Lemma**. *Let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver. A sequence of morphisms $L \xrightarrow{f} M \xrightarrow{g} N$ in $\mathrm{GMod}\Lambda$ is exact if and only if the sequences $L_i(x) \xrightarrow{f_{i,x}} M_i(x) \xrightarrow{g_{i,x}} N_{i,x}(x)$ are exact in $\mathrm{Mod}k$, for all $(i,x) \in \mathbb{Z} \times Q_0$.*

As in the classical case where graded algebras have an identity; see [52, page 7], the abelian category $\mathrm{GMod}\Lambda$ has arbitrary direct sums and arbitrary products.

3.2.3 **Proposition**. *Let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver. Consider a family $\{M_\sigma\}_{\sigma\in\Sigma}$ of modules in $\mathrm{GMod}\Lambda$.*

(1) *The direct sum $M = \oplus_{\sigma\in\Sigma} M_\sigma$ exists in $\mathrm{GMod}\Lambda$ such that $M_i(x) = \oplus_{\sigma\in\Sigma}(M_\sigma)_i(x)$ for all $(i,x) \in \mathbb{Z} \times Q_0$.*

(2) *The product $N = \Pi_{\sigma\in\Sigma} M_\sigma$ exists in $\mathrm{GMod}\Lambda$ such that $N_i(x) = \Pi_{\sigma\in\Sigma}(M_\sigma)_i(x)$ for all $(i,x) \in \mathbb{Z} \times Q_0$.*

Proof. We shall only prove Statement (2), since the proof of Statement (1) is dual. Put $N = \oplus_{(i,x)\in\mathbb{Z}\times Q_0} N_i(x)$, where $N_i(x) = \Pi_{\sigma\in\Sigma}(M_\sigma)_i(x)$. For any $u \in e_x \Lambda_j e_y$ and $(m_\sigma)_{\sigma\in\Sigma} \in N_i(x)$, we define $u\cdot(m_\sigma)_{\sigma\in\Sigma} = (um_\sigma)_{\sigma\in\Sigma} \in N_{i+j}(y)$. This defines a graded Λ-module structure of N. Let $\{p_\sigma : N \to M_\sigma\}_{\sigma\in\Sigma}$ be the family of canonical projections, in which $p_\pi : N \to M_\mu; (m_\sigma)_{\sigma\in\Sigma} \mapsto m_\mu$ for all $\mu \in \Sigma$. Assume that L is a graded Λ-module with a family $\{f_\sigma : L \to M_\sigma\}_{\sigma\in\Sigma}$ of graded Λ-linear morphisms. Fix $(i,x) \in \mathbb{Z}\times Q_0$. Note that, by definition, $N_i(x)$ is a product of $(M_\sigma)_i(x)$. Then there is a unique k-linear map $g_{i,x} : L_i(x) \to M_i(x)$ such that for each $\sigma \in \Sigma$ the diagram

$$\begin{array}{ccc} (M_\sigma)_i(x) & \xleftarrow{(p_\sigma)_{i,x}} & N_i(x) = \Pi_{\sigma\in\Sigma}(M_\sigma)_i(x) \\ {\scriptstyle (f_\sigma)_{i,x}}\uparrow & \nearrow{\scriptstyle g_{i,x}} & \\ L_i(x) & & \end{array}$$

commutes. Thus, we obtain a unique k-linear map $g = (g_{i,x})_{(i,x)} : L \to M$ such that the diagram

$$\begin{array}{ccc} M_\sigma & \xleftarrow{p_\sigma} & N \\ {\scriptstyle f_\sigma}\uparrow & \nearrow{\scriptstyle g} & \\ L & & \end{array}$$

commutes, for each $\sigma \in \Sigma$. Next, we show that g is Λ-linear, or equivalently, the diagram

$$\begin{array}{ccc} L_{i+j}(y) & \xrightarrow{g_{i+j,y}} & M_{i+j}(y) \\ {\scriptstyle u}\uparrow & & \uparrow{\scriptstyle u} \\ L_i(x) & \xrightarrow{g_{i,x}} & M_i(x) \end{array}$$

commutes for all $u \in e_y \Lambda_j e_x$ with $(j,y) \in \mathbb{Z}\times Q_0$. Given $m \in L_i(x)$, we have

$$u(f_{i,x}(m)) = u(((p_\sigma)_{i,x} \circ g_{i,x})(m)) = u((p_\sigma)_{i,x}(g_{i,x}(m))) = (p_\sigma)_{i+j,y}(u(g_{i,x}(m)))$$

and

$$f_{i+j,y}(um) = (p_\sigma)_{i+j,y} \circ g_{i+j,y}(um) = (p_\sigma)_{i+j,y}(g_{i+j,y}(um))$$

for all $\sigma \in \Sigma$. Since $u(f_{i,x}(m)) = f_{i+j,y}(um)$, we have

$$(p_\sigma)_{i+j,y}(u(g_{i,x}(m))) = (p_\sigma)_{i+j,y}(g_{i+j,y}(um))$$

for all $\sigma \in \Sigma$. This implies that $u(g_{i,x}(m)) = g_{i+j,y}(um)$. Therefore, g is a Λ-linear map. The proof of the proposition is completed.

In the sequel, we shall need to consider various types of graded Λ-modules as defined below.

3.2.4 **Definition**. Let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver. Let $M \in \mathrm{GMod}\Lambda$ with $M = \oplus_{i\in\mathbb{Z}} M_i = \oplus_{x\in Q_0} M(x) = \oplus_{(i,x)\in\mathbb{Z}\times Q_0} M_i(x)$. We shall say that M is

(1) **finitely supported** if $M(x) = 0$ for all but finitely many $x \in Q_0$;

(2) **finitely piece-supported** if $M_i(x) = 0$ for all but finitely many $(i, x) \in \mathbb{Z} \times Q_0$;

(3) **bounded above** if $M_i = 0$ for $i \gg 0$;

(4) **bounded below** if $M_i = 0$ for $i \ll 0$;

(5) **bounded** if $M_i = 0$ for all but finitely many $i \in \mathbb{Z}$.

(6) **locally finite dimensional** if M_i is finite dimensional for all $i \in \mathbb{Z}$;

(7) **piecewise finite dimensional** if $M_i(x)$ is finite dimensional for every $(i, x) \in \mathbb{Z} \times Q_0$.

The full subcategories of $\mathrm{GMod}\Lambda$ of finitely piece-supported modules, of bounded below modules and of bounded above modules will be written as $\mathrm{GMod}^b\Lambda$, $\mathrm{GMod}^+\Lambda$, $\mathrm{GMod}^-\Lambda$, respectively. Moreover, we shall denote by $\mathrm{gmod}\Lambda$ the full subcategory of $\mathrm{GMod}\Lambda$ of piecewise finite dimensional modules, while the full subcategories of $\mathrm{gmod}\Lambda$ of finite dimensional modules, of bounded below modules and of bounded above modules will be written as $\mathrm{gmod}^b\Lambda$, $\mathrm{gmod}^+\Lambda$, $\mathrm{gmod}^-\Lambda$, respectively.

3.2.5 **Lemma**. *Let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver. Then* $\mathrm{gmod}\,\Lambda$ *is an extension-closed abelian subcategory of* $\mathrm{GMod}\Lambda$.

Proof. It amount to show that given an exact sequence

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

in $\mathrm{GMod}\Lambda$, we have $M \in \mathrm{gmod}\Lambda$ if and only if $L, N \in \mathrm{gmod}\Lambda$. Indeed, by Lemma 3.2.2, the sequence $0 \longrightarrow L_i(x) \xrightarrow{f_{i,x}} M_i(x) \xrightarrow{g_{i,x}} N_i(x) \longrightarrow 0$ is exact for each $(i, x) \in \mathbb{Z} \times Q_0$. Therefore, $\dim_k M_i(x) < \infty$ for all $(i, x) \in \mathbb{Z} \times Q_0$ if and only if $\dim_k L_i(x) < \infty$ and $\dim_k N_i(x) < \infty$, for all $(i, x) \in \mathbb{Z} \times Q_0$. The proof of the lemma is completed.

3.3 The contravariant functor $\mathfrak{D}$

Throughout this section, $\Lambda = kQ/R$ is a graded algebra, where Q is a locally finite quiver. We shall construct a contravariant functor $\mathfrak{D} : \mathrm{GMod}\Lambda \to \mathrm{GMod}\Lambda^{\mathrm{o}}$, by applying $D = \mathrm{Hom}_k(-, k)$ to every piece of a graded Λ-module. This functor restricts to a duality $\mathfrak{D} : \mathrm{gmod}\Lambda \to \mathrm{gmod}\Lambda^{\mathrm{o}}$. Our functor $\mathfrak{D}$ is different from the similar functor defined in [42, page 70], where D is applied to every homogeneous component of a graded module, and the one defined in [16]; see also [26, 11], where D is applied to every point-component of a non-graded module.

Let M be a module in $\mathrm{GMod}\Lambda$. First, we write $\mathfrak{D}M = \oplus_{i\in\mathbb{Z}}(\mathfrak{D}M)_i$, where $(\mathfrak{D}M)_i = \oplus_{x\in Q_0} D(M_{-i}(x))$. Then for $\varphi \in D(M_{-i}(x))$ and $u \in e_x\Lambda_j e_y$, we define $u^{\mathrm{o}}\cdot \varphi \in D(M_{-i-j}(y))$ by setting

$$(u^{\mathrm{o}} \cdot \varphi)(m) = \varphi(um), \text{ for all } m \in M_{-i-j}(y).$$

It is easy to verify that $\mathfrak{D}M \in \mathrm{GMod}\Lambda^{\mathrm{o}}$ with $(\mathfrak{D}M)_i(x) = D(M_{-i}(x))$ for all $(i, x) \in \mathbb{Z} \times Q_0$. Sometimes, we shall consider $\mathfrak{D}M$ as a k-vector subspace of $DM = \mathrm{Hom}_k(M, k)$ in the following way: given $\varphi = \sum_{(i,x)\in\mathbb{Z}\times Q_0} \varphi_{i,x} \in \mathfrak{D}M$ with $\varphi_{i,x} \in D(M_i(x))$, for $m = \sum_{(j,y)\in\mathbb{Z}\times Q_0} m_{j,y} \in M$ with $m_{j,y} \in M_j(y)$, put $\varphi(m) = \sum_{(i,x)\in\mathbb{Z}\times Q_0} \varphi_{i,x}(m_{i,x})$. Moreover, given a morphism $f : M \to N$ in $\mathrm{GMod}\Lambda$, we obtain a morphism $\mathfrak{D}f : \mathfrak{D}N \to \mathfrak{D}M$ in $\mathrm{GMod}\Lambda^{\mathrm{o}}$ by setting $(\mathfrak{D}f)_{i,x} = D(f_{-i,x})$, for all $(i, x) \in \mathbb{Z} \times Q_0$. Clearly, this yields an additive contravariant functor $\mathfrak{D} : \mathrm{GMod}\Lambda \to \mathrm{GMod}\Lambda^{\mathrm{o}}$.

3.3.1 **Proposition**. *Let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver.*

(1) *If $M \in \mathrm{GMod}\Lambda$ and $s \in \mathbb{Z}$, then $\mathfrak{D}(M\langle s\rangle) = (\mathfrak{D}M)\langle -s\rangle$.*

(2) *Given $M \in \mathrm{GMod}\Lambda$ and $V \in \mathrm{Mod}k$, there exists a binatural morphism*

$$\theta : \mathfrak{D}M \otimes DV \to \mathfrak{D}(M \otimes V)$$

in $\mathrm{GMod}\Lambda$, which is an isomorphism in case $M \in \mathrm{gmod}\Lambda$ or $V \in \mathrm{mod}k$.

(3) *The functor $\mathfrak{D} : \mathrm{GMod}\Lambda \to \mathrm{GMod}\Lambda^{\mathrm{o}}$ is exact, and it restricts to functors $\mathfrak{D} : \mathrm{GMod}^+\Lambda \to \mathrm{GMod}^-\Lambda^{\mathrm{o}}$ and $\mathfrak{D} : \mathrm{GMod}^-\Lambda \to \mathrm{GMod}^+\Lambda^{\mathrm{o}}$.*

Proof. (1) Let $M \in \mathrm{GMod}\Lambda$. Given $s \in \mathbb{Z}$, by definition, we see that

$$(\mathfrak{D}M)\langle -s\rangle_i = (\mathfrak{D}M)_{i-s} = \oplus_{x\in Q_0} D(M_{s-i}(x))$$

and

$$(\mathfrak{D}M\langle s\rangle)_i = \oplus_{x\in Q_0} D(M\langle s\rangle_{-i}(x)) = \oplus_{x\in Q_0} D(M_{s-i}(x)),$$

for all $i \in \mathbb{Z}$. That is, $\mathfrak{D}(M\langle s\rangle) = (\mathfrak{D}M)\langle -s\rangle$.

(2) Let $M \in \mathrm{GMod}\Lambda$ and $V \in \mathrm{Mod}k$. For any $(i,x) \in \mathbb{Z}\times Q_0$, by definition

$$(\mathfrak{D}M \otimes DV)_i(x) = (\mathfrak{D}M)_i(x) \otimes DV = D(M_{-i}(x)) \otimes DV$$

and

$$(\mathfrak{D}(M\otimes V))_i(x) = D((M\otimes V)_{-i}(x)) = D(M_{-i}(x)\otimes V).$$

Now by Corollary 2.1.2(2), we have a binatural k-linear map

$$\theta_{i,x} : D(M_{-i}(x)) \otimes DV \to D(M_{-i}(x)\otimes V) : \varphi\otimes f \to \theta_{i,x}(\varphi\otimes f).$$

Let $\varphi \in D((M_{-i}(x))$, $f \in DV$ and $u \in e_x\Lambda_j e_y$ with $(i,x),(j,y) \in \mathbb{Z}\times Q_0$. For $m \in M_{-i-j}(y)$ and $v \in V$, as defined in Corollary 2.1.2(2), we have

$$(u^{\mathrm{o}}\cdot\theta_{i,x}(\varphi\otimes f))(m\otimes v) = \theta_{i,x}(\varphi\otimes f)((um)\otimes v) = \varphi(um)f(v)$$

and

$$\begin{aligned}\theta_{i+j,y}(u^{\mathrm{o}}\cdot(\varphi\otimes f))(m\otimes v) &= \theta_{i+j,y}((u^{\mathrm{o}}\cdot\varphi)\otimes f)(m\otimes v)\\ &= ((u^{\mathrm{o}}\cdot\varphi)(m))f(v)\\ &= \varphi(um)f(v).\end{aligned}$$

That is, $u^{\mathrm{o}}\cdot\theta_{i,x}(\varphi\otimes f) = \theta_{i+j,y}(u^{\mathrm{o}}\cdot(\varphi\otimes f))$. Therefore, $\theta = (\theta_{i,x})_{(i,x)\in\mathbb{Z}\times Q_0}$ is a morphism in $\mathrm{GMod}\Lambda$. Since the $\theta_{i,x}$ are natural in $M_{i,x}$ and V, we see that θ is natural in M and V. Finally, if $M \in \mathrm{gmod}\Lambda$ or $V \in \mathrm{mod}k$, then it follows from Corollary 2.1.2(2) that $\theta_{i,x}$ is a k-linear isomorphism for every $(i,x) \in \mathbb{Z}\times Q_0$. Thus, θ is an isomorphism.

(3) Let $L \xrightarrow{f} M \xrightarrow{g} N$ be an exact sequence in $\mathrm{GMod}\Lambda$. Given any $(i,x) \in \mathbb{Z}\times Q_0$, by Lemma 3.2.2, the sequence $L_i(x) \xrightarrow{f_{i,x}} M_i(x) \xrightarrow{g_{i,x}} N_{i,x}(x)$ is exact, and hence, the sequence $D(N_i(x)) \xrightarrow{D(g_{i,x})} D(M_i(x)) \xrightarrow{D(f_{i,x})} D(L_{i,x}(x))$ is exact. Again by Lemma 3.2.2, the sequence $\mathfrak{D}N \xrightarrow{\mathfrak{D}g} \mathfrak{D}M \xrightarrow{\mathfrak{D}f} \mathfrak{D}L$ is exact in $\mathrm{GMod}\Lambda^{\mathrm{o}}$. This proves the first part of Statement (3), and the second part follows immediately from the definition of $\mathfrak{D}$. The proof of the proposition is completed.

The following statement says that $\mathfrak{D}$ converts direct sums into direct products.

3.3.2 **Proposition**. *Let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver. Let $\{M_\omega\}_{\omega\in\Omega}$ be a family of modules in* GModΛ*. Then*

$$\mathfrak{D}(\oplus_{\omega\in\Omega}M_\omega) \cong \Pi_{\omega\in\Omega}\mathfrak{D}(M_\omega).$$

Proof. Write $M = \oplus_{\omega\in\Omega}M_\omega$ and $N = \Pi_{\omega\in\Omega}\mathfrak{D}(M_\omega)$. Fix $(i,x) \in \mathbb{Z}\times Q_0$. By Proposition 3.2.3, $M_i(x) = \oplus_{\omega\in\Omega}(M_\omega)_i(x)$ and $N_i(x) = \Pi_{\omega\in\Omega}(\mathfrak{D}(M_\omega))_i(x) = \Pi_{\omega\in\Omega}D((M_\omega)_{-i}(x))$. For each $\omega \in \Omega$, denote by $q_\omega : (M_\omega)_i(x) \to M_i(x)$ the canonical injection. It is well known that we have a k-linear isomorphism

$$\Phi_{i,x} : (\mathfrak{D}M)_i(x) = D(\oplus_{\omega\in\Omega}(M_\omega)_{-i}(x)) \to \Pi_{\omega\in\Omega}D((M_\omega)_{-i}(x)) = N_i(x)$$

such that $\Phi_{i,x}(f) = (f\circ q_\omega)_{\omega\in\Omega}$ for all $f \in (\mathfrak{D}M)_i(x)$. Given $f \in (\mathfrak{D}M)_i(x)$ and $u \in e_y\Lambda_je_x$, where $i,j \in \mathbb{Z}$ and $x,y \in Q_0$, it is a routine verification that $u\cdot\Phi_{i,x}(f) = \Phi_{i+j,y}(u\cdot f)$. Therefore, $\Phi = (\Phi_{i,x})_{(i,x)\in\mathbb{Z}\times Q_0} : \mathfrak{D}M \to N$ is a morphism in GModΛ. The proof of the proposition is completed.

The next statement is our promised duality.

3.3.3 **Proposition**. *Let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver.*

(1) *Given $M \in$ GModΛ, there is a natural monomorphism $\psi^M : M \to \mathfrak{D}^2M$ in* GModΛ*, which is an isomorphism in case $M \in$ gmodΛ.*

(2) *The contravariant functor $\mathfrak{D}$: gmod$\Lambda \to$ gmod$\Lambda^{\rm o}$ is a duality, which restricts to dualities $\mathfrak{D}$: gmod$^+\Lambda \to$ gmod$^-\Lambda^{\rm o}$ and $\mathfrak{D}$: gmod$^-\Lambda \to$ gmod$^+\Lambda^{\rm o}$.*

Proof. Let M be a module in GModΛ. Given $(i,x) \in \mathbb{Z}\times Q_0$, it is well known that there is a canonical k-linear monomorphim

$$\psi^M_{i,x} : M_i(x) \to D^2(M_i(x)) = (\mathfrak{D}^2M)_i(x)$$

given by the formula $\psi^M_{i,x}(m)(\varphi) = \varphi(m)$, for all $\varphi \in D(M_i(x))$ and $m \in M_i(x)$. We claim that this yields a graded Λ-linear morphism

$$\psi^M = (\psi^M_{i,x})_{(i,x)\in\mathbb{Z}\times Q_0} : M \to \mathfrak{D}^2M,$$

or equivalently, the following diagram commutes

$$\begin{array}{ccc} M_i(x) & \xrightarrow{\psi^M_{i,x}} & D^2(M_i(x)) \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle u} \\ M_{i+j}(y) & \xrightarrow{\psi^M_{i+j,y}} & D^2(M_{i+j}(y)) \end{array}$$

for $u \in e_y \Lambda_j e_x$ and $(i,x),(j,y) \in \mathbb{Z} \times Q_0$. Indeed, for $m \in M_i(x)$ and $\varphi \in D(M_{i+j}(y))$, we have

$$\begin{aligned}
\psi^M_{i+j,y}(um)(\varphi) &= \varphi(um) \\
&= (u^{\mathrm{o}} \cdot \varphi)(m) \\
&= \psi^M_{i,x}(m)(u^{\mathrm{o}} \cdot \varphi) \\
&= (u \cdot \psi^M_{i,x}(m))(\varphi).
\end{aligned}$$

This establishes our claim. Next, consider a morphism $f : M \to N$ in $\mathrm{GMod}\Lambda$. Fix $(i,x) \in \mathbb{Z} \times Q_0$. For $m \in M_i(x)$ and $\varphi \in D(N_i(x))$, we have

$$(\psi^N_{i,x} \circ f_{i,x})(m)(\varphi) = \psi^N_{i,x}(f_{i,x}(m))(\varphi) = \varphi(f_{i,x}(m)) = (\varphi \circ f_{i,x})(m)$$

and

$$\begin{aligned}
((\mathfrak{D}^2 f)_{i,x} \circ \psi^M_{i,x})(m)(\varphi) &= (D^2(f_{i,x})(\psi^M_{i,x}(m)))(\varphi) \\
&= ((\psi^M_{i,x}(m)) \circ D(f_{i,x}))(\varphi) \\
&= \psi^M_{i,x}(m)(D(f_{i,x})(\varphi)) \\
&= \psi^M_{i,x}(m)(\varphi \circ f_{i,x}) \\
&= (\varphi \circ f_{i,x})(m).
\end{aligned}$$

Thus, $(\psi^N_{i,x} \circ f_{i,x})(m) = ((\mathfrak{D}^2 f)_{i,x} \circ \psi^M_{i,x})(m)$. So, $\psi^N_{i,x} \circ f_{i,x} = (\mathfrak{D}^2 f)_{i,x} \circ \psi^M_{i,x}$. This shows that ψ is natural in M.

Finally, if $M \in \mathrm{gmod}\Lambda$, then $M_i(x) \in \mathrm{mod}k$, and consequently, $\psi^M_{i,x}$ is a k-linear isomorphism, for all $(i,x) \in \mathbb{Z} \times Q_0$. That is, ψ^M is a graded isomorphism. Thus, Statement (2) follows from Proposition 3.3.1(3). The proof of the proposition is completed.

Using the duality $\mathfrak{D} : \mathrm{gmod}\Lambda \to \mathrm{gmod}\Lambda^{\mathrm{o}}$, we have the following result.

3.3.4 **Lemma**. *Let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver.*

(1) *If $f : M \to N$ is left minimal in* $\mathrm{gmod}\Lambda$, *then $\mathfrak{D}f : \mathfrak{D}N \to \mathfrak{D}M$ is right minimal in* $\mathrm{gmod}\Lambda^{\mathrm{o}}$.

(2) *If $f : M \to N$ right minimal in* $\mathrm{gmod}\Lambda$, *then $\mathfrak{D}f : \mathfrak{D}N \to \mathfrak{D}M$ is left minimal in* $\mathrm{gmod}\Lambda^{\mathrm{o}}$.

Proof. We shall only prove Statement (1), since Statement(2) is dual. Suppose that $f : M \to N$ is left minimal in gmodΛ. Let $g : \mathfrak{D}N \to \mathfrak{D}N$ be a morphism in gmod$\Lambda^{\rm o}$ such that $\mathfrak{D}f \circ g = \mathfrak{D}f$. Then, $\mathfrak{D}g \circ \mathfrak{D}^2 f = \mathfrak{D}^2 f$. By proposition 3.3.3, we have $\psi^N \circ f = \mathfrak{D}^2 f \circ \psi^M$, where $\psi^M : M \to \mathfrak{D}^2 M$ and $\psi^N : N \to \mathfrak{D}^2 N$ are graded isomorphisms. Thus, we see that

$$\psi^N \circ f = \mathfrak{D}^2 f \circ \psi^M = \mathfrak{D}g \circ \mathfrak{D}^2 f \circ \psi^M = \mathfrak{D}g \circ \psi^N \circ f.$$

So, $f = (\psi^N)^{-1} \circ \mathfrak{D}g \circ \psi^N \circ f$, where $(\psi^N)^{-1}$ is an inverse of ψ^N. Since f is left minimal, $(\psi^N)^{-1} \circ \mathfrak{D}g \circ \psi^N$ is a graded automorphism, and hence, so is $\mathfrak{D}g$. This shows that $\mathfrak{D}f$ is right minimal. The proof of the Lemma is completed.

3.4 Graded projective modules and graded injective modules

Throughout this section, let $\Lambda = kQ/R$ be a graded k-algebra, where Q is a locally finite quiver. Under our general setting of Λ without an identity, we study graded projective Λ-modules and the graded injective Λ-modules. We shall give an explicit description of the morphisms between graded projective modules with a finitely piece-supported top and those between graded injective modules with a finitely piece-supported socle. Moreover, we shall prove that, as under the classical setting, GModΛ has enough projective objects and enough injective objects.

To start with, we put $P_a = \Lambda e_a$ for each $a \in Q_0$. Since Q is a locally finite, $P_a = \oplus_{i\in\mathbb{Z}}(P_a)_i \in$ gmodΛ, where $(P_a)_i = \Lambda_i e_a$. In particular, $(P_a)_i = 0$ for $i < 0$. In order to describe graded morphisms from such modules, we fix some notations which will be used for the rest of this thesis. Let M be a module in GModΛ. Given $u \in e_y \Lambda_s e_x$ with $s \in \mathbb{Z}$ and $x, y \in Q_0$, the left multiplication by u yields a k-linear map

$$M(u) : M_i(x) \to M_{i+s}(y); m \mapsto um,$$

for every $i \in \mathbb{Z}$. On the other hand, given $m \in M_s(a)$ with $s \in \mathbb{Z}$ and $a \in Q_0$, the right multiplication by m yields a graded Λ-linear morphism

$$M[m] : P_a\langle -s\rangle \to M; v \mapsto vm.$$

3.4.1 **Proposition**. *Let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver. Consider modules $M \in \mathrm{GMod}\Lambda$, $W \in \mathrm{Mod}k$ and $P_a\langle -s\rangle \otimes V$ with $(s,a) \in \mathbb{Z} \times Q_0$ and $V \in \mathrm{Mod}k$. We have a k-linear monomorphism*

$$\varphi_{_M} : M_s(a) \otimes \mathrm{Hom}_k(V,W) \to \mathrm{GHom}_\Lambda(P_a\langle -s\rangle \otimes V, M \otimes W); m \otimes f \mapsto M[m] \otimes f,$$

which is natural in M. Moreover, $\varphi_{_M}$ is an isomorphism in case $M_s(a)$ or V is finite dimensional.

Proof. Consider $M \in \mathrm{GMod}\Lambda$ and $W \in \mathrm{Mod}k$. We first show that the k-linear map $\varphi_{_M}$ stated in the proposition is natural in M. Given a morphism $g : M \to N$ in $\mathrm{GMod}\Lambda$, we have a commutative diagram

$$\begin{array}{ccc} M_s(a) \otimes \mathrm{Hom}_k(V,W) & \xrightarrow{\varphi_M} & \mathrm{GHom}_\Lambda(P_a\langle -s\rangle \otimes V, M \otimes W) \\ \downarrow{\scriptstyle g_{s,a}\otimes \mathrm{id}} & & \downarrow{\scriptstyle (g\otimes \mathrm{id}_W)_*} \\ N_s(a) \otimes \mathrm{Hom}_k(V,W) & \xrightarrow{\varphi_N} & \mathrm{GHom}_\Lambda(P_a\langle -s\rangle \otimes V, N \otimes W), \end{array}$$

where $(g \otimes \mathrm{id}_W)_* = \mathrm{GHom}_\Lambda(P_a\langle -s\rangle \otimes V, g \otimes \mathrm{id}_W)$. Indeed, we have

$$\begin{aligned} (\varphi_N \circ (g_{s,a} \otimes \mathrm{id}))(m \otimes f)(e_a \otimes v) &= \varphi_N(g_{s,a}(m) \otimes f)(e_a \otimes v) \\ &= (M[g_{s,a}(m)] \otimes f)(e_a \otimes v) \\ &= g_{s,a}(m) \otimes f(v) \end{aligned}$$

and

$$\begin{aligned} ((g \otimes \mathrm{id}_W)_* \circ \varphi_M)(m \otimes f)(e_a \otimes v) &= (g \otimes \mathrm{id}_W)_*(M[m] \otimes f)(e_a \otimes v) \\ &= (g \otimes \mathrm{id}_W) \circ (M[m] \otimes f)(e_a \otimes v) \\ &= (g \otimes \mathrm{id}_W)(m \otimes f(v)) \\ &= g_{s,a}(m) \otimes f(v) \end{aligned}$$

for all $m \in M_{s,a}$, $f \in \mathrm{Hom}_k(V,W)$ and $v \in V$; thus, we see that $\varphi_N \circ (g_{s,a} \otimes \mathrm{id}) = (g \otimes \mathrm{id}_W)_* \circ \varphi_M$.

Choose a basis $\{m_i \mid i \in \Omega\}$ of $M_{-s}(a)$. If $\omega = \sum_{i\in\Omega} m_i \otimes f_i \in \ker(\varphi_{_M})$ for some $f_i \in \mathrm{Hom}_k(V,W)$, then $\varphi_{_M}(\omega)(e_a \otimes v) = \sum_{i\in\Omega} m_i \otimes f_i(v) = 0$, and hence, $f_i(v) = 0$, for all $v \in V$ and $i \in \Omega$. That is, $\varphi_{_M}$ is a monomorphism.

Consider $f \in \mathrm{GHom}_\Lambda(P_a\langle -s\rangle \otimes V, M \otimes W)$. Given $v \in V$, since $e_a \otimes v \in P_a\langle -s\rangle_s \otimes V$, we see that $f(e_a \otimes v) \in (M \otimes V)_s(a) = M_s(a) \otimes V$, and hence, we can uniquely write $f(e_a \otimes v) = \sum_{i\in\Omega} m_i \otimes w_{i,v}$, for some $w_{i,v} \in W$. This yields, for every $i \in \Omega$, a k-linear map $f_i : V \to W$, sending v to $w_{i,v}$. In case Ω is finite, $f = \varphi_{_M}(\sum_{i\in\Omega} m_i \otimes f_i)$. Otherwise, choose a basis $\{v_j \,|\, j \in \Delta\}$ of V and a basis $\{w_i \,|\, i \in \Theta\}$ of W. For $(i,j) \in \Theta \times \Delta$, we have $\theta_{ij} \in \mathrm{Hom}_k(V, W)$ such that $\theta_{ij}(v_j) = w_i$ and $\theta_{ij}(v_p) = 0$ for $j \neq p$. For each $j \in \Delta$, we may write uniquely $f(e_a \otimes v_j) = \sum_{i\in\Theta} m_{ij} \otimes w_i$, where $m_{ij} \in M_s(a)$ with $m_{ij} = 0$ for all but finitely many $i \in \Theta$. In case Δ is finite, $m_{ij} = 0$ for all but finitely many $(i,j) \in \Theta \times \Delta$, and $f = \varphi_{_M}(\sum_{(i,j)\in\Theta\otimes\Delta} m_{ij} \otimes \theta_{ij})$. The proof of the proposition is completed.

Remark. By Proposition 3.4.1, $P_a\langle -s\rangle \otimes V$ with $(s,a) \in \mathbb{Z} \times Q_0$ and $V \in \mathrm{Mod}k$ are graded projective Λ-modules. Thus, the strictly full additive subcategory of $\mathrm{GMod}\Lambda$ generated by them will be written as $\mathrm{GProj}\Lambda$. Moreover, we denote by $\mathrm{gproj}\Lambda$ the strictly full additive subcategory of $\mathrm{GMod}\Lambda$ generated by $P_a\langle -s\rangle$ with $(s,a) \in \mathbb{Z} \times Q_0$.

As a special case of Proposition 3.4.1, we obtain the following useful result.

3.4.2 **Corollary**. *Let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver. Given $M \in \mathrm{GMod}\Lambda$ and $P_a\langle -s\rangle$ with $(s,a) \in \mathbb{Z}\times Q_0$, we obtain a k-linear isomorphism*

$$\varphi_{_M} : M_s(a) \to \mathrm{GHom}_\Lambda(P_a\langle -s\rangle, M); m \mapsto M[m],$$

whose inverse is given by $\eta_{_M} : \mathrm{GHom}_\Lambda(P_a\langle -s\rangle, M) \to M_s(a); f \mapsto f(e_a)$.

Proof. Let $f \in \mathrm{GHom}_\Lambda(P_a\langle -s\rangle, M)$. Observing that $e_a \in P_a\langle -s\rangle_s(a)$, we see that $m = f(e_a) \in M_s(a)$ such that $f = M[m]$. The proof of the corollary is completed.

We shall describe the morphisms in $\mathrm{GProj}\,\Lambda$; compare [9, (7.6)]. To simplify the notation, for $u \in e_a\Lambda_{s-t}e_b = P_b\langle -t\rangle_s$, the right multiplication by u will be simply written as

$$P[u] : P_a\langle -s\rangle \to P_b\langle -t\rangle; v \mapsto vu.$$

This notation will be used for the rest of this thesis. Note, however, that it does not distinguish $P[u]$ from its grading shifts.

3.4.3 **Proposition**. *Let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver. Consider $P_a\langle -s\rangle \otimes V$ and $P_b\langle -t\rangle \otimes W$ with $(s,a), (t,b) \in \mathbb{Z} \times Q_0$ and $V, W \in \mathrm{Mod}k$. Then, we obtain a k-linear isomorphism*

$$\varphi : e_a\Lambda_{s-t}e_b \otimes \mathrm{Hom}_k(V,W) \to \mathrm{GHom}_\Lambda(P_a\langle -s\rangle \otimes V, P_b\langle -t\rangle \otimes W); u \otimes f \mapsto P[u] \otimes f.$$

Proof. Write $M = P_b\langle -t\rangle \otimes W$. Then, $M_s(a) = P_b\langle -t\rangle_s(a) = e_a\Lambda_{s-t}e_b$, which is finite dimensional. By Proposition 3.4.1, we obtain a k-linear isomorphism φ as stated in the proposition. The proof of the proposition is completed.

Given $a \in Q_0$, we shall write $P_a^{\rm o} = \Lambda^{\rm o}e_a \in {\rm proj}\Lambda^{\rm o}$. Then $I_a = \mathfrak{D}P_a^{\rm o} \in {\rm gmod}\Lambda$ with a Λ-grading $I_a = \oplus_{i\in\mathbb{Z}}(I_a)_i$, where $(I_a)_i = \oplus_{x\in Q_0}(I_a)_i(x)$ with $(I_a)_i(x) = D(e_x\Lambda^{\rm o}_{-i}e_a)$, for $(i,x) \in \mathbb{Z}\times Q_0$. In particular, $(I_a)_i = 0$ for $i > 0$. If $f \in (I_a)_i(x)$ and $u \in e_y\Lambda_je_x$, then $(u\cdot f)(v^{\rm o}) = f(u^{\rm o}v^{\rm o})$ for any $v \in e_a\Lambda_{-i-j}e_y$. Therefore, $I_a(u) = D(P_a^{\rm o}(u^{\rm o})) : (I_a)_i(x) \to (I_a)_{i+j}(y)$.

3.4.4 **Proposition**. *Let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver. Given $M \in {\rm GMod}\Lambda$ and $I_a\langle s\rangle \otimes V$ with $(s,a) \in \mathbb{Z}\times Q_0$ and $V \in {\rm Mod}k$, we have a natural k-linear isomorphism*

$$\psi_{_M} : {\rm GHom}_\Lambda(M, I_a\langle s\rangle \otimes V) \to {\rm Hom}_k(M_{-s}(a), V).$$

Proof. First, we have a k-linear isomorphism $\theta_a : {\rm Hom}_k(e_a\Lambda_0^{\rm o}e_a, V) \to V; g \mapsto g(e_a)$. And given $(i,x) \in \mathbb{Z}\times Q_0$, by Corollary 2.1.2(1), we have a k-linear isomorphism

$$\sigma_{i,x} : I_a\langle s\rangle_i(x) \otimes V = D(e_x\Lambda^{\rm o}_{-i-s}e_a) \otimes V \to {\rm Hom}_k(e_x\Lambda^{\rm o}_{-i-s}e_a, V)$$

so that $\sigma_{i,x}(h\otimes v)(u^{\rm o}) = h(u^{\rm o})v$, for $h \in D(e_x\Lambda^{\rm o}_{-i-s}e_a)$, $u \in e_a\Lambda_{-i-s}e_x$ and $v \in V$. Furthermore, given any graded morphism $f : M \to I_a\langle s\rangle \otimes V$, we have a k-linear map $f_{-s,a} : M_{-s}(a) \to I_a\langle s\rangle_{-s}(a) \otimes V$. This yields a natural k-linear map

$$\psi_{_M} : {\rm GHom}_\Lambda(M, I_a\langle s\rangle \otimes V) \to {\rm Hom}_k(M_{-s}(a), V); f \mapsto \theta_a \circ \sigma_{-s,a} \circ f_{-s,a}.$$

Suppose that $\psi_{_M}(f) = 0$. Fix $(i,x) \in \mathbb{Z}\times Q_0$ and $m \in M_i(x)$. We may write $f_{i,x}(m) = \sum_{j=1}^r h_j \otimes v_j$, where $h_j \in D(e_x\Lambda^{\rm o}_{-i-s}e_a)$ and the v_j are k-linearly independent in V. If $u \in e_a\Lambda_{-i-s}e_x$, then $f_{-s,a}(um) = uf_{i,x}(m) = \sum_{j=1}^r uh_j \otimes v_j$. Observing that $uh_j \in I_a\langle s\rangle_{-s}(a)$, we obtain

$$0 = \psi_{_M}(f)(um) = \textstyle\sum_{j=1}^r \sigma_{-s,a}(uh_j \otimes v_j)(e_a) = \sum_{j=1}^r (uh_j)(e_a)v_j = \sum_{j=1}^r h_j(u^{\rm o})v_j.$$

Thus, $h_j(u^{\rm o}) = 0$, for $j = 1,\ldots,r$. Hence, $h_j = 0$, for $j = 1,\ldots,r$. In particular, $f_{i,x}(m) = 0$. Thus, $f_{i,x} = 0$ for all $(i,x) \in \mathbb{Z}\times Q_0$. So, $\psi_{_M}$ is a monomorphism.

Consider now a k-linear map $g : M_{-s}(a) \to V$. Given $(i,x) \in \mathbb{Z}\times Q_0$, we shall define a k-linear map $f_{i,x} : M_i(x) \to I_a\langle s\rangle_i(x) \otimes V$. For any $m \in M_i(x)$, we have a k-linear map $g_{i,x}(m) : e_x\Lambda^{\rm o}_{-i-s}e_a \to V$ such that $g_{i,x}(m)(u^{\rm o}) = g(um)$ for all $u \in e_a\Lambda_{-i-s}e_x$. This yields a k-linear map $f_{i,x} : M_i(x) \to I_a\langle s\rangle_i(x) \otimes V$, sending

m to $\sigma_{i,x}^{-1}(g_{i,x}(m))$. In other words, $\sigma_{i,x}(f_{i,x}(m)) = g_{i,x}(m)$, for all $m \in M_i(x)$. Let $v \in e_y \Lambda_j e_x$ and $m \in M_i(x)$. Given $u \in e_a \Lambda_{-i-j-s} e_y$, we obtain

$$\sigma_{i+j,y}(f_{i+j,y}(vm))(u^{\rm o}) = g_{i+j,y}(vm)(u^{\rm o}) = g(uvm) = g_{i,x}(m)((uv)^{\rm o}).$$

On the other hand, $\sigma_{i,x}^{-1}(g_{i,x}(m)) = \sum_{p=1}^{r} h_p \otimes v_p$, for some $h_p \in D(e_x \Lambda^{\rm o}_{-i-s} e_a)$ and $v_p \in V$. Thus, $v f_{i,x}(m) = \sum_{p=1}^{r} (v h_p) \otimes v_p$ with $v h_p \in D(e_y \Lambda^{\rm o}_{-i-j-s} e_a)$. So

$$\sigma_{i+j,y}(v f_{i,x}(m))(u^{\rm o}) = \textstyle\sum_{p=1}^{s} (v h_p)(u^{\rm o}) v_p = \sum_{p=1}^{s} \sigma_{i,x}(h_p \otimes v_p)((uv)^{\rm o}) = g_{i,x}(m)((uv)^{\rm o}).$$

Thus, $\sigma_{i+j,y}(v f_{i,x}(m)) = \sigma_{i+j,y}(f_{i+j,y}(vm))$. Hence, $f_{i+j,y}(vm) = v f_{i,x}(m)$. That is, we have a graded Λ-linear morphism $f = (f_{i,x})_{(i,x)\in \mathbb{Z}\times Q_0} : M \to I_a\langle s\rangle \otimes V$. Clearly, $\psi_{_M}(f) = g$. The proof of the proposition is completed.

Remark. By Proposition 3.4.4, $I_a\langle s\rangle \otimes V$ with $(s, a) \in \mathbb{Z} \times Q_0$ and $V \in {\rm Mod} k$ are graded injective Λ-modules. The strictly full additive subcategory of ${\rm GMod}\Lambda$ generated by them will be written as ${\rm GInj}\Lambda$. Moreover, the strictly full additive subcategory of ${\rm GMod}\Lambda$ generated by $I_a\langle s\rangle$ with $(s, a) \in \mathbb{Z} \times Q_0$ will be denoted by ${\rm ginj}\Lambda$.

Fix $a \in Q_0$. Observe that $(I_a)_0 = (I_a)_0(a) = D(e_a \Lambda_0^{\rm o} e_a) = D(k e_a)$. In the sequel, we shall always denote by $e_a^\star$ the k-linear form in $(I_a)_0(a)$ such that $e_a^*(e_a) = 1$. The following statement can be regarded as the dual statement of Corollary 3.4.2.

3.4.5 **Lemma**. *Let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver. Let M be a module in ${\rm GMod}\Lambda$. Given $m \in M_{-s}(a)$ with $(s, a) \in \mathbb{Z} \times Q_0$, there is a graded morphism $f : M \to I_a\langle s\rangle$, sending m to $e_a^\star$.*

Proof. Fix $m \in M_{-s}(a)$ with $(s, a) \in \mathbb{Z} \times Q_0$. In view of Proposition 3.4.4, we have a k-linear isomorphism

$$\psi_{_M} : {\rm GHom}_\Lambda(M, I_a\langle s\rangle) \to {\rm Hom}_k(M_{-s}(a), k); f \mapsto \theta_a \circ f_{-s,a},$$

where $\theta_a : {\rm Hom}_k(e_a \Lambda_0^{\rm o} e_a, k) \to k; g \mapsto g(e_a)$ is a k-linear isomorphism. Consider a k-linear map $h : M_{-s}(a) \to k$, sending m to 1. Then, there exists a graded morphism $f : M \to I_a\langle s\rangle$ such that $\psi_{_M}(f) = h$. Thus, we see that

$$1 = h(m) = \psi_{_M}(f)(m) = (\theta_a \circ f_{-s,a})(m) = \theta_a(f_{-s,a}(m)) = f_{-s,a}(m)(e_a).$$

That is, $f_{-s,a}(m) = e_a^\star$. The proof of the lemma is completed.

Now, we shall describe the morphisms in $\mathrm{GInj}\Lambda$. For this purpose, we fix some notations, which will be used for the rest of this thesis. Given $u \in e_a \Lambda_{t-s} e_b$ with $s,t \in \mathbb{Z}$ and $a,b \in Q_0$, the right multiplication by u^{o} yields a morphism $P[u^{\mathrm{o}}] : P_b^{\mathrm{o}}\langle -t\rangle \to P_a^{\mathrm{o}}\langle -s\rangle$ in $\mathrm{gproj}\Lambda^{\mathrm{o}}$. Applying the duality $\mathfrak{D} : \mathrm{gmod}\Lambda^{\mathrm{o}} \to \mathrm{gmod}\Lambda$, we obtain a morphism $I[u] = \mathfrak{D}(P[u^{\mathrm{o}}]) : I_a\langle s\rangle \to I_b\langle t\rangle$ in $\mathrm{ginj}\Lambda$. Note that this notation does not distinguish $I[u]$ from its grading shifts.

3.4.6 **Proposition**. *Let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver. Consider $I_a\langle s\rangle \otimes V$ and $I_b\langle t\rangle \otimes W$ with $(s,a),(t,b) \in \mathbb{Z} \times Q_0$ and $V,W \in \mathrm{Mod}k$. Then, we obtain a k-linear isomorphism*

$$\phi : e_a\Lambda_{t-s}e_b \otimes \mathrm{Hom}_k(V,W) \to \mathrm{GHom}_\Lambda(I_a\langle s\rangle \otimes V, I_b\langle t\rangle \otimes W); u \otimes f \mapsto I[u] \otimes f.$$

Proof. First, we have a canonical k-linear isomorphism

$$\eta : e_a\Lambda_{t-s}e_b \to D^2(e_b\Lambda^{\mathrm{o}}_{t-s}e_a); u \mapsto \eta(u),$$

such that $\eta(u)(g) = g(u^{\mathrm{o}})$ for all $u \in e_a\Lambda_{t-s}e_b$ and $g \in D(e_b\Lambda^{\mathrm{o}}_{t-s}e_a)$. And by 2.1.1, we obtain a k-linear isomorphism

$$\rho : D^2(e_b\Lambda^{\mathrm{o}}_{t-s}e_a) \otimes \mathrm{Hom}_k(V,W) \to \mathrm{Hom}_k(D(e_b\Lambda^{\mathrm{o}}_{t-s}e_a) \otimes V, W); \varphi \otimes f \mapsto \rho(\varphi \otimes f)$$

such that $\rho(\varphi \otimes f)(g \otimes v) = \varphi(g)f(v)$, for all $g \in D(e_b\Lambda^{\mathrm{o}}_{t-s}e_a)$ and $v \in V$.

As did in the proof of Proposition 3.4.4, we consider two k-linear isomorphisms $\theta_b : \mathrm{Hom}_k(e_b\Lambda^{\mathrm{o}}_0e_b, V) \to W; g \mapsto g(e_b)$ and

$$\sigma_{-t,b} : D(e_b\Lambda^{\mathrm{o}}_0e_b) \otimes W \to \mathrm{Hom}_k(e_b\Lambda^{\mathrm{o}}_0e_b, W); g \otimes w \mapsto \sigma_{-t,b}(g \otimes w)$$

such that $\sigma_{-t,b}(g \otimes w)(e_b) = g(e_b)w$. Since $(I_a\langle s\rangle \otimes V)_{-t}(b) = D(e_b\Lambda^{\mathrm{o}}_{t-s}e_a) \otimes V$, we obtain a k-linear isomorphism

$$\psi : \mathrm{GHom}_\Lambda(I_a\langle s\rangle \otimes V, I_b\langle t\rangle \otimes W) \to \mathrm{Hom}_k(D(e_b\Lambda^{\mathrm{o}}_{t-s}e_a) \otimes V, W); h \mapsto \theta_b \circ \sigma_{-t,b} \circ h_{-t,b}.$$

Now, we obtain a k-linear isomorphism

$$\phi = \psi^{-1} \circ \rho \circ (\eta \otimes \mathrm{id}) : e_a\Lambda_{t-s}e_b \otimes \mathrm{Hom}_k(V,W) \to \mathrm{GHom}_\Lambda(I_a\langle s\rangle \otimes V, I_b\langle t\rangle \otimes W).$$

Fix $u \in e_a\Lambda_{t-s}e_b$ and $f \in \mathrm{Hom}_k(V,W)$. We claim that $\phi(u \otimes f) = I[u] \otimes f$, or equivalently, $(\rho \circ (\eta \otimes \mathrm{id}))(u \otimes f) = \psi(I[u] \otimes f)$. Indeed, given $g \in D(e_b\Lambda^{\mathrm{o}}_{t-s}e_a)$ and $v \in V$, we have

$$\begin{aligned}(\rho \circ (\eta \otimes \mathrm{id}))(u \otimes f)(g \otimes v) &= \rho((\eta \otimes \mathrm{id})(u \otimes f))(g \otimes v) \\ &= \rho(\eta(u) \otimes f)(g \otimes v) \\ &= (\eta(u)(g))f(v) \\ &= g(u^{\mathrm{o}})f(v)\end{aligned}$$

and

$$\begin{aligned}\psi(I[u]\otimes f)(g\otimes v) &= (\theta_b\circ\sigma_b\circ(I[u]\otimes f)_{-t,b})(g\otimes v)\\ &= (\theta_b\circ\sigma_b)(I[u](g)\otimes f(v))\\ &= \sigma_b(I[u](g)\otimes f(v))(e_b)\\ &= (I[u](g)(e_a))f(v)\\ &= g(u^{\mathrm{o}})f(v).\end{aligned}$$

That is, $(\rho\circ(\eta\otimes \mathrm{id}))(u\otimes f)(g\otimes v) = \psi(I[u]\otimes f)(g\otimes v)$. This establishes our claim. The proof of the proposition is completed.

As an immediate consequence of Propositions 3.4.3 and 3.4.6, we obtain the following statement, which shows an important difference between the graded setting and the non-graded setting.

3.4.7 **Corollary**. *Let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver. Given $a\in Q_0$, we have* $\mathrm{GEnd}_\Lambda(P_a)\cong \mathrm{GEnd}_\Lambda(I_a)\cong k$; *and in particular, P_a and I_a are strongly indecomposable.*

The following result is well known in case Λ has an identity; see [52, page 7].

3.4.8 **Proposition**. *Let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver. Then,* $\mathrm{GMod}\Lambda$ *has enough projective objects and enough injective objects.*

Proof. Let $M\in \mathrm{GMod}\Lambda$. Given $(i,x)\in\mathbb{Z}\times Q_0$, it is clear that we have a graded morphism $f_{i,x}: P_x\langle -i\rangle\otimes M_i(x)\to M$ such that $f_{i,x}(u\otimes m) = um$ for $u\in P_x\langle -i\rangle$ and $m\in M_i(x)$. Consider $\oplus_{(i,x)\in\mathbb{Z}\times Q_0}P_x\langle -i\rangle\otimes M_i(x)$ with canonical inclusions $q_{s,a}: P_a\langle -s\rangle\otimes M_s(a)\to \oplus_{(i,x)\in\mathbb{Z}\times Q_0}P_x\langle -i\rangle\otimes M_i(x)$, with $(s,a)\in\mathbb{Z}\times Q_0$. By the universal property of direct sums, there exists a unique graded morphism $f:\oplus_{(i,x)\in\mathbb{Z}\times Q_0}P_x\langle -i\rangle\otimes M_i(x)\to M$ such that $f\circ q_{s,a} = f_{s,a}$ for all $(s,a)\in\mathbb{Z}\times Q_0$. Given $m\in M_i(x)$ with $(i,x)\in\mathbb{Z}\times Q_0$, we see that $f(q_{i,x}(e_x\otimes m)) = (f\circ q_{i,x})(e_x\otimes m) = f_{i,x}(e_x\otimes m) = m$. Thus, f is a graded Λ-linear epimorphism. That is, $\mathrm{GMod}\Lambda$ has enough projective objects.

Since $\mathrm{GMod}\Lambda^{\mathrm{o}}$ has enough projective objects, by the above construction, we have a graded epimorphism $g:\oplus_{(i,x)\in\mathbb{Z}\times Q_0}P^{\mathrm{o}}_x\langle i\rangle\otimes D(M_i(x))\to \mathfrak{D}M$. Applying the exact functor $\mathfrak{D}$, we obtain a graded monomorphism

$$\mathfrak{D}(g):\mathfrak{D}^2M\to\mathfrak{D}(\oplus_{(i,x)\in\mathbb{Z}\times Q_0}P^{\mathrm{o}}_x\langle i\rangle\otimes D(M_i(x))).$$

In view of Propositions 3.3.1 and 3.3.2, we have

$$\begin{aligned}\mathfrak{D}(\oplus_{(i,x)\in\mathbb{Z}\times Q_0}P_x^{\mathrm{o}}\langle -i\rangle\otimes D(M_i(x))) &\cong \Pi_{(i,x)\in\mathbb{Z}\times Q_0}\mathfrak{D}(P_x^{\mathrm{o}}\langle -i\rangle\otimes D(M_i(x)))\\ &\cong \Pi_{(i,x)\in\mathbb{Z}\times Q_0}\mathfrak{D}(P_x^{\mathrm{o}}\langle -i\rangle)\otimes D^2(M_i(x))\\ &= \Pi_{(i,x)\in\mathbb{Z}\times Q_0}I_x\langle i\rangle\otimes D^2(M_i(x)),\end{aligned}$$

which is graded injective by Proposition 3.4.4. Moreover, by Proposition 3.3.3(1), there exists a graded monomorphism $\psi : M \to \mathfrak{D}^2 M$. This yields a desired graded monomorphism $\mathfrak{D}(g)\circ\psi : M \to \mathfrak{D}(\oplus_{(i,x)\in\mathbb{Z}\times Q_0}P_x^{\mathrm{o}}\langle i\rangle\otimes D(M_i(x)))$. The proof of the proposition is completed.

3.5 Graded socle and graded radical

In this section, let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver. Put $J = \oplus_{n>0}\Lambda_n$, which is a graded ideal of Λ. Under this setting, we shall be able to describe the graded radical and the graded socle of a graded Λ-modules. Moreover, we will provide some sufficient conditions for the graded socle to be graded essential and the graded radical to be graded superfluous.

To start with, we shall describe all graded simple modules in GModΛ; compare [2, (I.5.17)]. Fix $a \in Q_0$. We put $S_a = P_a/Je_a$, where Je_a is a graded submodule of P_a by Lemma 3.1.3(3). Since $P_a = Je_a + ke_a$, we have $S_a = k(e_a + Je_a)$. Thus, $S_a\langle n\rangle$ is graded simple for each $n \in \mathbb{Z}$.

3.5.1 **Proposition**. *Let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver. Then every graded simple module in* GModΛ *is isomorphic to $S_a\langle -n\rangle$, for some $(n, a) \in \mathbb{Z}\times Q_0$.*

Proof. Let S be a graded simple module in GModΛ. Then, there exists a non-zero element $m \in S_n(a)$ for some $(n, a) \in \mathbb{Z}\times Q_0$. By Corollary 3.1.4, Λm is a non-zero graded submodule of S, and hence, $S = \Lambda m$. By Corollary 3.4.2, we have a graded Λ-linear epimorphism $p : P_a\langle -n\rangle \to S$ such that $p(e_a) = m$. On the other hand, by Lemma 3.1.3, Jm is also a graded submodule of S. Since $Jm \subseteq \oplus_{i\ge n+1}S_i$, we see that $Jm \ne S$, and hence, $Jm = 0$. This implies that $(JP_a)\langle -n\rangle \subseteq \mathrm{Ker}(p)$. Therefore, p induces a graded epimorphism $\bar{p} : P_a\langle -n\rangle/(JP_a)\langle -n\rangle \to S$. Since $P_a\langle -n\rangle/(JP_a)\langle -n\rangle = S_a\langle -n\rangle$, which is graded simple, we see that $S \cong S_a\langle -n\rangle$. The proof of the Proposition is completed.

The following statement describes the graded socle of any graded Λ-module.

3.5.2 **Lemma**. *Let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver. Consider a module $M = \oplus_{i\in\mathbb{Z}} M_i = \oplus_{(i,x)\in\mathbb{Z}\times Q_0} M_i(x)$ in* $\mathrm{GMod}\Lambda$.

(1) $\mathrm{soc}M = \oplus_{(i,x)\in\mathbb{Z}\times Q_0}(\mathrm{soc}M)_i(x)$ *with* $(\mathrm{soc}M)_i(x) = \{m \in M_i(x) \mid Jm = 0\}$.

(2) $\mathrm{soc}M$ *is contained in any essential graded submodule of* M.

(3) *If* $M \in \mathrm{GMod}^{-}\Lambda$ *is non-zero, then* $\mathrm{soc}M$ *is graded essential in* M.

Proof. (1) Write $N_{i,x} = \{m \in M_i(x) \mid Jm = 0\}$ for all $(i,x) \in \mathbb{Z}\times Q_0$. By Proposition 3.5.1, we see that $(\mathrm{soc}M)_i(x) \subseteq N_{i,x}$. On the other hand, let $m \in N_{i,x}$ with $(i,x) \in \mathbb{Z}\times Q_0$. By Corollary 3.1.4, Λm is a graded submodule of M. Since $\Lambda e_x = Je_x + ke_x$, we see that $\Lambda m = \Lambda e_x m = km$. Thus, $\Lambda m = 0$ or Λm is a graded simple submodule of M. So, $\Lambda m \subseteq \mathrm{soc}M$, and consequently, $m \in (\mathrm{soc}M)_i(x)$. Thus, $N_{i,x} \subseteq \mathrm{soc}M$, and hence, $(\mathrm{soc}M)_i(x) = N_{i,x}$.

(2) Let L be an essential graded submodule of M. If S is a simple graded submodule of M, then $L \cap S$ is a non-zero graded submodule of S. Hence, $L \cap S = S$, that is, $S \subseteq L$. Therefore, $\mathrm{soc}M \subseteq L$.

(3) Assume that M is bounded above. Let N be a non-zero graded submodule of M. Then N contains a non-zero element $m_i \in N_i$ for some i. Since N is bounded above, there exists some $j \geq 0$ such that $\Lambda_j m_i \neq 0$ but $\Lambda_{j+1} m_i = 0$. Observing that $\Lambda_{j+1} = J\Lambda_j$, we have $J(\Lambda_j m_i) = 0$. By Statement (1), $0 \neq \Lambda_j m_i \subseteq \mathrm{soc}M$. This shows that $\mathrm{soc}M$ is graded essential in M. The proof of the lemma is completed.

Example. Consider the graded algebra $\Lambda = kQ/R$, where

$$Q:\ \alpha \circlearrowleft 1 \xrightarrow{\ \beta\ } 2$$

and $R = \langle \beta\alpha \rangle$. Consider $P_1 = k\langle e_1, \bar{\beta}, \bar{\alpha}, \cdots, \bar{\alpha}^i, \cdots \rangle$. By Lemma 3.5.2(1), $\mathrm{soc}P_1 = k\langle \bar{\beta} \rangle$. Considering the graded submodule $k\langle \bar{\alpha}, \cdots, \bar{\alpha}^i, \cdots \rangle$ of P_1, we see that $\mathrm{soc}P_1$ is not graded essential in P_1. Indeed, P_1 is not bounded above.

The following statement is well known for non-graded modules.

3.5.3 **Proposition**. *Let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver. If $f : M \to N$ is a morphism in* $\mathrm{GMod}\Lambda$, *then* $f(\mathrm{soc}M) \subseteq \mathrm{soc}N$.

Proof. Let $f : M \to N$ be a morphism in $\mathrm{GMod}\Lambda$. Consider a pure element $m \in \mathrm{soc}M$. Then $f(m)$ is a pure element in N. By Lemma 3.5.2(1), $Jm = 0$, and hence, $Jf(m) = f(Jm) = 0$. By Lemma 3.5.2(1) again, $f(m) \in \mathrm{soc}N$. The proof of the proposition is completed.

Recall that $e_a^\star$ with $a \in Q_0$ stands for the k-linear form in $(I_a)_0(a)$ such that $e_a^*(e_a) = 1$.

3.5.4 **Corollary**. *Let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver. If $a \in Q_0$, then $\mathrm{soc}I_a = ke_a^\star$, which is graded essential in I_a.*

Proof. Fix $a \in Q_0$. Note that $(I_a)_0 = (\mathfrak{D}P_a^{\mathrm{o}})_0 = D(\Lambda_0^{\mathrm{o}} e_a) = ke_a^\star = (I_a)_0(a)$. Since $(I_a)_i = 0$ for $i > 0$, we have $Je_a^\star = 0$. By Lemma 3.5.2(1), $e_a^\star \in (\mathrm{soc}I_a)_0(a)$. So, $(\mathrm{soc}I_a)_0 = ke^\star$. Next, suppose that f is a non-zero function in $(I_a)_{-i}(x) = D(e_a\Lambda_i e_x)$ for some $i > 0$ and $x \in Q_0$. By definition, there exists some $u \in e_a\Lambda_i e_x$ such that $f(u^{\mathrm{o}}) \neq 0$. Then, $(u \cdot f)(e_a) = f(u^{\mathrm{o}}) \neq 0$. That is, $u \cdot f \neq 0$. Observe that $u \in J$. Thus, by Lemma 3.5.2(1), $f \notin \mathrm{soc}I_a$. Therefore, $\mathrm{soc}I_a = (\mathrm{soc}I_a)_0 = ke_a^\star$. Finally, since I_a is bounded above, by Lemma 3.5.2(3), $\mathrm{soc}I_a$ is graded essential in I_a. The proof of the corollary is completed.

It is well known that a finitely generated non-zero module (not necessarily graded) admits a maximal submodule; see, for example, [1, (2.8)]. We shall describe all possible maximal graded submodules of a module M in $\mathrm{GMod}\Lambda$. First, by Lemma 3.1.3, JM is a graded submodule of M. An element m in M is called a **top-element** if $m \in M_n(a)\backslash JM$ for some $(n, a) \in \mathbb{Z} \times Q_0$. In this case, by Lemma 2.1.4, there exists a k-vector subspace $L_{n,a}$ of $M_n(a)$ containing $M_n(a) \cap JM$ such that $M_n(a) = L_{n,a} \oplus km$. Setting $L_{i,x} = M_i(x)$ for $(i, x) \in \mathbb{Z} \times Q_0$ with $(i, x) \neq (n, a)$, we obtain a k-vector subspace $L(m) = \sum_{(i,x)\in\mathbb{Z}\times Q_0} L_{i,x}$ of M. Note that this construction of $L(m)$ is not unique.

3.5.5 **Lemma**. *Let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver. Let M be a module in $\mathrm{GMod}\Lambda$. If L is a graded submodule of M, then it is graded maximal in M if and only if $L = L(m)$ for some top-element $m \in M$; and in this case, $JM \subseteq L$.*

Proof. Suppose that m is a top-element in $M_n(a)$, where $(n, a) \in \mathbb{Z} \times Q_0$. We claim that $L(m) = \sum_{(i,x)\in\mathbb{Z}\times Q_0} L_{i,x}$ is a Λ-submodule of M. If this is not the case, then there exist some $m' \in L_{i,x}$ and $u \in e_y\Lambda_j e_x$, where $i, j \in \mathbb{Z}$ and $x, y \in Q_0$, such that $um' \in M_{i+j,y}\backslash L_{i+j,y}$. In view of the definition of $L(m)$, we see that $(i+j,y)=(n,a)$. If $j>0$, then $um'\in M_n(a)\cap JM\subseteq L_{n,a}=L_{i+j,y}$, a contradiction. If $j=0$, then $(i,x)=(n,a)$ and $u\in e_a\Lambda_0e_a=ke_a$, and consequently, $um'\in km'\subseteq L_{i,x}=L_{i+j,y}$, a contradiction. Hence, our claim holds. By Lemma 3.2.1, $L(m)$ is a graded submodule of M. By definition, $L(m)$ contains JM and $M/L(m)$ is one-dimensional. By Lemma 3.1.6, $L(m)$ is a maximal graded submodule of M.

Conversely, assume that L is a maximal graded submodule of M. This yields a graded simple Λ-module $M/L=\oplus_{(i,x)\in\mathbb{Z}\times Q_0}(M_i(x)+L)/L$. By Proposition 3.5.1, $M/L\cong S_a\langle -n\rangle$ for some $(n,a)\in\mathbb{Z}\times Q_0$. Therefore, $(M_n(a)+L)/L=k(m+L)$ with $m\in M_n(a)\backslash L_n(a)$, and $L_i(x)=M_i(x)$ for all $(i,x)\in\mathbb{Z}\times Q_0$ with $(i,x)\neq(n,a)$. As a consequence, $M_n(a)=L_n(a)+km$ and $M_n(a)\cap JM\subseteq L_n(a)$. Since $m\notin L_n(a)$, we see that $M_n(a)=L_n(a)\oplus km$. By the above construction, we have $L=L(m)$. The proof of the lemma is completed.

Applying Lemma 3.5.5, we obtain the following important property of the graded radical of a module in GModΛ, which is known in case Q is finite and $M\in$ GMod$^+\Lambda$; see, for example, [42, Page 70].

3.5.6 **Proposition**. *Let $\Lambda=kQ/R$ be a graded algebra, where Q is a locally finite quiver. Then,* rad$M=JM$ *for any module $M\in$ GModΛ.*

Proof. Let M be a module in GModΛ with Λ-grading $M=\oplus_{i\in\mathbb{Z}}M_i$. Suppose first that M has no maximal graded submodule. By definition, rad$M=M$. On the other hand, by Lemma 3.5.5, M contains no top-element, that is, $M_i(x)=M_i(x)\cap JM$ for all $(i,x)\in\mathbb{Z}\times Q_0$. Thus, $M=JM$.

Suppose now that M has maximal graded submodules. By Lemma 3.5.5, JM is contained in every maximal graded submodule of M, and hence, $JM\subseteq$ rad$M$. On the other hand, assume that $m\in M\backslash JM$. Write $m=\sum_{(i,x)\in\mathbb{Z}\times Q_0}m_{i,x}$ with $m_{i,x}\in M_i(x)$. By Lemma 3.2.1(3), $m_{n,a}\in M_n(a)\backslash JM$ for some $(n,a)\in\mathbb{Z}\times Q_0$, that is, $m_{n,a}$ is a top-element in $M_n(a)$. By Lemma 3.5.5, we may construct a maximal graded submodule $L(m_{n,a})$ of M such that $m_{n,a}\notin L(m_{n,a})_{n,a}$. By Lemma 3.2.1(1), $m\notin L(m_{n,a})$, and consequently, $m\notin$ rad$M$. This shows that rad$M\subseteq JM$, and consequently, $JM=$ rad$M$. The proof of the proposition is completed.

As an immediate consequence of Proposition 3.5.6, we obtain the following statement, which is well known in case Q is finite.

3.5.7 **Corollary**. *Let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver. Then,* $\text{rad}({}_{\Lambda}\Lambda) = J$*. Moreover for any $a \in Q_0$,* $\text{rad}P_a = Je_a$*, which is the unique maximal graded submodule of P_a.*

The following statement is well known for non-graded modules.

3.5.8 **Corollary**. *Let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver. If $f : M \to N$ is a morphism in* GModΛ*, then* $f(\text{rad}M) \subseteq \text{rad}N$.

Proof. Let $f : M \to N$ be a morphism in GModΛ. By Proposition 3.5.6, $f(\text{rad}M) = f(JM) = Jf(M) \subseteq \text{rad}N$. The proof of the corollary is completed.

It is well known that a finitely generated non-graded module is semisimple if and only if it is artin and its radical is zero; see [1, (10.15)]. The following statement is a generalized graded version of this fact.

3.5.9 **Proposition**. *Let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver. A non-zero module M in* GModΛ *is graded semisimple if and only if* $\text{rad}M = 0$.

Proof. Let M be a non-zero module in GModΛ. By Proposition 3.5.6, $\text{rad}M = JM$. Now, M is graded semisimple if and only if $\text{soc}M = M$ if and only if $(\text{soc}M)_i(x) = M_i(x)$, for all $(i, x) \in \mathbb{Z} \times Q_0$. By Lemma 3.5.2(1), this is equivalent to $JM_i(x) = 0$, for all $(i, x) \in \mathbb{Z} \times Q_0$, that is, $JM = 0$. The proof of the proposition is completed.

3.5.10 **Corollary**. *Let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver. A nonzero module M in* GModΛ *is graded semisimple if and only if $\mathfrak{D}M$ is graded semisimple.*

Proof. Let M be a nonzero module in GModΛ. Suppose that $JM = 0$. Then, given any $f \in (\mathfrak{D}M)_i(x) = D(M_{-i}(x))$ with $(i, x) \in \mathbb{Z} \times Q_0$ and $u^{\text{o}} \in e_y(\Lambda^{\text{o}})_j e_x$ with $j \geq 1$ and $y \in Q_0$, we have $(u^{\text{o}} \cdot f)(m) = f(um) = 0$, for all $m \in M_{-i-j}(y)$. That is, $J^{\text{o}} \cdot \mathfrak{D}M = 0$. Suppose that $JM \neq 0$. Then, $um \neq 0$ for some $m \in M_i(x)$ with $(i, x) \in \mathbb{Z} \times Q_0$ and $u \in e_y \Lambda_j e_x$ with $j \geq 1$ and $y \in Q_0$. Then, $f(um) \neq 0$ for some $f \in D(M_{i+j}(y))$. That is, $(u^{\text{o}} \cdot f)(m) = f(um) \neq 0$, where $f \in (\mathfrak{D}M)_{-i-j}(y)$ and $u^{\text{o}} \in e_x(\Lambda^{\text{o}})_j e_y$. Thus, $J^{\text{o}} \cdot \mathfrak{D}M \neq 0$. In view of Propositions 3.5.6 and 3.5.9, we see that M is graded semisimple if and only if $\mathfrak{D}M$ is graded semisimple. The proof of the corollary is completed.

In general, the graded radical of a graded module M is not necessarily graded superfluous in M. However, this is the case when M is bounded below; compare [1, (9.18)].

3.5.11 **Proposition**. *Let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver. Consider a module $M \in \mathrm{GMod}^{+}\Lambda$.*

(1) $\mathrm{rad}M$ *contains any superfluous graded submodule* M.

(2) $\mathrm{rad}M$ *is graded superfluous in* M.

Proof. (1) Let N be a superfluous graded submodule of M. If $N \nsubseteq \mathrm{rad}M$, then there exists a non-zero pure element $m \in N$ but $m \notin \mathrm{rad}M$. By definition, m is a top-element in M, and by Lemma 3.5.5, we may construct a maximal graded submodule $L(m)$ of M. Since $m \notin L(m)$, we obtain $N + L(m) = M$, contrary to N being graded superfluous. Therefore, $N \subseteq \mathrm{rad}M$.

(2) Let N be a graded submodule of M such that $\mathrm{rad}M + N = M$. Suppose that $N \neq M$. Since M is bounded below, there exists a minimal s such that $N_s \neq M_s$. So, there exists $m \in M_s(a) \backslash N_s(a)$ for some $a \in Q_0$. By the minimality of s, we see that $M_s(a) \cap JM \subseteq N_s(a)$. In particular, m is a top-element. By Lemma 2.1.4, $M_s(a) = L_{s,a} \oplus km$, where $L_{s,a}$ is a subspace of $M_s(a)$ containing $N_s(a)$. Since $M_s(a) \cap JM \subseteq L_{s,a}$, by Lemma 3.5.5, $L(m)$ is a maximal graded submodule of M. Moreover, since $N_s(a) \subseteq L_{s,a}$, we have $N \subseteq L(m)$. Since $\mathrm{rad}M \subseteq L(m)$, we obtain $M = L(m)$, a contradiction. Thus, $\mathrm{rad}M$ is graded superfluous in M. The proof of the proposition is completed.

3.5.12 **Corollary**. *Let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver. Let M be a module in $\mathrm{GMod}^{+}\Lambda$.*

(1) *If* $\mathrm{rad}M = M$, *then* $M = 0$.

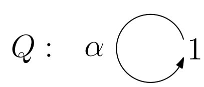
(2) *If* M *is non-zero, then* $\mathrm{top}M$ *is graded semisimple.*

Proof. (1) Assume that $\mathrm{rad}M = M$. In particular, $\mathrm{rad}M + 0 = M$. Since $\mathrm{rad}M$ is graded superfluous in M by Proposition 3.5.11(2), $M = 0$.

(2) Suppose that $M \neq 0$. By Statement (1), $\mathrm{top}M \neq 0$, and by Proposition 3.5.6, $\mathrm{rad}(\mathrm{top}M) = J(M/JM) = 0$. By Lemma 3.5.9, $\mathrm{top}M$ is graded semisimple. The proof of the corollary is completed.

Remark. As will be shown in the following section, finitely generated graded modules are bounded below. Thus, Corollary 3.5.12(1) includes the graded version of Nakayama's Lemma, which is known for positively graded rings with an identity; compare [53, (2.9.2)]. In case Q is finite, this is stated without a proof in [42, Page 70].

Example. Consider the graded algebra $\Lambda = kQ$, where



Then, $\Lambda = k\langle \alpha^i \mid i \geq 0\rangle$, where $\alpha^0 = e_1$. Thus, $\mathfrak{D}\Lambda = \oplus_{i\geq 0}(\mathfrak{D}\Lambda)_{-i}$ with $(\mathfrak{D}\Lambda)_{-i} = k\langle(\alpha^i)^\star\rangle$, where $\langle(\alpha^i)^\star\rangle$ are dual bases of $\langle \alpha^i\rangle$. In particular, $\mathfrak{D}\Lambda \notin \mathrm{GMod}^+\Lambda$. Observe that $\alpha^{\mathrm{o}} \cdot (\alpha^i)^\star = (\alpha^{i-1})^\star$ for all $i \geq 1$. By Proposition 3.5.6, $\mathrm{rad}\,\mathfrak{D}\Lambda = J \cdot \mathfrak{D}\Lambda = \mathfrak{D}\Lambda$. Thus, $\mathrm{rad}\,\mathfrak{D}\Lambda$ is not graded superfluous in $\mathfrak{D}\Lambda$ and $\mathrm{top}\,\mathfrak{D}\Lambda = 0$.

3.6 Finitely generated and finitely cogenerated graded modules

Throughout this section, let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver. In this section, we will give explicit descriptions of finitely generated modules and finitely cogenerated modules in $\mathrm{GMod}\Lambda$.

A module M in $\mathrm{GMod}\Lambda$ is called **finitely generated** if $M = \Lambda m_1 + \cdots + \Lambda m_r$, where $m_1, \ldots, m_r$ are homogeneous elements in M, or equivalently, $M = \Lambda m_1 + \cdots + \Lambda m_r$, where $m_1, \ldots, m_r$ are pure elements in M; see (3.2.1). The following result is essential to describe finitely generated graded modules.

3.6.1 **Lemma**. *Let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver. Consider a module M in $\mathrm{GMod}\Lambda$. If $\mathrm{rad}M$ is graded superfluous in M, then a set $\{m_1, \ldots, m_r\}$ of elements in M is a generating set for M if and only if $\{m_1 + \mathrm{rad}M, \ldots, m_r + \mathrm{rad}M\}$ is a generating set for $\mathrm{top}M$.*

Proof. Suppose that $\mathrm{rad}M$ is graded superfluous in M. We only need to show the sufficiency. Suppose that $\{m_1, \ldots, m_r\}$ is a set of pure elements such that $\{m_1 + \mathrm{rad}M, \ldots, m_r + \mathrm{rad}M\}$ is a generating set for $\mathrm{top}M$. Given any $m \in M$, we have $m + \mathrm{rad}M = \sum_{i=1}^r (\lambda_i m_i + \mathrm{rad}M)$ with $\lambda_i \in k$, namely, $m - \sum_{i=1}^r \lambda_i m_i \in$

rad$M$. Thus, rad$M + \sum_{i=1}^{r} \Lambda m_i = M$, where $\sum_{i=1}^{r} \Lambda m_i$ is a graded submodule of M by Corollary 3.1.4. Since rad$M$ is graded superfluous, $M = \sum_{i=1}^{r} \Lambda m_i$. The proof of the lemma is completed.

3.6.2 **Lemma**. *Let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver. Given a module M in* GModΛ*, the following statements are equivalent*:

(1) *M is finitely generated.*

(2) *$M \in$ GMod$^+\Lambda$ such that* top$M$ *is finite dimensional.*

(3) rad$M$ *is graded superfluous in M such that* top$M$ *is finite dimensional.*

Proof. Assume that M is finitely generated, say $M = \sum_{i=1}^{r} \Lambda m_i$, where $m_i \in M_{s_i}(a_i)$ with $(s_i, a_i) \in \mathbb{Z} \times Q_0$. Then, top$M$ is generated by $m_1 + \text{rad}M, \ldots, m_r + \text{rad}M$, and hence, it is finite dimensional. Now, $\Lambda m_i = \oplus_{j\ge 0} \Lambda_j m_i = \oplus_{j \ge s_i} (\Lambda m_i)_j$, for $i = 1, \ldots, r$. Thus, $M \in$ GMod$^+\Lambda$. This shows that Statement (1) implies Statement (2). By Proposition 3.5.11, Statement (2) implies Statement (3). And by Lemma 3.6.1, Statement (3) implies Statement (1). The proof of the lemma is completed.

Dually, a module M in GModΛ is called **finitely cogenerated** if soc$M$ is graded essential in M and is finite dimensional; compare [1, (10.4), (10.6)].

3.6.3 **Lemma**. *Let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver. A module M in* GModΛ *is finitely cogenerated if and only if $M \in$* GMod$^-\Lambda$ *and* soc$M$ *is finite dimensional.*

Proof. Let $M \in$ GModΛ. The sufficiency follows from Lemma 3.5.2(3). Assume that M is finitely cogenerated. By Lemma 3.5.2(1), we see that soc$M$ has a k-basis $\{m_1, \ldots, m_r\}$, where $m_i \in M_{s_i}(a_i)$ with $(s_i, a_i) \in \mathbb{Z} \times Q_0$. We may suppose that $s_1 \le \cdots \le s_r$. Then, soc$M \cap M_j = 0$ for all $j > s_r$. Suppose that $M_s \ne 0$ for some $s > s_r$. Let $m \in M_s$ be non-zero. Since soc$M$ is graded essential in M, there is some $u \in \Lambda_t$ with $t \ge 0$ such that $0 \ne um \in$ soc$M$. So, soc$M \cap M_{t+s} \ne 0$ with $t + s > s_r$, a contradiction. This shows $M_j = 0$ for all $j > s_r$. Therefore, $M \in$ GMod$^-\Lambda$. The proof of the lemma is completed.

Example. In view of Lemma 3.6.3 and Corollary 3.5.4, we see that every module in ginjΛ is finitely cogenerated.

The following notions are important for our later study of graded projective covers and graded injective envelopes.

3.6.4 **Definition**. Let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver. Consider a module M in GModΛ. A set $\{m_1, \ldots, m_r\}$ of pure elements in M is called

(1) a **top-basis** for M if $\{m_1 + \text{rad}M, \ldots, m_r + \text{rad}M\}$ is a k-basis of top$M$ and M is generated by $m_1, \ldots, m_r$;

(2) a **socle-basis** for M if $\{m_1, \ldots, m_r\}$ is a k-basis of soc$M$, and soc$M$ is graded essential in M.

EXAMPLE. Given $a \in Q_0$, it is clear that $\{e_a\}$ is a top-basis for P_a, and by Corollary 3.5.4, $\{e_a^\star\}$ is a socle-basis for I_a.

3.6.5 **Lemma**. *Let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver. Consider a module M in* GModΛ *with $\{m_1, \ldots, m_r\}$ a set of pure elements in M.*

(1) *If $M \in$ GMod$^+\Lambda$, then $\{m_1, \ldots, m_r\}$ is a minimal generating set of M if and only if $\{m_1 + \text{rad}M, \ldots, m_r + \text{rad}M\}$ is a k-basis of* top$M$; *and in this case, $\{m_1, \ldots, m_r\}$ is a top-basis for M.*

(2) *If $M \in$ GMod$^-\Lambda$, then $\{m_1, \ldots, m_r\}$ is a socle-basis for M if and only if it is a k-basis of* soc$M$.

Proof. (1) Assume that $M \in$ GMod$^+\Lambda$. By Proposition 3.5.11, rad$M$ is graded superfluous in M. In view of Lemma 3.6.1, we see that $\{m_1, \ldots, m_r\}$ is a minimal generating set of M if and only if $\{m_1 + \text{rad}M, \ldots, m_r + \text{rad}M\}$ is a minimal generating set of top$M$. Since top$M$ is graded semisimple by Corollary 3.5.12(2), the latter condition is equivalent to $\{m_1 + \text{rad}M, \ldots, m_r + \text{rad}M\}$ being a k-basis of top$M$.

(2) Let M be a module in GMod$^-\Lambda$. By Lemma 3.5.2(3), soc$M$ is graded essential in M. Thus, Statement (2) follows from the definition of a socle-basis. The proof of the lemma is completed.

3.6.6 **Proposition**. *Let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver. Let M be a non-zero module in* GModΛ. *Then,*

(1) *M is finitely generated if and only if it admits a top-basis*;

(2) *M is finitely cogenerated if and only if it admits a socle-basis.*

Proof. (1) The sufficiency follows from the definition. Suppose that M is finitely generated. Then, M has a minimal generating set $\{m_1, \ldots, m_r\}$ of pure elements. By Lemma 3.6.2, $M \in \mathrm{GMod}^+\Lambda$, and by Lemma 3.6.5(1), $\{m_1, \ldots, m_r\}$ is a top-basis for M.

(2) The sufficiency follows from the definition of a socle-basis. Assume that M is finitely cogenerated. By Lemma 3.2.1(3), $\mathrm{soc}M$ has a k-basis $\{m_1, \ldots, m_r\}$ of pure elements. By Lemma 3.6.3, $M \in \mathrm{GMod}^-\Lambda$, and by Lemma 3.6.5(2), $\{m_1, \ldots, m_r\}$ is a socle-basis for M. The proof of the proposition is completed.

Further, given a module $M \in \mathrm{GMod}\Lambda$, we shall say that M is **finitely generated in degree** s if it is generated by finitely many homogeous elements of degree s. and **finitely cogenereated in degree** $-s$ if M is finitely cogenerated and $\mathrm{soc}M \subseteq M_{-s}$. For instance, $P_a\langle -s\rangle$ is finitely generated in degree s and $I_a\langle s\rangle$ is finitely cogenerated in degree $-s$, for any $(s, a) \in \mathbb{Z} \times Q_0$.

The following statements will be useful in our later construction of linear projective resolutions and colinear injective coresolutions.

3.6.7 **Lemma**. *Let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver. Consider a module M in $\mathrm{GMod}\Lambda$. Then,*

(1) *M is finitely generated in degree s if and only if M_s contains a top-basis for M; and in this case, $M = \oplus_{i\ge s} M_i$ and $\mathrm{rad}M = \oplus_{i>s} M_i$.*

(2) *M is finitely cogenerated in degree $-s$ if and only if M_{-s} contains a socle-basis for M; and in this case, $M = \oplus_{i\le -s} M_i$ and $\mathrm{soc}M = M_{-s}$.*

Proof. (1) Assume that M_s contains a top-basis $\{m_1, \ldots, m_r\}$ for M. By definition, M is generated by $m_1, \ldots, m_r \in M_s$. That is, M is finitely generated in degree s. Conversely, suppose that M is finitely generated in degree s. Then, M_s contains a minimal generating set $\{m_1, \ldots, m_r\}$ of M, which consists of pure elements. By Lemma 3.6.2, $M \in \mathrm{GMod}^+\Lambda$, and by Lemma 3.6.5, $\{m_1, \ldots, m_r\}$ is a top-basis for M. In particular, $M = \Lambda M_s = \oplus_{j\ge 0}(\Lambda_j M_s)$. In particular, $M_i = 0$ for all $i < s$. Thus, $M = \oplus_{i\ge s} M_i$. This yields $M_i = \Lambda_{i-s} M_s \subseteq JM_s$ for all $i > s$. As a consequence, $\oplus_{i>s} M_i \subseteq JM_s \subseteq \oplus_{i>s} M_i$. By Proposition 3.5.6, $\mathrm{rad}M = JM = JM_s = \oplus_{i>s} M_i$.

(2) Suppose that M_{-s} contains a socle-basis $\{m_1, \ldots, m_r\}$ for M. By definition, soc$M$ is graded essential in M and soc$M = km_1 + \cdots + km_r \subseteq M_{-s}$. Thus, M is finitely cogenerated in degree $-s$. Conversely, assume that M is finitely cogenerated in degree $-s$. By definition, soc$M$ is graded essential in M, finite dimensional and contained in M_{-s}. By Lemma 3.2.1, soc$M$ contains a k-basis $\{m_1, \ldots, m_r\}$ of pure elements in M_{-s}. By definition, $\{m_1, \ldots, m_r\}$ is a socle-basis for M contained in M_{-s}. As in the proof of Lemma 3.6.3, we see that $M_j = 0$ for $i > -s$. Thus, $M = \oplus_{i \leq -s} M_i$ and $JM_{-s} = 0$. By Lemma 3.5.2(1), $M_{-s} \subseteq$ soc$M$. Therefore, soc$M = M_{-s}$. The proof of the lemma is completed.

The following easy statement will be needed later.

3.6.8 **Lemma**. *Let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver.*

(1) *A graded projective module P in* gprojΛ *is generated in degree s if and only if $P \cong P_{a_1}\langle -s\rangle \oplus \cdots \oplus P_{a_r}\langle -s\rangle$ for some $a_1, \ldots, a_r \in Q_0$.*

(2) *A graded injective module I in* ginjΛ *is cogenerated in degree $-s$ if and only if $I \cong I_{a_1}\langle s\rangle \oplus \cdots \oplus I_{a_r}\langle s\rangle$ for some $a_1, \ldots, a_r \in Q_0$.*

Proof. (1) Let $P \in$ gprojΛ. We may assume that $P = P_{a_1}\langle -s_1\rangle \oplus \cdots \oplus P_{a_r}\langle -s_r\rangle$, where $s_i \in \mathbb{Z}$ and $a_i \in Q_0$. Observe that $P_{a_i}\langle -s_i\rangle$ is generated in degree s_i for $i = 1, \ldots, r$. Now, $P \in$ gprojΛ is generated in degree s if and only if $P_{a_i}\langle -s_i\rangle$ is generated in degree s for $i = 1, \ldots, r$ if and only $s_i = s$, for $i = 1, \ldots, r$. This establishes Statement (1), and Statement (2) follows dually. The proof of the lemma is completed.

3.7 Superfluous graded epimorphisms and essential graded monomorphisms

In this section, let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver. The objective of this section is to study superfluous epimorphisms and essential monomorphisms in GModΛ.

The following statement is well known for non-graded modules, see [1, (5.13) and (5.15)]. For the reader's convenience, we shall include a proof.

3.7.1 **Proposition**. *Let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver.*

(1) *An epimorphism $f : M \to N$ in* $\mathrm{GMod}\Lambda$ *is superfluous if and only if* $\mathrm{Ker}(f)$ *is graded superfluous in M; and in this case,* $f^{-1}(\mathrm{rad}N) = \mathrm{rad}M$.

(2) *A monomorphism $f : M \to N$ in* $\mathrm{GMod}\Lambda$ *is essential if and only if* $\mathrm{Im}(f)$ *is graded essential in N; and in this case,* $\mathrm{soc}N = f(\mathrm{soc}M)$.

Proof. (1) Let $f : M \to N$ be an epimorphism in $\mathrm{GMod}\Lambda$. By Proposition 3.5.6, $\mathrm{rad}N = JN = Jf(M) = f(\mathrm{rad}M)$. Thus, $f^{-1}(\mathrm{rad}N) = \mathrm{rad}M + \mathrm{Ker}(f)$. Suppose that f is superfluous. Let L be a graded submodule of M such that $\mathrm{Ker}(f) + L = M$. Then, $f \circ q : L \to N$ is a graded epimorphism, where $q : L \to M$ is the inclusion morphism. Thus, q is a graded epimorphism, that is, $L = M$. So, $\mathrm{Ker}(f)$ is graded superfluous in M. By Propositions 3.5.11(1), $\mathrm{Ker}(f) \subseteq \mathrm{rad}M$, and consequently, $f^{-1}(\mathrm{rad}N) = \mathrm{rad}M$.

Conversely, assume that $\mathrm{Ker}(f)$ is graded superfluous in M. Let $g : L \to M$ be a graded morphism such that $f \circ g$ is a graded epimorphism. It is easy to see that $\mathrm{Ker}(f) + \mathrm{Im}(g) = M$. Since $\mathrm{Im}(g)$ is a graded submodule of M by Proposition 3.1.8, we obtain $\mathrm{Im}(g) = M$. So, f is superfluous.

(2) Let $f : M \to N$ be a monomorphism in $\mathrm{GMod}\Lambda$. Assume that f is essential. Let L be a graded submodule of N such that $\mathrm{Im}(f) \cap L = 0$. It is easy to see that $p \circ f : M \to N/L$ is a monomorphism, where $p : N \to N/L$ is the canonical projection. Thus, p is a graded monomorphism, that is, $L = 0$. So, $\mathrm{Im}(f)$ is graded essential in N. By Lemma 3.5.2(2), $\mathrm{soc}N \subseteq \mathrm{Im}(f)$. So, for any pure element $n \in \mathrm{soc}N$, there exists a pure element $m \in M$ such that $n = f(m)$. Now, $f(Jm) = Jf(m) = Jn = 0$. Since f is a monomorphism, $Jm = 0$, and by Lemma 3.5.2(1), $m \in \mathrm{soc}M$. This shows that $\mathrm{soc}N \subseteq f(\mathrm{soc}M)$. On the other hand, by Proposition 3.5.3, $f(\mathrm{soc}M) \subseteq \mathrm{soc}N$. Hence, $\mathrm{soc}N = f(\mathrm{soc}M)$.

Conversely, suppose that $\mathrm{Im}(f)$ is graded essential in N. Let $h : N \to L$ be a graded morphism such that $h \circ f$ is a graded monomorphism. Then, we have $\mathrm{Im}(f) \cap \mathrm{Ker}(h) = 0$. Since $\mathrm{Ker}(h)$ is a graded submodule of M by Proposition 3.1.8, we obtain $\mathrm{Ker}(h) = 0$, that is, h is a graded monomorphism. So, f is essential. The proof of the proposition is completed.

Remark. Proposition 3.7.1(1) says that a superfluous graded epimorphism $f : M \to N$ induces a graded isomorphism $\bar{f} : \mathrm{top}M \to \mathrm{top}N$.

The following statement is well known for finite dimensional non-graded modules.

3.7.2 **Corollary**. *Let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver.*

(1) *An epimorphism $f : M \to N$ in* $\mathrm{GMod}^+\Lambda$ *is superfluous if and only if* $\mathrm{Ker}(f) \subseteq \mathrm{rad}M$.

(2) *A monomorphism $f : M \to N$ in* $\mathrm{GMod}^-\Lambda$ *is essential if and only if* $\mathrm{soc}N \subseteq \mathrm{Im}(f)$.

Proof. We shall only prove Statement (1), since the proof of Statement (2) is dual. Let $f : M \to N$ be an epimorphism in $\mathrm{GMod}^+\Lambda$. By Proposition 3.5.11, $\mathrm{rad}M$ is the largest superfluous graded submodule of M. By Lemma 3.7.1(1), f is superfluous if and only if $\mathrm{Ker}(f)$ is graded superfluous in M; and by Lemma 3.1.5(1), this is equivalent to $\mathrm{Ker}(f) \subseteq \mathrm{rad}M$. The proof of the corollary is completed.

The following statement is interesting.

3.7.3 **Proposition**. *Let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver.*

(1) *Let $f : M \to N$ be an epimorphism in* $\mathrm{GMod}\Lambda$. *If M is finitely generated (in degree s), then N is finitely generated (in degree s). And the converse holds if f is superfluous.*

(2) *Let $f : M \to N$ be a monomorphism in* $\mathrm{GMod}\Lambda$. *If N is finitely cogenerated (in degree $-s$), then M is finitely cogenerated (in degree $-s$). And the converse holds if f is essential.*

Proof. (1) The first part of the statement is evident. Suppose that f is superfluous and that N is finitely generated. Write $N = \Lambda n_1 + \cdots \Lambda n_r$, where $n_i \in N_{s_i}$ with $s_i \in \mathbb{Z}$. Since f is a graded epimorphism, $n_i = f(m_i)$ with $m_i \in M_{s_i}$, for $i = 1, \ldots, r$. Then, $M = \mathrm{Ker}(f) + \sum_{i=1}^{r} \Lambda m_i$. Since $\mathrm{Ker}(f)$ is graded superfluous in M by Proposition 3.7.1(1), $M = \sum_{i=1}^{r} \Lambda m_i$. That is, M is finitely generated. If N is finitely generated in degree s, then $s_1 = \cdots = s_r = s$. That is, M is finitely generated in degree s.

(2) Suppose that N is finitely cogenerated. By Lemma 3.6.3, $N \in \text{GMod}^{-}\Lambda$ and $\dim_k \text{soc}N < \infty$. Since f is a monomorphism, $M \in \text{GMod}^{-}\Lambda$. So, by Lemma 3.5.2(3), soc$M$ is graded essential in M. Moreover since $f(\text{soc}M) \subseteq \text{soc}N$ by Proposition 3.5.3, $\dim_k \text{soc}M < \infty$. Again by Lemma 3.6.3, M is finitely cogenerated. If N is finitely cogenerated in degree $-s$, then $\text{soc}N \subseteq N_{-s}$, and hence, $\text{soc}M \subseteq M_{-s}$. That is, M is finitely cogenerated in degree $-s$.

Assume that f is essential and that M is finitely cogenerated. Then soc$M$ is graded essential in M. Since f is a monomorphism, $f(\text{soc}M)$ is graded essential in $f(M)$. And since $f(M)$ is graded essential in N by Proposition 3.7.1(2), we see that $\text{soc}N = f(\text{soc}M)$ is graded essential in N. Moreover, by Lemma 3.6.3, soc$M$ is finite dimensional, and therefore, soc$N$ is finite dimensional. Again by Lemma 3.6.3, N is finitely cogenerated. If M is finitely cogenerated in degree $-s$, then $\text{soc}M \subseteq M_{-s}$, and hence, $\text{soc}N \subseteq N_{-s}$. That is, N is finitely cogenerated in degree $-s$. The proof of the proposition is completed.

To conclude this section, we shall concentrate on piecewise finite dimensional graded Λ-modules.

3.7.4 **Lemma**. *Let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver. Consider a morphism $f : M \to N$ in* gmodΛ.

(1) *If f is a superfluous epimorphism, then $\mathfrak{D}f : \mathfrak{D}N \to \mathfrak{D}M$ is an essential monomorphism.*

(2) *If f is an essential monomorphism, then $\mathfrak{D}f : \mathfrak{D}N \to \mathfrak{D}M$ is a superfluous epimorphism.*

Proof. We shall only prove Statement (1), since Statement(2) is dual. By Proposition 3.3.3(2), we have a duality $\mathfrak{D} : \text{gmod}\Lambda \to \text{gmod}\Lambda^{\text{o}}$. Let $f : M \to N$ be a superfluous epimorphism in gmodΛ. Then, $\mathfrak{D}f : \mathfrak{D}N \to \mathfrak{D}M$ is a graded monomorphism in gmodΛ^{o}. Assume that $g : \mathfrak{D}M \to L$ is a morphism in gmodΛ^{o} such that $g \circ \mathfrak{D}f$ is a graded monomorphism. Then, $\mathfrak{D}^2 f \circ \mathfrak{D}g$ is a graded epimorphism. By Proposition 3.3.3(1), we have a commutative diagram

$$\begin{array}{ccccc} \mathfrak{D}L & \xrightarrow{\mathfrak{D}g} & \mathfrak{D}^2M & \xrightarrow{\mathfrak{D}^2 f} & \mathfrak{D}^2N \\ & & \downarrow{\scriptstyle \phi^M} & & \downarrow{\scriptstyle \phi^N} \\ & & M & \xrightarrow{f} & N, \end{array}$$

where ϕ^M and ϕ^N are graded isomorphisms. Thus, $f \circ \phi^M \circ \mathfrak{D}g$ is a graded epimorphism. Since f is superfluous, $\mathfrak{D}g$ is a graded epimorphism, and consequently, g is a graded monomorphism. Therefore, $\mathfrak{D}f$ is essential in gmod$\Lambda^{\rm o}$. The proof of the lemma is completed.

The following statement is well known for finite dimensional non-graded modules.

3.7.5 **Proposition**. *Let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver.*

(1) *If* $M \in \text{gmod}^+\Lambda$*, then* $\text{soc}(\mathfrak{D}M) \cong \mathfrak{D}(\text{top}M)$ *and* $\mathfrak{D}M/\text{soc}(\mathfrak{D}M) \cong \mathfrak{D}(\text{rad}M)$.

(2) *If* $M \in \text{gmod}^-\Lambda$*, then* $\text{top}(\mathfrak{D}M) \cong \mathfrak{D}(\text{soc}M)$ *and* $\text{rad}(\mathfrak{D}M) \cong \mathfrak{D}(M/\text{soc}M)$.

Proof. We shall only prove Statement (1), since the proof of Statement (2) is dual. Let M be a nonzero module in gmod$^+\Lambda$. Consider the canonical short exact sequence

$$0 \longrightarrow \text{rad}M \xrightarrow{g} M \xrightarrow{f} \text{top}M \longrightarrow 0$$

in gmod$^+\Lambda$. This yields a short exact sequence

$$0 \longrightarrow \mathfrak{D}(\text{top}M) \xrightarrow{\mathfrak{D}f} \mathfrak{D}M \xrightarrow{\mathfrak{D}g} \mathfrak{D}(\text{rad}M) \longrightarrow 0$$

in gmod$^-\Lambda^{\rm o}$. By Proposition 3.7.2(1), f is superfluous, and by Lemma 3.7.4(1), $\mathfrak{D}f$ is essential. Thus, $\text{soc}(\mathfrak{D}M) \subseteq \text{Im}(\mathfrak{D}f)$ by Lemma 3.7.1(2). On the other hand, by Corollary 3.5.12, top$M$ is graded semi-simple, and by Corollary 3.5.10, so is $\mathfrak{D}(\text{top}M)$. Then, by Proposition 3.5.3, $\text{Im}(\mathfrak{D}f) \subseteq \text{soc}(\mathfrak{D}M)$. Therefore, $\text{soc}(\mathfrak{D}M) = \text{Im}(\mathfrak{D}f) \cong \mathfrak{D}(\text{top}M)$. As a consequence, $\mathfrak{D}(\text{rad}M) \cong \mathfrak{D}M/\text{soc}(\mathfrak{D}M)$. The proof of the proposition is completed.

3.8 Graded projective covers and graded injective envelopes

Throughout this section, let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver. The objective of this section is to study projective covers and injective envelopes in GModΛ, which will be called **graded projective covers** and **graded injective envelopes**, respectively. We start with an immediate consequence of Corollary 3.7.2 as follows.

3.8.1 **Lemma**. *Let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver.*

(1) *An epimorphism $f : P \to M$ in* $\mathrm{GMod}\Lambda$ *with P a bounded below projective module is a graded projective cover of M if and only if* $\mathrm{Ker}(f) \subseteq \mathrm{rad}P$.

(2) *A monomorphism $f : M \to I$ in* $\mathrm{GMod}\Lambda$ *with I a bounded above injective module is an injective envelope of M if and only if* $\mathrm{soc}I \subseteq \mathrm{Im}(f)$.

In particular, we may describe the graded projective cover and the graded injective envelope for each graded simple module. This will be used frequently in the sequel.

3.8.2 **Lemma**. *Let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver. Given any $a \in Q_0$, the graded simple module S_a admits*

(1) *a graded projective cover $p_a : P_a \to S_a$, sending e_a to $e_a + Je_a$;*

(2) *a graded injective envelope $q_a : S_a \to I_a$, sending $e_a + Je_a$ to $e_a^\star$.*

Proof. Let $a \in Q_0$. Since $\mathrm{Ker}(p_a) = \mathrm{rad}P_a$, by Lemma 3.8.1(1), p_a is a graded projective cover. Next, $S_a = (S_a)_0(a) = k(e_a + Je_a)$. By Lemma 3.4.5, we have a graded morphism $q_a : S_a \to I_a$, which is necessarily a monomorphism, sending $e_a + \mathrm{rad}P_a$ to $e_a^\star$. By Corollary 3.5.4, $\mathrm{Im}(q_a) = ke_a^\star = \mathrm{soc}I_a$, and by Lemma 3.8.1(2), q_a is a graded injective envelope. The proof of the lemma is completed.

The above result can be extended to any bounded below semisimple modules.

3.8.3 **Lemma**. *Let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver. If $M \in \mathrm{GMod}^+\Lambda$ is graded semisimple, then it has a graded projective cover*

$$f : \oplus_{(i,x)\in\mathbb{Z}\times Q_0} P_x\langle -i\rangle \otimes M_i(x) \to M$$

such that $f(e_x \otimes m_{i,x}) = m_{i,x}$, where $e_x \in P_x\langle -i\rangle_i(x)$ and $m_{i,x} \in M_i(x)$.

Proof. Let $M \in \mathrm{GMod}^+\Lambda$ be graded semisimple. By Lemma 3.5.2, $JM = 0$. Thus, $M = \oplus_{(i,x)\in\mathbb{Z}\times Q_0} M_i(x)$, where $M_i(x)$ is a semisimple graded submodule of M concentrated in the (i,x)-piece. Clearly, for any $(i,x) \in \mathbb{Z}\times Q_0$, we have a graded isomorphism $g(i,x) : S_x\langle -i\rangle \otimes M_i(x) \to M_i(x)$, sending $(e_x + Je_x) \otimes m_{i,x}$

to $m_{i,x}$, where $(e_x + Je_x) \in S_x\langle -i\rangle_i(x)$ and $m_{i,x} \in M_i(x)$. This yields a graded isomorphism

$$g = \oplus_{(i,x)\in\mathbb{Z}\otimes Q_0} g(i,x) : \oplus_{(i,x)\in\mathbb{Z}\times Q_0} S_x\langle -i\rangle \otimes M_i(x) \to M.$$

Consider the short exact sequence

$$0 \longrightarrow (\mathrm{rad}P_x)\langle -i\rangle \longrightarrow P_x\langle -i\rangle \xrightarrow{p_x\langle -i\rangle} S_x\langle -i\rangle \longrightarrow 0$$

in GModΛ, where $p_x : P_x \to S_x$ is the canonical projection, we obtain a short exact sequence

$$0 \longrightarrow \oplus_{(i,x)\in\mathbb{Z}\times Q_0}(\mathrm{rad}P_x)\langle -i\rangle \otimes M_i(x) \longrightarrow \oplus_{(i,x)\in\mathbb{Z}\times Q_0} P_x\langle -i\rangle \otimes M_i(x)$$

$$\xrightarrow{\oplus_{(i,x)\in\mathbb{Z}\times Q_0} p_x\langle -i\rangle} \oplus_{(i,x)\in\mathbb{Z}\times Q_0} S_x\langle -i\rangle \otimes M_i(x) \longrightarrow 0$$

in GModΛ. This gives rise to a graded epimorphism

$$f = g \circ (\oplus_{(i,x)\in\mathbb{Z}\times Q_0} p_x\langle -i\rangle) : \oplus_{(i,x)\in\mathbb{Z}\times Q_0} P_x\langle -i\rangle \otimes M_i(x) \to M$$

with $\mathrm{Ker}(f) = \oplus_{(i,x)\in\mathbb{Z}\times Q_0}(\mathrm{rad}P_x)\langle -i\rangle \otimes M_i(x)$. Now, by Proposition 3.5.6,

$$\begin{aligned}\mathrm{rad}(\oplus_{(i,x)\in\mathbb{Z}\times Q_0} P_x\langle -i\rangle \otimes M_i(x)) &= J(\oplus_{(i,x)\in\mathbb{Z}\times Q_0} P_x\langle -i\rangle \otimes M_i(x))\\ &= \oplus_{(i,x)\in\mathbb{Z}\times Q_0}(JP_x)\langle -i\rangle \otimes M_i(x)\\ &= \oplus_{(i,x)\in\mathbb{Z}\times Q_0}(\mathrm{rad}P_x)\langle -i\rangle \otimes M_i(x).\end{aligned}$$

That is, $\mathrm{Ker}(f) = \mathrm{rad}(\oplus_{(i,x)\in\mathbb{Z}\otimes Q_0} P_x\langle -i\rangle \otimes M_i(x))$. Moreover, since $M_i(x) = 0$ for $i \ll 0$, we see that $\oplus_{(i,x)\in\mathbb{Z}\otimes Q_0} P_x\langle -i\rangle \otimes M_i(x)$ is bounded below. By Lemma 3.8.1(1), f is a graded projective cover of M. The proof of the lemma is completed.

Now, we are ready to construct graded projective covers for bounded below graded modules.

3.8.4 **Proposition**. *Let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver. If $M \in$ GMod$^+\Lambda$, then M admits a graded projective cover $p_{_M} : P \to M$, where P is a graded projective module in* GMod$^+\Lambda$.

Proof. Let M be a non-zero module in GMod$^+\Lambda$. By Corollary 3.5.12(2), top$M$ is a semisimple graded module in GMod$^+\Lambda$. In view of Lemma 3.8.3, we have a graded projective cover $f : P = \oplus_{(i,x)\in\mathbb{Z}\times Q_0} P_x\langle -i\rangle \otimes V_{i,x} \to \mathrm{top}M$ with $\mathrm{Ker}(f) = \mathrm{rad}P$, where $V_{i,x} = (\text{top}M)_i(x)$. Then, there exists a graded morphism $p_{_M} : P \to M$ such that the diagram

$$\begin{array}{ccc} P & \overset{p_M}{\dashrightarrow} & M \\ & \searrow^{f} & \downarrow{\scriptstyle g} \\ & & \text{top}M \end{array}$$

commutes, where g is the canonical projection with $\text{Ker}(g) = \text{rad}M$. Since M is bounded below, by Proposition 3.7.2(1), g is superfluous. Since $g \circ p_{_M}$ is a graded epimorphism, so is $p_{_M}$. On the other hand, $\text{Ker}(p_{_M}) \subseteq \text{Ker}(f) = \text{rad}P$. By Lemma 3.8.1(1), p_M is a graded projective cover of M. The proof of the proposition is completed.

Remark. We do not know if every module in $\text{GMod}^{-}\Lambda$ admits an injective envelope.

Next, we shall describe how to construct graded projective covers for finitely generated graded modules and graded injective envelopes for finitely co-generated graded modules; compare [37, (1.1)].

3.8.5 **Proposition**. *Let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver. Then, a module M in* $\text{GMod}\Lambda$ *admits*

(1) *a graded projective cover $f : P_{a_1}\langle -s_1\rangle \oplus \cdots \oplus P_{a_r}\langle -s_r\rangle \to M$, sending e_{a_i} to m_i if and only if $\{m_1, \ldots, m_r\}$ with $m_i \in M_{s_i}(a_i)$ is a top-basis for M;*

(2) *a graded injective envelope $g : M \to I_{a_1}\langle s_1\rangle \oplus \cdots \oplus I_{a_r}\langle s_r\rangle$, sending m_i to $e^{\star}_{a_i}$ if and only if $\{m_1, \ldots, m_r\}$ with $m_i \in M_{-s_i}(a_i)$ is a socle-basis for M.*

Proof. (1) Let $\{m_1, \ldots, m_r\}$ be a top-basis for M, where $m_i \in M_{s_i}(a_i)$. In view of Corollary 3.4.2, we obtain a graded morphism $f : P_{a_1}\langle -s_1\rangle \oplus \cdots \oplus P_{a_r}\langle -s_r\rangle \to M$, sending e_{a_i} to m_i. Since $M = \sum_{i=1}^{r} \Lambda m_i$, we see that f is a graded epimorphism. Let $u \in \text{Ker}(f)$. Write $u = \sum_{i=1}^{r} (\lambda_i e_{a_i} + u_i)$ with $\lambda_i \in k$ and $u_i \in JP_{a_i}\langle -s_i\rangle$. Then $\sum_{i=1}^{r} (\lambda_i m_i + u_i m_i) = f(u) = 0$, that is, $\sum_{i=1}^{r} \lambda_i m_i \in \text{rad}M$. Hence, $\sum_{i=1}^{r} \lambda_i (m_i + \text{rad}M) = 0$ in $M/\text{rad}M$, and consequently, $\lambda_i = 0$ for all $i = 1, \ldots, r$. So, $u = \sum_{i=1}^{r} u_i \in \text{rad}(P_{a_1}\langle -s_1\rangle \oplus \cdots \oplus P_{a_r}\langle -s_r\rangle)$. Therefore, by Lemma 3.8.1(1), f is a graded projective cover.

Conversely, suppose that $f : P = P_{a_1}\langle -s_1\rangle \oplus \cdots \oplus P_{a_r}\langle -s_r\rangle \to M$ is a graded projective cover, sending e_{a_i} to m_i. Obviously, M is generated by $\{m_1, \ldots, m_r\}$, and hence, $\text{top}M$ is generated by $\{m_1 + \text{rad}M, \ldots, m_r + \text{rad}M\}$. Assume that

$\sum_{i=1}^r \lambda_i(m_i + \mathrm{rad}M) = 0$, that is, $\sum_{i=1}^r \lambda_i m_i \in \mathrm{rad}M$, where $\lambda_i \in k$. By Proposition 3.7.1, $\mathrm{rad}P = f^{-1}(\mathrm{rad}M)$. Thus, $\sum_{i=1}^r \lambda_i m_i = f(u)$ for some $u \in \mathrm{rad}P$. Then, $\sum_{i=1}^r \lambda_i e_{a_i} - u \in \mathrm{Ker}(f)$. By Lemma 3.8.1(1), $\mathrm{Ker}(f) \subseteq \mathrm{rad}P$. Thus, $\sum_{i=1}^r \lambda_i e_{a_i} \in \mathrm{rad}P = \oplus_{i=1}^r \mathrm{rad}(P_{a_i}\langle -s_i\rangle)$. Therefore, $\lambda_i e_{a_i} \in \mathrm{rad}(P_{a_i}\langle -s_i\rangle)$, and consequently, $\lambda_i = 0$, for all $i = 1, \ldots, r$. This shows that $\{m_1, \ldots, m_r\}$ is a top basis for M.

(2) First, suppose that $g : M \to I_{a_1}\langle s_1\rangle \oplus \cdots \oplus I_{a_r}\langle s_r\rangle = I$ is a graded injective envelope, sending m_i to $e^\star_{a_i}$. In view of Corollary 3.5.4, we see that $\{e^\star_{a_1}, \ldots, e^\star_{a_r}\}$ is a k-basis of $\oplus_{i=1}^r \mathrm{soc}(I_{a_i}\langle s_i\rangle) = \mathrm{soc}I$. Now, since I is bounded above, so is M. Thus, $\mathrm{soc}M$ is graded essential in M by Lemma 3.5.2(3). We claim that $\{m_1, \ldots, m_r\}$ is a k-basis of $\mathrm{soc}M$. Assume that $\sum_{i=1}^r \lambda_i m_i = 0$, $\lambda_i \in k$. Then, $g(\sum_{i=1}^r \lambda_i m_i) = \sum_{i=1}^r \lambda_i e^\star_{a_i} = 0$, and hence, $\lambda_i = 0$ for all $i = 1, \ldots, r$. Moreover, if $m \in \mathrm{soc}M$, then $g(m) \in \mathrm{soc}I$ by Corollary 3.5.3. Hence, $g(m) = \sum_{i=1}^r \lambda_i e^\star_{a_i} = \sum_{i=1}^r \lambda_i g(m_i)$, where $\lambda_i \in k$. Hence, $m = \sum_{i=1}^r \lambda_i m_i$. This establishes our claim. Therefore, $\{m_1, \ldots, m_r\}$ is a socle-basis for M.

Conversely, suppose that $\mathrm{soc}M$ is graded essential in M and has a k-basis $\{m_1, \ldots, m_s\}$, where $m_i \in M_{-s_i}(a_i)$ with $(s_i, a_i) \in \mathbb{Z} \times Q_0$, for $i = 1, \ldots, r$. Then, $\mathrm{soc}M = km_1 \oplus \cdots \oplus km_r$, where km_i is a graded simple submodule of M. In view of Lemma 3.4.5, there exists a graded monomorphism $q : \mathrm{soc}M \to I_{a_1}\langle s_1\rangle \oplus \cdots \oplus I_{a_r}\langle s_r\rangle = I$, sending m_i to $e^\star_{a_i}$. Thus, there exists a graded morphism $g : M \to I$ such that $g \circ h = q$, where $h : \mathrm{soc}M \to M$ is the inclusion map. In particular, $g(m_i) = e^\star_{a_i}$, for $i = 1, \ldots, r$, and hence, $\mathrm{soc}I = ke^\star_{a_1} \oplus \cdots \oplus ke^\star_{a_r} \subseteq \mathrm{Im}(g)$. Moreover, since M is bounded above, by Proposition 3.7.2(2), h is an essential graded monomorphism. Since $g \circ h = q$, we see that g is a graded monomorphism. By Lemma 3.8.1(2), g is a graded injective envelope of M. The proof of the proposition is completed.

As a consequence of Propositions 3.8.5 and 3.7.3, we obtain the following useful statement.

3.8.6 **Corollary**. *Let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver.*

(1) *A module $M \in \mathrm{GMod}\Lambda$ is finitely generated if and only if it admits a graded projective cover over* $\mathrm{gproj}\Lambda$; *and in this case, M is a locally finite dimensional module in* $\mathrm{gmod}^+\Lambda$.

(2) *A module $M \in \mathrm{GMod}\Lambda$ is finitely cogenerated if and only if it admits a*

graded injective envelope over $\text{ginj}\Lambda$; *and in this case,* M *is a locally finite dimensional module in* $\text{gmod}^{-}\Lambda$.

Proof. (1) By Proposition 3.6.6(1), a module $M \in \text{GMod}\Lambda$ is finitely generated if and only if M admits a top-basis; and by Proposition 3.8.5(1), this is equivalent to M admitting a graded projective cover over $\text{gproj}\Lambda$. Since Q is locally finite, every P_a with $a \in Q_0$ is a locally finite dimensional module in $\text{gmod}^{+}\Lambda$, and so is every module P in $\text{gproj}\Lambda$. Thus, if $f : P \to M$ is a projective cover with $P \in \text{gproj}\Lambda$, then M is a locally finite dimensional module in $\text{gmod}^{+}\Lambda$.

(2) Similarly, we deduce from Propositions 3.6.6(2) and 3.8.5(2) that a module $M \in \text{GMod}\Lambda$ is finitely cogenerated if and only if there exists a graded injective envelope $g : M \to I$, where $I \in \text{ginj}\Lambda$. In this case, $I \cong \mathfrak{D}P$, for some $P \in \text{gproj}\Lambda^{\text{o}}$. As seen above, P is a locally finite dimensional module in $\text{gmod}^{+}\Lambda^{\text{o}}$. Thus, I is a locally finite dimensional module in $\text{gmod}^{-}\Lambda$, and so is M. The proof of the corollary is completed.

The following statement will be useful for our later investigation.

3.8.7 **Lemma**. *Let* $\Lambda = kQ/R$ *be a graded algebra, where* Q *is a locally finite quiver. Consider* M *a module in* $\text{GMod}\Lambda$.

(1) *If* $f : P \to M$ *is a graded projective cover of* M *over* $\text{gproj}\Lambda$*, then* $\mathfrak{D}f : \mathfrak{D}M \to \mathfrak{D}P$ *is a graded injective envelope of* $\mathfrak{D}M$ *over* $\mathfrak{D}P \in \text{ginj}\Lambda^{\text{o}}$.

(2) *If* $g : M \to I$ *is a graded injective envelope of* M *over* $\text{ginj}\Lambda$*, then* $\mathfrak{D}g : \mathfrak{D}I \to \mathfrak{D}M$ *is a graded projective cover of* $\mathfrak{D}M$ *over* $\text{gproj}\Lambda^{\text{o}}$.

Proof. Let $f : P \to M$ be a graded projective cover of M, where $P \in \text{gproj}\Lambda$. By 3.7.4(1), $\mathfrak{D}f : \mathfrak{D}M \to \mathfrak{D}P$ is an essential monomorphism in $\text{gmod}\Lambda^{\text{o}}$, where $\mathfrak{D}P \in \text{ginj}\Lambda^{\text{o}}$. By definition, $\mathfrak{D}f$ is a graded injective envelope of $\mathfrak{D}M$ over $\mathfrak{D}P \in \text{ginj}\Lambda^{\text{o}}$. This establishes Statement (1). And Statement (2) follows dually. The proof of the lemma is completed.

3.9 Hom-finite Krull-Schmidt exact categories of graded modules

Throughout this section let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver. The main objective of this section is to introduce several

Hom-finite Krull-Schmidt k-subcategories of GModΛ, which will play an important role in our later investigation. We start with the following important result, which does not hold under the non-graded setting.

3.9.1 **Proposition**. *Let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver. Then*

(1) gprojΛ *is Hom-finite Krull-Schmidt and contains all finitely generated projective modules in* GModΛ.

(2) ginjΛ *is Hom-finite Krull-Schmidt and contains all finitely cogenerated injective modules in* GModΛ.

Proof. We shall only prove Statement (1), since the proof of Statement (2) is dual. In view of Proposition 3.4.3, we see that gprojΛ is Hom-finite. And being generated by the strongly indecomposable objects $P_a\langle -s\rangle$ with $(s,a) \in \mathbb{Z}\times Q_0$; see (3.4.7), gproj$\Lambda$ is Krull-Schmidt. Finally, let P be a finitely generated projective module in GModΛ. By Corollary 3.8.6(1), P admits a graded projective cover $f : U \to P$ with $U \in$ gprojΛ. Since $\mathrm{id}_P : P \to P$ is also a graded projective cover of P, by Lemma 1.3.6, $P \cong U$. The proof of the proposition is completed.

More generally, we have the following statement.

3.9.2 **Lemma**. *Let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver. Consider a module $M \in$* gmodΛ.

(1) *If $P \in$ gprojΛ, then* $\dim_k \mathrm{GHom}_\Lambda(P, M) < \infty$.

(2) *If $I \in$ ginjΛ, then* $\dim_k \mathrm{GHom}_\Lambda(M, I) < \infty$.

Proof. (1) Let $P \in$ gprojΛ. We may assume that $P = \oplus_{i=1}^r P_{a_i}\langle -s_i\rangle$, where $(s_i, a_i) \in \mathbb{Z} \times Q_0$. By Corollary 3.4.2, we have

$$\mathrm{GHom}_\Lambda(P, M) \cong \oplus_{i=1}^r \mathrm{GHom}_\Lambda(P_{a_i}\langle -s_i\rangle, M) \cong \oplus_{i=1}^r M_{s_i}(a_i).$$

Since M is piecewise finite dimensional, we have

$$\dim_k \mathrm{GHom}_\Lambda(P, M) \le \sum_{i=1}^r \dim_k M_{s_i}(a_i) < \infty.$$

(2) Let $I \in \mathrm{ginj}\Lambda$. Since $\mathfrak{D}I \in \mathrm{gproj}\Lambda^{\mathrm{o}}$ and $\mathfrak{D}M \in \mathrm{gmod}\Lambda^{\mathrm{o}}$, by Statement (1), $\dim_k \mathrm{GHom}_{\Lambda^{\mathrm{o}}}(\mathfrak{D}I, \mathfrak{D}M) < \infty$. In view of Proposition 3.3.3(2), we have a k-linear isomorphism

$$\mathrm{GHom}_{\Lambda}(M, I) \cong \mathrm{GHom}_{\Lambda^{\mathrm{o}}}(\mathfrak{D}I, \mathfrak{D}M).$$

Therefore, $\dim_k \mathrm{GHom}_{\Lambda}(M, N) < \infty$. The proof of the lemma is completed.

In the sequel, we shall write $\mathrm{gmod}^{+,b}\Lambda$ and $\mathrm{gmod}^{-,b}\Lambda$ for the full subcategories of $\mathrm{GMod}\Lambda$ of finitely generated modules and of finitely cogenerated modules, respectively. By Corollary 3.8.6, both are subcategories of $\mathrm{gmod}\Lambda$.

3.9.3 **Proposition**. *Let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver. Then, the duality $\mathfrak{D} : \mathrm{gmod}\Lambda \to \mathrm{gmod}\Lambda^{\mathrm{o}}$ restricts to two dualities $\mathfrak{D} : \mathrm{gmod}^{+,b}\Lambda \to \mathrm{gmod}^{-,b}\Lambda^{\mathrm{o}}$ and $\mathfrak{D} : \mathrm{gmod}^{-,b}\Lambda \to \mathrm{gmod}^{+,b}\Lambda^{\mathrm{o}}$.*

Proof. By Proposition 3.3.3(2), we have a duality $\mathfrak{D} : \mathrm{gmod}\Lambda \to \mathrm{gmod}\Lambda^{\mathrm{o}}$. Let $M \in \mathrm{gmod}^{+,b}\Lambda$. By Corollary 3.8.6(1), M has a graded projective $f : P \to M$ with $P \in \mathrm{gproj}\Lambda$. Then, by Lemma 3.8.7(1), $\mathfrak{D}(f) : \mathfrak{D}M \to \mathfrak{D}P$ is a graded injective envelope with $\mathfrak{D}P \in \mathrm{ginj}\Lambda^{\mathrm{o}}$. Thus, $\mathfrak{D}M$ is finitely cogenerated by Corollary 3.8.6(2). So, $\mathfrak{D} : \mathrm{gmod}^{+,b}\Lambda \to \mathrm{gmod}^{-,b}\Lambda^{\mathrm{o}}$ is a duality. Dually, one can prove that the second part of the proposition. The proof of the proposition is completed.

Recall that $\mathrm{gmod}^{b}\Lambda$ denotes the full subcategory of $\mathrm{GMod}\Lambda$ of finite dimensional modules.

3.9.4 **Proposition**. *Let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver. Then $\mathrm{gmod}^{+,b}\Lambda$ and $\mathrm{gmod}^{-,b}\Lambda$ are Hom-finite Krull-Schmidt extension-closed subcategories of $\mathrm{GMod}\Lambda$ such that their intersection is $\mathrm{gmod}^{b}\Lambda$.*

Proof. Let $M, N \in \mathrm{gmod}^{+,b}\Lambda$. By Corollary 3.8.6(1), M has a graded projective cover $f : P \to M$ with $P \in \mathrm{gproj}\Lambda$. Applying the left exact functor $\mathrm{GHom}_{\Lambda}(-, N)$, we obtain the exact sequence

$$0 \longrightarrow \mathrm{GHom}_{\Lambda}(M, N) \xrightarrow{\mathrm{GHom}_{\Lambda}(f,N)} \mathrm{GHom}_{\Lambda}(P, N).$$

By Lemma 3.9.2(1), $\dim_k \mathrm{GHom}_{\Lambda}(P, N) < \infty$, and hence, $\dim_k \mathrm{GHom}_{\Lambda}(M, N) < \infty$. So, $\mathrm{gmod}^{+,b}\Lambda$ is Hom-finite. It is evident that $\mathrm{gmod}^{+,b}\Lambda$ is closed under direct summands. By Proposition 1.3.2, $\mathrm{gmod}^{+,b}\Lambda$ is Krull-Schmidt.

Let $0 \longrightarrow X \xrightarrow{g} Y \xrightarrow{h} Z \longrightarrow 0$ be an exact sequence in $\mathrm{GMod}\Lambda$, where $X, Z \in \mathrm{gmod}^{+,b}\Lambda$. Assume that $X = \sum_{i=1}^{s} \Lambda m_i'$ and $Z = \sum_{i=1}^{t} \Lambda m_i''$, where the

m_i' and the m_i'' are homogeneous elements. Since h is a graded epimorphism, $m_i'' = g(m_i)$ for some homogeneous element $m_i \in Y$, $i = 1, \ldots, t$. It is easy to check that $Y = \sum_{i=1}^{t} \Lambda m_i + \sum_{i=1}^{s} \Lambda g(m_i')$, that is, Y is finitely generated.

Finally, in view of Lemmas 3.6.2 and 3.6.3, we see that $\mathrm{gmod}^b\Lambda$ is contained in both $\mathrm{gmod}^{+,b}\Lambda$ and $\mathrm{gmod}^{-,b}\Lambda$. If $M \in \mathrm{GMod}\Lambda$ is finitely generated and finitely cogenerated, then it follows from Corollary 3.8.6 that M is locally finite dimensional and bounded, and consequently, M is finite dimensional. The proof of the proposition is completed.

Remark. By Proposition 3.9.4, $\mathrm{gmod}^{+,b}\Lambda$ and $\mathrm{gmod}^{-,b}\Lambda$ are exact k-categories. Note that they are not abelian in general.

A module M in $\mathrm{GMod}\Lambda$ is called **noetherian** if every graded submodule of M is finitely generated. Note that, by Theorem 5.4.7 in [53], a graded module is noetherian in $\mathrm{GMod}\Lambda$ if and only if it is noetherian as a non-graded module.

3.9.5 **Proposition**. *Let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver.*

(1) *If Λ is locally left noetherian, then* $\mathrm{gmod}^{+,b}\Lambda$ *is abelian.*

(2) *If Λ is locally right noetherian, then* $\mathrm{gmod}^{-,b}\Lambda$ *is abelian.*

Proof. (1) Suppose that Λ is locally left noetherian. Consider a morphism $f : M \to N$ in $\mathrm{gmod}^{+,b}\Lambda$. In view of Proposition 3.7.3, the cokernel of f lies in $\mathrm{gmod}^{+,b}\Lambda$. By Corollary 3.8.6(1), M admits a graded projective cover $g : \oplus_{i=1}^{r} P_{a_i}\langle -s_i\rangle \to M$, where $(s_i, a_i) \in \mathbb{Z} \times Q_0$. By the hypothesis, we see that $\oplus_{i=1}^{r} P_{a_i}\langle -s_i\rangle$ is noetherian, and hence, so is M; see, for example, [1, (10.12)]. In particular, the kernel of f lies in $\mathrm{gmod}^{+,b}\Lambda$. Therefore, $\mathrm{gmod}^{+,b}\Lambda$ is an abelian subcategory of $\mathrm{GMod}\Lambda$.

(2) Assume that Λ is locally right noetherian. That is, Λ^{o} is locally left noetherian. By Statement (1), $\mathrm{gmod}^{+,b}\Lambda^{\mathrm{o}}$ is abelian, and by Proposition 3.9.3, so is $\mathrm{gmod}^{-,b}\Lambda$. The proof of the proposition is completed.

Recall that Λ is locally left bounded if Λe_a with $a \in Q_0$ are finite dimensional, and it is locally right bounded if $\Lambda^{\mathrm{o}} e_a$ with $a \in Q_0$ are finite dimensional. The following statement will be useful in our later study.

3.9.6 Proposition. *Let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver.*

(1) *If Λ is locally left bounded, then* $\mathrm{gmod}^{+,b}\Lambda = \mathrm{gmod}^b\Lambda$.

(2) *If Λ is locally right bounded, then* $\mathrm{gmod}^{-,b}\Lambda = \mathrm{gmod}^b\Lambda$.

Proof. (1) Suppose that Λ is locally left bounded. Let $M \in \mathrm{gmod}^{+,b}\Lambda$. Write $M = \sum_{i=1}^{r} \Lambda m_i$, where $m_i \in M_{s_i}(a_i)$ with $(s_i, a_i) \in \mathbb{Z} \times Q_0$. Since Λe_{a_i} is finite dimensional, so is $\Lambda m_i = \Lambda e_{a_i} m_i$. Thus, $M \in \mathrm{gmod}^b\Lambda$, and hence, $\mathrm{gmod}^{+,b}\Lambda = \mathrm{gmod}^b\Lambda$.

(2) Assume that Λ is locally right bounded. That is, Λ^{o} is locally left bounded. By Statement (1), $\mathrm{gmod}^{+,b}\Lambda^{\mathrm{o}}$ contains only finite dimensional modules, and by Proposition 3.9.3, so does $\mathrm{gmod}^{-,b}\Lambda$. That is, $\mathrm{gmod}^{-,b}\Lambda = \mathrm{gmod}^b\Lambda$. The proof of the proposition is completed.

Next, we shall introduce finitely presented and finitely co-presented graded modules, which will play an essential role in our later study of the existence of almost split sequences.

3.9.7 Definition. Let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver. A module $M \in \mathrm{GMod}\Lambda$ is said to be

(1) **finitely presented** if it admits a graded projective presentation

$$P^{-1} \xrightarrow{d^{-1}} P^0 \xrightarrow{d^0} M \longrightarrow 0,$$

where $P^0, P^{-1} \in \mathrm{gproj}\Lambda$.

(2) **finitely copresented** if it admits a graded injective co-presentation

$$0 \longrightarrow M \xrightarrow{d^0} I^0 \xrightarrow{d^1} I^1,$$

where $I^0, I^1 \in \mathrm{ginj}\Lambda$.

The following statement will be needed in our later study.

3.9.8 Lemma. *Let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver.*

(1) *A finitely presented graded module in* GModΛ *admits a minimal graded projective presentation over* gprojΛ.

(2) *A finitely copresented graded module in* GModΛ *admits a minimal graded injective copresentation over* ginjΛ.

Proof. We shall only prove Statement (1), since the proof of Statement (2) is dual. Let $M \in \mathrm{GMod}\Lambda$ with a graded projective presentation

$$P^{-1} \xrightarrow{d^{-1}} P^0 \xrightarrow{d^0} M \longrightarrow 0,$$

where $P^{-1}, P^0 \in \mathrm{gproj}\Lambda$. In particular, M and $\mathrm{Ker}(d^0)$ are finitely generated. By Corollary 3.8.6(1), M has a graded projective cover $u^0 : U^0 \to M$ with $U^0 \in \mathrm{gproj}\Lambda$. Consider the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Ker}(u^0) & \xrightarrow{v} & U^0 & \xrightarrow{u^0} & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} & & \| & & \\
0 & \longrightarrow & \mathrm{Ker}(d^0) & \xrightarrow{w} & P^0 & \xrightarrow{d^0} & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle f'} & & \| & & \\
0 & \longrightarrow & \mathrm{Ker}(u^0) & \xrightarrow{v} & U^0 & \xrightarrow{u^0} & M & \longrightarrow & 0.
\end{array}$$

In particular, $u^0 \circ f' \circ f = u^0$. Since u^0 is right minimal by Lemma 1.3.6(1), $f' \circ f$ is an automorphism of U^0, and consequently, $g' \circ g$ is an automorphism of $\mathrm{Ker}(u^0)$. Therefore, $\mathrm{Ker}(u^0)$ is a direct summand of $\mathrm{Ker}(d^0)$. Since $\mathrm{Ker}(d^0)$ is finitely generated, so is $\mathrm{Ker}(u^0)$. By Corollary 3.8.6(1), $\mathrm{Ker}(u^0)$ admits a graded projective cover $h : U^{-1} \to \mathrm{Ker}(u^0)$ with $U^{-1} \in \mathrm{gproj}\Lambda$. By Lemma 1.3.7, we obtains a right minimal graded morphism $u^{-1} = j \circ h : U^{-1} \to U^0$, where $j : \mathrm{Ker}(u^0) \to P^0$ is the inclusion map. That is, we have a minimal graded projective presentation $U^{-1} \xrightarrow{u^{-1}} U^0 \xrightarrow{u^0} M \longrightarrow 0$ over gprojΛ. The proof of the lemma is completed.

In the sequel, we shall denote by $\mathrm{gmod}^{+,p}\Lambda$ (respectively, $\mathrm{gmod}^{-,i}\Lambda$) the full additive subcategory of gmodΛ generated by the finitely presented (respectively, co-presented) modules.

3.9.9 **Proposition**. *Let* $\Lambda = kQ/R$ *be a graded algebra, where* Q *is a locally finite quiver. Then, the duality* $\mathfrak{D} : \mathrm{gmod}\Lambda \to \mathrm{gmod}\Lambda^{\mathrm{o}}$ *restricts to two dualities* $\mathfrak{D} : \mathrm{gmod}^{+,p}\Lambda \to \mathrm{gmod}^{-,i}\Lambda^{\mathrm{o}}$ *and* $\mathfrak{D} : \mathrm{gmod}^{-,p}\Lambda \to \mathrm{gmod}^{+,i}\Lambda^{\mathrm{o}}$.

Proof. By proposition 3.3.3(2), we have a duality $\mathfrak{D} : \mathrm{gmod}\Lambda \to \mathrm{gmod}\Lambda^{\mathrm{o}}$. Let $M \in \mathrm{gmod}^{+,p}\Lambda$ with a graded projective presentation

$$P^{-1} \xrightarrow{d^{-1}} P^0 \xrightarrow{d^0} M \longrightarrow 0,$$

where $P^{-1}, P^0 \in \mathrm{gproj}\Lambda$. Applying the duality $\mathfrak{D}$, we obtain an injective copresentation

$$0 \longrightarrow \mathfrak{D}M \xrightarrow{\mathfrak{D}(d^0)} \mathfrak{D}P^0 \xrightarrow{\mathfrak{D}(d^{-1})} \mathfrak{D}P^{-1}$$

of $\mathfrak{D}M$ over $\mathrm{ginj}\Lambda^{\mathrm{o}}$. So, the duality $\mathfrak{D} : \mathrm{gmod}\Lambda \to \mathrm{gmod}\Lambda^{\mathrm{o}}$ restricts to a functor $\mathfrak{D} : \mathrm{gmod}^{+,p}\Lambda \to \mathrm{gmod}^{-,i}\Lambda^{\mathrm{o}}$. Similarly, the duality $\mathfrak{D} : \mathrm{gmod}\Lambda^{\mathrm{o}} \to \mathrm{gmod}\Lambda$ restricts to a functor $\mathfrak{D} : \mathrm{gmod}^{-,i}\Lambda^{\mathrm{o}} \to \mathrm{gmod}^{+,p}\Lambda$. Therefore, $\mathfrak{D} : \mathrm{gmod}^{+,p}\Lambda \to \mathrm{gmod}^{-,i}\Lambda^{\mathrm{o}}$ is a duality. Dually, we have a duality $\mathfrak{D} : \mathrm{gmod}^{-,p}\Lambda \to \mathrm{gmod}^{+,i}\Lambda^{\mathrm{o}}$. The proof of the proposition is completed.

The following statement does not hold under the general non-graded setting.

3.9.10 **Proposition**. *Let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver. Then,* $\mathrm{gmod}^{+,p}\Lambda$ *and* $\mathrm{gmod}^{-,i}\Lambda$ *are Hom-finite Krull-Schmidt extension-closed subcategories of* $\mathrm{GMod}\Lambda$ *such that their intersection is* $\mathrm{gmod}^{b}\Lambda$.

Proof. By Proposition 3.9.4, $\mathrm{gmod}^{+,p}\Lambda$ is Hom-finite, and by Proposition 2.1 in [3], $\mathrm{gmod}^{+,p}\Lambda$ is extension-closed in $\mathrm{GMod}\Lambda$. We claim that it is closed under direct summands. Indeed, let $M \in \mathrm{gmod}^{+,p}\Lambda$ with a graded projective presentation $P^{-1} \xrightarrow{d^{-1}} P^0 \xrightarrow{d^0} M \longrightarrow 0$ over $\mathrm{gproj}\Lambda$. Then, $\mathrm{Ker}(u^0)$ is finitely generated. Assume that $M = L \oplus N$ with a canonical injection $q : L \to M$ and a canonical projection $p : M \to L$. In particular, $L \in \mathrm{gmod}^{+,b}\Lambda$. By Corollary 3.8.6(1), L has a graded projective cover $u^0 : U^0 \to L$ with $U^0 \in \mathrm{gproj}\Lambda$. Hence, we obtain a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Ker}(u^0) & \xrightarrow{v} & U^0 & \xrightarrow{u^0} & L & \longrightarrow & 0 \\
 & & \downarrow g & & \downarrow f & & \downarrow q & & \\
0 & \longrightarrow & \mathrm{Ker}(d^0) & \xrightarrow{w} & P^0 & \xrightarrow{d^0} & M & \longrightarrow & 0 \\
 & & \downarrow g' & & \downarrow f' & & \downarrow p & & \\
0 & \longrightarrow & \mathrm{Ker}(u^0) & \xrightarrow{v} & U^0 & \xrightarrow{u^0} & L & \longrightarrow & 0.
\end{array}$$

In particular, $u^0 \circ f' \circ f = u^0$. Since u^0 is right minimal by Lemma 1.3.6(1), $f' \circ f$ is an automorphism of U^0, and hence, $g' \circ g$ is an automorphism of $\mathrm{Ker}(u^0)$. So,

$\mathrm{Ker}(u^0)$ is a direct summand of $\mathrm{Ker}(d^0)$, and hence, it is finitely generated. Therefore, L admits a graded projective presentation over gprojΛ. This establishes our claim. By Proposition 1.3.2, $\mathrm{gmod}^{+,p}\Lambda$ is Krull-Schmidt. Moreover, in view of Proposition 3.9.9, we see that $\mathrm{gmod}^{-,i}\Lambda$ is also a Hom-finite Krull-Schmidt and extension-closed subcategory of GModΛ.

Finally, by Proposition 3.9.4, the intersection of $\mathrm{gmod}^{+,p}\Lambda$ and $\mathrm{gmod}^{-,i}\Lambda$ lies in $\mathrm{gmod}^b\Lambda$. Conversely, let $M \in \mathrm{gmod}^b\Lambda$. In particular, there exists some $t \in \mathbb{Z}$ such that $M_i = 0$ for all $i \geq t$. By Corollary 3.9.5, M admits a graded projective cover $f : P \to M$ with $P \in \mathrm{gproj}\Lambda$. Write $L = \mathrm{Ker}(f)$. Then $L = \oplus_{i\in\mathbb{Z}} L_i$, where $L_i \subseteq P_i$ for all $i \in \mathbb{Z}$. Since f is a graded morphism, $L_i = P_i$ for all $i \geq t$. If $i > t$, then $L_i = P_i = J_{i-t}P_t \subseteq \mathrm{rad}L$. Therefore, $\mathrm{top}L = \oplus_{i\leq t}(L_i + \mathrm{rad}L)/\mathrm{rad}L$. On the other hand, since P is locally finite dimensional and bounded below; see (3.8.6), $\oplus_{i\leq t}P_i$ is finite dimensional. As a consequence, $\oplus_{i\leq t}L_i$ is finite dimensional, and so is top$L$. Being bounded below, L is finitely generated by Lemma 3.6.2. By Corollary 3.8.6(1), $\mathrm{Ker}(f)$ admits a graded projective cover over gprojΛ. So, $M \in \mathrm{gmod}^{+,p}\Lambda$. Dually, we may show that $M \in \mathrm{gmod}^{-,i}\Lambda$. The proof of the proposition is completed.

Remark. By Proposition 3.9.10, $\mathrm{gmod}^{+,p}\Lambda$ and $\mathrm{gmod}^{-,i}\Lambda$ are exact k-categories. Note that they are not abelian in general.

As an analogue of Proposition 3.9.5, we have the following statement.

3.9.11 **Proposition**. *Let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver.*

(1) *If Λ is locally left noetherian, then* $\mathrm{gmod}^{+,p}\Lambda$ *is abelian.*

(2) *If Λ is locally right noetherian, then* $\mathrm{gmod}^{-,i}\Lambda$ *is abelian.*

Proof. (1) Suppose that Λ is locally left noetherian. Let $f : P \to P'$ be a morphism in gprojΛ. By Proposition 3.9.5(1), $\mathrm{Ker}(f) \in \mathrm{gmod}^{+,b}\Lambda$. Thus, by Corollary 3.8.6(1), we have a graded projective cover $g : U \to \mathrm{Ker}(f)$ with $U \in \mathrm{gproj}\Lambda$. This yields an exact sequence $U \xrightarrow{q\circ g} P \xrightarrow{f} P'$ in gprojΛ, where $q : \mathrm{Ker}(f) \to P$ is the inclusion morphism. Now, by Proposition 2.1 in [3], $\mathrm{gmod}^{+,p}\Lambda$ is closed under kernels and cokernels. Therefore, $\mathrm{gmod}^{+,p}\Lambda$ is abelian.

(2) Assume that Λ is locally right noetherian. That is, Λ^{o} is locally left noetherian. By Statement (1), $\mathrm{gmod}^{+,p}\Lambda^{\mathrm{o}}$ is abelian, and by Proposition 3.9.9, so is $\mathrm{gmod}^{-,i}\Lambda$. The proof of the proposition is completed.

We conclude this section with the following result, which will be useful in the study of almost split sequences for graded representations of any locally finite quiver.

3.9.12 **Proposition**. *Let Q be a locally finite quiver. Then* $\mathrm{gmod}^{+,p}(kQ)$ *and* $\mathrm{gmod}^{-,i}(kQ)$ *are Hom-finite Krull-Schmidt hereditary abelian subcategories of* $\mathrm{gmod}kQ$ *such that their intersection is* $\mathrm{gmod}^{b}(kQ)$.

Proof. By Proposition 3.9.10, $\mathrm{gmod}^{+,p}(kQ)$ is Hom-finite and Krull-Schmidt. It is well known that the category of all unitary kQ-modules is hereditary; see [20, (8.2)]. In particular, $\mathrm{GExt}^2_{kQ}(M,N) = 0$ for all $M, N \in \mathrm{gmod}^{+,p}(kQ)$. That is, $\mathrm{gmod}^{+,p}(kQ)$ is hereditary. Let $f : P \to P'$ be a morphism in $\mathrm{gproj}\Lambda$. Since $\mathrm{GMod}^{+,p}(kQ)$ has enough projective objects by Proposition 3.4.8, $\mathrm{Im}(f)$ is projective, and hence, $P \cong \mathrm{Ker}(f) \oplus \mathrm{Im}(f)$. Since $\mathrm{gproj}\Lambda$ is Krull-Schmidt, $\mathrm{Ker}(f) \in \mathrm{gproj}\Lambda$. Now, by Proposition 2.1 in [3], $\mathrm{gmod}^{+,p}(kQ)$ is closed under kernels and cokernels. Therefore, $\mathrm{gmod}^{+,p}(kQ)$ is abelian. This establishes the first part of the statement, and the second part follows dually. The proof of the proposition is completed.

Chapter 4

Auslander-Reiten theory over a graded algebra arising from a locally finite quiver

In this chapter, we shall study the existence of almost split sequences in GModΛ and almost split triangles in the derived categories of graded modules. For this purpose, we shall first construct a Nakayama functor for graded modules. Using the Nakayama functor, we give a novel, more categorical, proof of Auslander-Reiten formulas, eliminating the need for tensor product functors, compare [31]. As a consequence, we prove that GModΛ has almost split sequences. In the final section, by using the link between the Nakayama functor and the almost split triangles, we investigate the existence of almost split triangles in the bounded derived category of piecewise finite dimensional graded Λ-modules for bounded complexes of finitely generated Λ-modules and for bounded complexes of finitely cogenerated Λ-modules.

4.1 Graded Nakayama functor

The objective of this section is to construct a Nakayama functor for graded modules, which is essential for the existence of almost split sequences in the graded module category and almost split triangles in derived categories of graded modules.

We start by constructing the contravariant functor $(-)^t$ as follows. Given $M \in \mathrm{GMod}\Lambda$, put $M^t = \oplus_{i\in\mathbb{Z}}(M^t)_i$, where $(M^t)_i = \oplus_{x\in Q_0}\mathrm{GHom}_\Lambda(M\langle -i\rangle, P_x)$.

For $u \in e_x \Lambda_j e_y$ and $f \in \mathrm{GHom}_\Lambda(M\langle -i\rangle, P_x)$, put $f\langle -j\rangle : M\langle -i-j\rangle \to P_x\langle -j\rangle$, considering the morphism $P[u] : P_x\langle -j\rangle \to P_y$, we define

$$u^{\mathrm{o}} \cdot f = P[u] \circ f\langle -j\rangle \in \mathrm{GHom}_\Lambda(M\langle -i-j\rangle, P_y),$$

that is, $(u^{\mathrm{o}} \cdot f)(m) = (f\langle -j\rangle)(m)u$, for all $m \in M\langle -i\rangle$. In this way, $M^t \in \mathrm{GMod}\Lambda^{\mathrm{o}}$. Given a morphism $g : M \to N$, we define a morphism $g^t : N^t \to M^t$ by setting $(g^t)_{i,x} = \mathrm{GHom}_\Lambda(g\langle -i\rangle, P_x)$, for all $(i,x) \in \mathbb{Z} \times Q_0$. This yields a contravariant functor $(-)^t : \mathrm{GMod}\Lambda \to \mathrm{GMod}\Lambda^{\mathrm{o}}$.

4.1.1 **Lemma**. *Let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver. Then the contravariant functor $(-)^t : \mathrm{GMod}\Lambda \to \mathrm{GMod}\Lambda^{\mathrm{o}}$ is left exact.*

Proof. Let $L \longrightarrow M \longrightarrow N \longrightarrow 0$ be an exact sequence in $\mathrm{GMod}\Lambda$. Fix $(i,x) \in \mathbb{Z} \times Q_0$. Then, $L\langle -i\rangle \longrightarrow M\langle -i\rangle \longrightarrow N\langle -i\rangle \longrightarrow 0$ is an exact sequence in $\mathrm{GMod}\Lambda$. Applying the left exact functor $\mathrm{GHom}_\Lambda(-, P_x)$, we obtain an exact sequence

$$0 \longrightarrow \mathrm{GHom}_\Lambda(N\langle -i\rangle, P_x) \longrightarrow \mathrm{GHom}_\Lambda(M\langle -i\rangle, P_x) \longrightarrow \mathrm{GHom}_\Lambda(L\langle -i\rangle, P_x).$$

That is, the sequence $0 \longrightarrow (N^t)_i(x) \longrightarrow (M^t)_i(x) \longrightarrow (L^t)_i(x)$ is eaxct. By Proposition 3.2.2, $0 \longrightarrow N^t \longrightarrow M^t \longrightarrow L^t$ is an exact sequence in $\mathrm{GMod}\Lambda^{\mathrm{o}}$. The proof of the lemma is completed.

4.1.2 **Lemma**. *Let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver.*

(1) *If $M \in \mathrm{GMod}\Lambda$ and $V \in \mathrm{mod}k$, then $((M \otimes V)\langle s\rangle)^t \cong M^t\langle -s\rangle \otimes DV$, for $s \in \mathbb{Z}$.*

(2) *We have a duality $(-)^t : \mathrm{gproj}\Lambda \to \mathrm{gproj}\Lambda^{\mathrm{o}}$ such that $P_a^t \cong P_a^{\mathrm{o}}$, for all $a \in Q_0$.*

Proof. Let $M \in \mathrm{GMod}\Lambda$ and $V \in \mathrm{mod}k$. Using first the adjunction isomorphism and then applying Lemma 2.1.2, we obtain

$$\begin{aligned}\mathrm{GHom}_\Lambda(M \otimes V, P_x) &\cong \mathrm{Hom}_k(V, \mathrm{GHom}_\Lambda(M, P_x)) \\ &\cong \mathrm{GHom}_\Lambda(M, P_x) \otimes DV,\end{aligned}$$

for $x \in Q_0$. Now, in view of the definition of $(-)^t$, we see that

$$(M \otimes V)^t \cong M^t \otimes DV$$

and $M\langle s\rangle^t = M^t\langle -s\rangle$. Then, Statement (1) follows.

Fix $a \in Q_0$. Given any $(i, x) \in \mathbb{Z} \times Q_0$, by Proposition 3.4.3, we obtain a k-linear isomorphism

$$f^a_{i,x} : e_x \Lambda^{\mathrm{o}}_i e_a \to \mathrm{GHom}_\Lambda(P_a\langle -i\rangle, P_x) : v^{\mathrm{o}} \to P[v].$$

It is easy to see that $f^a_{i+j,y}(u^{\mathrm{o}}v^{\mathrm{o}}) = u^{\mathrm{o}} f^a_{i,x}(v^{\mathrm{o}})$, for $u \in e_x\Lambda_j e_y$ and $v \in e_a \Lambda_i e_x$. That is, $f^a = (f^a_{i,x})_{(i,x)\in\mathbb{Z}\times Q_0} : P^{\mathrm{o}}_a \to P^t_a$ is an isomorphism in $\mathrm{gproj}\Lambda^{\mathrm{o}}$. Similarly, we may construct an isomorphism $g^a : (P^o_a)^t \to (P^{\mathrm{o}}_a)^{\mathrm{o}} = P_a$ in $\mathrm{proj}\Lambda$. This yields an isomorphism $\zeta_a = g^a \circ (f^a)^t : P^{tt}_a \to P_a$ in $\mathrm{gproj}\Lambda$.

Fix $u \in e_a\Lambda_s e_b$. Consider the graded morphisms $P[u] : P_a \to P_b\langle s\rangle$ and $P[u^{\mathrm{o}}] : P^{\mathrm{o}}_b\langle -s\rangle \to P^{\mathrm{o}}_a$, the right multiplication by u and u^{o} respectively. Given $v \in e_b\Lambda_{i-s}e_x$, we have $P[uv] = P[v] \circ P[u]\langle -i\rangle$, that is,

$$f^a_{i,x}(P[u^{\mathrm{o}}]_{i,x}(v^{\mathrm{o}})) = \mathrm{GHom}(P[u]\langle -i\rangle, P_x)(f^b_{i-s,x}(v^{\mathrm{o}})).$$

So, $f^a_{i,x} \circ P[u^{\mathrm{o}}]_{i,x} = P[u]^t_{i,x} \circ f^b\langle -s\rangle_{i,x}$, and hence, $f^a \circ P[u^{\mathrm{o}}] = P[u]^t \circ f^b\langle -s\rangle$. Similarly, $P[u] \circ g^a = g^b\langle s\rangle \circ P[u^{\mathrm{o}}]^t$. This implies $P[u] \circ \zeta^a = \zeta^b\langle s\rangle \circ P[u]^{tt}$. Since every morphism in $\mathrm{Hom}_\Lambda(P_a, P_b\langle s\rangle)$ is of the form $P[u]$; see (3.4.3), ζ^a is natural in P_a.

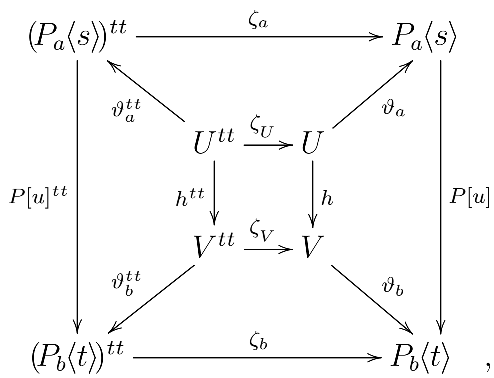
Now, let U and V be two indecomposable projective modules in $\mathrm{gproj}\Lambda$. Then there are two graded isomorphisms $\vartheta_a : U \to P_a\langle s\rangle$ and $\vartheta_b : V \to P_b\langle t\rangle$ for some $a, b \in Q_0$ and $s, t \in \mathbb{Z}$. This yields two commutative diagram

$$\begin{array}{ccc} (P_a\langle s\rangle)^{tt} & \xrightarrow{\zeta_a} & P_a\langle s\rangle \\ \uparrow{\scriptstyle \vartheta_a^{tt}} & & \uparrow{\scriptstyle \vartheta_a} \\ U^{tt} & \xrightarrow{\zeta_U} & U \end{array} \qquad \text{and} \qquad \begin{array}{ccc} (P_b\langle t\rangle)^{tt} & \xrightarrow{\zeta_b} & P_b\langle t\rangle \\ \uparrow{\scriptstyle \vartheta_b^{tt}} & & \uparrow{\scriptstyle \vartheta_b} \\ V^{tt} & \xrightarrow{\zeta_V} & V \end{array}$$

for some graded isomorphisms $\zeta_U : U^{tt} \to U$ and $\zeta_V : V^{tt} \to V$. On the other hand, given a graded morphism $h : U \to V$, by Corollary 3.4.3, we have a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{\vartheta_a} & P_a\langle s\rangle \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle P[u]} \\ V & \xrightarrow{\vartheta_b} & P_b\langle t\rangle \end{array}$$

for some $u \in e_a \Lambda_{t-s} e_b$. Now, observe the following diagram



we see that

$$\begin{aligned} \vartheta_b \circ h \circ \zeta_U &= P[u] \circ \vartheta_a \circ \zeta_U \\ &= P[u] \circ \zeta_a \circ \vartheta_a^{tt} \\ &= \zeta_b \circ P[u]^{tt} \circ \vartheta_a^{tt} \\ &= \zeta_b \circ \vartheta_b^{tt} \circ h^{tt} \\ &= \vartheta_b \circ \zeta_V \circ h^{tt}. \end{aligned}$$

Since ϑ_b is a graded isomorphism, $h \circ \zeta_U = \zeta_V \circ h^{tt}$. Therefore, ζ_a extends to a natural isomorphism $\zeta_U : U^{tt} \to U$ for each $U \in \text{proj}\Lambda$. Thus, $\text{id} \cong (-)^t \circ (-)^t$. The proof of the lemma is completed.

Composing the functors $(-)^t$ and $\mathfrak{D}$ yields two functors $\nu = \mathfrak{D} \circ (-)^t$: $\text{GMod}\Lambda \to \text{GMod}\Lambda$ and $\nu^- = (-)^t \circ \mathfrak{D} : \text{GMod}\Lambda \to \text{GMod}\Lambda$. By Proposition 3.3.3 and Lemma 4.1.2, they restrict respectively to functors $\nu : \text{gproj}\Lambda \to \text{ginj}\Lambda$ and $\nu^- : \text{ginj}\Lambda \to \text{gproj}\Lambda$.

4.1.3 **Theorem**. *Let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver.*

(1) *The restricted functors $\nu : \text{gproj}\Lambda \to \text{ginj}\Lambda$ and $\nu^- : \text{ginj}\Lambda \to \text{gproj}\Lambda$ are mutually quasi-inverse such that $\nu(P_a\langle s\rangle \otimes V) \cong I_a\langle s\rangle \otimes V$, for $s \in \mathbb{Z}$ and $V \in \text{mod}k$.*

(2) *Given $M \in \text{GMod}\Lambda$ and $P \in \text{gproj}\Lambda$, there exists a binatural k-linear isomorphism*

$$\Phi_{P,M} : \text{GHom}_\Lambda(M, \nu P) \to D(\text{GHom}_\Lambda(P, M)).$$

Proof. By Proposition 3.3.3 and Lemma 4.1.2, both $\mathfrak{D} : \text{gproj}\Lambda^{\text{o}} \to \text{ginj}\Lambda$ and $(-)^t : \text{gproj}\Lambda \to \text{gproj}\Lambda^{\text{o}}$ are dualities. Thus, $\nu : \text{gproj}\Lambda \to \text{ginj}\Lambda$ and $\nu^- : \text{ginj}\Lambda \to \text{gproj}\Lambda$ are mutually quasi-inverse. Given $V \in \text{mod}k$, in view of Proposition 3.3.1(2) and Lemma 4.1.2, we see that

$$\nu(P_a\langle s\rangle \otimes V) \cong \mathfrak{D}(P_a^{\text{o}}\langle -s\rangle \otimes DV) \cong I_a\langle s\rangle \otimes V.$$

This establishes Statement (1).

Fix $M \in \text{GMod}\Lambda$ and $P_a\langle s\rangle$ with $(s, a) \in \mathbb{Z} \times Q_0$. By Lemma 4.1.2, we have a natural isomorphism $f^a\langle -s\rangle : P_a^{\text{o}}\langle -s\rangle \to P_a^t\langle -s\rangle$ in $\text{gproj}\Lambda^{\text{o}}$, and hence, a natural isomorphism $\mathfrak{D}(f^a)\langle s\rangle : (\nu P_a)\langle s\rangle \to I_a\langle s\rangle$ in $\text{ginj}\Lambda$. This gives rise to a k-linear isomorphism

$$\rho_M^{s,a} = \text{GHom}_\Lambda(M, \mathfrak{D}(f^a)\langle s\rangle) : \text{GHom}_\Lambda(M, (\nu P_a)\langle s\rangle) \to \text{GHom}_\Lambda(M, I_a\langle s\rangle),$$

which is binatural in M and $P_a\langle s\rangle$. Next, by Corollary 3.4.2, we obtain a k-linear isomorphism $\eta_M^{s,a} : \text{GHom}_\Lambda(P_a\langle s\rangle, M) \to M_{-s}(a)$, which is clearly binatural in M and $P_a\langle s\rangle$. This yields a binatural k-isomorphism

$$D\eta_M^{s,a} : DM_{-s}(a) \to D\text{GHom}_\Lambda(P_a\langle s\rangle, M).$$

Finally, we have a k-linear isomorphism $\theta_a : D(e_a\Lambda_0^{\text{o}}e_a) \to k : h \mapsto h(e_a)$. Applying Proposition 3.4.4 for the case $V = k$, we get a k-linear isomorphism

$$\psi_M^{s,a} : \text{GHom}_\Lambda(M, I_a\langle s\rangle) \to D(M_{-s}(a)); g \mapsto \theta_a \circ g_{-s,a},$$

which is clearly natural in M. Consider $u \in e_a\Lambda_{t-s}e_b$ and the graded morphisms $P[u] : P_a\langle s\rangle \to P_b\langle t\rangle$ and $P[u^{\text{o}}] : P_b^{\text{o}}\langle -t\rangle \to P_a^{\text{o}}\langle -s\rangle$. Setting $I[u] = \mathfrak{D}(P[u^{\text{o}}])$, we claim that the following diagram commutes

$$\begin{array}{ccc} \text{GHom}_\Lambda(M, I_a\langle s\rangle) & \xrightarrow{\psi_M^{s,a}} & D(M_{-s}(a)) \\ {\scriptstyle \text{GHom}_\Lambda(M, I[u])}\downarrow & & \downarrow{\scriptstyle D(M(u))} \\ \text{GHom}_\Lambda(M, I_b\langle t\rangle) & \xrightarrow{\psi_M^{t,b}} & D(M_{-t}(b)). \end{array}$$

Consider $g \in \text{GHom}_\Lambda(M, I_a\langle s\rangle)$. By definition, we have $\psi_M^{t,b}(\text{GHom}_\Lambda(M, I[u])(g)) = \theta_b \circ I[u]_{-t,b} \circ g_{-t,b}$ and $D(M(u))(\psi_M^{s,a}(g)) = \theta_a \circ g_{-s,a} \circ M(u)$. Moreover, given $m \in M_{-t}(b)$, we have

$$(\theta_a \circ g_{-s,a} \circ M(u))(m) = g_{-s,a}(um)(e_a) = (ug_{-t,b}(m))(e_a) = g_{-t,b}(m)(u^{\text{o}}).$$

On the other hand,

$$(I[u]_{-t,b} \circ g_{-t,b})(m) = D(P[u^{\rm o}])(g_{-t,b}(m)) = g_{-t,b}(m) \circ P[u^{\rm o}],$$

and hence,

$$(\theta_b \circ I[u]_{-t,b} \circ g_{-t,b})(m) = (g_{-t,b}(m) \circ P[u^{\rm o}])(e_b) = g_{-t,b}(m)(u^{\rm o}).$$

Our claim holds. Since every morphism in $\mathrm{Hom}_\Lambda(P_b^{\rm o}\langle -t\rangle, P_a^{\rm o}\langle -s\rangle)$ is of the form $P[u^{\rm o}]$; see (3.4.3), $\psi_{_M}^{s,a}$ is natural in $P_a\langle s\rangle$. Thus, we obtain a binatural k-linear isomorphism

$$\Phi_{P_a\langle s\rangle,\, M} = D(\eta_{_M}^{s,a}) \circ \psi_{_M}^{s,a} \circ \rho_{_M}^{s,a} : \mathrm{GHom}_\Lambda(M, \nu P_a\langle s\rangle) \to D(\mathrm{GHom}_\Lambda(P_a\langle s\rangle, M)).$$

Now, let $U \in \mathrm{gproj}\Lambda$ be indecomposable. Suppose that $\vartheta_a : U \to P_a\langle s\rangle$ is a graded isomorphism for some $(s,a) \in \mathbb{Z} \times Q_0$ and δ_a is an inverse of ϑ_a. We shall show that $\Phi_{P_a\langle s\rangle,\, M}$ extends to a binatural k-linear isomorphism $\Phi_{U,M}$. The composite σ of graded isomorphisms

$$\begin{array}{ccc} \nu U & \xrightarrow{\quad\sigma\quad} & \mathfrak{D}(U^{\rm o}) \\ {\scriptstyle \nu(\vartheta_a)}\downarrow & & \uparrow{\scriptstyle \mathfrak{D}(\delta_a^{\rm o})} \\ (\nu P_a)\langle -s\rangle & \xrightarrow{\mathfrak{D}(f^a)\langle s\rangle} & \mathfrak{D}(P_a^{\rm o}\langle -s\rangle) = I_a\langle s\rangle \end{array}$$

gives rise to a k-isomorphism

$$\rho_{_M}^{U} = \mathrm{GHom}_\Lambda(M, \sigma) : \mathrm{GHom}_\Lambda(M, \nu U) \to \mathrm{GHom}_\Lambda(M, \mathfrak{D}(U^{\rm o})),$$

which is binatural in M and U. Composing $D\eta_{_M}^{s,a}$ and $D(\mathrm{GHom}_\Lambda(\delta_a, M))$ yields a binatural k-linear isomorphism

$$\varphi_{_M}^{U} = D(\mathrm{GHom}_\Lambda(\delta_a, M)) \circ D\eta_{_M}^{s,a} : DM_{-s}(a) \to D\mathrm{GHom}_\Lambda(U, M);$$

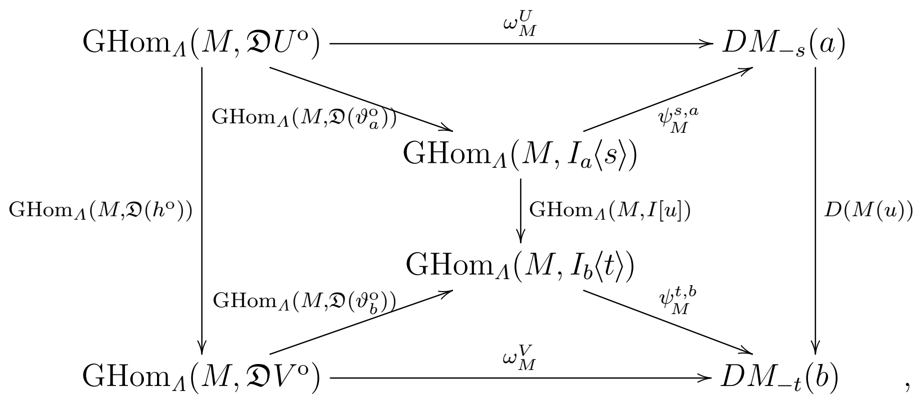
and composing $\mathrm{GHom}_\Lambda(M, \mathfrak{D}(\vartheta_a^{\rm o}))$ and $\psi_{_M}^{s,a}$ yields a k-linear isomorphism

$$\omega_M^U = \psi_{_M}^{s,a} \circ \mathrm{GHom}_\Lambda(M, \mathfrak{D}(\vartheta_a^{\rm o})) : \mathrm{GHom}_\Lambda(M, \mathfrak{D}U^{\rm o}) \to DM_{-s}(a),$$

which is natural in M. Let $h : U \to V$ be a graded morphism, where U and V are indecomposable projective modules. Considering a graded isomorphism $\vartheta_b : V \to P_b\langle t\rangle$ with $(t,b) \in \mathbb{Z} \times Q_0$, by Corollary 3.4.3, we have a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{\vartheta_a} & P_a\langle s\rangle \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle P[u]} \\ V & \xrightarrow{\vartheta_b} & P_b\langle t\rangle \end{array}$$

for some $u \in e_a \Lambda_{t-s} e_b$. Considering the following diagram



,

we see that

$$\begin{aligned} D(M(u)) \circ \omega_M^U &= D(M(u)) \circ \psi_M^{s,a} \circ \mathrm{GHom}_\Lambda(M, \mathfrak{D}(\vartheta_a^{\rm o})) \\ &= \psi_M^{t,b} \circ \mathrm{GHom}_\Lambda(M, I[u]) \circ \mathrm{GHom}_\Lambda(M, \mathfrak{D}(\vartheta_a^{\rm o})) \\ &= \psi_M^{t,b} \circ \mathrm{GHom}_\Lambda(M, \mathfrak{D}(\vartheta_b^{\rm o})) \circ \mathrm{GHom}_\Lambda(M, \mathfrak{D}(h^{\rm o})) \\ &= \omega_M^V \circ \mathrm{GHom}_\Lambda(M, \mathfrak{D}(h^{\rm o})). \end{aligned}$$

Hence, ω_M^U is natural in U. Therefore, we obtain a binatural k-linear isomorphism

$$\Phi_{U,M} = \varphi_M^U \circ \omega_M^U \circ \rho_M^U : \mathrm{GHom}_\Lambda(M, \nu U) \to D(\mathrm{GHom}_\Lambda(U, M)).$$

It is well-known that $\Phi_{P_a\langle s\rangle, M}$ extends to a binatural k-linear isomorphism $\Phi_{U,M}$ for all $U \in \mathrm{proj}\Lambda$. The proof of the theorem is completed.

Remark. By Theorem 4.1.3, the functor $\nu : \mathrm{gproj}\Lambda \to \mathrm{GMod}\Lambda$ is a **Nakayama functor** as defined in [38, (5.4)].

4.2 Almost split sequences for graded modules

In case Q is a finite quiver, Martinez-Villa established the existence of almost split sequences in the category of finitely presented graded Λ-modules; [44, (1.7.1)]. In this section, we supply a new, more categorical, proof of Auslander-Reiten formulas which does not rely on the tensor product functors but the Nakayama functor; compare [7, (IV.4.5)]. As a consequence, we shall extend Martinez-Villa's result to the locally finite case.

We shall denote by $\underline{\text{GMod}}\Lambda$ the projectively stable category of GModΛ, and by $\overline{\text{GMod}}\Lambda$ the injectively stable category of GModΛ. By Lemma 1.5.1 and Proposition 3.4.8, we see that $\underline{\text{GMod}}\Lambda$ is the quotient category of GModΛ modulo the morphisms factoring through graded projective modules in GModΛ, and $\overline{\text{GMod}}\Lambda$ is the quotient category of GModΛ modulo the morphisms factoring through graded injective modules in GModΛ. Given $M, N \in \text{GMod}\Lambda$, we shall write

$$\underline{\text{GHom}}_\Lambda(M,N) = \text{GHom}_\Lambda(M,N)/\mathcal{P}(M,N)$$

and

$$\overline{\text{GHom}}_\Lambda(M,N) = \text{GHom}_\Lambda(M,N)/\mathcal{I}(M,N).$$

Moreover, we shall write

$$\underline{\text{GEnd}}_\Lambda(M) = \underline{\text{GHom}}_\Lambda(M,M) \text{ and } \overline{\text{GEnd}}_\Lambda(M) = \overline{\text{GHom}}_\Lambda(M,M).$$

The following easy statement is well known. For the convenience of the reader, we will provide a brief proof.

4.2.1 **Lemma**. *Let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver. Consider $M, N \in$ GModΛ.*

(1) *If there exists an epimorphism $f : P \to N$ in* GModΛ *with P graded projective, then there exists an exact sequence*

$$\text{GHom}_\Lambda(M,P) \xrightarrow{\text{GHom}_\Lambda(M,f)} \text{GHom}_\Lambda(M,N) \longrightarrow \underline{\text{GHom}}_\Lambda(M,N) \longrightarrow 0.$$

(2) *If there exits a monomorphism $f : M \to I$ in* GModΛ *with I graded injective, then there exists an exact sequence*

$$\text{GHom}_\Lambda(I,N) \xrightarrow{\text{GHom}_\Lambda(f,N)} \text{GHom}_\Lambda(M,N) \longrightarrow \overline{\text{GHom}}_\Lambda(M,N) \longrightarrow 0.$$

Proof. We shall only prove Statement (2), since the proof of Statement (1) is dual. Let $f : M \to I$ be a monomorphism in GModΛ with I graded injective. Consider the following sequence

$$\text{GHom}_\Lambda(I,N) \xrightarrow{\text{GHom}_\Lambda(M,f)} \text{GHom}_\Lambda(M,N) \xrightarrow{p} \overline{\text{GHom}}_\Lambda(M,N) \longrightarrow 0,$$

where p is the canonical epimorphism. Since I is graded injective, we have $\text{Im}(\text{GHom}_\Lambda(M,f)) \subseteq \mathcal{I}(M,N)$. On the other hand, given $g \in \mathcal{I}(M,N)$, we have

$g = g_1 \circ g_2$ for some graded morphisms $g_1 : I' \to N$ and $g_2 : N \to I'$ in $\mathrm{GMod}\Lambda$ with I' injective module. Since f is a monomorphism, there exists a graded morphism $h : I \to I'$ such that $g_2 = h \circ f$. That is, $\mathrm{GHom}_\Lambda(M, f)(g_1 \circ h) = g$. Hence, $\mathrm{Im}(\mathrm{GHom}_\Lambda(M, f)) \supseteq \mathcal{I}(M, N)$. Therefore, $\mathrm{Im}(\mathrm{GHom}_\Lambda(M, f)) = \mathcal{I}(M, N)$. The proof of the lemma is completed.

We denote by $\underline{\mathrm{gmod}}^{+,p}\Lambda$ the full subcategory of $\underline{\mathrm{GMod}}\Lambda$ generated by the finitely presented graded modules without non-zero graded projective direct summands, and by $\overline{\mathrm{gmod}}^{-,i}\Lambda$ the full subcategory of $\overline{\mathrm{GMod}}\Lambda$ generated by the finitely co-presented graded modules without non-zero graded injective direct summands. By Lemma 4.2.1, we see that $\underline{\mathrm{gmod}}^{+,p}\Lambda$ is a dense full subcategory of the quotient category of $\mathrm{gmod}^{+,p}\Lambda$ modulo the morphisms factoring through modules in $\mathrm{gproj}\Lambda$, and $\overline{\mathrm{gmod}}^{-,i}\Lambda$ is a dense full subcategory of the quotient category of $\mathrm{gmod}^{-,i}\Lambda$ modulo the morphisms factoring through modules in $\mathrm{ginj}\Lambda$.

4.2.2 **Proposition**. *Let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver. Then $\underline{\mathrm{gmod}}^{+,p}\Lambda$ and $\overline{\mathrm{gmod}}^{-,i}\Lambda$ are Hom-finite and Krull-Schmidt.*

Proof. By Proposition 3.9.10, $\mathrm{gmod}^{+,p}\Lambda$ and $\mathrm{gmod}^{-,i}\Lambda$ are Hom-finite and Krull-Schmidt. So are their quotient categories by Lemma 1.2.4. The proof of the proposition is completed.

For each $M \in \mathrm{gmod}^{+,p}\Lambda$, by Lemma 3.9.8(1), we fix a minimal graded projective presentation

$$P^{-1} \xrightarrow{d^{-1}} P^0 \xrightarrow{d^0} M \longrightarrow 0,$$

where $P^{-1}, P^0 \in \mathrm{gproj}\Lambda$. Applying the functor $(-)^t$, by Lemma 4.1.1, we obtain an exact sequence

$$0 \longrightarrow M^t \xrightarrow{(d^0)^t} (P^0)^t \xrightarrow{(d^{-1})^t} (P^{-1})^t \longrightarrow \mathrm{Coker}(d^{-1})^t \longrightarrow 0$$

in $\mathrm{gmod}\Lambda^{\mathrm{o}}$. We define the **transpose** of M to be $\mathrm{Tr}M = \mathrm{Coker}(d^{-1})^t$.

4.2.3 **Lemma**. *Let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver. Consider $M \in \mathrm{gmod}^{+,p}\Lambda$.*

(1) *M is graded projective if and only if $\mathrm{Tr}M = 0$.*

(2) *If $M = X \oplus Y$, then $\mathrm{Tr}M \cong \mathrm{Tr}X \oplus \mathrm{Tr}Y$.*

Proof. (1) Suppose that M is graded projective. Then the sequence

$$0 \longrightarrow M \xrightarrow{\text{id}} M \longrightarrow 0$$

is a minimal graded projective presentation of M. By definition, $\text{Tr}M$ is a cokernel of the morphism $M^t \to 0$, which is zero.

Suppose that $\text{Tr}M = 0$. Let $P^{-1} \xrightarrow{d^{-1}} P^0 \xrightarrow{d^0} M \longrightarrow 0$ be a minimal graded projective presentation of M. By definition, we get a graded projective presentation $(P^0)^t \xrightarrow{(d^{-1})^t} (P^{-1})^t \longrightarrow \text{Tr}M \longrightarrow 0$. Since $\text{Tr}M = 0$ and $(P^{-1})^t$ is projective, $(d^{-1})^t$ is a retraction. Because $(-)^t$ is a duality by Lemma 4.1.2(2), d^{-1} is a section. This yields a split short exact sequence

$$0 \longrightarrow P^{-1} \xrightarrow{d^{-1}} P^0 \xrightarrow{d^0} M \longrightarrow 0.$$

In particular, M is graded projective.

(2) Assume that $M = X \oplus Y$. Consider two minimal graded projective presentations $V^{-1} \xrightarrow{v^{-1}} V^0 \xrightarrow{v^0} X \longrightarrow 0$ and $W^{-1} \xrightarrow{w^{-1}} W^0 \xrightarrow{w^0} Y \longrightarrow 0$. Then, M admits a minimal graded projective presentation

$$V^{-1} \oplus W^{-1} \xrightarrow{\begin{pmatrix} v^{-1} & 0 \\ 0 & w^{-1} \end{pmatrix}} V^0 \oplus W^0 \xrightarrow{\begin{pmatrix} v^0 & 0 \\ 0 & w^0 \end{pmatrix}} X \oplus Y \longrightarrow 0.$$

Thus, by definition, it easy to see that $\text{Tr}M \cong \text{Tr}X \oplus \text{Tr}Y$. The proof of the lemma is completed.

4.2.4 **Proposition**. *Let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver. Let $M \in \text{gmod}^{+,p}\Lambda$ admitting a minimal graded projective presentation*

$$P^{-1} \xrightarrow{d^{-1}} P^0 \xrightarrow{d^0} M \longrightarrow 0.$$

(1) *The graded projective presentation $(P^0)^t \xrightarrow{(d^{-1})^t} (P^{-1})^t \longrightarrow \text{Tr}M \longrightarrow 0$ is minimal.*

(2) $\text{Tr}M$ *has no non-zero graded projective direct summands.*

(3) $\vartheta_M : M \to \text{Tr}^2 M$ *is a graded isomorphism.*

(4) *M is indecomposable and non-projective if and only if so is* $\text{Tr}M$.

Proof. (1) Assume that $(P^0)^t \xrightarrow{(d^{-1})^t} (P^{-1})^t \xrightarrow{u} \mathrm{Tr}M \longrightarrow 0$ is not minimal. Consider first the case where u is not a graded projective cover. By Lemma 3.8.2, there exists a graded projective cover $f : V^{-1} \to \mathrm{Tr}M$. Thus, we may assume that

$$u = (f, 0) : (P^{-1})^t = V^{-1} \oplus W^{-1} \to \mathrm{Tr}M,$$

where $W^{-1} \neq 0$. Then $\mathrm{Im}(d^{-1})^t = \mathrm{Ker}(u) = \mathrm{Ker}\, f \oplus W^{-1}$, and hence,

$$(d^{-1})^t = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} : (P^0)^t \to V^{-1} \oplus W^{-1},$$

where $h_2 : (P^0)^t \to W^{-1}$ is a graded epimorphism. Since W^{-1} is graded projective, we may write

$$h_2 = (0, g) : (P^0)^t = V^0 \oplus W^0 \to W^{-1},$$

where $g : W^0 \to W^{-1}$ is a graded isomorphism. Therefore,

$$(d^{-1})^t = \begin{pmatrix} p & q \\ 0 & g \end{pmatrix} : (P^0)^t = V^0 \oplus W^0 \to V^{-1} \oplus W^{-1} = (P^{-1})^t.$$

Since $(-)^t$ is a duality by Lemma 4.1.2(2), d^{-1} is isomorphic to the graded morphism

$$\begin{pmatrix} p^t & 0 \\ q^t & g^t \end{pmatrix} : (V^{-1})^t \oplus (W^{-1})^t \to (V^0)^t \oplus (W^0)^t.$$

Therefore, the minimal graded projective presentation of M is isomorphic to an exact sequence

$$(V^{-1})^t \oplus (W^{-1})^t \xrightarrow{\begin{pmatrix} p^t & 0 \\ q^t & g^t \end{pmatrix}} (V^0)^t \oplus (W^0)^t \xrightarrow{\begin{pmatrix} d_1 & d_2 \end{pmatrix}} M \longrightarrow 0.$$

Since g^t is a graded isomorphism, $d_2 = 0$ and $(W^0)^t \neq 0$. This is contrary to d^0 being a graded projective cover.

Next, suppose that the co-restriction of $(d^{-1})^t$ to $\mathrm{Im}(d^{-1})^t$ is not a graded projective cover. Since $\mathrm{Im}(d^{-1})^t$ is finitely generated, there exists a graded projective cover $v : V^0 \to \mathrm{Im}(d^{-1})^t$ by Lemma 3.8.2. Thus, $(d^{-1})^t$ is isomorphic to a morphism $(v, 0) : V \oplus W \to (P^{-1})^t$, where $W \neq 0$. Since $(-)^t$ is a duality by Lemma 4.1.2(2), d^{-1} is isomorphic to the morphism

$$\begin{pmatrix} v^t \\ 0 \end{pmatrix} : P^{-1} \to V^t \oplus W^t.$$

Therefore, the minimal graded projective presentation of M is isomorphic to the following exact sequence

$$P^{-1} \xrightarrow{\begin{pmatrix} v^t \\ 0 \end{pmatrix}} V^t \oplus W^t \xrightarrow{u'} M \longrightarrow 0.$$

This yields $M \cong \text{Coker}(v^t) \oplus W^t$, where W^t is a non-zero projective module, a contradiction to the assumption on M. This establishes the second part of Statement (1).

(2) Assume that $\text{Tr}M = V \oplus X$, where V is a non-zero graded projective module. By Proposition 3.9.10, X admits a minimal graded projective presentation $W^{-1} \xrightarrow{w^{-1}} W^0 \xrightarrow{w^0} X \longrightarrow 0$ over $\text{gproj}\Lambda^{\text{o}}$. Then, $\text{Tr}M$ admits a minimal graded projective presentation

$$W^{-1} \xrightarrow{\begin{pmatrix} 0 \\ w^{-1} \end{pmatrix}} V \oplus W^0 \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & w^0 \end{pmatrix}} V \oplus X \longrightarrow 0.$$

By Statement (1), it is isomorphic to

$$(P^0)^t \xrightarrow{(d^{-1})^t} (P^{-1})^t \longrightarrow \text{Tr}M \longrightarrow 0.$$

Since $(-)^t$ is a duality, d^{-1} is isomorphic to the morphism

$$(0, (w^{-1})^t) : V^t \oplus (W^0)^t \to (W^{-1})^t.$$

So, the restriction of d^{-1} to a non-zero direct summand V^t of P^{-1} is zero, a contradiction.

(3) Since $(-)^t : \text{gproj}\Lambda \to \text{gproj}\Lambda^{\text{o}}$ is a duality by Lemma 4.1.2(2), we have a graded morphism $\vartheta_M : M \to \text{Tr}^2 M$ making the following diagram

$$\begin{array}{ccccccc}
P^{-1} & \xrightarrow{d^{-1}} & P^0 & \xrightarrow{d^0} & M & \longrightarrow & 0 \\
\downarrow \cong & & \downarrow \cong & & \vdots \, \vartheta_M & & \\
(P^{-1})^{tt} & \xrightarrow{(d^{-1})^{tt}} & (P^0)^{tt} & \xrightarrow{(d^0)^{tt}} & \text{Tr}^2 M & \longrightarrow & 0
\end{array}$$

commute, where rows are exact and columns are graded isomorphisms. In particular, ϑ_M is a graded isomorphism.

(4) By Statement (3), it suffices to prove the necessity. Assume that M is indecomposable and non-projective, and assume to the contrary that $\text{Tr}M =$

$X \oplus Y$ is a nontrivial direct sum decomposition. By Statement (2), X and Y are not graded projective. By Statement (3) and Lemma 4.2.3, $M \cong \mathrm{Tr}^2 M \cong \mathrm{Tr}X \oplus \mathrm{Tr}Y$, where $\mathrm{Tr}X \neq 0$ and $\mathrm{Tr}Y \neq 0$. This is a contradiction. So, $\mathrm{Tr}M$ is indecomposable, and by Statement (2), it is non-projective. The proof of the proposition is completed.

The following statement is important in our investigation, which corresponds to Auslander's result for an arbitrary ring with an identity; see [4, Chapter I, Section 3]. Here, we provide a detailed proof.

4.2.5 **Proposition**. *Let $\varLambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver. Then, we have a duality* $\mathrm{Tr} : \underline{\mathrm{gmod}}^{+,p}\varLambda \to \underline{\mathrm{gmod}}^{+,p}\varLambda^{\mathrm{o}}$.

Proof. For each non-zero object $M \in \underline{\mathrm{gmod}}^{+,p}\varLambda$, we fix a minimal graded projective presentation

$$P^{-1} \xrightarrow{d^{-1}} P^0 \xrightarrow{d^0} M \longrightarrow 0.$$

By Proposition 4.2.4, $\mathrm{Tr}M$ has no non-zero projective summands and has a minimal graded projective presentation in $\mathrm{gmod}\varLambda^{\mathrm{o}}$ as follows:

$$(P^0)^t \xrightarrow{(d^{-1})^t} (P^{-1})^t \longrightarrow \mathrm{Tr}M \longrightarrow 0.$$

Consider a morphism $\underline{f} = f + \mathcal{P}(M, N)$ in $\underline{\mathrm{gmod}}^{+,p}\varLambda$, where $f \in \mathrm{GHom}_{\varLambda}(M, N)$. We shall define $\mathrm{Tr}(\underline{f})$ as follows. First, we have a commutative diagram

$$\begin{array}{ccccccc}
P^{-1} & \xrightarrow{d^{-1}} & P^0 & \xrightarrow{d^0} & M & \longrightarrow & 0 \\
\downarrow{\scriptstyle f^{-1}} & & \downarrow{\scriptstyle f^0} & & \downarrow{\scriptstyle f} & & \\
L^{-1} & \xrightarrow{q^{-1}} & L^0 & \xrightarrow{q^0} & N & \longrightarrow & 0,
\end{array}$$

where the lower row is a minimal graded projective presentation of N. Applying the contravariant functor $(-)^t$, we obtain a commutative diagram with exact rows

$$\begin{array}{ccccccc}
(P^0)^t & \xrightarrow{(d^{-1})^t} & (P^{-1})^t & \longrightarrow & \mathrm{Tr}M & \longrightarrow & 0 \\
\uparrow{\scriptstyle (f^0)^t} & & \uparrow{\scriptstyle (f^{-1})^t} & & \vdots{\scriptstyle f'} & & \\
(L^0)^t & \xrightarrow{(q^{-1})^t} & (L^{-1})^t & \longrightarrow & \mathrm{Tr}N & \longrightarrow & 0.
\end{array}$$

Set $\mathrm{Tr}(\underline{f}) = f' + \mathcal{P}(\mathrm{Tr}N, \mathrm{Tr}M)$. We claim that $\mathrm{Tr}(\underline{f})$ is well-defined. Indeed, assume that $\underline{f} = g + \mathcal{P}(M, N)$ for some $g \in \mathrm{GHom}_{\varLambda}(M, N)$ and that we have a commutative diagram

$$\begin{array}{ccccccc} P^{-1} & \xrightarrow{d^{-1}} & P^0 & \xrightarrow{d^0} & M & \longrightarrow & 0 \\ \downarrow{\scriptstyle g^{-1}} & & \downarrow{\scriptstyle g^0} & & \downarrow{\scriptstyle g} & & \\ L^{-1} & \xrightarrow{q^{-1}} & L^0 & \xrightarrow{q^0} & N & \longrightarrow & 0, \end{array}$$

which induces a commutative diagram

$$\begin{array}{ccccccc} (P^0)^t & \xrightarrow{(d^{-1})^t} & (P^{-1})^t & \xrightarrow{u} & \mathrm{Tr}M & \longrightarrow & 0 \\ \uparrow{\scriptstyle (g^0)^t} & & \uparrow{\scriptstyle (g^{-1})^t} & & \uparrow{\scriptstyle g'} & & \\ (L^0)^t & \xrightarrow{(q^{-1})^t} & (L^{-1})^t & \xrightarrow{v} & \mathrm{Tr}N & \longrightarrow & 0. \end{array}$$

Since L^0 is graded projective, by Lemma 4.2.1, $f - g = q^0 \circ h$ for some graded morphism $h : M \to L_0$. Since $q^0 \circ (f^0 - g^0) = (f - g) \circ d^0 = q^0 \circ h \circ d^0$, there is a graded morphism $h^0 : P^0 \to L^{-1}$ such that $q^{-1} \circ h^0 = f^0 - g^0 - h \circ d^0$, and thus, $(h^0)^t \circ (q^{-1})^t = (f^0)^t - (g^0)^t - (d^0)^t \circ h^t$. Observe that

$$((f^{-1})^t - (g^{-1})^t) \circ (q^{-1})^t = (d^{-1})^t \circ ((f^{-0})^t - (g^{-0})^t) = (d^{-1})^t \circ (h^0)^t \circ (q^{-1})^t.$$

Then, there exists a graded morphism $w : \mathrm{Tr}N \to (P^{-1})^t$ such that

$$w \circ v = (f^{-1})^t - (g^{-1})^t) - (d^{-1})^t \circ (h^0)^t.$$

Moreover, we see that

$$(f' - g') \circ v = u \circ ((f^{-1})^t - (g^{-1})^t) = u \circ (w \circ v + (d^{-1})^t \circ (h^0)^t) = u \circ w \circ v.$$

Since v is a graded epimorphism, $f' - g' = u \circ w$, that is, $f' - g'$ factors through $(P^{-1})^t$. Thus, $f' + \mathcal{P}(\mathrm{Tr}N, \mathrm{Tr}M) = g' + \mathcal{P}(\mathrm{Tr}N, \mathrm{Tr}M)$. This establishes our claim. This defines a contravariant functor $\mathrm{Tr} : \underline{\mathrm{gmod}}^{+,p}\Lambda \to \underline{\mathrm{gmod}}^{+,p}\Lambda^{\mathrm{o}}$. Similarly, we have a contravariant functor $\mathrm{Tr} : \underline{\mathrm{gmod}}^{+,p}\Lambda^{\mathrm{o}} \to \underline{\mathrm{gmod}}^{+,p}\Lambda$.

We shall construct a natural isomorphism $\vartheta : \mathrm{id}_{\underline{\mathrm{gmod}}^{+,p}\Lambda} \to \mathrm{Tr}^2$. Indeed, by Proposition 4.2.4(2), we obtain a minimal graded projective presentation

$$(P^{-1})^{tt} \xrightarrow{(d^{-1})^{tt}} (P^0)^{tt} \xrightarrow{(d^0)^{tt}} \mathrm{Tr}^2 M \longrightarrow 0$$

over $\mathrm{gproj}\Lambda$. By Proposition 4.2.4(3), we have a graded isomorphism $\vartheta_M : M \to \mathrm{Tr}^2 M$. This induces a graded isomorphism $\underline{\vartheta}_M : M \to \mathrm{Tr}^2 M$ in $\underline{\mathrm{gmod}}^{+,p}\Lambda$. It remains to show that $\underline{\vartheta}_M$ is natural in M. Let $\underline{f} : M \to N$ be a morphism in

$\underline{\mathrm{gmod}}^{+,p}\Lambda$, where $f \in \mathrm{GHom}_\Lambda(M,N)$. Then, we have a commutative diagram with exact rows

$$\begin{array}{ccccccc}
P^{-1} & \xrightarrow{d^{-1}} & P^0 & \xrightarrow{d^0} & M & \longrightarrow & 0 \\
{\scriptstyle f^{-1}}\downarrow & & {\scriptstyle f^0}\downarrow & & \downarrow{\scriptstyle f} & & \\
L^{-1} & \xrightarrow{q^{-1}} & L^0 & \xrightarrow{q^0} & N & \longrightarrow & 0.
\end{array}$$

We have obtained a commutative diagram with exact rows

$$\begin{array}{ccccccc}
(P^0)^t & \xrightarrow{(d^{-1})^t} & (P^{-1})^t & \longrightarrow & \mathrm{Tr}M & \longrightarrow & 0 \\
{\scriptstyle (f^0)^t}\uparrow & & {\scriptstyle (f^{-1})^t}\uparrow & & {\scriptstyle f'}\,\vdots\uparrow & & \\
(L^0)^t & \xrightarrow{(q^{-1})^t} & (L^{-1})^t & \longrightarrow & \mathrm{Tr}N & \longrightarrow & 0
\end{array}$$

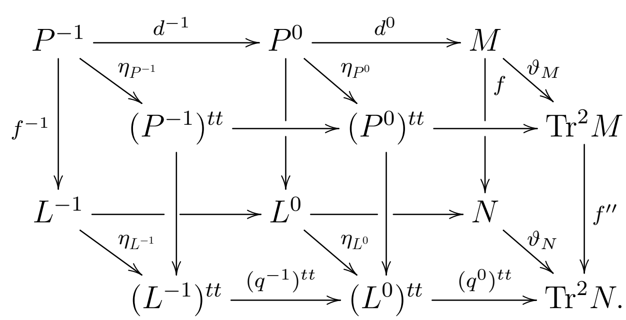
such that $\mathrm{Tr}(\underline{f}) = f' + \mathcal{P}(\mathrm{Tr}N, \mathrm{Tr}M)$. Applying the contravariant functor $(-)^t$ yields a commutative diagram with exact rows

$$\begin{array}{ccccccc}
(P^{-1})^{tt} & \xrightarrow{(d^{-1})^{tt}} & (P^0)^{tt} & \xrightarrow{(d^0)^{tt}} & \mathrm{Tr}^2M & \longrightarrow & 0 \\
{\scriptstyle (f^{-1})^{tt}}\downarrow & & {\scriptstyle (f^0)^{tt}}\downarrow & & \downarrow{\scriptstyle f''} & & \\
(L^{-1})^{tt} & \xrightarrow{(q^{-1})^{tt}} & (L^0)^{tt} & \xrightarrow{(q^0)^{tt}} & \mathrm{Tr}^2N & \longrightarrow & 0.
\end{array}$$

By definition, $\mathrm{Tr}^2(\underline{f}) = f'' + \mathcal{P}(M,N)$. Now, as seen above, we have a commutative diagram with exact rows

$$\begin{array}{ccccccc}
L^{-1} & \xrightarrow{q^{-1}} & L^0 & \xrightarrow{q^0} & N & \longrightarrow & 0 \\
{\scriptstyle \eta_{L^{-1}}}\downarrow\cong & & {\scriptstyle \eta_{L^0}}\downarrow\cong & & {\scriptstyle \vartheta_N}\,\vdots\downarrow\cong & & \\
(L^{-1})^{tt} & \xrightarrow{(q^{-1})^{tt}} & (L^0)^{tt} & \xrightarrow{(q^0)^{tt}} & \mathrm{Tr}^2N & \longrightarrow & 0
\end{array}$$

in $\mathrm{gmod}\Lambda$, where the upper row is a minimal graded projective presentation. Consider the following diagram



In view of this diagram, we see that

$$\begin{aligned}\vartheta_N \circ f \circ d^0 &= \vartheta_N \circ q^0 \circ f^0 \\ &= (q^0)^{tt} \circ \eta_{L^0} \circ f^0 \\ &= (q^0)^{tt} \circ (f^0)^{tt} \circ \eta_{P^0} \\ &= f'' \circ (d^0)^{tt} \circ \eta_{P^0} \\ &= f'' \circ \vartheta_M \circ d^0.\end{aligned}$$

Since d^0 is a graded epimorphism, $\vartheta_N \circ f = f'' \circ \vartheta_M$. This yields a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\underline{\vartheta}_M} & \mathrm{Tr}^2 M \\ {\scriptstyle \underline{f}}\downarrow & & \downarrow{\scriptstyle \mathrm{Tr}^2(\underline{f})} \\ N & \xrightarrow{\underline{\vartheta}_N} & \mathrm{Tr}^2 N. \end{array}$$

That is, we have a natural isomorphism $\underline{\vartheta} : \mathrm{id}_{\underline{\mathrm{gmod}}^{+,p}\Lambda} \to \mathrm{Tr}^2$. Similarly, we have a natural isomorphism $\underline{\varrho} : \mathrm{id}_{\underline{\mathrm{gmod}}^{+,p}\Lambda^{\mathrm{o}}} \to \mathrm{Tr}^2$. The proof of the proposition is completed.

Remark. In case Q is a finite quiver, Proposition 4.2.5 was stated by Martinez-Villa in [4, Section 1.4] without a proof.

The following statement is important.

4.2.6 **Proposition**. *Let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver.*

(1) *Let $M \in \mathrm{gmod}^{+,p}\Lambda$ with a minimal graded projective presentaion*

$$P^{-1} \xrightarrow{d^{-1}} P^0 \xrightarrow{d^0} M \longrightarrow 0$$

over $\mathrm{gproj}\Lambda$. *Then there exists in* $\mathrm{gmod}\Lambda$ *an exact sequence*

$$0 \longrightarrow \mathfrak{D}\mathrm{Tr}M \longrightarrow \nu P^{-1} \xrightarrow{\nu(d^{-1})} \nu P^0 \xrightarrow{\nu(d^0)} \nu M \longrightarrow 0.$$

(2) *Let $N \in \mathrm{gmod}^{-,i}\Lambda$ with a minimal graded injective co-presentation*

$$0 \longrightarrow N \xrightarrow{d^0} I^0 \xrightarrow{d^1} I^1$$

over $\mathrm{ginj}\Lambda$. *Then there exists in* $\mathrm{gmod}\Lambda$ *an exact sequence*

$$0 \longrightarrow \nu^- N \longrightarrow \nu^- I^0 \xrightarrow{\nu^-(d^{-1})} \nu^- I^1 \xrightarrow{\nu^-(d^0)} \mathrm{Tr}\mathfrak{D}N \longrightarrow 0.$$

Proof. (1) Since the functor $(-)^t$ is left exact by Lemma 4.1.1, we obtain an exact sequence

$$0 \longrightarrow M^t \xrightarrow{(d^0)^t} (P^0)^t \xrightarrow{(d^{-1})^t} (P^{-1})^t \longrightarrow \mathrm{Tr}M \longrightarrow 0,$$

which lies in gmodΛ^{o} by Lemma 3.2.5. Recalling that $\nu M = \mathfrak{D}M^t$, by Proposition 3.3.1(3), we obtain an exact sequence

$$0 \longrightarrow \mathfrak{D}(\mathrm{Tr}M) \longrightarrow \nu P^{-1} \xrightarrow{\nu(d^{-1})} \nu P^0 \xrightarrow{\nu(d^0)} \nu M \longrightarrow 0$$

in gmodΛ.

(2) Applying the exact functor $\mathfrak{D}$, in view of Lemma 3.8.7(2), we obtain a minimal graded projective presentation

$$\mathfrak{D}I^1 \xrightarrow{\mathfrak{D}(d^1)} \mathfrak{D}I^0 \xrightarrow{\mathfrak{D}(d^0)} \mathfrak{D}N \longrightarrow 0$$

in gmodΛ^{o} over gprojΛ^{o}. Applying the left exact functor $(-)^t$, by Lemma 4.1.1, we obtain an exact sequence

$$0 \longrightarrow \nu^- N \longrightarrow \nu^- I^0 \xrightarrow{\nu^-(d^0)} \nu^- I^1 \xrightarrow{\nu^-(d^1)} \mathrm{Tr}(\mathfrak{D}N) \longrightarrow 0$$

in gmodΛ, where $\nu^- N = (\mathfrak{D}N)^t$. The proof of the proposition is completed.

Combing Proposition 4.2.5 and 4.2.6, we obtain immediately the following result.

4.2.7 **Proposition**. *Let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver. Then, we have two mutually quasi-inverse equivalences*

$$\tau = \mathfrak{D} \circ \mathrm{Tr} : \underline{\mathrm{gmod}}^{+,p}\Lambda \to \overline{\mathrm{gmod}}^{-,i}\Lambda$$

and

$$\tau^- = \mathrm{Tr} \circ \mathfrak{D} : \overline{\mathrm{gmod}}^{-,i}\Lambda \to \underline{\mathrm{gmod}}^{+,p}\Lambda,$$

called the **Auslander-Reiten translations***.*

We shall need the following statement to prove the graded Auslander-Reiten formula.

4.2.8 **Proposition**. *Let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver. Consider a short exact sequence*

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

in GModΛ. *If $X \in \mathrm{gmod}^{+,p}\Lambda$, then we have an exact sequence of k-vector spaces*

$$0 \longrightarrow \mathrm{GHom}_\Lambda(N, \tau X) \xrightarrow{\mathrm{GHom}_\Lambda(g,\tau X)} \mathrm{GHom}_\Lambda(M, \tau X) \xrightarrow{\mathrm{GHom}_\Lambda(f,\tau X)} \mathrm{GHom}_\Lambda(L, \tau X)$$

$$\longrightarrow D\mathrm{GHom}_\Lambda(X, N) \longrightarrow D\mathrm{GHom}_\Lambda(X, M) \longrightarrow D\mathrm{GHom}_\Lambda(X, L) \longrightarrow 0.$$

Proof. Let $X \in \mathrm{gmod}^{+,p}\Lambda$ with a minimal graded projective presentaion

$$P^{-1} \xrightarrow{d^{-1}} P^0 \xrightarrow{d^0} X \longrightarrow 0$$

over gprojΛ. By Proposition 4.2.6(1), we have an exact sequence

$$0 \longrightarrow \tau X \longrightarrow \nu P^{-1} \xrightarrow{\nu(d^{-1})} \nu P^0.$$

Fix $Y \in \mathrm{GMod}\Lambda$. Applying the left exact functor $\mathrm{GHom}_\Lambda(Y, -)$ yields an exact sequence

$$0 \longrightarrow \mathrm{GHom}_\Lambda(Y, \tau X) \longrightarrow \mathrm{GHom}_\Lambda(Y, \nu P^{-1}) \xrightarrow{\mathrm{GHom}_\Lambda(Y,\nu(d^{-1}))} \mathrm{GHom}_\Lambda(Y, \nu P^0).$$

On the other hand, applying the right exact functor $D\mathrm{GHom}_\Lambda(-, Y)$ to the first exact sequence, in view of Theorem 4.1.3(2), we obtain a commutative diagram with exact lower row

$$\begin{array}{ccccccc}
\mathrm{GHom}_\Lambda(Y, \nu P^{-1}) & \xrightarrow{\mathrm{GHom}_\Lambda(Y,\nu(d^{-1}))} & \mathrm{GHom}_\Lambda(Y, \nu P^0) & & & & \\
\cong\downarrow & & \cong\downarrow & & & & \\
D\mathrm{GHom}_\Lambda(P^{-1}, Y) & \xrightarrow{D\mathrm{Hom}_\Lambda(d^{-1},Y)} & D\mathrm{GHom}_\Lambda(P^0, Y) & \xrightarrow{D(d^0)^*} & D\mathrm{GHom}_\Lambda(X, Y) & \longrightarrow & 0,
\end{array}$$

where $(d^0)^* = \mathrm{GHom}_\Lambda(d^0, Y)$. This yields an exact sequence

$$\mathrm{GHom}_\Lambda(Y, \nu P^{-1}) \xrightarrow{\mathrm{GHom}_\Lambda(Y,\nu(d^{-1}))} \mathrm{GHom}_\Lambda(Y, \nu P^0) \longrightarrow D\mathrm{GHom}_\Lambda(X, Y) \longrightarrow 0.$$

Now, since νP^{-1} and νP^0 are graded injective, we obtain a commutative diagram with exact rows and exact columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{GHom}_\Lambda(N,\tau X) & \longrightarrow & \mathrm{GHom}_\Lambda(M,\tau X) & \longrightarrow & \mathrm{GHom}_\Lambda(L,\tau X) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{GHom}_\Lambda(N,\nu P^{-1}) & \longrightarrow & \mathrm{GHom}_\Lambda(M,\nu P^{-1}) & \longrightarrow & \mathrm{GHom}_\Lambda(L,\nu P^{-1}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{GHom}_\Lambda(N,\nu P^{0}) & \longrightarrow & \mathrm{GHom}_\Lambda(M,\nu P^{0}) & \longrightarrow & \mathrm{GHom}_\Lambda(L,\nu P^{0}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & D\mathrm{GHom}_\Lambda(X,N) & \longrightarrow & D\mathrm{GHom}_\Lambda(X,M) & \longrightarrow & D\mathrm{GHom}_\Lambda(X,L) & \longrightarrow & 0. \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Using the Snake Lemma, we obtain the desired exact sequence stated in the proposition. The proof of the proposition is completed.

We also need the following statement.

4.2.9 **Proposition**. *Let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver. Consider a short exact sequence*

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

in GModΛ. *If $X \in \mathrm{gmod}^{-,i}\Lambda$, then we have an exact sequence of k-vector spaces*

$$0 \longrightarrow D^2\mathrm{GHom}_\Lambda(\tau^-X, L) \longrightarrow D^2\mathrm{GHom}_\Lambda(\tau^-X, M) \longrightarrow D^2\mathrm{GHom}_\Lambda(\tau^-X, N)$$

$$\longrightarrow D\mathrm{GHom}_\Lambda(L, X) \longrightarrow D\mathrm{GHom}_\Lambda(M, X) \longrightarrow D\mathrm{GHom}_\Lambda(N, X) \longrightarrow 0.$$

Proof. Let $X \in \mathrm{gmod}^{-,i}\Lambda$ with a minimal graded projective presentaion

$$0 \longrightarrow X \xrightarrow{d^0} I^0 \xrightarrow{d^1} I^1$$

over ginjΛ. By Proposition 4.2.6(2), we have an exact sequence

$$\nu^- I^0 \xrightarrow{\nu^-(d^1)} \nu^- I^1 \longrightarrow \tau^- X \longrightarrow 0.$$

Fix $Y \in \mathrm{GMod}\Lambda$. Applying the left exact functor $D^2\mathrm{GHom}_\Lambda(-, Y)$ yields an exact sequence

$$0 \to D^2\mathrm{GHom}_\Lambda(\tau^- X, Y) \to D^2\mathrm{GHom}_\Lambda(\nu^- I^1, Y) \xrightarrow{D^2\mathrm{GHom}_\Lambda(\nu^- d^1, Y)} D^2\mathrm{GHom}_\Lambda(\nu^- I^0, Y).$$

On the other hand, applying the right exact functor $D\mathrm{GHom}_\Lambda(Y, -)$ to the first exact sequence, in view of Theorem 4.1.3(2), we obtain a commutative diagram with exact upper row

$$\begin{array}{ccccccc}
D\mathrm{GHom}_\Lambda(Y, I^1) & \xrightarrow{D\mathrm{GHom}_\Lambda(Y, d^1)} & D\mathrm{GHom}_\Lambda(Y, I^0) & \xrightarrow{Dd^0_*} & D\mathrm{GHom}_\Lambda(Y, X) & \to & 0 \\
\uparrow \cong & & \uparrow \cong & & & & \\
D^2\mathrm{GHom}_\Lambda(\nu^- I^1, Y) & \xrightarrow{D^2\mathrm{GHom}_\Lambda(\nu^- d^1, Y)} & D^2\mathrm{GHom}_\Lambda(\nu^- I^0, Y), & & & &
\end{array}$$

where $d^0_* = \mathrm{GHom}_\Lambda(Y, d^0)$. This yields an exact sequence

$$D^2\mathrm{GHom}_\Lambda(\nu^- I^1, Y) \xrightarrow{D^2\mathrm{GHom}_\Lambda(\nu^- d^1, Y)} D^2\mathrm{GHom}_\Lambda(\nu^- I^0, Y) \to D\mathrm{GHom}_\Lambda(Y, X) \to 0.$$

Now, since $\nu^- I^0$ and $\nu^- I^1$ are projective, we obtain a commutative diagram with exact rows and exact columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & D^2\mathrm{GHom}_\Lambda(\tau^- X, L) & \to & D^2\mathrm{GHom}_\Lambda(\tau^- X, M) & \to & D^2\mathrm{GHom}_\Lambda(\tau^- X, N) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & D^2\mathrm{GHom}_\Lambda(\nu^- I^1, L) & \to & D^2\mathrm{GHom}_\Lambda(\nu^- I^1, M) & \to & D^2\mathrm{GHom}_\Lambda(\nu^- I^1, N) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & D^2\mathrm{GHom}_\Lambda(\nu^- I^0, L) & \to & D^2\mathrm{GHom}_\Lambda(\nu^- I^0, M) & \to & D^2\mathrm{GHom}_\Lambda(\nu^- I^0, N) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & D\mathrm{GHom}_\Lambda(L, X) & \longrightarrow & D\mathrm{GHom}_\Lambda(M, X) & \longrightarrow & D\mathrm{GHom}_\Lambda(N, X) & \longrightarrow & 0. \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Using the Snake Lemma, we obtain the desired exact sequence stated in the proposition. The proof of the proposition is completed.

We are ready to obtain the graded Auslander-Reiten formula. The classical approach to this well known formula consists of the following two identifications;

see [4, (I.3.4)] and the corollary to [44, (1.6.3)]. First, the covariant stable Hom functor given by a finitely presented module is identified with the Tor^1 functor given by its transpose; see [4, (I.3.2)] and [44, (1.6.3)]. Secondly, the dual of the Tor^1 functor given by a module is identified by the adjunction isomorphism with the contravariant Ext^1 functor given by its dual; see [4, (I.3.3)], [18, (VI.5.1)] and [44, (1.6.1)]. Our approach is to apply the Nakayama functor; see (4.1.3), which does not pass through the Tor^1 functor and does not involve the tensor product over the algebra or the adjunction isomorphism.

4.2.10 **Theorem**. *Let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver. Consider $M, N \in \mathrm{GMod}\Lambda$.*

(1) *If $M \in \mathrm{gmod}^{+,p}\Lambda$, then there exists a k-linear isomorphism*

$$D\underline{\mathrm{GHom}}_\Lambda(M, N) \cong \mathrm{GExt}^1_\Lambda(N, \tau M),$$

which is natural in N.

(2) *If $N \in \mathrm{gmod}^{-,i}\Lambda$, then there exists a k-linear isomorphism*

$$D^2\mathrm{GExt}^1_\Lambda(\tau^- N, M) \cong D\overline{\mathrm{GHom}}_\Lambda(M, N),$$

which is natural in M.

Proof. (1) Assume that $M \in \mathrm{gmod}^{+,p}\Lambda$. By Proposition 3.4.8, there exists a short exact sequence $0 \longrightarrow L \xrightarrow{q} P \xrightarrow{p} N \longrightarrow 0$ in $\mathrm{GMod}\Lambda$ with P being graded projective. Applying $\mathrm{GHom}_\Lambda(-, \tau M)$ yields an exact sequence

$$0 \longrightarrow \mathrm{GHom}_\Lambda(N, \tau M) \xrightarrow{p^*} \mathrm{GHom}_\Lambda(P, \tau M) \xrightarrow{q^*} \mathrm{GHom}_\Lambda(L, \tau M)$$

$$\longrightarrow \mathrm{GExt}^1_\Lambda(N, \tau M) \longrightarrow 0,$$

where $p^* = \mathrm{GHom}_\Lambda(p, \tau M)$ and $q^* = \mathrm{GHom}_\Lambda(q, \tau M)$. In particular, we obtain an isomorphism $\mathrm{Coker}(q^*) \cong \mathrm{GExt}^1_\Lambda(N, \tau M)$, which is natural in M. On the other hand, in view of Lemma 4.2.1(1), we have an exact sequence

$$0 \longrightarrow D\underline{\mathrm{GHom}}_\Lambda(M, N) \longrightarrow D\mathrm{GHom}_\Lambda(M, N) \xrightarrow{D\mathrm{GHom}_\Lambda(M,p)} D\mathrm{GHom}_\Lambda(M, P).$$

This yields an isomorphism $D\underline{\mathrm{GHom}}_\Lambda(M, N) \cong \mathrm{Ker}(D\mathrm{GHom}_\Lambda(M, p))$, which is natural in M. Further, by Proposition 4.2.8, we have an exact sequence

$$0 \longrightarrow \mathrm{GHom}_\Lambda(N, \tau M) \xrightarrow{p^*} \mathrm{GHom}_\Lambda(P, \tau M) \xrightarrow{q^*} \mathrm{GHom}_\Lambda(L, \tau M) \xrightarrow{\eta}$$

$$D\mathrm{GHom}_\Lambda(M,N)\xrightarrow{D\mathrm{GHom}_\Lambda(M,p)}D\mathrm{GHom}_\Lambda(M,P)\longrightarrow D\mathrm{GHom}_\Lambda(M,L)\longrightarrow 0.$$

This yields an isomorphism

$$\mathrm{Ker}(D\mathrm{GHom}_\Lambda(M,p))=\mathrm{Im}(\eta)\cong\mathrm{Coker}(q^*),$$

which is clearly natural in M. As a consequence, we obtain a natural isomorphism

$$D\underline{\mathrm{GHom}}_\Lambda(M,N)\cong\mathrm{GExt}^1_\Lambda(N,\tau M).$$

(2) Assume that $N\in \mathrm{gmod}^{-,i}\Lambda$. By Proposition 3.4.8, there exists a short exact sequence $0\longrightarrow M\xrightarrow{q} I\xrightarrow{p} L\longrightarrow 0$ in $\mathrm{GMod}\Lambda$ with I being graded injective. Put $U=\tau^-N$. Applying the functor $\mathrm{GHom}_\Lambda(\tau^-N,-)$, since I is graded injective, we obtain an exact sequence

$$0\longrightarrow \mathrm{GHom}_\Lambda(\tau^-N,M)\xrightarrow{q_*}\mathrm{GHom}_\Lambda(\tau^-N,I)\xrightarrow{p_*}\mathrm{GHom}_\Lambda(\tau^-N,L)$$

$$\longrightarrow \mathrm{GExt}^1_\Lambda(\tau^-N,M)\longrightarrow 0,$$

where $p_*=\mathrm{GHom}_\Lambda(\tau^-N,p)$ and $q_*=\mathrm{GHom}_\Lambda(\tau^-N,q)$. Applying the exact functor D^2 yields an exact sequence

$$0\longrightarrow D^2\mathrm{GHom}_\Lambda(\tau^-N,M)\xrightarrow{D^2(q_*)}D^2\mathrm{GHom}_\Lambda(\tau^-N,I)\xrightarrow{D^2(p_*)}D^2\mathrm{GHom}_\Lambda(\tau^-N,L)$$

$$\longrightarrow D^2\mathrm{GExt}^1_\Lambda(\tau^-N,M)\longrightarrow 0.$$

Hence, we have an isomorphism $D^2\mathrm{GExt}^1_\Lambda(\tau^-N,M)\cong\mathrm{Coker}(D^2(p_*))$, which is natural in N. On the other hand, we deduce from Lemma 4.2.1(2) that

$$0\longrightarrow D\overline{\mathrm{GHom}}_\Lambda(M,N)\longrightarrow D\mathrm{GHom}_\Lambda(M,N)\xrightarrow{D\mathrm{GHom}_\Lambda(M,q)}D\mathrm{GHom}_\Lambda(I,N)$$

is an exact sequence. In particular, we obtain an isomorphism $D\mathrm{G}\underline{\mathrm{Hom}}_\Lambda(M,N)\cong\mathrm{Ker}(D\mathrm{GHom}_\Lambda(M,q))$, which is natural in N. Further, by Proposition 4.2.9, we have an exact sequence

$$D^2\mathrm{GHom}_\Lambda(\tau^-N,M)\longrightarrow D^2\mathrm{GHom}_\Lambda(\tau^-N,I)\xrightarrow{D^2(p_*)}D^2\mathrm{GHom}_\Lambda(\tau^-N,L)\xrightarrow{\eta}$$

$$D\mathrm{GHom}_\Lambda(M,N)\xrightarrow{D\mathrm{GHom}_\Lambda(M,q)}D\mathrm{GHom}_\Lambda(I,N)\longrightarrow D\mathrm{GHom}_\Lambda(L,N)\longrightarrow 0.$$

This yields an isomorphism

$$\mathrm{Ker}(D\mathrm{GHom}_\Lambda(M,q)) = \mathrm{Im}(\eta) \cong \mathrm{Coker}(D^2(p_*)),$$

which is clearly natural in N. As a consequence, we obtain a natural isomorphism

$$D\underline{\mathrm{GHom}}_\Lambda(M,N) \cong D^2\mathrm{GExt}^1_\Lambda(\tau^- N, M).$$

The proof of the theorem is completed.

Remark. (1) The non-graded version of Theorem 4.2.10(1) was established by Auslander and Reiten for modules over any ring; see [4, (I. 3.4)].
(2) In case Q is finite, Theorem 4.2.10(1) was established by Martinez-Villa in case M is finitely presented and N is locally finite dimensional; see [44, Page 42].

We shall also need the following easy statement.

4.2.11 **Lemma**. *Let Σ be a local k-algebra. Then $D\Sigma$ has a non-zero socle as a left Σ-module and as a right Σ-module.*

Proof. We shall consider only the left Σ-module $D\Sigma$. Let $p : \Sigma \to \Sigma/\mathrm{rad}\Sigma$ be the canonical projection. Applying the left exact functor D, we obtain an injection $D(p) : D(\Sigma/\mathrm{rad}\Sigma) \to D(\Sigma)$. Fix a non-zero element $\varphi \in D(\Sigma/\mathrm{rad}\Sigma)$. Then $D(p)(\varphi) = \varphi \circ p$ is a non-zero element in $D(\Sigma)$. For any $u \in \mathrm{rad}\Sigma$ and $v \in \Sigma$, we have $(u \cdot (\varphi \circ p))(v) = (\varphi \circ p)(vu) = \varphi(p(vu)) = 0$. That is, $0 \neq \varphi p \in \mathrm{soc}(D\Sigma)$. The proof of the lemma is completed.

The following is one of the main results of this thesis.

4.2.12 **Theorem**. *Let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver.*

(1) *If $M \in \mathrm{gmod}^{+,p}\Lambda$ is indecomposable and non-projective, then there exists an almost split sequence*

$$0 \longrightarrow \tau M \longrightarrow E \longrightarrow M \longrightarrow 0$$

in $\mathrm{GMod}\Lambda$*, which is contained in* $\mathrm{gmod}\Lambda$.

(2) *If $N \in \mathrm{gmod}^{-,i}\Lambda$ is indecomposable and non-injective, then there exists an almost split sequence*

$$0 \longrightarrow N \longrightarrow E \longrightarrow \tau^- N \longrightarrow 0$$

in $\mathrm{GMod}\Lambda$*, which is contained in* $\mathrm{gmod}\Lambda$.

Proof. (1) Let $M \in \mathrm{gmod}^{+,p}\Lambda$ be indecomposable and non-projective. Then, by Proposition 4.2.4(4), $\mathrm{Tr}M \in \mathrm{gmod}^{+,p}\Lambda^{\mathrm{o}}$ is indecomposable and non-projective. By Proposition 3.9.9, τM is indecomposable and non-injective in $\mathrm{gmod}^{-,i}\Lambda$. So, by Proposition 3.9.10, $\mathrm{GEnd}_\Lambda(M)$ and $\mathrm{GEnd}_\Lambda(\tau M)$ are local. Now, by Theorem 4.2.10(1), there exists a natural isomorphism

$$\Psi_M : \mathrm{GExt}^1_\Lambda(-, \tau M) \to D\underline{\mathrm{GHom}}_\Lambda(M, -).$$

In particular, $\Psi_{M,M} : \mathrm{GExt}^1_\Lambda(M, \tau M) \to D\underline{\mathrm{GEnd}}_\Lambda(M)$ is a right $\underline{\mathrm{GEnd}}_\Lambda(M)$-linear isomorphism. Since $\underline{\mathrm{GEnd}}_\Lambda(M)$ is local, by Lemma 4.2.11, $D\underline{\mathrm{GEnd}}_\Lambda(M)$ has a non-zero $\underline{\mathrm{GEnd}}_\Lambda(M)$-socle. In particular, the intersection of the image of $\Psi_{M,M}$ and the socle of $D\underline{\mathrm{GEnd}}_\Lambda(M)$ is non-zero. By Theorem 3.6 in [38], there exists an almost split sequence as stated in Statement (1).

(2) Let $N \in \mathrm{gmod}^{-,i}\Lambda$ be indecomposable and non-injective. Then, $\mathfrak{D}N \in \mathrm{gmod}^{+,p}\Lambda^{\mathrm{o}}$ is indecomposable and non-projective by Proposition 3.9.9, and thus, $\tau^- N \in \mathrm{gmod}^{+,p}\Lambda$ is indecomposable and non-projective by Proposition 4.2.4(4). So, by Proposition 3.9.10, $\mathrm{GEnd}_\Lambda(N)$ and $\mathrm{GEnd}_\Lambda(\tau^- N)$ are local. Now, by Theorem 4.2.10(2), there exists a functorial isomophism

$$\Psi_N : D^2\mathrm{GExt}^1_\Lambda(\tau^- N, -) \to D\overline{\mathrm{GHom}}_\Lambda(-, N).$$

In particular, $\Psi_N : D^2\mathrm{GExt}^1_\Lambda(\tau^- N, N) \to D\overline{\mathrm{GEnd}}_\Lambda(N)$ is a left $\overline{\mathrm{GEnd}}_\Lambda(N)$-linear isomorphism. On the other hand, it is well known that there exists a natural monomorphism $\Phi_N : \mathrm{GExt}^1_\Lambda(\tau^- N, -) \to D^2\mathrm{GExt}^1_\Lambda(\tau^- N, -)$. This yields a natural monomorphism $\Theta_N = \Phi_N \circ \Psi_N : \mathrm{GExt}^1_\Lambda(\tau^- N, -) \to D\overline{\mathrm{GHom}}_\Lambda(-, N)$. Since $\overline{\mathrm{GEnd}}_\Lambda(N)$ is finite k-dimensional by Proposition 4.2.2, so are $D\overline{\mathrm{GEnd}}_\Lambda(N)$ and $D^2\mathrm{GExt}^1_\Lambda(\tau^- N, N)$. Thus, $\Phi_{N,N} : \mathrm{GExt}^1_\Lambda(\tau^- N, N) \to D^2\mathrm{GExt}^1_\Lambda(\tau^- N, N)$ is an isomorphism. This yields a left $\overline{\mathrm{GEnd}}(N)$-linear isomorphism

$$\Theta_{N,N} : \mathrm{GExt}^1_\Lambda(\tau^- N, N) \to D\overline{\mathrm{GEnd}}_\Lambda(N).$$

Observing that $\overline{\mathrm{GEnd}}_\Lambda(N)$ is local, by Lemma 4.2.11, the left $\overline{\mathrm{GEnd}}_\Lambda(N)$-module $D\overline{\mathrm{GEnd}}_\Lambda(N)$ has a non-zero socle. In particular, the intersection of the image of $\Theta_{N,N}$ and the socle of $D\overline{\mathrm{GEnd}}_\Lambda(N)$ is non-zero. By Theorem 3.6 in [38, 3.6], there exists an almost split sequence as stated in Statement (2). The proof of the theorem is completed.

Remark. (1) To the best of our knowledge, the non-graded version of Theorem 4.2.12(2) is only known under certain finiteness condition on Λ; see [8, (V.1.15)]; and compare also [26].

(2) In case Q is a finite quiver; see , Martinez-Villa stated an almost split sequence in GModΛ as stated in Theorem 4.2.12(1); [44, (1.7.1)]. However, the proof given there shows only that the almost split sequence is in gmodΛ; see [44, (1.6.1)]; compare also [19, (3.5)].

As a special case of Theorem 4.2.12, $\mathrm{gmod}^{+,p}\Lambda$ has almost split sequences on the left for finite dimensional modules, and $\mathrm{gmod}^{-,i}\Lambda$ has almost split sequences on the right for finite dimensional modules.

4.2.13 **Corollary**. *Let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver, and let M be an indecomposable finite dimensional module in* GModΛ.

(1) *If M is non-projective, then* $\mathrm{gmod}^{-,i}\Lambda$ *has an almost split sequence*

$$0 \longrightarrow \tau M \longrightarrow E \longrightarrow M \longrightarrow 0.$$

(2) *If M is non-injective, then* $\mathrm{gmod}^{+,p}\Lambda$ *has an almost split sequence*

$$0 \longrightarrow M \longrightarrow E \longrightarrow \tau^{-}M \longrightarrow 0.$$

Proof. We shall prove only Statement (1), since the proof of Statement (2) is dual. By Proposition 3.9.10, $M \in \mathrm{gmod}^{+,p}\Lambda \cap \mathrm{gmod}^{-,i}\Lambda$. If M is not projective, by Theorem 4.2.12, there exists an almost split sequence

$$0 \longrightarrow \tau M \longrightarrow L \longrightarrow M \longrightarrow 0$$

in gmodΛ, where $\tau M \in \mathrm{gmod}^{-,i}\Lambda$ by Proposition 4.2.7. Since $L \in \mathrm{gmod}^{-,i}\Lambda$ by Proposition 3.9.10, this is an almost split sequence in $\mathrm{gmod}^{-,i}\Lambda$. The proof of the corollary is completed.

We shall strengthen the existence of almost split sequences in $\mathrm{gmod}^{+,p}\Lambda$ or $\mathrm{gmod}^{-,i}\Lambda$ under the locally right or left bounded setting.

4.2.14 **Theorem**. *Let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver.*

(1) *If Λ is locally right bounded, then* $\mathrm{gmod}^{+,p}\Lambda$ *has almost split sequences on the right; and an indecomposable non-injective module is the starting term of an almost split sequence if and only if it is finite dimensional.*

(2) *If Λ is locally left bounded, then* $\mathrm{gmod}^{-,i}\Lambda$ *has almost split sequences on the left*; *and an indecomposable non-projective module is the ending term of an almost split sequence if and only if it is finite dimensional.*

(3) *If Λ is locally bounded, then* $\mathrm{gmod}^{b}\Lambda$ *has almost split sequences.*

Proof. We shall prove only Statement (1), since Statement (2) is dual and Statement (3) follows immediately from the first two statements. Assume that Λ is locally right bounded. By Proposition 3.9.6(2), $\mathrm{gmod}^{-,i}\Lambda = \mathrm{gmod}^{b}\Lambda$, and hence, by Proposition 3.9.10, $\mathrm{gmod}^{-,i}\Lambda \subseteq \mathrm{gmod}^{+,p}\Lambda$.

Now, let $M \in \mathrm{gmod}^{+,p}\Lambda$ be indecomposable. If M is not graded projective, then, by Theorem 4.2.12, there is an almost split sequence

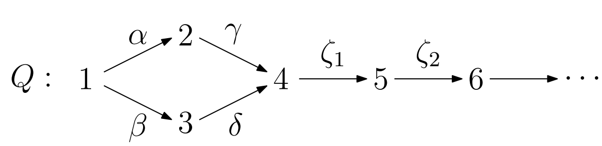
$$0 \longrightarrow \tau M \longrightarrow L \longrightarrow M \longrightarrow 0$$

in $\mathrm{gmod}\Lambda$. Since $\tau M \in \mathrm{gmod}^{-,i}\Lambda \subseteq \mathrm{gmod}^{+,p}\Lambda$, this is an almost split sequence in $\mathrm{gmod}^{+,p}\Lambda$. So, the first part of Statement (1) holds. And the sufficiency of the second part follows directly from Corollary 4.2.13(2). Assume that there exists an almost split sequence

$$0 \longrightarrow M \longrightarrow L \longrightarrow N \longrightarrow 0$$

in $\mathrm{gmod}^{+,p}\Lambda$. In particular, $N \in \mathrm{gmod}^{+,p}\Lambda$, and hence, $M = \tau N \in \mathrm{gmod}^{-,i}\Lambda$. By Proposition 3.9.10, M is finite dimensional. The proof of the theorem is completed.

Example. Consider the graded algebra $\Lambda = kQ/R$, where



and $R = k\langle \gamma\alpha - \delta\beta \rangle$. Since Q has no infinite path with an ending point, Λ is locally right bounded. By Theorem 4.2.14(1), $\mathrm{gmod}^{-,i}\Lambda$ has almost split sequences on the right. Clearly, S_1 has a minimal graded projective resolution

$$0 \longrightarrow P_4 \xrightarrow{P[\bar\delta - \bar\gamma]} P_2 \oplus P_3 \xrightarrow{\left(P[\bar\alpha]\ \ P[\bar\beta]\right)} P_1 \longrightarrow S_1 \longrightarrow 0.$$

Then, $\mathrm{rad}P_1 \in \mathrm{gmod}^{+,p}\Lambda$ is not graded injective. Moreover, since $\mathrm{rad}^2 P_1 \cong P_4$, we see that $\mathrm{rad}P_1$ is indecomposable of infinite dimension. By Theorem 4.2.14(1), there exists no almost split sequence in $\text{gmod}^{+,p}\Lambda$ starting with $\text{rad}P_1$. Thus, $\text{gmod}^{+,i}\Lambda$ does not have almost split sequences on the left. In a dual fashion, one can construct graded algebras Λ such that $\text{gmod}^{-,i}\Lambda$ has almost split sequences on the left but not on the right.

In case Q is a strongly locally finite quiver, the existence of almost split sequences in the category of finitely presented representations has been studied in [11, (3.7)]. As a special case of Theorem 4.2.14, we obtain the following result.

4.2.15 **Theorem**. *Let Q be a locally finite quiver.*

(1) *If Q has no infinite path with an ending point, then $\text{gmod}^{+,p}(kQ)$ has almost split sequences on the right.*

(2) *If Q has no infinite path with a starting point, then $\text{gmod}^{-,i}(kQ)$ has almost split sequences on the left.*

(3) *If Q has no infinite path, then $\text{gmod}^{b}(kQ)$ has almost split sequences.*

Proof. If Q has no infinite path with an ending point(respectively, starting point), then kQ is locally right (respectively, left) bounded. Thus, Statements (1) and (2) follow from Theorem 4.2.14 (1) and (2), respectively. Finally, Statement (3) follows immediately from the first two statements. The proof of the theorem is completed.

4.3 Graded almost split triangles

The objective of this section is to study the existence of almost split triangles in the derived categories of graded modules.

4.3.1 **Lemma**. *Let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver.*

(1) *The categories $K^b(\text{gproj}\Lambda)$ and $K^b(\text{ginj}\Lambda)$ are Hom-finite and Krull-Schmidt.*

(2) *The Nakayama functor induces two mutually quasi-inverse triangle equivalences $\nu : K^b(\text{gproj}\Lambda) \to K^b(\text{ginj}\Lambda)$ and $\nu^- : K^b(\text{ginj}\Lambda) \to K^b(\text{gproj}\Lambda)$.*

Proof. Since gprojΛ is Hom-finite and Krull-Schmmidt; see (3.9.1), $C^b(\text{gproj}\Lambda)$ is Hom-finite and closed under taking direct summands. In particular, the idempotents in $C^b(\text{gproj}\Lambda)$ split. Hence, $C^b(\text{gproj}\Lambda)$ is Krull-Schmidt. And consequently, $K^b(\text{gproj}\Lambda)$ is Hom-finite and Krull-Schmidt; see [36, page 431]. Next, the Nakayama functor $\nu : \text{gproj}\Lambda \to \text{GMod}\Lambda$ induces two mutually quasi-inverse equivalences $\nu : \text{gproj}\Lambda \to \text{ginj}\Lambda$ and $\nu^- : \text{ginj}\Lambda \to \text{gproj}\Lambda$; see (4.1.3). Applying them componentwise, we obtain two mutually quasi-inverse triangle equivalences $\nu : K^b(\text{gproj}\Lambda) \to K^b(\text{ginj}\Lambda)$ and $\nu^- : K^b(\text{ginj}\Lambda) \to K^b(\text{gproj}\Lambda)$. The proof of the lemma is completed.

Note that $K^b(\text{gproj}\Lambda)$ and $K^b(\text{ginj}\Lambda)$ are full triangulated subcategories of each of $D^b(\text{gmod}\Lambda)$, $D(\text{gmod}\Lambda)$ and $D(\text{GMod}\Lambda)$ by Lemma 1.7.9. On the other hand, $D(\text{gmod}\Lambda)$ is not necessarily a triangulated subcategory of $D(\text{GMod}\Lambda)$.

4.3.2 **Theorem**. *Let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver. If $P^\bullet \in K^b(\text{gproj}\Lambda)$ is indecomposable, then there exists an almost split triangle*

$$\nu P^\bullet[-1] \longrightarrow M^\bullet \longrightarrow P^\bullet \longrightarrow \nu P^\bullet$$

in each of $D^b(\text{gmod}\Lambda)$, $D(\text{gmod}\Lambda)$ and $D(\text{GMod}\Lambda)$.

Proof. Let $P^\bullet$ be an indecomposable complex in $K^b(\text{gproj}\Lambda)$. By Lemma 4.3.1, both $P^\bullet$ and $\nu P^\bullet$ are strongly indecomposable. Consider the Nakayama functors $\nu : \text{gproj}\Lambda \to \text{gmod}\Lambda$ and $\nu : \text{gproj}\Lambda \to \text{GMod}\Lambda$; see (4.1.3). By the result stated in [38, (5.8)], we obtain a desired almost split triangle in each of $D^b(\text{gmod}\Lambda)$, $D(\text{gmod}\Lambda)$, and $D(\text{GMod}\Lambda)$. The proof of the theorem is completed.

We shall study the existence of almost split triangles in the bounded derived category of piecewise finite dimensional graded Λ-modules for bounded complexes of finitely generated Λ-modules and for bounded complexes of finitely cogenerated Λ-modules.

4.3.3 **Lemma**. *Let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver.*

(1) *If Λ is locally left noetherian, then $D^b(\text{gmod}^{+,b}\Lambda)$ is a full Krull-Schmidt triangulated subcategory of $D(\text{gmod}\Lambda)$.*

(2) *If Λ is locally right noetherian, then $D^b(\text{gmod}^{-,b}\Lambda)$ is a full Krull-Schmidt triangulated subcategory of $D(\text{gmod}\Lambda)$.*

Proof. Suppose that Λ is locally left noetherian. By Proposition 3.9.5(1), $\mathrm{gmod}^{+,b}\Lambda$ is an abelian subcategory of $\mathrm{gmod}\Lambda$, and by Corollary 3.8.6, every module in $\mathrm{gmod}^{+,b}\Lambda$ has a graded projective resolution over $\mathrm{gproj}\Lambda$. Thus, $D^b(\mathrm{gmod}^{+,b}\Lambda)$ is a full triangulated subcategory of $D(\mathrm{gmod}\Lambda)$; see [9, (1.11)]. Given $M, N \in \mathrm{gmod}^{+,b}\Lambda$, we deduce from Lemma 3.9.2 that $\mathrm{GExt}^i_\Lambda(M, N)$ is finite dimensional for all $i \geq 0$. As a consequence, $D^b(\mathrm{gmod}^{+,b}\Lambda)$ is Hom-finite and Krull-Schmidt; see [32, Corollary B]. This establishes Statement (1).

Suppose that Λ is locally right noetherian. By Proposition 3.9.3, we have a duality $\mathfrak{D} : \mathrm{gmod}^{+,b}\Lambda^{\mathrm{o}} \to \mathrm{gmod}^{-,b}\Lambda$. Thus, Statement (2) follows from Statement (1). The proof of the lemma is completed.

The following result is analogous to Happel's result under the finite dimensional non-graded setting; see [28, (2.3)].

4.3.4 **Theorem**. *Let $\Lambda = kQ/R$ be a graded algebra with Q a locally finite quiver such that Λ is locally left and right noetherian.*

(1) *Given an indecomposable complex $M^\bullet \in D^b(\mathrm{gmod}^{+,b}\Lambda)$, there exists an almost split triangle $N^\bullet \longrightarrow L^\bullet \longrightarrow M^\bullet \longrightarrow N^\bullet[1]$ in $D^b(\mathrm{gmod}\Lambda)$ if and only if $M^\bullet$ has a finite projective resolution over* $\mathrm{gproj}\Lambda$; *and in this case,* $N^\bullet \in D^b(\mathrm{gmod}^{-,b}\Lambda)$.

(2) *Given an indecomposable complex $N^\bullet \in D^b(\mathrm{gmod}^{-,b}\Lambda)$, there exists an almost split triangle $N^\bullet \longrightarrow L^\bullet \longrightarrow M^\bullet \longrightarrow N^\bullet[1]$ in $D^b(\mathrm{gmod}\Lambda)$ if and only if $N^\bullet$ has a finite injective coresolution over* $\mathrm{ginj}\Lambda$; *and in this case,* $M^\bullet \in D^b(\mathrm{gmod}^{+,b}\Lambda)$.

Proof. We shall prove only Statement (1). Assume that $M^\bullet$ is an indecomposable complex in $D^b(\mathrm{gmod}^{+,b}\Lambda)$. Since $\mathrm{gmod}^{+,b}\Lambda$ is abelian with enough projective objects, $M^\bullet$ has a truncated projective resolution $P^\bullet \in C^-(\mathrm{gproj}\Lambda)$; see [25, (7.5)]. If $P^\bullet \in C^b(\mathrm{gproj}\Lambda)$, then $M^\bullet \cong P^\bullet$ in $D^b(\mathrm{gmod}\Lambda)$. By Theorem 4.3.2, there exists an almost split triangle

$$\nu N^\bullet \longrightarrow L^\bullet \longrightarrow M^\bullet \longrightarrow \nu N^\bullet[1]$$

in $D^b(\mathrm{gmod}\Lambda)$, where $N = \nu P^\bullet[-1]$ is a complex over $\mathrm{ginj}\Lambda \subseteq \mathrm{gmod}^{-,b}\Lambda$. Conversely, if there exists an almost split triangle

$$N^\bullet \longrightarrow L^\bullet \longrightarrow M^\bullet \longrightarrow N^\bullet[1]$$

in $D^b(\text{gmod}\Lambda)$, then it follows from Theorem 5.2 in [38] that $M^\bullet$ has a bounded projective resolution over $\text{gproj}\Lambda$. The proof of the theorem is completed.

REMARK. In case Λ is multi-serial, it is locally right and left noetherian. Hence, both statements in Theorem 4.3.4 hold.

Next, we shall study the existence of almost split triangles in the bounded derived category of finite dimensional graded Λ-modules. The following statement is analogous to Happel's result stated in [28, (1.5)].

4.3.5 **Theorem**. *Let $\Lambda = kQ/R$ be a locally bounded graded algebra, where Q is a locally finite quiver. Consider an indecomposable complex $M^\bullet$ in $D^b(\text{gmod}^b\Lambda)$.*

(1) *There exists in $D^b(\text{gmod}^b\Lambda)$ an almost split triangle*

$$N^\bullet \longrightarrow L^\bullet \longrightarrow M^\bullet \longrightarrow N^\bullet[1]$$

if and only if $M^\bullet$ has a finite projective resolution over $\text{gproj}\Lambda$.

(2) *There exists in $D^b(\text{gmod}^b\Lambda)$ an almost split triangle*

$$M^\bullet \longrightarrow L^\bullet \longrightarrow N^\bullet \longrightarrow M^\bullet[1]$$

if and only if $M^\bullet$ has a finite injective coresolution over $\text{ginj}\Lambda$.

Proof. Since Λ is locally bounded, both $\text{gproj}\Lambda$ and $\text{ginj}\Lambda$ are contained in $\text{gmod}^b\Lambda$. Thus, $\text{gmod}^b\Lambda$ is an abelian category with enough projective objects and enough injective objects. Thus, the necessity stated in Statements (1) and (2) follow immediately from Corollary 5.3 in [38]. On the other hand, by Theorem 4.1.3, we have a Nakayama functor $\nu : \text{gproj}\Lambda \to \text{gmod}^b\Lambda$, which restricts to an equivalence $\nu : \text{gproj}\Lambda \to \text{ginj}\Lambda$. Now the sufficiency stated in Statements (1) and (2) follow from Theorem 5.8 in [38]. The proof of the theorem is completed.

To conclude this section, we shall specialize in the case where $\Lambda = kQ$.

4.3.6 **Theorem**. *Let Q be a locally finite quiver.*

(1) *If Q has no infinite path with an ending point, then $D^b(\text{gmod}^{+,p}(kQ))$ has almost split triangles on the right.*

(2) *If Q has no infinite path with a starting point, then $D^b(\mathrm{gmod}^{-,i}(kQ))$ has almost split sequences on the left.*

(3) *If Q has no infinite path, then $\mathrm{gmod}^b(kQ)$ has almost split triangles.*

Proof. Suppose that Q has no infinite path with an ending point. Then kQ is locally right bounded. In view of Proposition 3.9.12, we see that $\mathrm{gmod}^{+,p}(kQ)$ is a hereditary abelian subcategory of $\mathrm{gmod}\Lambda$ with enough projective objects. Since $\mathrm{ginj}\Lambda$ is contained in $\mathrm{gmod}^{+,p}(kQ)$, by Theorem 4.1.3, we have a Nakayama functor $\nu : \mathrm{gproj}\Lambda \to \mathrm{gmod}^{+,p}(kQ)$. Let $M^\bullet$ be an indecomposable complex in $D^b(\mathrm{gmod}^{+,p}(kQ))$. Since $\mathrm{gmod}^{+,p}(kQ)$ is hereditary, $M^\bullet$ is isomorphic to a stalk complex; see [33, (3.1)]. By Proposition 3.9.12, $M^\bullet \cong P^\bullet$, where $P^\bullet$ is a 2-term complex over $\mathrm{gproj}\Lambda$. By Theorem 5.8 in [38], $D^b(\mathrm{gmod}^{+,p}(kQ))$ has an almost split triangle

$$\nu P^\bullet \longrightarrow L^\bullet \longrightarrow M^\bullet \longrightarrow \nu P^\bullet[1].$$

This establishes Statement (1). Dually, Statement (2) holds. Finally, if Q has no infinite path, then $\mathrm{gmod}^{+,p}(kQ) = \mathrm{gmod}^{-,i}(kQ) = \mathrm{gmod}^b(kQ)$. Therefore, Statement (3) follows immediately from the first two statements. The proof of the theorem is completed.

Chapter 5

Koszul algebras

The objective of this chapter is to provide a combinatorial description of the local Koszul complexes and the quadratic dual. Using this local viewpoint, we can describe the linear projective resolution and the colinear injective coresolution of a graded simple module in terms of subspaces of the quadratic dual if they exist. This enables us to show that a quadratic algebra is Koszul if and only if every graded simple module has a colinear injective coresolution if and only if the opposite algebra or the quadratic dual is Koszul. This generalizes Beilinson, Ginzburg and Soergel's results stated in [13, (2.2.1), (2.9.1)]. We shall also include two applications: a new class of Koszul algebras and a stronger version of the Extension Conjecture for finite dimensional Koszul algebras with a noetherian Koszul dual.

5.1 Linear projective resolutions and colinear injective coresolutions

In this section, let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver. We shall introduce the notions of linear projective n-presentation and colinear n-copresentations.

5.1.1 **Definition**. Let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver. Consider $M \in \mathrm{gmod}\Lambda$ and $n \geq 1$.

(1) In case M is generated in degree s, a projective n-presentation

$$P^{-n} \xrightarrow{d^{-n}} P^{1-n} \longrightarrow \cdots \longrightarrow P^{-1} \xrightarrow{d^{-1}} P^{0} \xrightarrow{d^{0}} M \longrightarrow 0$$

of M over gprojΛ is called **linear** in case P^{-i} is generated in degree $s+i$ for $i=0,\ldots,n$, and d^{-n} is right minimal.

(2) In case M is cogenerated in degree $-t$, an injective n-copresentation

$$0 \longrightarrow M \xrightarrow{d^0} I^0 \xrightarrow{d^1} I^1 \longrightarrow \cdots \longrightarrow I^{n-1} \xrightarrow{d^n} I^n$$

of M over ginjΛ is called **colinear** in case I^i is cogenerated in degree $-t-i$, for $i=0,\ldots,n$, and d^n is left minimal.

REMARK. It is easy to see that a linear projective n-presentation and a colinear injective n-copresentation are minimal.

The following statement is important for our later investigation.

5.1.2 **Lemma**. *Let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver. Consider $M \in$ gmodΛ. A sequence*

$$P^{-n} \xrightarrow{d^{-n}} P^{-n+1} \longrightarrow \cdots \longrightarrow P^{-1} \xrightarrow{d^{-1}} P^0 \xrightarrow{d^0} M \longrightarrow 0$$

is a linear projective n-presentation of M over gprojΛ *if and only if the sequence*

$$0 \longrightarrow \mathfrak{D}M \xrightarrow{\mathfrak{D}d^0} \mathfrak{D}P^0 \xrightarrow{\mathfrak{D}d^{-1}} \mathfrak{D}P^{-1} \longrightarrow \cdots \longrightarrow \mathfrak{D}P^{1-n} \xrightarrow{\mathfrak{D}d^{-n}} \mathfrak{D}P^{-n}$$

is a colinear injective n-copresentation of $\mathfrak{D}M$ over ginj$\Lambda^{\rm o}$.

Proof. By Proposition 3.3.3(2), one of the two sequences stated in the lemma is exact if and only if the other one is exact. By Lemma 3.3.4, d^{-n} is right minimal if and only if $\mathfrak{D}d^{-n}$ is left minimal. Moreover, by Corollary 3.6.8, M is generated in degree s if and only if $\mathfrak{D}M$ is cogenerated in degree $-s$, and P^{-i} is generated in degree $s+i$ if and only if $\mathfrak{D}P^{-i}$ is cogenerated in degree $-s-i$, for $i=0,1,\ldots,n$. The proof of the lemma is completed.

As an example, we have the following well known statement. For the convenience of the reader, we will provide a brief proof.

5.1.3 **Lemma**. *Let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver. Consider a graded simple module S_a with $a \in Q_0$.*

(1) *If $Q_1(a,-) = \{\alpha_i : a \to b_i \mid i = 1,\ldots,r\}$, then S_a admits a linear projective presentation*

$$P_{b_1}\langle -1\rangle \oplus \cdots \oplus P_{b_r}\langle -1\rangle \xrightarrow{(P[\bar\alpha_1],\cdots,P[\bar\alpha_r])} P_a \xrightarrow{p_a} S_a \longrightarrow 0.$$

(2) *If* $Q_1(-,a)=\{\beta_j : c_j \to a \mid j = 1, \ldots, s\}$, *then* S_a *admits a colinear injective copresentation*

$$0 \longrightarrow S_a \xrightarrow{q_a} I_a \xrightarrow{(I[\bar{\beta}_1], \cdots, I[\bar{\beta}_s])^T} I_{c_1}\langle 1\rangle \oplus \cdots \oplus I_{c_s}\langle 1\rangle.$$

Proof. (1) Assume that $Q_1(a,-) = \{\alpha_i : a \to b_i \mid i = 1, \ldots, r\}$. Let $p_a : P_a \to S_a$ be the canonical projection. In particular, $\mathrm{Ker}(p_a) = \mathrm{rad}P_a$. Clearly, $\mathrm{rad}P_a$ has a top-basis $\{\bar{\alpha}_1, \cdots, \bar{\alpha}_r\}$, where $\bar{\alpha}_i \in (\mathrm{rad}P_a)_1(b_i)$. By Proposition 3.8.5(1), we have a graded projective cover $f : P = P_{b_1}\langle -1\rangle \oplus \cdots \oplus P_{b_r}\langle -1\rangle \to \mathrm{Ker}(p_a)$ such that $(P[\bar{\alpha}_1], \cdots, P[\bar{\alpha}_r]) = g \circ f$, where $g : \mathrm{Ker}(p_a) \to P_a$ is the inclusion morphism. Moreover, since P is generated in degree 1, we obtain a desired linear projective presentation of S_a.

(2) By Lemma 5.1.2, applying $\mathfrak{D}$ to the linear projective presentation of S_a^{o} yields a desired colinear injective copresentation of S_a. The proof of the lemma is completed

For our purpose, we quote the following statement from [16, (2.13)]; see also [15].

5.1.4 **Theorem**. *Let* $\Lambda = kQ/R$ *be a graded algebra with* Q *a locally finite quiver. Then* Λ *is a quadratic algebra if and only if every graded simple* Λ*-module admits a linear projective* 2*-presentation.*

Now, we introduce the notions of a linear resolution and a colinear coresolution.

5.1.5 **Definition**. Let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver. Consider a module $M \in \mathrm{gmod}\Lambda$.

(1) In case M is generated in degree s, a graded projective resolution

$$\cdots \longrightarrow P^{-n} \xrightarrow{d^{-n}} P^{1-n} \longrightarrow \cdots \longrightarrow P^{-1} \xrightarrow{d^{-1}} P^0 \xrightarrow{d^0} M \longrightarrow 0$$

of M over $\mathrm{gproj}\Lambda$ is called **linear** if P^{-n} is generated in degree $s+n$ for all $n \geq 0$.

(2) In case M is cogenerated in degree $-t$, a graded injective coresolution

$$0 \longrightarrow I^0 \xrightarrow{d^1} I^1 \longrightarrow \cdots \longrightarrow I^{n-1} \xrightarrow{d^n} I^n \longrightarrow \cdots$$

of M over $\mathrm{ginj}\Lambda$ is called **colinear** if I^n is cogenerated in degree $-t-n$ for all $n \geq 0$.

Remark. It is easy to see that a linear projective resolution and a colinear injective coresolution are minimal.

We are ready to recall the notion of a Koszul algebra from [16, (2.14)], which is essentially the same as the classical one; see [13, (1.2.1)] and [48, (5.4)].

5.1.6 **Definition**. Let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver. We shall call Λ a **Koszul algebra** if every graded simple Λ-module admits a linear projective resolution over $\text{proj}\Lambda$.

Remark. (1) By Theorem 5.1.4, a Koszul algebra is quadratic; compare [13, (2.3.3)].
(2) It is clear that Λ is Koszul if and only if S_a admits a linear projective resolution over $\text{proj}\Lambda$, for every $a \in Q_0$.

Example. The path algebra kQ of any locally finite quiver Q is a Koszul algebra. Indeed, for any $a \in Q_0$ with $Q_1(a, -) = \{\alpha_i : a \to b_i \,|\, i = 1, \ldots, r\}$, we see that

$$\cdots \longrightarrow 0 \longrightarrow P_{b_1}\langle -1\rangle \oplus \cdots \oplus P_{b_r}\langle -1\rangle \xrightarrow{(P[\bar{\alpha}_1], \cdots, P[\bar{\alpha}_r])} P_a \longrightarrow S_a \longrightarrow 0$$

is a linear projective resolution of S_a.

5.2 Local Koszul complexes

Most of the content of this section is taken with a slight modification from [16, Section 2]; see also [15]. The main objective is to describe explicitly the local Koszul complexes. Throughout this section, $\Lambda = kQ/R$ is a quadratic algebra, where Q is a locally finite quiver.

We start with some notations and terminology. Given $\alpha : y \to x \in Q_1$, we have a **left derivation** $\partial_\alpha : kQ \to kQ$, the k-linear map sending a path ρ to δ if $\rho = \alpha\delta$; and to 0 if α is not a terminal arrow of ρ; and a **right derivation** $\partial^\alpha : kQ \to kQ$, the k-linear map sending a path ρ to δ if $\rho = \delta\alpha$; and to 0 if α is not an initial arrow of ρ. In particular, ∂_α and ∂^α send kQ_n to kQ_{n-1} for $n > 0$ and vanishes on kQ_0.

Fix $a \in Q_0$ and $n > 0$. For $\alpha \in Q_1(y, x)$, we have a graded Λ-linear morphism $P[\bar{\alpha}] : P_x\langle -n\rangle \to P_y\langle 1 - n\rangle$ and a k-linear map $\partial_\alpha : kQ_n(a, x) \to kQ_{n-1}(a, y)$. In view of Lemma 2.1.1, we obtain a k-linear map

$$\partial_a^{-n}(y,x) = \sum_{\alpha\in Q_1(y,x)} P[\bar{\alpha}]\otimes\partial_\alpha : P_x\langle -n\rangle \otimes kQ_n(a,x) \to P_y\langle 1{-}n\rangle \otimes kQ_{n-1}(a,y),$$

which is clearly a morphism in gprojΛ, for any $x, y \in Q_0$. The following statement is useful for later calculation, which is quoted from [16, (3.2)].

5.2.1 **Lemma**. *Let $\Lambda = kQ/R$ be a quadratic algebra, where Q is a locally finite quiver. Consider the graded morphism $\partial_a^{-n}(y,x)$ with $a, x, y \in Q_0$ and $n > 0$ as defined above. Given $u \in P_x\langle -n\rangle$, $\delta \in kQ_{n-1}(a,y)$ and $\zeta \in kQ_1(y,x)$, we have*

$$\partial_a^{-n}(y,x)(u\otimes\zeta\delta) = u\bar{\zeta}\otimes\delta.$$

Fix $a, x \in Q_0$. We put $R^{(n)}(a,x) = kQ_n(a,x)$ for $n = 0, 1$, and

$$R^{(n)}(a,x) = \cap_{0\le j\le n-2}\, kQ_{n-2-j}(-,x)\cdot R_2\cdot kQ_j(a,-)$$

for $n \ge 2$. In particular, $R^{(2)}(a,x) = R_2(a,x)$. Put $R^{(n)}(a,-) = \oplus_{x\in Q_0} R^{(n)}(a,x)$ for $n \ge 0$. The following statement collects some important properties of these subspaces.

5.2.2 **Lemma**. *Let R be a quadratic ideal of kQ, where Q is a locally finite quiver. Consider $a, x \in Q_0$ with $Q_1(-,x) = \{\alpha_i : y_i \to x \mid i = 1, \ldots, r\}$ and $n \ge 1$.*

1. *If $\gamma \in R^{(n)}(a,x)$ and $\alpha \in Q_1(y,x)$, then $\partial_\alpha(\gamma) \in R^{(n-1)}(a,y)$; and consequently, $\gamma = \sum_{i=1}^r \alpha_i\gamma_i$, for some $\gamma_i \in R^{(n-1)}(a,y_i)$.*
2. *If $\rho = \sum_{i=1}^r \zeta_i\rho_i$ with $\rho_i \in R^{(n-1)}(a,y_i)$ and $\zeta_i \in kQ_1(y_i,x)$, then $\rho \in R^{(n)}(a,x)$ if and only if $\rho \in R_2(-,x)\cdot kQ_{n-2}(a,-)$.*

Proof. The proof of the first part of Statement (1) is presented in [16, (3.1)], and the second part follows immediately from the first part. Moreover, Statement (2) follows directly from the definition of $R^{(n)}(a,x)$. The proof of the lemma is completed.

Fix $a \in Q_0$. Given $n \ge 0$, since Q is locally finite, we obtain a module

$$\mathcal{K}_a^{-n} = \oplus_{x\in Q_0} P_x\langle -n\rangle \otimes R^{(n)}(a,x) \in \text{gproj}\Lambda.$$

For $n \geq 1$, we write $\mathcal{K}_a^{1-n} = \oplus_{y \in Q_0} P_y\langle 1{-}n\rangle \otimes R^{(n-1)}(a,y)$; and by Lemma 5.2.2(1), we obtain a graded morphism

$$\partial_a^{-n}(y,x) = \textstyle\sum_{\alpha \in Q_1(y,x)} P[\bar{\alpha}] \otimes \partial_\alpha : P_x\langle -n\rangle \otimes R^{(n)}(a,x) \to P_y\langle 1{-}n\rangle \otimes R^{(n-1)}(a,y),$$

for $x, y \in Q_0$. This yields a graded Λ-linear morphism

$$\partial_a^{-n} = (\partial_a^{-n}(y,x))_{(y,x) \in Q_0 \times Q_0} : \mathcal{K}_a^{-n} \to \mathcal{K}_a^{1-n},$$

for $n \geq 1$. The following statement is quoted from [16, (3.3)], see also [15].

5.2.3 **Lemma**. *Let $\Lambda = kQ/R$ be a quadratic algebra, where Q is a locally finite quiver. Given any $a \in Q_0$, the above construction yields a complex*

$$\mathcal{K}_a^{\bullet}: \qquad \cdots \longrightarrow \mathcal{K}_a^{-n} \xrightarrow{\partial_a^{-n}} \mathcal{K}_a^{1-n} \longrightarrow \cdots \longrightarrow \mathcal{K}_a^{-1} \xrightarrow{\partial_a^{-1}} \mathcal{K}_a^{0} \longrightarrow 0 \longrightarrow \cdots$$

over gprojΛ *such that* $\mathrm{Ker}(\partial_a^{-n}) \subseteq \mathrm{rad}\mathcal{K}_a^{-n}$, *if $n > 0$; and otherwise, $\mathcal{K}_a^{-n} = 0$.*

In the sequel, we shall call $\mathcal{K}_a^{\bullet}$ the **local Koszul complex** at a for Λ. Since $\mathcal{K}_a^0 = P_a \otimes k\varepsilon_a$, we have a graded projective cover $\partial_a^0 : \mathcal{K}_a^0 \to S_a$, sending $e_a \otimes \varepsilon_a$ to $e_a + \mathrm{rad}P_a$. The following statement is a reformulation of Theorem 3.4 in [16]; see also [15].

5.2.4 **Proposition**. *Let $\Lambda = kQ/R$ be a quadratic algebra, where Q is a locally finite quiver. If $a \in Q_0$ and $n > 0$, then S_a has a linear projective n-presentation over* gprojΛ *if and only if*

$$\mathcal{K}_a^{-n} \xrightarrow{\partial_a^{-n}} \mathcal{K}_a^{1-n} \longrightarrow \cdots \longrightarrow \mathcal{K}_a^{-1} \xrightarrow{\partial_a^{-1}} \mathcal{K}_a^{0} \xrightarrow{\partial_a^{0}} S_a \longrightarrow 0$$

is a linear projective n-presentation of S_a.

Combining Theorem 5.1.4 and Proposition 5.2.4, we obtain immediately the following result; compare [13, (2.6.1)].

5.2.5 **Theorem**. *Let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver. Then Λ is Koszul if and only if Λ is quadratic and $\mathcal{K}_a^{\bullet}$ is a graded projective resolution of S_a, for every $a \in Q_0$.*

5.3 Quadratic dual

Throughout this section, $\Lambda = kQ/R$ is a quadratic algebra, where Q is a locally finite quiver. We shall recall the definition of the quadratic dual of Λ from [16, Section 3]. Note that the classical quadratic dual is defined to be the tensor algebra of the dual space of the generating space under a finiteness condition; see [13, (1.2.4), (2.8.1)].

We start with some notation. Fix $n \geq 0$. Given $\xi \in Q_n$, let $\xi^* \in D(kQ_n)$ such that $\xi^*(\eta) = 1$ if $\eta = \xi$; and $\xi^*(\eta) = 0$ otherwise, for any $\eta \in Q_n$. Given $\gamma = \sum \lambda_i \xi_i \in kQ_n$ with $\lambda_i \in k$ and $\xi_i \in Q_n$, we write $\gamma^* = \sum \lambda_i \xi_i^*$. This yields a k-linear isomorphism

$$\psi_n : kQ_n^{\rm o} \to D(kQ_n) : \gamma^{\rm o} \to \gamma^*.$$

Given $\xi \in kQ_n(x, y)$, for the sake of simplicity, the restriction of ξ^* to $kQ_n(x, y)$ is also written as ξ^*. Since Q is locally finite, $\{\xi^* \mid \xi \in Q_n(x, y)\}$ is the dual basis of $Q_n(x, y)$ in $D(kQ_n(x, y))$. We shall need the following easy statement for later calculation, which is quoted from [16, (3.5)].

5.3.1 **Lemma**. *Let Q be a locally finite quiver with $x, y, z \in Q_0$ and $s, t \geq 0$.*

(1) *If $\xi \in kQ_s(x, y)$ and $\zeta \in Q_1(y, z)$, then $(\zeta\xi)^*(\eta) = \xi^*(\partial_\zeta(\eta))$, for all $\eta \in kQ_{s+1}$.*

(2) *If $\xi \in kQ_s(x, y)$ and $\zeta \in kQ_t(y, z)$, then $(\zeta\xi)^*(\gamma\delta) = \zeta^*(\gamma)\xi^*(\delta)$, for all $\delta \in kQ_s$ and $\gamma \in kQ_t$.*

Given $x, y \in Q_0$ and $n \geq 2$, we denote by $R_2^!(y, x)$ the k-vector subspace of $kQ_2^{\rm o}(y, x)$ spanned by the elements $\rho^{\rm o}$ with $\rho \in kQ_2(x, y)$ such that ρ^* vanishes on $R_2(x, y)$. The **quadratic ideal** $R^!$ of R is the two-sided ideal of $kQ^{\rm o}$ generated by the $R_2^!(y, x)$ with $x, y \in Q_0$. We are ready to quote the following definition from [16, (3.7)]; compare [13, (2.8.1)], [42, Section 1] and [48, Section 4.1].

5.3.2 **Definition**. Let $\Lambda = kQ/R$ be a quadratic algebra, where Q is a locally finite quiver. The **quadratic dual** of Λ is the algebra

$$\Lambda^! = kQ^{\rm o}/R^!$$

where $R^!$ is the quadratic dual of R.

The following statement is quoted from [16, (3.8)].

5.3.3 **Proposition**. *Let $\Lambda = kQ/R$ be a quadratic algebra, where Q is a locally finite quiver. Then $\Lambda^!$ and $\Lambda^{\rm o}$ are quadratic with $(\Lambda^!)^! = \Lambda$ and $(\Lambda^{\rm o})^! = (\Lambda^!)^{\rm o}$.*

5.4 A characterization of Koszul algebras

In this section, we shall show that a quadratic algebra is Koszul if and only if its quadratic dual or its opposite algebra is Koszul if and only if every graded simple module admits a co-linear injective coresolution. This generalizes the result stated in [16, (3.13)] where the algebra is assumed to be locally finite dimensional and extends the results in [13, Section 2] where the graded algebra is assumed to have an identity.

Throughout this section, unless otherwise explicitly stated, $\Lambda = kQ/R$ is a quadratic algebra, where Q is a locally finite quiver. First, we reformulate an alternative description of the local Koszul complexes for Λ in terms of the quadratic dual $\Lambda^! = kQ^{\rm o}/R^!$ from [16, Section 3]. For this purpose, we fix some notations for $\Lambda^!$. We shall write $\bar\gamma^! = \gamma^{\rm o} + R^!$ for $\gamma \in kQ^+$, but $e_x = \varepsilon_x + R^!$ for $x \in Q_0$. Then, $\Lambda^!$ is graded as $\Lambda^! = \oplus_{n\in\mathbb{Z}}(\Lambda^!)_n$, where $(\Lambda^!)_n = \{\bar\gamma^! \mid \gamma \in kQ_n\}$ for $n \geq 0$, and $\Lambda^!_n = 0$ for $n < 0$. Given $x \in Q_0$, we write $P^!_x = \Lambda^! e_x$, and $S^!_x = P^!_x/{\rm rad}P^!_x$, and $I^!_x = \mathfrak{D}((\Lambda^!)^{\rm o}e_x)$.

Fix $a \in Q_0$. Given $n \in \mathbb{Z}$, we set $\mathcal{P}_a^{-n} = \oplus_{x\in Q_0}(P_x\langle -n\rangle \otimes D(e_a\Lambda^!_n e_x)) \in {\rm gproj}\Lambda$. For $n > 0$, we write $\mathcal{P}_a^{1-n} = \oplus_{y\in Q_0}(P_y\langle 1-n\rangle \otimes D(e_a\Lambda^!_{n-1}e_y))$. Given $\alpha \in Q_1(y,x)$, the right multiplication by $\bar\alpha^!$ gives rise to a k-linear map $P[\bar\alpha^!] : e_a\Lambda^!_{n-1}e_y \to e_a\Lambda^!_n e_x$, and hence, a k-linear map $DP[\bar\alpha^!] : D(e_a\Lambda^!_n e_x) \to D(e_a\Lambda^!_{n-1}e_y)$. In view of Lemma 2.1.1, we obtain a k-linear map

$$P[\bar\alpha] \otimes DP[\bar\alpha^!] : P_x\langle -n\rangle \otimes D(e_a\Lambda^!_n e_x) \to P_y\langle 1-n\rangle \otimes D(e_a\Lambda^!_{n-1}e_y),$$

which is clearly a graded Λ-linear morphism. Thus, we have a graded Λ-linear morphism

$$\ell_a^{-n}(y,x) = \textstyle\sum_{\alpha\in Q_1(y,x)} P[\bar\alpha]\otimes DP[\bar\alpha^!] : P_x\langle -n\rangle\otimes D(e_a\Lambda^!_n e_x) \to P_y\langle 1-n\rangle\otimes D(e_a\Lambda^!_{n-1}e_y)$$

for $x, y \in Q_0$. This yields a graded Λ-linear morphism

$$\ell_a^{-n} = (\ell_a^{-n}(y,x))_{(y,x)\in Q_0\times Q_0} : \mathcal{P}_a^{-n} \to \mathcal{P}_a^{1-n}$$

for $n \geq 1$. Otherwise, $\mathcal{P}_a^{-n} = 0$ for all $n < 0$. Therefore, we get a double infinite sequence

$$\mathcal{P}_a^{\bullet}: \quad \cdots \longrightarrow \mathcal{P}_a^{-n} \xrightarrow{\ell_a^{-n}} \mathcal{P}_a^{1-n} \longrightarrow \cdots \longrightarrow \mathcal{P}_a^{-1} \xrightarrow{\ell_a^{-1}} \mathcal{P}_a^{0} \longrightarrow 0 \longrightarrow \cdots,$$

of morphisms in gprojΛ. This is indeed a complex gprojΛ by the following statement, whose proof is similar to that of Lemma 3.9 in [16].

5.4.1 **Lemma**. *Let $\Lambda = kQ/R$ be a quadratic algebra, where Q is a locally finite quiver. Given $a \in Q_0$, the sequence $\mathcal{P}_a^{\bullet}$ as constructed above is isomorphic to the local Koszul complex $\mathcal{K}_a^{\bullet}$ at a.*

Next, we shall reformulate an explicit description of the colinear injective coresolution, if it exists, of a graded simple module from [16, Section 3]. For the sake of simplicity, we put $\hat{\Lambda} = (\Lambda^!)^{\rm o} = kQ/(R^!)^{\rm o}$. Write $\hat{\gamma} = \gamma + (R^!)^{\rm o}$ for $\gamma \in kQ^+$ and $e_x = \varepsilon_x + (R^!)^{\rm o}$ for $x \in Q_0$. In this way, we have $\hat{\Lambda} = \oplus_{n\in\mathbb{Z}}\hat{\Lambda}_n$, where $\hat{\Lambda}_n = \{\hat{\gamma} \mid \gamma \in kQ_n\}$ for $n \geq 0$, and $\hat{\Lambda}_n = 0$ for $n < 0$. Moreover, we put $\hat{P}_x = \hat{\Lambda}e_x$ and $\hat{S}_x = \hat{P}_x/{\rm rad}\hat{P}_x$, for all $x \in Q_0$.

Fix $a \in Q_0$. Given $n \in \mathbb{Z}$, we set $\mathcal{I}_a^n = \oplus_{x\in Q_0}(I_x\langle n\rangle \otimes e_x\Lambda_n^! e_a) \in$ ginjΛ. For $n > 0$, we write $\mathcal{I}_a^{n-1} = \oplus_{y\in Q_0}(I_y\langle n-1\rangle \otimes e_y\Lambda_{n-1}^! e_a)$. For each $\alpha \in Q_1(x,y)$, we have a morphism $I[\bar{\alpha}] : I_y\langle n-1\rangle \to I_x\langle n\rangle$ in ginjΛ; see (3.4.6) and a k-linear morphism $P_a^!(\bar{\alpha}^!) : e_y\Lambda_{n-1}^! e_a \to e_x\Lambda_n^! e_a$, that is the left multiplication by $\bar{\alpha}^!$. In view of Lemma 2.1.1, we have a k-linear map

$$I[\bar{\alpha}] \otimes P_a^!(\alpha^!) : I_y\langle n-1\rangle \otimes e_y\Lambda_{n-1}^! e_a \to I_x\langle n\rangle \otimes e_x\Lambda_n^! e_a,$$

which is clearly a morphism in ginjΛ. Thus, we have a morphism

$$d_a^n(x,y) = \textstyle\sum_{\alpha\in Q_1(x,y)} I[\bar{\alpha}] \otimes P_a^!(\alpha^!) : I_y\langle n-1\rangle \otimes e_y\Lambda_{n-1}^! e_a \to I_x\langle n\rangle \otimes e_x\Lambda_n^! e_a$$

in ginjΛ, for each $(x,y) \in Q_0 \times Q_0$. This yields a morphism

$$d_a^n = (d_a^n(x,y))_{(x,y)\in Q_0\times Q_0} : \mathcal{I}_a^{n-1} \to \mathcal{I}_a^n$$

for $n > 0$. Otherwise, $\mathcal{I}_a^n = 0$ for all $n < 0$. And consequently, we have a double infinite sequence

$$\mathcal{I}_a^{\bullet}: \qquad \cdots \longrightarrow 0 \longrightarrow \mathcal{I}_a^0 \xrightarrow{d_a^1} \mathcal{I}_a^1 \longrightarrow \cdots \longrightarrow \mathcal{I}_a^{n-1} \xrightarrow{d_a^n} \mathcal{I}_a^n \longrightarrow \cdots$$

of morphisms in ginjΛ. The following statement is a reformulation of Lemma 3.11 in [16, (3.5)] with a slightly detailed proof.

5.4.2 **Proposition**. *Let $\Lambda = kQ/R$ be a quadratic algebra, where Q is a locally finite quiver. Given $a \in Q_0$, the sequence $\mathcal{I}_a^{\bullet}$ as constructed above is a complex over* ginjΛ*, which is a truncated graded injective coresolution of S_a if and only if S_a admits a colinear injective coresolution over* ginjΛ.

Proof. Fix $a \in Q_0$. By Proposition 5.3.3, $(\Lambda^{\rm o})^! = \hat{\Lambda}$. Let $\mathcal{P}_{a^{\rm o}}^{\bullet}$ be the complex over gproj$\Lambda^{\rm o}$ for $S_a^{\rm o}$ as stated in Lemma 5.4.1, that is the complex

$$\cdots \longrightarrow \mathcal{P}_{a^{\rm o}}^{-n} \xrightarrow{\ell^{-n}} \mathcal{P}_{a^{\rm o}}^{1-n} \longrightarrow \cdots \longrightarrow \mathcal{P}_{a^{\rm o}}^{-1} \xrightarrow{\ell^{-1}} \mathcal{P}_{a^{\rm o}}^{0} \longrightarrow 0 \longrightarrow \cdots,$$

where $\mathcal{P}_{a^{\rm o}}^{-n} = \oplus_{x\in Q_0}(P_x^{\rm o}\langle -n\rangle \otimes D(e_a\hat{\Lambda}_n e_x))$, $\mathcal{P}_{a^{\rm o}}^{1-n} = \oplus_{y\in Q_0}(P_y^{\rm o}\langle 1-n\rangle \otimes D(e_a\hat{\Lambda}_{n-1} e_y))$ and $\ell^{-n} = (\sum_{\alpha\in Q_1(x,y)} P[\bar{\alpha}^{\rm o}] \otimes DP[\hat{\alpha}])_{(y,x)\in Q_0\times Q_0}$.

First, we shall show that $\mathfrak{D}(\mathcal{P}_{a^{\rm o}}^{\bullet}) \cong \mathcal{I}_a^{\bullet}$. Given any $n \geq 0$, since Q is locally finite, $e_a\hat{\Lambda}_n$ is finite dimensional. In view of Proposition 3.3.1(2), $\mathfrak{D}(\mathcal{P}_{a^{\rm o}}^{-n}) = \oplus_{x\in Q_0}(I_x\langle n\rangle \otimes D^2(e_a\hat{\Lambda}_n e_x))$ and $\mathfrak{D}(\ell^{-n}) = (I[\bar{\alpha}] \otimes D^2P[\hat{\alpha}])_{(y,x)\in Q_0\times Q_0}$. Moreover, since $\hat{\Lambda} = (\Lambda^!)^{\rm o}$, we have a k-linear isomorphism $\sigma_x^n : e_a\hat{\Lambda}_n e_x \to e_x\Lambda_n^! e_a$, sending $\hat{\gamma} \mapsto \bar{\gamma}^!$. Composing this with the canonical k-linear isomorphism $\varphi_x^n : D^2(e_a\hat{\Lambda}_n e_x) \to e_a\hat{\Lambda}_n e_x$, we obtain a k-linear isomorphism

$$\theta_x^n = \sigma_x^n \circ \varphi_x^n : D^2(e_a\hat{\Lambda}_n e_x) \to e_x\Lambda_n^! e_a.$$

Given $\alpha \in Q_1(x,y)$, it is easy to verify that we have a commutative diagram

$$\begin{array}{ccccc} D^2(e_a\hat{\Lambda}_{n-1}e_y) & \xrightarrow{\varphi_y^{n-1}} & e_a\hat{\Lambda}_{n-1}e_y & \xrightarrow{\sigma_y^{n-1}} & e_y\Lambda_{n-1}^! e_a \\ \downarrow{\scriptstyle D^2P[\hat{\alpha}]} & & \downarrow{\scriptstyle P[\hat{\alpha}]} & & \downarrow{\scriptstyle P_a^!(\bar{\alpha}^!)} \\ D^2(e_a\hat{\Lambda}_n e_x) & \xrightarrow{\varphi_x^n} & e_a\hat{\Lambda}_n e_x & \xrightarrow{\sigma_x^n} & e_x\Lambda_n^! e_a \end{array}$$

This yields a complex isomorphism $\mathfrak{D}(\mathcal{P}_{a^{\rm o}}^{\bullet}) \cong \mathcal{I}_a^{\bullet}$, given by the graded isomorphisms

$$\oplus_{x\in Q_0}(\mathrm{id} \otimes \theta_x^n) : \oplus_{x\in Q_0}(I_x\langle n\rangle \otimes D^2(e_a\hat{\Lambda}_n e_x)) \to \oplus_{x\in Q_0}(I_x\langle n\rangle \otimes e_x\Lambda_n^! e_a)$$

with $n \in \mathbb{Z}$. Now, suppose that S_a has a colinear injective coresolution $\mathcal{I}^{\bullet}$ over ginjΛ. In view of Proposition 5.1.2, we see that $\mathfrak{D}(\mathcal{I}^{\bullet})$ is a linear projective resolution of $S_a^{\rm o}$ over gproj$\Lambda^{\rm o}$. By Theorem 5.2.5 and Lemma 5.4.1, we see that $\mathfrak{D}(\mathcal{I}^{\bullet}) \cong \mathcal{P}_{a^{\rm o}}^{\bullet}$. Therefore, $\mathcal{I}^{\bullet} \cong \mathfrak{D}^2(\mathcal{I}^{\bullet}) \cong \mathfrak{D}(\mathcal{P}_{a^{\rm o}}^{\bullet}) \cong \mathcal{I}_a^{\bullet}$. The proof of the proposition is completed.

The following statement is a generalization of Theorem 3.13 in [16], which is under the assumption that $e_x\Lambda e_y$ is finite dimensional for all $x, y \in Q_0$.

5.4.3 **Theorem**. *Let $\Lambda = kQ/R$ be a graded algebra, where Q is a locally finite quiver. The following statements are equivalent.*

(1) *The algebra Λ is Koszul.*

(2) *The opposite algebra $\Lambda^{\rm o}$ is Koszul.*

(3) *The algebra Λ is quadratic and $\Lambda^!$ is Koszul.*

(4) *Every graded simple Λ-module has a colinear injective coresolution over* ginjΛ.

Proof. In view of Lemma 5.1.2, we see that Statements (4) and (2) are equivalent. Assume now that Λ is Koszul. Fix $a \in Q_0$. Since $\Lambda^{\rm o}$ is quadratic; see (5.1.4), in view of Theorem 5.1.4 and Lemma 5.1.2, we may assume that S_a has a colinear injective $(n-1)$-copresentation

$$0 \longrightarrow S_a \xrightarrow{d^0} I^0 \xrightarrow{d^1} I^1 \longrightarrow \cdots \longrightarrow I^{n-2} \xrightarrow{d^{n-1}} I^{n-1}$$

over ginjΛ, for some $n \geq 2$. We denote by $c^n : I^{n-1} \to C^n$ the cokernel of d^{n-1}. Given $b \in Q_0$ and $p \in \mathbb{Z}$, since soc$I^{n-1} \subseteq {\rm Ker}(c^n)$, it is well known that

$${\rm GExt}^n_\Lambda(S_b\langle p\rangle, S_a) \cong {\rm GHom}_\Lambda(S_b\langle p\rangle, C^n);$$

see (1.4.5). Since Λ is Koszul, S_b has a linear projective resolution

$$\cdots \longrightarrow P^{-i} \longrightarrow P^{1-i} \longrightarrow \cdots \longrightarrow P^{-1} \longrightarrow P_b \longrightarrow S_b \longrightarrow 0$$

over projΛ. So,

$${\rm GHom}_\Lambda(S_b\langle p\rangle, C^n) \cong {\rm GExt}^n_\Lambda(S_b, S_a\langle -p\rangle) \cong {\rm GHom}_\Lambda(P^{-n}, S_a\langle -p\rangle).$$

Since P^{-n} is generated in degree n and $S_a\langle -p\rangle$ is generated in degree p, we have ${\rm GHom}_\Lambda(S_b\langle p\rangle, C^n) = 0$ for all but finitely many $b \in Q_0$ and for all $p \neq n$. Hence, soc$\,C^n$ is finitely generated in degree $-n$. Since I^{n-1} is finitely cogenerated, socC^n is graded essential in C^n. That is, C^n is finitely cogenerated in degree $-n$. By Proposition 3.8.5(2), there exists a graded injective envelope $q^n : C^n \to I^n$, where $I^n \in$ ginjΛ is cogenerated in degree $-n$. Thus, S_a has a colinear injective n-copresentation over ginjΛ. By induction, S_a has a colinear injective coresolution over ginjΛ. Thus Statement (4) holds, and consequently, Statement (2) holds.

We shall also establish Statement (3). Indeed, by Proposition 5.3.3, $\Lambda^!$ is quadratic with $(\Lambda^!)^! = \Lambda$. By Lemma 5.4.1, the local Koszul complex at a of $\Lambda^!$ is isomorphic to the complex

$$\mathcal{P}^{\bullet}_{a^!}: \qquad \cdots \longrightarrow \mathcal{P}^{-n}_{a^!} \xrightarrow{\ell^{-n}} \mathcal{P}^{1-n}_{a^!} \longrightarrow \cdots \longrightarrow \mathcal{P}^{-1}_{a^!} \xrightarrow{\ell^{-1}} \mathcal{P}^{0}_{a^!} \longrightarrow 0 \longrightarrow \cdots,$$

where $\mathcal{P}^{-n}_{a^!} = \oplus_{x\in Q_0}(P^!_x\langle -n\rangle \otimes D(e_a\Lambda_n e_x))$ and $\mathcal{P}^{1-n}_{a^!} = \oplus_{y\in Q_0}(P^!_y\langle 1-n\rangle \otimes D(e_a\Lambda_{n-1}e_y))$, and $\ell^{-n} = (\sum_{\alpha\in Q_1(x,y)} P[\bar\alpha^!] \otimes DP[\bar\alpha])_{(y,x)\in Q_0\times Q_0}$. Fix $n > 0$. We claim that $\mathcal{P}^{\bullet}_{a^!}$ is exact in degree $-n$. That is, for any $(s,b) \in \mathbb{Z} \times Q_0$ the sequence

$$(*) \qquad \oplus_{z\in Q_0}(e_b\Lambda^!_{s-1}e_z \otimes D(e_a\Lambda_{n+1}e_z)) \xrightarrow{\ell^{-n-1}_{n+s,b}} \oplus_{x\in Q_0}(e_b\Lambda^!_s e_x \otimes D(e_a\Lambda_n e_x))$$
$$\xrightarrow{\ell^{-n}_{n+s,b}} \oplus_{y\in Q_0}(e_b\Lambda^!_{s+1}e_y \otimes D(e_a\Lambda_{n-1}e_y))$$

is exact, where $\ell^{-n}_{n+s,b} = (\sum_{\alpha\in Q_1(x,y)} P[\bar\alpha^!] \otimes DP[\bar\alpha])_{(y,x)\in Q_0\times Q_0}$.

If $s < 0$, then $e_b\Lambda^!_s e_x = 0$, and $(*)$ is evidently exact. In case $s = 0$, it becomes

$$0 \longrightarrow e_b\Lambda^!_0 e_b \otimes D(e_a\Lambda_n e_b) \xrightarrow{\ell^{-n}_{n,b}} \oplus_{y\in Q_0}(e_b\Lambda^!_1 e_y \otimes D(e_a\Lambda_{n-1}e_y)),$$

where $\ell^{-n}_{n,b} = (\ell^{-n}_{n,b}(y,b))_{y\in Q_0}$ with $\ell^{-n}_{n,b}(y,b) = \sum_{\alpha\in Q_1(b,y)} P[\bar\alpha^!] \otimes DP[\bar\alpha]$. Consider $0 \neq f \in D(e_a\Lambda_n e_b)$. Since $n > 0$, there exist $\beta \in Q_1(b,y)$ and $u \in e_a\Lambda_{n-1}e_y$ with $y \in Q_0$ such that $f(u\bar\beta) \neq 0$, that is, $(DP[\bar\beta])(f)(u) \neq 0$. So, $(DP[\bar\beta])(f) \neq 0$. Now, $\ell^{-n}_{n,b}(y,b)(e_b \otimes f) = \sum_{\alpha\in Q_1(b,y)} \bar\alpha^! \otimes (DP[\bar\alpha])(f)$, which is non-zero. Thus, $\ell^{-n}_{n,b}$ is a monomorphism. That is, the sequence $(*)$ is exact in this case.

Let $s > 0$. By Proposition 5.2.5 and Lemma 5.4.1, S_b has a linear graded projective resolution $\mathcal{P}^{\bullet}_b$, which is exact in degree $-s$. Writing $\mathcal{P}^{-s}_b = \oplus_{x\in Q_0}(P_x\langle -s\rangle \otimes D(e_b\Lambda^!_s e_x))$, and restricting it to the $(n+s,a)$-piece, we obtain an exact sequence

$$(**) \qquad \oplus_{y\in Q_0}(e_a\Lambda_{n-1}e_y \otimes D(e_b\Lambda^!_{s+1}e_y)) \xrightarrow{\ell^{-s-1}_{b,n+s,a}} \oplus_{x\in Q_0}(e_a\Lambda_n e_x \otimes D(e_b\Lambda^!_s e_x))$$
$$\xrightarrow{\ell^{-s}_{b,n+s,a}} \oplus_{z\in Q_0}(e_a\Lambda_{n+1}e_z \otimes D(e_b\Lambda^!_{s-1}e_z)),$$

where $\ell^{-s-1}_{b,n+s,a} = (\sum_{\alpha\in Q_1(x,y)} P[\bar\alpha] \otimes DP[\bar\alpha^!])_{(x,y)\in Q_0\times Q_0}$. Applying D to $(**)$, we obtain an exact sequence isomorphic to $(*)$; see (2.1.3). Our claim holds. Thus, $\mathcal{P}^{\bullet}_{a^!}$ is a linear projective resolution of $S^!_a$; see (5.2.4). So, $\Lambda^!$ is Koszul.

As we have shown, Statement (1) implies Statements (2) and (3). In case Λ is quadratic, $(\Lambda^{\rm o})^{\rm o} = \Lambda$ and $(\Lambda^!)^! = \Lambda$. Thus, each of Statements (2) and (3) implies Statement (1). The proof of the theorem is completed.

Remark. (1) In case Q_0 is finite, the equivalence of the first three conditions is due to Beilinson, Ginzburg and Soergel; see [13, (2.2.1), (2.10.2)].
(2) In case Λ is Koszul, one calls $\Lambda^!$ the **Koszul dual** of Λ.

Example. Consider $\Lambda = kQ/(kQ^+)^2$, where Q is a locally finite quiver. Then $\Lambda^! = kQ^{\rm op}$, which is Koszul. By Theorem 5.4.3, Λ is Koszul.

5.5 A new class of Koszul algebras

It is well known that some special classes of algebras are Koszul, including hereditary algebras and radical square zero algebras; see [42, 1.6], exterior algebras and symmetric algebras; see [13, page 476], and monomial algebras; see [42, 2.19]. In this section, we shall apply the Koszul complexes to study when a quadratic multi-serial algebra is Koszul. As a consequence, we obtain a new class of Koszul algebras.

To start with, we shall introduce a condition $(*)$ for a quadratic multi-serial algebra $\Lambda = kQ/R$ as follows: Given a polynomial relation $\sum_{i=1}^{s} \lambda_i\beta_i\alpha_i \in R_2(x,z)$ with $\lambda_i \in k$ and $\alpha_i, \beta_i \in Q_1$, if there exist arrows $\gamma \in Q_1(a,x)$ and $\zeta \in Q_1(z,b)$ with $\zeta\beta_1 \notin R_2$, then $\alpha_i\gamma$ is a summand of a relation in $R_2(a,-)$, for all $i = 2, \ldots, s$.

5.5.1 **Lemma**. *Let $\Lambda = kQ/R$ be a quadratic multi-serial algebra, satisfying the condition $(*)$. Consider a polynomial relation $\sum_{i=1}^{s} \lambda_i\beta_i\alpha_i \in R_2(x,z)$ with $\lambda_i \in k$, $\alpha_i \in Q_1(x,y_i)$ and $\beta_i \in Q_1(y_i,z)$, such that $\zeta\beta_1 \notin R_2$ for some $\zeta \in Q_1(z,-)$. If $\xi \in R^{(n-1)}(a,x)$ with $n \geq 1$ then, for each $2 \leq i \leq s$, there exists $\eta_i \in kQ_n(a,y_i)$ such that $\beta_i\xi + \eta_i \in R^{(n)}(a,y_i)$ and $\alpha_i\eta_i \in R_2(-,z) \cdot kQ_{n-1}(a,-)$.*

Proof. Let $\xi \in R^{(n-1)}(a,x)$. If $n = 1$, then we take $\eta_i = 0$, for each $1 < i \leq s$. Let $n \geq 2$. Choose a k-basis $\{\xi_1, \ldots, \xi_t\}$ with $\xi_j \in R^{(n-2)}(a,b_j)$ of $R^{(n-2)}(a,-)$. Then, $\xi = \sum_{j=1}^{t} \sigma_j\,\xi_j$ with $\sigma_j \in kQ_1(b_j,x)$; see (5.2.2).

Fix $1 < i \leq s$. If $\alpha_i\sigma_j \in R_2(b_i,y_l)$ for all $1 \leq j \leq t$, then $\alpha_i\xi \in R^{(n)}(a,y_i)$; see (5.2.2), and we take $\eta_i = 0$. Otherwise, let J_i be the set of $j \in \{1,\ldots,t\}$ such that $\alpha_i\sigma_j \notin R_2(b_j,y_i)$. Fix $j \in J_i$. Since Λ is multi-serial, $\sigma_j = \lambda_j\theta_j + \delta_j$, where $\lambda_j \in k$ and $\theta_j, \delta_j \in Q_1(b_j,x)$ such that $\lambda_j\alpha_i\theta_j \notin R_2(b_j,y_i)$ and $\alpha_i\delta_j \in R_2(b_j,y_i)$. By the condition $(*)$, there exists a polynomial relation $\omega_j = \lambda_j\alpha_i\theta_j + \sum_{l=1}^{r_j} \lambda_{jl}\alpha_{il}\theta_{jl}$ in $R_2(b_j,y_i)$, where $\lambda_{jl} \in k$; $\theta_{jl} \in Q_1(b_j,c_{jl})$ and $\alpha_{il} \in Q_1(c_{jl},y_i)$. Since Λ is multi-serial, $\alpha_{il} \neq \alpha_i$ for $1 \leq l \leq r_j$. Since $\beta_i\alpha_i \notin R_2(x,z)$, we have $\beta_i\alpha_{il} \in R_2(c_{jl},z)$ for $1 \leq l \leq r_j$. By the induction hypothesis, there exists $\eta_{jl} \in kQ_{n-1}(a,c_{jl})$ such that

$\xi_{jl} = \lambda_{jl}\theta_{jl}\xi_j + \eta_{jl} \in R^{(n-1)}(a, c_{jl})$ and $\alpha_{il}\eta_{jl} \in R_2(-, y_i)\cdot kQ_{n-2}(a, -)$, for $1 \le l \le r_j$. Set $\eta_i = \sum_{j\in J_i; 1\le l\le r_j} \alpha_{il}\xi_{jl} \in kQ_n(a, y_i)$. Then, $\beta_i\eta_i \in R_2(-, z)\cdot kQ_{n-1}(a, -)$. Put $\chi_i = \alpha_i\xi + \eta_i$. Then, $\chi_i = \alpha_i\xi + \sum_{j\in J_i; 1\le l\le r_j} \alpha_{jl}\xi_{jl}$, where $\xi, \xi_{jl} \in R^{(n-1)}(a, -)$. On the other hand, we can verify that

$$\chi_i = \sum_{j\in I_i}(\omega_j + \alpha_i\delta_j)\xi_j + \sum_{j\notin I_i}\alpha_i\sigma_j\xi_j + \sum_{j\in J_i; 1\le l\le r_j}\alpha_{il}\eta_{jl} \in R_2(-, y_i)\cdot kQ_{n-2}(a, -).$$

By Lemma 5.2.2(2), $\chi_i \in R^{(n)}(a, y_i)$. The proof of the lemma is completed.

The following is the promised new class of Koszul algebras; compare [24, (2.2)].

5.5.2 **Theorem**. *Let $\Lambda = kQ/R$ be a quadratic multi-serial algebra with Q a locally finite quiver. If the condition $(*)$ or its dual is satisfied, then Λ is a Koszul algebra.*

Proof. It is evident that Λ satisfies the dual condition of $(*)$ if and only if $\Lambda^{\rm o}$ satisfies the condition $(*)$. By Theorem 5.4.3, we may assume that the condition $(*)$ is satisfied. By Theorem 5.2.5, it amounts to show for any $a \in Q_0$ that $\mathcal{K}_a^{\bullet}$ is exact in all negative degrees. By Theorem 5.1.4 and Proposition 5.2.4, we may assume $n \ge 2$. It suffices to prove that $\mathrm{Ker}\,(\partial_a^{-n}) \subseteq \mathrm{Im}\,(\partial_a^{-n-1})$. Recall that

$$\mathcal{K}_a^{-n} = \oplus_{y\in Q_0} P_y\langle -n\rangle \otimes R^{(n)}(a, y).$$

Consider a non-zero element u in $\mathrm{Ker}(\partial_a^{-n}) \subseteq \mathrm{rad}\mathcal{K}_a^{-n}$. Since $\mathcal{K}_a^{-n}$ s generated in degree n, we may assume that $u \in \mathrm{Ker}(\partial_a^{-n})_m(b)$ for $b \in Q_0$ and $m > n$. Then,

$$u \in \oplus_{y\in Q_0} P_y\langle -n\rangle_m(b) \otimes R^{(n)}(a, y) = \oplus_{y\in Q_0} e_b\Lambda_{m-n}e_y \otimes R^{(n)}(a, y).$$

Let s be minimal such that $u = \sum_{l=1}^{s} \bar\theta_l \otimes \rho_l$, where $\theta_l \in Q_{m-n}(y_l, b)$ and $\rho_l \in R^{(n)}(a, y_l)$. Note that $\bar\theta_1, \ldots, \bar\theta_s$ are k-linear independent in $e_b\Lambda_{m-n}$. Choose a k-basis $\{\xi_1, \ldots, \xi_t\}$ of $R^{(n-1)}(a, -)$, where $\xi_j \in R^{(n-1)}(a, x_j)$; and since Λ is multi-serial, $e_b\Lambda_{m-n-1}$ has a k-basis $\{\bar\eta_1, \ldots, \bar\eta_r\}$, where $\eta_i \in Q_{m-n-1}(z_i, b)$ with $z_i \in Q_0$. Then, $\rho_l = \sum_{j=1}^{t} \zeta_{lj}\,\xi_j$ with $\zeta_{lj} \in kQ_1(x_j, y_l)$; see (5.2.2) and $\bar\theta_l = \sum_{i=1}^{r} \bar\eta_i\bar\delta_{il}$ with $\delta_{il} \in kQ_1(y_l, z_i)$, for $l = 1, \ldots, s$.

(1) *For any $1 \le j \le t$, we have $\sum_{l=1}^{s} \bar\theta_l\bar\zeta_{lj} = \sum_{i=1}^{r}\sum_{l=1}^{s} \bar\eta_i\bar\delta_{il}\bar\zeta_{lj} = 0$.*

Indeed since $u = \sum_{j=1}^{t}\sum_{l=1}^{s} \bar\theta_l \otimes \zeta_{lj}\xi_j$, we have $\partial_a^{-n}(u) = \sum_{j=1}^{t}(\sum_{l=1}^{s} \bar\theta_l\bar\zeta_{lj}) \otimes \xi_j = 0$; see (5.2.1). Since the ξ_j are k-linearly independent,

$$\sum_{l=1}^{s} \bar\theta_l\bar\zeta_{lj} = \sum_{i=1}^{r}\sum_{l=1}^{s} \bar\eta_i\bar\delta_{il}\bar\zeta_{lj} = 0,$$

for $1 \leq j \leq t$. This establishes Statement (1).

(2) *If $m = n+1$, then $u \in \mathrm{Im}(\partial_a^{-n-1})$.*

If $m = n+1$, then $e_b\Lambda_{m-n-1} = e_b\Lambda_0 = ke_b$. In particular, $r = 1$ and $\eta_1 = \varepsilon_b$. By Statement (1), we obtain

$$\sum_{l=1}^{s} \delta_{1l}\zeta_{lj} = \sum_{i=1}^{r} \eta_i(\sum_{l=1}^{s} \delta_{il}\zeta_{lj}) \in R_2(x_j, b), \text{ for } j = 1, \ldots, t.$$

Put $\chi_1 = \sum_{l=1}^{s} \delta_{1l}\rho_l = \sum_{l=1}^{s}(\sum_{j=1}^{t} \delta_{1l}\zeta_{lj})\xi_j \in R_2(-, z_1) \cdot kQ_{n-1}(x_j, -)$. Then, by Lemma 5.2.2(2), $\chi_1 \in R^{(n+1)}(a, z_1)$ such that $\partial_a^{-n-1}(\bar{\eta}_1 \otimes \chi_1) = \sum_{l=1}^{s} \bar{\theta}_l \otimes \rho_l = u$. This establishes Statement (2).

We assume now that $m \geq n+2$. Then, $\bar{\theta}_l = \sum_{i=1}^{r} \bar{\eta}_i\bar{\delta}_{il}$, for $\delta_{il} \in kQ_1(y_l, z_i)$ and non-trivial $\eta_i \in Q_{m-n-1}(z_i, b)$. Since Λ is multi-serial, we may assume that δ_{il} is a monomial, for $i = 1, \ldots, r; l = 1, \ldots, s$. We shall consider another derivation $\partial^\alpha : kQ \to kQ$ for any $\alpha \in Q_1$, which sends a path ρ to η if $\rho = \eta\alpha$; and 0 if α is not an initial arrow of ρ.

(3) *If $\delta_{il}\zeta_{lj} \notin R_2(x_j, z_i)$, then ζ_{lj} has a summand $\lambda_{lj}\alpha_{lj}$, where $\lambda_{lj} \in k$ and $\alpha_{lj} \in Q_1(x_j, y_l)$, such that $\lambda_{lj}\delta_{il}\alpha_{lj}$ is a summand of a polynomial relation in R_2.*

Suppose that $\delta_{ip}\zeta_{pq} \notin R_2(x_q, z_i)$ for some $1 \leq i \leq r; 1 \leq p \leq s; 1 \leq q \leq t$. Then, δ_{ip} is a non-zero monomial in $kQ_1(y_p, z_i)$ and ζ_{pq} has a non-zero summand $\lambda_{pq}\alpha_{pq}$, where $\alpha_{pq} \in k$ and $\alpha_{pq} \in Q_1(x_q, y_p)$, such that $\lambda_{pq}\delta_{ip}\alpha_{pq} \notin R_2(x_q, z_i)$. By Statement (1), we may write $\sum_{l=1}^{s} \theta_l\zeta_{lq} = \sum_{j=1}^{h} \nu_j\omega_j\kappa_j$, where $\kappa_j \in Q_{n_j}(x_q, -)$ with n_j some non-negative integer, $\omega_j \in R_2$ and $\nu_j \in kQ_{m-n-n_j-1}(-, b)$.

Assume, for each $1 \leq j \leq h$, that $n_j > 0$ or $\partial^{\alpha_{pq}}(\omega_j) = 0$. Applying $\partial^{\alpha_{pq}}$ to the above equation, we obtain $\sum_{l=1}^{s} \lambda_l\theta_l \in R_{m-n}(-, b)$, where $\lambda_l = \partial^{\alpha_{pq}}(\zeta_{lq}) \in k$. Since $\lambda_p = \lambda_{pq} \neq 0$, contrary to $\bar{\theta}_1, \ldots, \bar{\theta}_s$ being k-linearly independent. Thus, we may assume that $n_1 = 0$ and α_{pq} is the initial arrow of a monomial summand of $\omega_1 \in R_2(x_q, -)$. Since Λ is multi-serial with $\lambda_{pq}\delta_{ip}(\alpha_{pq} \notin R_2(x_q, z_i)$, we see that $\lambda_{pq}\delta_{ip}(\alpha_{pq})$ is a summand of ω_1, which is a polynomial relation in $R_2(x_q, z_i)$. This establishes Statement (3).

(4) *For each $1 \leq i \leq r$, there exists some element $\chi_i \in R^{(n+1)}(a, z_i)$ such that $\partial_a^{-n-1}(\bar{\eta}_i \otimes \chi_i) = \sum_{l=1}^{s} \bar{\eta}_i\bar{\delta}_{il} \otimes \rho_l$.*

Fix $1 \leq i \leq r$. If $\bar{\eta}_i\bar{\delta}_{i1} = 0$ for all $l = 1, \ldots, s$, then we take $\chi_i = 0$. Otherwise, denote by L the set of l with $1 \leq l \leq s$ such that $\bar{\eta}_i\bar{\delta}_{il} \neq 0$; and for $l \in L$, denote by J_l the set of j with $1 \leq j \leq t$ such that $\delta_{il}\zeta_{lj} \notin R_2(x_j, z_i)$. Fix $(j, l) \in L \times J_l$. Since Λ is multi-serial, we may write $\zeta_{lj} = \alpha_{lj} + \sigma_{lj}$ such that $\delta_{il}\sigma_{lj} \in R_2(x_j, z_i)$ and α_{lj} is a monomial with $\delta_{il}\alpha_{lj} \notin R_2(x_j, z_i)$. By Statement (3), we have a polynomial relation $\omega_{lj} = \delta_{il}\alpha_{lj} + \sum_{p=1}^{r_{lj}} \gamma_{il}^p \beta_{lj}^p$ in $R_2(x_j, z_i)$, where $\beta_{lj}^p \in Q_1(x_j, c_{lj}^p)$ with $c_{lj}^p \in Q_0$ and γ_{il}^p is a monomial in $kQ_1(c_{lj}^p, z_i)$. Since η_i is a non-trivial path with $\bar{\eta}_i\bar{\delta}_{il} \neq 0$, we have $\bar{\eta}_i\bar{\gamma}_{il}^p = 0$ for all $1 \le p \le r_{lj}$. And by Lemma 5.5.1, there exists $\xi_{lj}^p \in kQ_n(a, c_{lj}^p)$ such that $\rho_{lj}^p = \beta_{lj}^p\xi_j + \xi_{lj}^p \in R^{(n)}(a, c_{lj}^p)$ and $\gamma_{il}^p\xi_{lj}^p \in R_2(-, z_i) \cdot kQ_{n-1}(a, -)$, for each $1 \le p \le r_{lj}$. Now, we put

$$\chi_i = \textstyle\sum_{l\in L}\delta_{il}\rho_l + \sum_{l\in L; j\in J_l; 1\le p\le r_{lj}} \gamma_{il}^p \rho_{lj}^p,$$

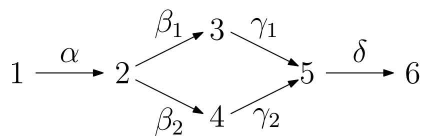
where $\rho_l, \rho_{lj}^p \in R^{(n)}(a, -)$. Observing that $\rho_l = \sum_{j=1}^t \zeta_{lj}\, \xi_j$, we can verify that

$$\chi_i = \textstyle\sum_{l\in L; j\in J_l}(\omega_{lj} + \delta_{il}\sigma_{lj})\xi_j + \sum_{l\in L; j\notin J_l}\delta_{il}\zeta_{lj}\xi_j + \sum_{l\in L; j\in J_l; 1\le p\le t_{lj}}\gamma_{il}^p\xi_{lj}^p.$$

Since $\delta_{il}\zeta_{lj} \in R_2(x_j, z_i)$ for $(l, j) \in L \times J_l$, we get $\chi_i \in R_2(-, z_i) \cdot kQ_{n-1}(a, -)$. By Lemma 5.2.2(2), $\chi_i \in R^{(n+1)}(a, z_i)$, and hence, $\bar{\eta}_i \otimes \varphi_i \in \mathcal{K}_a^{-n-1}$. Further, since $\bar{\eta}_i\bar{\gamma}_{il}^p = 0$ for $(l, j) \in L \times J_l$ and $1 \le p \le r_{lj}$ and $\bar{\eta}_i\bar{\delta}_{il} = 0$ for $l \notin L_i$, we deduce that $\partial_a^{-n-1}(\bar{\eta}_i \otimes \chi_i) = \sum_{l\in L}\bar{\eta}_i\bar{\delta}_{il} \otimes \rho_l = \sum_{l=1}^s \bar{\eta}_i\bar{\delta}_{il} \otimes \rho_l$. This proves Statement (4).

Finally, $w = \sum_{i=1}^r \bar{\eta}_i \otimes \chi_i \in K_a^{-n-1}$ is such that $\partial_a^{-n-1}(w) = \sum_{l=1}^s \bar{\theta}_l \otimes \rho_l = u$. The proof of the theorem is completed.

Example. Consider the quadratic special biserial algebra $\Lambda = kQ/R$, where

$Q:$


(1) If $R = \langle \beta_2\alpha, \delta\gamma_1, \gamma_1\beta_1 + \gamma_2\beta_2\rangle$, then Λ does not satisfies the condition $(\star)$ and its dual. Clearly, S_1 has a minimal graded projective resolution

$$0 \longrightarrow P_6\langle -4\rangle \xrightarrow{P[\bar{\delta}\bar{\gamma}_2]} P_4\langle -2\rangle \xrightarrow{P[\bar{\beta}_2]} P_2\langle -1\rangle \xrightarrow{P[\bar{\alpha}]} P_1 \xrightarrow{P[e_1]} S_1 \longrightarrow 0.$$

Since $P_6\langle -4\rangle$ is generated in degree 4, Λ is not a Koszul algebra.

(2) If $R = \langle \beta_2\alpha, \delta\gamma_2, \gamma_1\beta_1 + \gamma_2\beta_2\rangle$, then Λ satisfies the condition $(\star)$ and its dual. By Theorem 5.5.2, Λ is Koszul. Indeed, as shown below, every graded simple module in GModΛ admits a linear graded projective resolution

- $0 \longrightarrow P_4\langle -2\rangle \longrightarrow P_2\langle -1\rangle \longrightarrow P_1 \longrightarrow S_1 \longrightarrow 0,$
- $0 \longrightarrow P_5\langle -2\rangle \longrightarrow P_3\langle -1\rangle \oplus P_2\langle -1\rangle \longrightarrow P_2 \longrightarrow S_2 \longrightarrow 0,$
- $0 \longrightarrow P_5\langle -1\rangle \longrightarrow P_3 \longrightarrow S_3 \longrightarrow 0,$
- $0 \longrightarrow P_6\langle -2\rangle \longrightarrow P_5\langle -1\rangle \longrightarrow P_4 \longrightarrow S_4 \longrightarrow 0,$
- $0 \longrightarrow P_6\langle -1\rangle \longrightarrow P_5 \longrightarrow S_5 \longrightarrow 0.$

5.6 Extension Conjecture for Koszul algebras

The objective of this section is to apply our previous results to establish a stronger version of the Extension Conjecture for Koszul algebras whose Koszul dual is locally left or right noetherian. In this section, we shall mainly consider non-graded modules over Koszul algebras. We recall the **Extension Conjecture** from [29, (2.6)] as follows.

5.6.1 **Conjecture**. *Let A be an artin algebra, and let S be a simple A-module. If $\mathrm{Ext}^1_A(S,S) \neq 0$, then $\mathrm{Ext}^n_A(S,S) \neq 0$ for infinitely many integers $n > 0$.*

Let $\Lambda = kQ/R$ be a Koszul algebra, where Q is locally finite. Recall that Λ is locally left noetherian if P_x is noetherian as a non-graded left Λ-module for any $x \in Q_0$; and locally right noetherian if $e_x\Lambda$ is noetherian as non-graded right Λ-module for any $x \in Q_0$.

5.6.2 **Lemma**. *Let $\Lambda = kQ/R$ be a locally left noetherian Koszul algebra, where Q is a locally finite quiver. Let $a \in Q_0$ such that $\Lambda^! e_a$ is finite dimensional. Then the e_a-trace is defined for every endomorphism in* $\mathrm{gmod}^{+,b}\Lambda$.

Proof. By the assumption, $\Lambda^!_t e_a = 0$ for some $t > 0$. By Theorem 5.4.3 and Proposition 5.4.2, S_a has a graded injective coresolution

$$I^\bullet: \quad 0 \longrightarrow S_a \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots \longrightarrow I^n \longrightarrow \cdots$$

where $I^n = \oplus_{x\in Q_0}(I_x\langle n\rangle \otimes e_x\Lambda^!_n e_a)$, for all $n \in \mathbb{Z}$. In particular, $I^n_a = 0$ for $n > t$. Consider $M \in \mathrm{gmod}^{+,b}\Lambda$. Since Λ is locally left noetherian, by Proposition 3.9.5(1), $\mathrm{gmod}^{+,b}\Lambda$ is abelian. In view of Corollary 3.8.6(1), we see that M has a minimal graded projective resolution

$$P^\bullet: \quad \cdots \longrightarrow P^{-n} \longrightarrow \cdots \longrightarrow P^{-1} \longrightarrow P^0 \longrightarrow M \longrightarrow 0$$

over $\mathrm{gproj}\Lambda$. For $s \in \mathbb{Z}$, $\mathrm{GHom}_\Lambda(P^{-n}, S_a\langle s\rangle) \cong \mathrm{GExt}^n_\Lambda(M, S_a\langle s\rangle)$. By Proposition 1.4.6, we see that $\mathrm{GHom}_\Lambda(P^{-n}, S_a\langle s\rangle)$ is a sub-quotient of $\mathrm{GHom}_\Lambda(M, I^n\langle s\rangle)$. Thus, given $n > t$, $\mathrm{GHom}_\Lambda(P^{-n}, S_a\langle s\rangle) = 0$, that is, $P_a\langle s\rangle$ is not a direct summand of P^{-n} for any $s \in \mathbb{Z}$. Forgetting the graduation, we see that $P^\bullet$ is a projective resolution of M over $\mathrm{proj}\Lambda$ such that P_a is not direct summand of P^{-n} for all $n > t$. In other words, $P^\bullet$ is an e_a-bounded projective resolution of M over $\mathrm{proj}\Lambda$. Thus, the e_a-trace is defined for every endomorphism of M. The proof of the lemma is completed.

The following statement is a local version of Lenzing's result in [34] for locally left noetherian Koszul algebras.

5.6.3 **Proposition**. *Let $\Lambda = kQ/R$ be a locally left noetherian Koszul algebra, where Q is a locally finite quiver with a loop σ at a vertex a. If $\Lambda^! e_a$ is finite dimensional, then $\bar{\sigma}$ is not nilpotent.*

Proof. Assume that $\Lambda^! e_a$ is finite dimensional with $\bar{\sigma}^r = 0$ for some $r > 0$. Consider $M^{(i)} = \Lambda \bar{\sigma}^i \in \mathrm{gmod}^{+,b}\Lambda$ for $i \geq 0$, where $\bar{\sigma}^0 = e_a$. Let $\varphi^{(i)} : M^{(i)} \to M^{(i)}$ be the right multiplication by $\bar{\sigma}$, for $i \geq 0$. By Lemma 5.6.2, $\mathrm{tr}_a(\varphi^{(i)})$ is defined for $i \geq 0$. Now, since $\varphi^{(i)}(M^{(i)}) \subseteq M^{(i+1)}$, we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M^{(i+1)} & \longrightarrow & M^{(i)} & \longrightarrow & M^{(i)}/M^{(i+1)} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi^{(i+1)}} & & \downarrow{\scriptstyle \varphi^{(i)}} & & \downarrow{\scriptstyle 0} & & \\ 0 & \longrightarrow & M^{(i+1)} & \longrightarrow & M^{(i)} & \longrightarrow & M^{(i)}/M^{(i+1)} & \longrightarrow & 0, \end{array}$$

and by Proposition 2.5.4(2), $\mathrm{tr}_a(\varphi^{(i)}) = \mathrm{tr}_a(\varphi^{(i+1)})$ for all $i \geq 0$. In view of Lemma 2.5.1(1), we see that

$$(\bar{\sigma} + \textstyle\sum_{x\in Q\setminus\{a\}} \Lambda e_x \Lambda) + [\Lambda_a, \Lambda_a] = \mathrm{tr}_a(\varphi^{(0)}) = \mathrm{tr}_a(\varphi^{(r)}) = 0.$$

That is, $\bar{\sigma} + \sum_{x\in Q_0\setminus\{a\}} \Lambda e_x \Lambda \in [\Lambda_a, \Lambda_a]$, and hence, $\bar{\sigma} \in \sum_{x\in Q_0\setminus\{a\}} \Lambda e_x \Lambda + [\Lambda, \Lambda]$. Observing that $e_x \Lambda e_y \subseteq \mathrm{rad}\Lambda$ if $x \neq y$ and $e_a u = u e_a$ for any $u \in e_a \Lambda e_a$, we deduce that $\bar{\sigma} \in \mathrm{rad}^2\Lambda$, contrary to R being generated in degree 2. Thus, $\bar{\sigma}$ is not nilpotent in Λ. The proof of the proposition is completed.

We are ready to obtain the main result of this section.

5.6.4 **Theorem**. *Let $\Lambda = kQ/R$ be a Koszul algebra such that $\Lambda^!$ is locally left noetherian, where Q is a locally finite quiver containing a loop σ at some vertex a. If Λe_a is finite dimensional, then $\mathrm{Ext}^n_\Lambda(S_a, S_a) \neq 0$ for $n \geq 1$.*

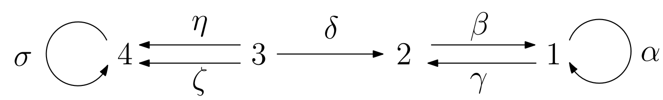
Proof. By Theorem 5.2.5 and Lemma 5.4.1, S_a has a linear projective resolution

$$P^\bullet : \quad \cdots \longrightarrow P^{-n} \longrightarrow \cdots \longrightarrow P^{-1} \longrightarrow P^0 \longrightarrow S_a \longrightarrow 0,$$

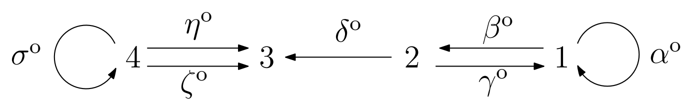
where $P^{-n} = \oplus_{x\in Q_0} (P_x\langle -n\rangle \otimes D(e_a \Lambda^!_n e_x))$ for $n \geq 0$. Assume that Λe_a is finite dimensional. Since $\Lambda^!$ is locally left noetherian with $(\Lambda^!)^! = \Lambda$, by Proposition 5.6.3, $0 \neq (\bar{\sigma}^!)^n \in e_a \Lambda^!_n e_a$ for all $n \geq 0$. Since $D(e_a \Lambda^!_n e_a) \neq 0$, we see that $P_a\langle -n\rangle$

is a direct summand of P^{-n}, for $n \geq 1$. That is, $\mathrm{GExt}^n_\Lambda(S_a, S_a\langle -n\rangle) \neq 0$, and hence, $\mathrm{Ext}^n_\Lambda(S_a, S_a) \neq 0$ for $n \geq 1$. The proof of the theorem is completed.

Example. Consider the algebra $\Lambda = kQ/R$, where Q is the quiver



and $R = \langle \alpha^2 + \beta\gamma, \alpha\beta, \gamma\beta, \gamma\alpha, \sigma\eta, \sigma\zeta\rangle$. It is easy to see that Λ is a multi-serial algebra satisfying the condition $(*)$. By Theorem 5.5.2, Λ is Koszul. Note that $\Lambda^! = kQ^{\rm o}/R^!$, where $Q^{\rm o}$ is the quiver



and $R^! = \langle (\alpha^{\rm o})^2 - \gamma^{\rm o}\beta^{\rm o}, \delta^{\rm o}\beta^{\rm o}, (\sigma^{\rm o})^2\rangle$. We claim that $\Lambda^!$ is left noetherian. Indeed, $P^!_3 = k\langle e_3\rangle$ and $P^!_4 = k\langle e_4, \bar{\sigma}^!, \bar{\eta}^!, \bar{\zeta}^!, \bar{\eta}^!\bar{\sigma}^!, \bar{\zeta}^!\bar{\sigma}^!\rangle$, which are finite dimensional.

Since $\bar{\gamma}^!\bar{\beta}^! = (\bar{\alpha}^!)^2$ and $\bar{\delta}^!\bar{\beta}^! = 0$, we see that $P^!_1 = k\langle \bar{\beta}^!, (\bar{\alpha}^!)^n, \bar{\beta}^!(\bar{\alpha}^!)^n \mid n \geq 0\rangle$, where $(\bar{\alpha}^!)^0 = e_1$. Thus, every element of $P^!_1$ is of the form $\bar{\beta}^! f(\bar{\alpha}^!) + g(\bar{\alpha}^!)$, where f and g are polynomials. Let M be a non-zero graded $\Lambda^!$-submodule of $P^!_1$. Since $\bar{\gamma}^!\bar{\alpha}^! = 0$ and $\bar{\beta}^!\bar{\beta}^! = 0$, there exists a non-zero polynomial g such that $g(\bar{\alpha}^!) \in M$. We may assume that g is of minimal degree s. Then $M_{\leq s} = \oplus_{0 \leq i \leq s} M_s$ is finite dimensional such that $M = \Lambda M_{\leq s}$. So $P^!_1$ is noetherian. Again since $\bar{\gamma}^!\bar{\beta}^! = (\bar{\alpha}^!)^2$, we see that

$$P^!_2 = k\langle e_2, \bar{\delta}^!, \bar{\gamma}^!, \bar{\beta}^!\bar{\gamma}^!, (\bar{\alpha}^!)^n\bar{\gamma}^!, \bar{\beta}^!(\bar{\alpha}^!)^n\bar{\gamma}^! \mid n \geq 1\rangle.$$

Similarly, we can prove that $P^!_2$ is noetherian. This establishes our claim. Now, since Λe_1 is finite dimensional, by Theorem 5.6.4, $\mathrm{Ext}^i_\Lambda(S_1, S_1) \neq 0$ for all $i \geq 1$. On the other hand, $\mathrm{Ext}^i_\Lambda(S_4, S_4) = 0$ for all $i \geq 2$. Indeed, Λe_4 is infinite dimensional.

As a consequence of Theorem 5.6.4, we obtain the following statement, which is stronger than the Extension Conjecture for finite dimensional Koszul algebras with a noetherian Koszul dual.

5.6.5 **Theorem**. *Let $\Lambda = kQ/R$ be a finite dimensional Koszul algebra such that $\Lambda^!$ is left or right noetherian. If Q contains a loop at some vertex a, then $\mathrm{Ext}^n_\Lambda(S_a, S_a) \neq 0$, for every $n \geq 1$.*

Proof. Let σ be a loop in Q at some vertex a. Suppose that $\Lambda^!$ is left noetherian. Since Λe_a is finite dimensional, by Theorem 5.6.4, $\mathrm{Ext}^n_\Lambda(S_a, S_a) \neq 0$, for every $n \geq 1$. Suppose now that $\Lambda^!$ is locally right noetherian. Then, $(\Lambda^!)^{\mathrm{o}}$ is locally left noetherian. By Theorem 5.4.3 and Proposition 5.3.3, Λ^{o} is a finite dimensional Koszul algebra with $(\Lambda^{\mathrm{o}})^! = (\Lambda^!)^{\mathrm{o}}$. Since σ^{o} is a loop in Q^{o} at a, $\mathrm{Ext}^n_{\Lambda^{\mathrm{o}}}(S^{\mathrm{o}}_a, S^{\mathrm{o}}_a) \neq 0$, for every $n \geq 1$. Since Λ is finite dimensional, we have a duality $D = \mathrm{Hom}_k(-, k)$: $\mathrm{mod}^b\Lambda^{\mathrm{o}} \to \mathrm{mod}^b\Lambda$. This yields $\mathrm{Ext}^n_\Lambda(S_a, S_a) \neq 0$, for every $n \geq 1$. The proof of the theorem is completed.

Remark. Since a multi-serial algebra is locally left and right noetherian; see (2.4.2), the Extension Conjecture holds for finite dimensional Koszul algebras with a multi-serial Koszul dual.

Chapter 6

Generalized Koszul dualities

The main objective of this chapter is to describe our generalized Koszul dualities for a Koszul algebra, which include the classical Koszul duality of Beilinson, Ginzburg and Soergel stated in [13, (2.12.1)]; see also [48, Theorem 30]. In the locally bounded Koszul case, we shall obtain two equivalences of bounded derived categories, one for finitely piece-supported graded modules and one for finite dimensional graded modules. This generalizes the result stated in [13, (2.12.6)].

6.1 Koszul Functors

In this section, we shall construct two Koszul functors for a quadratic algebra given by a locally finite quiver, which are adapted from the Koszul functors for a quadratic algebra given by a locally finite gradable quiver in the non-graded setting; see [16, Section 5].

Throughout this section, $\Lambda = kQ/R$ is a quadratic algebra, where Q is a locally finite quiver. Let us recall some notations which will be frequently used in this chapter. First, $\Lambda^! = kQ^{\rm o}/R^! = \{\bar{\gamma}^! \mid \gamma \in kQ\}$, where $\bar{\gamma}^! = \gamma^{\rm o} + R^!$; and $\hat{\Lambda} = (\Lambda^!)^{\rm o} = kQ/(R^!)^{\rm o} = \{\hat{\gamma} \mid \gamma \in kQ\}$, where $\hat{\gamma} = \gamma + (R^!)^{\rm o}$. Then, for $x \in Q_0$, we have $P^!_x = \Lambda^! e_x$ and $I^!_x = \mathfrak{D}\hat{P}_x$, where $\hat{P}_x = \hat{\Lambda} e_x$.

We start with defining the **right Koszul functor** $\mathcal{F} : {\rm GMod}\Lambda \to C({\rm GMod}\Lambda^!)$ as follows; compare [16, page 35]; and also [15, page 63]. Let $M \in {\rm GMod}\Lambda$ and $n \in \mathbb{Z}$. We put

$$\mathcal{F}(M)^n = \oplus_{x \in Q_0} (P^!_x\langle n\rangle \otimes M_n(x)).$$

Writing $\mathcal{F}(M)^{n+1} = \oplus_{y\in Q_0}(P^!_y\langle n+1\rangle \otimes M_{n+1}(y))$, we define

$$d^n_{\mathcal{F}(M)} = (d^n_{\mathcal{F}(M)}(y,x))_{(y,x)\in Q_0\times Q_0} : \mathcal{F}(M)^n \to \mathcal{F}(M)^{n+1}$$

with

$$d^n_{\mathcal{F}(M)}(y,x) = \textstyle\sum_{\alpha\in Q_1(x,y)} P[\bar{\alpha}^!] \otimes M(\bar{\alpha}) : P^!_x\langle n\rangle \otimes M_n(x) \to P^!_y\langle n+1\rangle \otimes M_{n+1}(y),$$

where $P[\bar{\alpha}^!] : P^!_x\langle n\rangle \to P^!_y\langle n+1\rangle$ is the $\Lambda^!$-linear morphism given by the right multiplication by $\bar{\alpha}^!$, and $M(\bar{\alpha}) : M_n(x) \to M_{n+1}(y)$ is the k-linear map given by left multiplication by $\bar{\alpha}$. Given a morphism $f : M \to N$ in $\mathrm{GMod}\Lambda$ and $n \in \mathbb{Z}$, we put

$$\mathcal{F}(f)^n = \oplus_{x\in Q_0}(\mathrm{id} \otimes f_{n,x}) : \oplus_{x\in Q_0}(P^!_x\langle n\rangle \otimes M_n(x)) \to \oplus_{x\in Q_0}(P^!_x\langle n\rangle \otimes N_n(x)),$$

where $f_{n,x} : M_n(x) \to N_n(x)$ is the k-linear map obtained by restricting f.

We define the **left Koszul functor** $\mathcal{G} : \mathrm{GMod}\Lambda \to C(\mathrm{GMod}\Lambda^!)$ in a similar fashion as follows. Let $M \in \mathrm{GMod}\Lambda$ and $n \in \mathbb{Z}$. We set

$$\mathcal{G}(M)^n = \oplus_{x\in Q_0}(I^!_x\langle n\rangle \otimes M_n(x)).$$

Writing $\mathcal{G}(M)^{n+1} = \oplus_{y\in Q_0}(I^!_y\langle n+1\rangle \otimes M_{n+1}(y))$, we put

$$d^n_{\mathcal{G}(M)} = (d^n_{\mathcal{G}(M)}(y,x))_{(y,x)\in Q_0\times Q_0} : \mathcal{G}(M)^n \to \mathcal{G}(M)^{n+1}$$

with

$$d^n_{\mathcal{G}(M)}(y,x) = \textstyle\sum_{\alpha\in Q_1(x,y)} I[\bar{\alpha}^!] \otimes M(\bar{\alpha}) : I^!_x\langle n\rangle \otimes M_n(x) \to I^!_y\langle n+1\rangle \otimes M_{n+1}(y),$$

where $I[\bar{\alpha}^!] : I^!_x\langle n\rangle \to I^!_y\langle n+1\rangle$ is the $\mathfrak{D}$-dual of $P[\hat{\alpha}] : \hat{P}_y\langle -n-1\rangle \to \hat{P}_x\langle -n\rangle$. And given a morphism $f : M \to N$ in $\mathrm{GMod}\Lambda$ and $n \in \mathbb{Z}$, we set

$$\mathcal{G}(f)^n = \oplus_{x\in Q_0}(\mathrm{id} \otimes f_{n,x}) : \oplus_{x\in Q_0}(I^!_x\langle n\rangle \otimes M_n(x)) \to \oplus_{x\in Q_0}(I^!_x\langle n\rangle \otimes N_n(x)).$$

The following statement is a generalization of Proposition 5.1 in [16] where the quiver Q is assumed to be gradable.

6.1.1 **Proposition**. *Let $\Lambda = kQ/R$ be a quadratic algebra, where Q is a locally finite quiver. The above construction yields functors $\mathcal{F} : \mathrm{GMod}\Lambda \to C(\mathrm{GMod}\Lambda^!)$ and $\mathcal{G} : \mathrm{GMod}\Lambda \to C(\mathrm{GMod}\Lambda^!)$, which are full, faithful and exact.*

Proof. We only verify that $\mathcal{G}$ is an exact functor. Fix $M \in \mathrm{GMod}\Lambda$ and $n \in \mathbb{Z}$. Write $\mathcal{G}(M)^n = \oplus_{x\in Q_0}(I^!_x\langle n\rangle \otimes M_n(x))$ and $\mathcal{G}(M)^{n+2} = \oplus_{z\in Q_0}(I^!_z\langle n{+}2\rangle \otimes M_{n+2}(z))$. For $(z,x) \in Q_0 \times Q_0$, write $Q_2(x,z) = \{\alpha_1\beta_1, \ldots, \alpha_s\beta_s\}$, where $\alpha_i, \beta_i \in Q_1$. In view of the definition of $\mathcal{G}$, we see that $d^{n+1}_{\mathcal{G}(M)} \circ d^n_{\mathcal{G}(M)} = (d^n_{z,x})_{(z,x)\in Q_0\times Q_0}$, where

$$d^n_{z,x} = \textstyle\sum_{i=1}^s I^!_x[\bar\beta^!_i\bar\alpha^!_i] \otimes M(\bar\alpha_i\bar\beta_i) : I^!_x\langle n\rangle \otimes M_n(x) \to I^!_z\langle n{+}2\rangle \otimes M_{n+2}(z).$$

Choose a k-basis $\{\rho_1, \ldots, \rho_r, \rho_{r+1}, \ldots, \rho_s\}$ of $kQ_2(x,z)$, where $\{\rho_1, \ldots, \rho_r\}$ is a k-basis of $R_2(x,z)$. There exists a k-basis $\{\eta_1, \ldots, \eta_r, \eta_{r+1}, \ldots, \eta_s\}$ of $kQ_2(x,z)$ such that $\{\eta^\star_1, \ldots, \eta^\star_r, \eta^\star_{r+1}, \ldots, \eta^\star_s\}$ is the dual basis of $\{\rho_1, \ldots, \rho_r, \rho_{r+1}, \ldots, \rho_s\}$. Then, $\{\eta^\star_1, \ldots, \eta^\star_s\}$ is the dual basis of $\{\rho_1, \ldots, \rho_s\}$ and $\{\eta^{\mathrm{o}}_{r+1}, \ldots, \eta^{\mathrm{o}}_s\}$ is a k-basis of $R^!_2(z,x)$. Observe that $\bar\rho_i = 0$ for $1 \le i \le r$, and $\bar\eta^!_j = 0$ and for $r < j \le s$. On the other hand, by Corollary 2.1.2, we have a k-linear isomorphism

$$\sigma : D(kQ_2(x,z)) \otimes kQ_2(x,z) \to \mathrm{End}_k(kQ_2(x,z));\, f \otimes \gamma \mapsto [\delta \mapsto f(\delta)\gamma].$$

Since $\{\alpha_1\beta_1, \ldots, \alpha_s\beta_s\}$ and $\{\rho_1, \ldots, \rho_s\}$ are k-bases of $kQ_2(x,z)$, we see that $\sigma(\sum_{i=1}^s(\alpha_i\beta_i)^\star \otimes \alpha_i\beta_i) = \mathrm{id} = \sigma(\sum_{i=1}^s \eta^\star_i \otimes \rho_i)$. So, $\sum_{i=1}^s(\alpha_i\beta_i)^\star \otimes \alpha_i\beta_i = \sum_{i=1}^s \eta^\star_i \otimes \rho_i$. In view of the canonical k-linear isomorphism $D(kQ_2(x,z)) \to kQ^{\mathrm{o}}_2(z,x)$, we obtain $\sum_{i=1}^s(\alpha_i\beta_i)^{\mathrm{o}} \otimes \alpha_i\beta_i = \sum_{i=1}^s \eta^{\mathrm{o}}_i \otimes \rho_i$. Applying the tensor product of the canonical projections $kQ_2(x,z) \to e_z\Lambda_2 e_x$ and $kQ^{\mathrm{o}}_2(z,x) \to e_x\Lambda^!_2 e_z$, we obtain

$$\textstyle\sum_{i=1}^s \bar\beta^!_i\bar\alpha^!_i \otimes \bar\alpha_i\bar\beta_i = \sum_{i=1}^s \bar\eta^!_i \otimes \bar\rho_i.$$

Moreover, we clearly have a k-linear morphism

$$\psi : e_x\Lambda^!_2 e_z \otimes e_z\Lambda_2 e_x \to \mathrm{Hom}_\Lambda(I^!_x\langle n\rangle, I^!_z\langle n{+}2\rangle) \otimes \mathrm{Hom}_k(M_n(x), M_{n+2}(z)),$$

sending $\bar\eta^! \otimes \bar\rho$ to $I[\bar\eta^!] \otimes M(\bar\rho)$. Applying this to the above equation, we obtain

$$\textstyle\sum_{i=1}^s I[\bar\beta^!_i\bar\alpha^!_i] \otimes M(\bar\alpha_i\bar\beta_i) = \sum_{i=1}^s I[\bar\eta^!_i] \otimes M(\bar\rho_i) = 0.$$

Therefore, $d^{n+1}_{\mathcal{G}(M)} \circ d^n_{\mathcal{G}(M)} = 0$. That is, $\mathcal{G}(M)^\bullet$ is a complex in $C(\mathrm{GMod}\Lambda^!)$.

Next, given a morphism $f : M \to N$ in $\mathrm{GMod}\Lambda$, it is easy to verify that $\mathcal{G}(f)^{n+1} \circ d^n_{\mathcal{G}(M)} = d^n_{\mathcal{G}(N)} \circ \mathcal{G}(f)^n$, for $n \in \mathbb{Z}$. This yields a morphism of complexes $\mathcal{G}(f)^\bullet : \mathcal{G}(M)^\bullet \to \mathcal{G}(N)^\bullet$. Since the tensor product is over a field, $\mathcal{G}$ is exact and faithful. Finally, consider a morphism $f^\bullet : \mathcal{G}(M)^\bullet \to \mathcal{G}(N)^\bullet$ in $C(\mathrm{GMod}\Lambda^!)$, where $M, N \in \mathrm{GMod}\Lambda$. Write

$$f^n = (f^n(y,x))_{(y,x)\in Q_0\times Q_0} : \oplus_{x\in Q_0}(I^!_x\langle n\rangle \otimes M_n(x)) \to \oplus_{y\in Q_0}(I^!_y\langle n\rangle \otimes N_n(y)),$$

where $f^n(y,x) : I^!_x\langle n\rangle \otimes M_n(x) \to I^!_y\langle n\rangle \otimes N_n(y)$ is a graded morphism. In view of Proposition 3.4.6, we see that $f^n(y,x) = 0$ if $y \neq x$; and otherwise, $f^n(y,x) = \mathrm{id}_{I^!_x} \otimes g_{n,x}$, for some $g_{n,x} \in \mathrm{Hom}_k(M_n(x), N_n(x))$. This implies

$$f^n = \oplus_{x\in Q_0}(\mathrm{id}_{I^!_x} \otimes g_{n,x}) : \oplus_{x\in Q_0}(I^!_x\langle n\rangle \otimes M_n(x)) \to \oplus_{x\in Q_0}(I^!_x\langle n\rangle \otimes N_n(x)).$$

Let $(z,x) \in Q_0 \times Q_0$ with $Q_1(x,z) \neq 0$. We deduce from the equation $f^{n+1} \circ d^n_{\mathcal{G}(M)} = d^n_{\mathcal{G}(N)} \circ f^n$ that

$$(f^{n+1} \circ d^n_{\mathcal{G}(M)})(z,x) = (d^n_{\mathcal{G}(N)} \circ f^n)(z,x) : I^!_x\langle n\rangle \otimes M_n(x) \to I^!_z\langle n{+}1\rangle \otimes N_{n+1}(z),$$

namely, $\sum_{\alpha\in Q_1(x,z)} I[\bar{\alpha}^!] \otimes (N(\bar{\alpha}) \circ g_{n,x}) = \sum_{\alpha\in Q_1(x,z)} I[\bar{\alpha}^!] \otimes (g_{n+1,z} \circ M(\bar{\alpha}))$. By the uniqueness stated in Proposition 3.4.6, $N(\bar{\alpha}) \circ g_{n,x} = g_{n+1,z} \circ M(\bar{\alpha})$, for every arrow $\alpha \in Q_1(x,z)$. This yields $g = (g_{n,x})_{(n,x)\in\mathbb{Z}\times Q_0}$ is a Λ-linear graded morphism from M to N such that $\mathcal{G}(g) = f$. That is, $\mathcal{G}$ is full. The proof of the proposition is completed.

Let $X^\bullet$ be a complex in $C(\mathrm{GMod}\Lambda)$. Given $s \in \mathbb{Z}$, we define the **grading** s-**shift** $X^\bullet\langle s\rangle$ by setting $(X^\bullet\langle s\rangle)^n = X^n\langle s\rangle$ and $d^n_{X^\bullet\langle s\rangle} = d^n_X\langle s\rangle$ for all $n \in \mathbb{Z}$.

6.1.2 **Lemma**. *Let $\Lambda = kQ/R$ be a quadratic algebra, where Q is a locally finite quiver. If $X^\bullet$ is a complex in $C(\mathrm{GMod}\Lambda)$, then*

(1) $(X^\bullet\langle s\rangle)\langle t\rangle = X^\bullet\langle s+t\rangle$, *for all* $s,t \in \mathbb{Z}$;

(2) $\mathrm{H}^n(X^\bullet\langle s\rangle) = \mathrm{H}^n(X^\bullet)\langle s\rangle$, *for all* $n,s \in \mathbb{Z}$.

Proof. Statement (1) is evident. Let $X^\bullet \in C(\mathrm{GMod}\Lambda)$. Given $s,n \in \mathbb{Z}$, in view of Lemma 3.1.9, we have

$$\begin{aligned}\mathrm{H}^n(X^\bullet\langle s\rangle) &= \mathrm{Ker}(d^n_{X^\bullet\langle s\rangle})/\mathrm{Im}(d^{n-1}_{X^\bullet\langle s\rangle}) \\ &= \mathrm{Ker}(d^n_X\langle s\rangle)/\mathrm{Im}(d^{n-1}_X\langle s\rangle) \\ &= (\mathrm{Ker}(d^n_X)/\mathrm{Im}(d^{n-1}_X))\langle s\rangle \\ &= \mathrm{H}^n(X^\bullet)\langle s\rangle.\end{aligned}$$

The proof of the lemma is completed.

The following statement tells us how the Koszul functors $\mathcal{F}$ and $\mathcal{G}$ are related to the grading shift of graded modules and the shift of complexes; compare [48, Proposition 20].

6.1.3 **Lemma**. *Let $\Lambda = kQ/R$ be a quadratic algebra, where Q is a locally finite quiver. If $M \in \mathrm{GMod}\Lambda$ and $s \in \mathbb{Z}$, then $\mathfrak{t}^s(\mathcal{F}(M)^{\bullet}[s]) = \mathcal{F}(M\langle s\rangle)^{\bullet}\langle s\rangle$ and $\mathfrak{t}^s(\mathcal{G}(M)^{\bullet}[s]) = \mathcal{G}(M\langle s\rangle)^{\bullet}\langle s\rangle$, where $\mathfrak{t}$ is the twist functor.*

Proof. We shall only prove the first part, since the proof of the second part is similar. Let $M \in \mathrm{GMod}\Lambda$ and $s, n \in \mathbb{Z}$. By definition,

$$\mathfrak{t}^s(\mathcal{F}(M)^{\bullet}[s])^n = (\mathcal{F}(M)^{\bullet}[s])^n = \mathcal{F}(M)^{n+s} = \oplus_{x\in Q_0}(P_x^!\langle n+s\rangle \otimes M_{n+s}(x))$$

and $d^n_{\mathfrak{t}^s(\mathcal{F}(M)^{\bullet}[s])} = (-1)^s d^n_{\mathcal{F}(M)^{\bullet}[s]} = d^{n+s}_{\mathcal{F}(M)}$. On the other hand, by definition,

$$\mathcal{F}(M\langle s\rangle)^n = \oplus_{x\in Q_0}(P_x^!\langle n\rangle \otimes M_{n+s}(x)).$$

Writing $\mathcal{F}(M\langle s\rangle)^{n+1} = \oplus_{y\in Q_0}(P_y^!\langle n+1\rangle \otimes M_{n+s+1}(y))$, we have

$$d^n_{\mathcal{F}(M\langle s\rangle)} = (d^n_{\mathcal{F}(M\langle s\rangle)}(y,x))_{(y,x)\in Q_0\times Q_0} : \mathcal{F}(M\langle s\rangle)^n \to \mathcal{F}(M\langle s\rangle)^{n+1},$$

where

$$d^n_{\mathcal{F}(M\langle s\rangle)}(y,x) = \textstyle\sum_{\alpha\in Q_1(x,y)} P[\bar{\alpha}^!] \otimes M(\bar{\alpha}) : P_x^!\langle n\rangle \otimes M_{n+s}(x) \to P_y^!\langle n+1\rangle \otimes M_{n+s+1}(y).$$

Therefore,

$$(\mathcal{F}(M\langle s\rangle)^{\bullet}\langle s\rangle)^n = \mathcal{F}(M\langle s\rangle)^n\langle s\rangle = \oplus_{x\in Q_0}(P_x^!\langle n+s\rangle \otimes M_{n+s}(x)) = \mathfrak{t}^s(\mathcal{F}(M)^{\bullet}[s])^n,$$

and

$$d^n_{\mathcal{F}(M\langle s\rangle)^{\bullet}\langle s\rangle} = d^n_{\mathcal{F}(M\langle s\rangle)}\langle s\rangle = (d^n_{\mathcal{F}(M\langle s\rangle)}(y,x)\langle s\rangle)_{(y,x)\in Q_0\times Q_0},$$

where $d^n_{\mathcal{F}(M\langle s\rangle)}(y,x)\langle s\rangle$ is the morphism

$$\textstyle\sum_{\alpha\in Q_1(x,y)} P[\bar{\alpha}^!] \otimes M(\bar{\alpha}) : P_x^!\langle n+s\rangle \otimes M_{n+s}(x) \to P_y^!\langle n+s+1\rangle \otimes M_{n+s+1}(y).$$

So, $d^n_{\mathcal{F}(M\langle s\rangle)}(y,x)\langle s\rangle = d^{n+s}_{\mathcal{F}(M)}(y,x)$ for all $(y,x) \in Q_0 \times Q_0$, and consequently, $d^n_{\mathcal{F}(M\langle s\rangle)}\langle s\rangle = d^{n+s}_{\mathcal{F}(M)}$. Therefore, $d^n_{\mathcal{F}(M\langle s\rangle)^{\bullet}\langle s\rangle} = d^n_{\mathfrak{t}^s(\mathcal{F}(M)^{\bullet}[s])}$, for all $n \in \mathbb{Z}$. This shows that $\mathfrak{t}^s(\mathcal{F}(M)^{\bullet}[s]) = \mathcal{F}(M\langle s\rangle)^{\bullet}\langle s\rangle$. And similarly, $\mathfrak{t}^s(\mathcal{G}(M)^{\bullet}[s]) = \mathcal{G}(M\langle s\rangle)^{\bullet}\langle s\rangle$. The proof of the lemma is completed.

We shall need the following statement.

6.1.4 **Corollary**. *Let $\Lambda = kQ/R$ be a quadratic algebra, where Q is a locally finite quiver. If M is a module in $\mathrm{GMod}\Lambda$, then $\mathrm{H}^{n-s}(\mathcal{F}(M\langle s\rangle)^{\bullet}) = \mathrm{H}^n(\mathcal{F}(M)^{\bullet})\langle -s\rangle$ and $\mathrm{H}^{n-s}(\mathcal{G}(M\langle s\rangle)^{\bullet}) = \mathrm{H}^n(\mathcal{G}(M)^{\bullet})\langle -s\rangle$, for all $n, s \in \mathbb{Z}$.*

Proof. Let $M \in \mathrm{GMod}\Lambda$. Fix $s, n \in \mathbb{Z}$. Since $\mathfrak{t}$ is an automorphism of $C(\mathrm{GMod}\Lambda)$, applying Lemmas 6.1.3 and 6.1.2(2), we see that

$$\begin{aligned} \mathrm{H}^n(\mathcal{F}(M)^\bullet) &= \mathrm{H}^{n-s}(\mathcal{F}(M)^\bullet[s]) \\ &= \mathrm{H}^{n-s}(\mathfrak{t}^s(\mathcal{F}(M)^\bullet[s])) \\ &= \mathrm{H}^{n-s}(\mathcal{F}(M\langle s\rangle)^\bullet\langle s\rangle) \\ &= \mathrm{H}^{n-s}(\mathcal{F}(M\langle s\rangle)^\bullet)\langle s\rangle. \end{aligned}$$

By Lemma 6.1.2(1), we have $\mathrm{H}^{n-s}(\mathcal{F}(M\langle s\rangle)^\bullet) = \mathrm{H}^n(\mathcal{F}(M)^\bullet)\langle -s\rangle$. Similarly, $\mathrm{H}^{n-s}(\mathcal{G}(M\langle s\rangle)^\bullet) = \mathrm{H}^n(\mathcal{G}(M)^\bullet)\langle -s\rangle$. The proof of the Corollary is completed.

To conclude this section, we shall show that the Koszul functors are compatible with tensor products and arbitrary direct sums. Let $(X^\bullet, d_X^\bullet)$ be a complex in $C(\mathrm{GMod}\Lambda)$. Given $V \in \mathrm{Mod}k$, we shall define $X^\bullet \otimes V$ to be the complex such that $(X^\bullet \otimes V)^n = X^n \otimes V$ and $d^n_{X\otimes V} = d^n_X \otimes \mathrm{id}_V$ for all $n \in \mathbb{Z}$.

6.1.5 **Lemma**. *Let $\Lambda = kQ/R$ be a quadratic algebra, where Q is a locally finite quiver.*

1. *If $M \in \mathrm{GMod}\Lambda$ and $V \in \mathrm{Mod}k$, then $\mathcal{F}(M\otimes V)^\bullet = \mathcal{F}(M)^\bullet \otimes V$ and $\mathcal{G}(M\otimes V)^\bullet = \mathcal{G}(M)^\bullet \otimes V$.*
2. *If M_σ with $\sigma \in \Sigma$ are modules in $\mathrm{GMod}\Lambda$, then $\mathcal{F}(\oplus_{\sigma\in\Sigma} M_\sigma)^\bullet = \oplus_{\sigma\in\Sigma}\mathcal{F}(M_\sigma)^\bullet$ and $\mathcal{G}(\oplus_{\sigma\in\Sigma} M_\sigma)^\bullet = \oplus_{\sigma\in\Sigma}\mathcal{G}(M_\sigma)^\bullet$.*

Proof. (1) Let $M \in \mathrm{GMod}\Lambda$ and $V \in \mathrm{Mod}k$. Fix $n \in \mathbb{Z}$. By definition,

$$\begin{aligned} \mathcal{F}(M\otimes V)^n &= \oplus_{x\in Q_0}(P^!_x\langle n\rangle \otimes M_n(x) \otimes V) \\ &= (\oplus_{x\in Q_0} P^!_x\langle n\rangle \otimes M_n(x)) \otimes V \\ &= \mathcal{F}(M)^n \otimes V. \end{aligned}$$

Writing $\mathcal{F}(M\otimes V)^{n+1} = \oplus_{y\in Q_0}(P^!_y\langle n+1\rangle \otimes M_{n+1}(y)\otimes V)$, we have

$$d^n_{\mathcal{F}(M\otimes V)} = (d^n_{\mathcal{F}(M\otimes V)}(y,x))_{(y,x)\in Q_0\times Q_0} : \mathcal{F}(M\otimes V)^n \to \mathcal{F}(M\otimes V)^{n+1},$$

where $d^n_{\mathcal{F}(M\otimes V)}(y,x)$ is the morphism

$$\textstyle\sum_{\alpha\in Q_1(x,y)} P[\bar\alpha^!] \otimes (M\otimes V)(\bar\alpha) : P^!_x\langle n\rangle \otimes M_n(x)\otimes V \to P^!_y\langle n+1\rangle \otimes M_{n+1}(y)\otimes V.$$

On the other hand, writing $\mathcal{F}(M)^{n+1}\otimes V = (\oplus_{y\in Q_0} P^!_y\langle n+1\rangle \otimes M_{n+1}(x))\otimes V$, we have

$$d^n_{\mathcal{F}(M)\otimes V} = (d^n_{\mathcal{F}(M)\otimes V}(y,x))_{(y,x)\in Q_0\times Q_0} : \mathcal{F}(M)^n\otimes V \to \mathcal{F}(M)^{n+1}\otimes V,$$

where $d^n_{\mathcal{F}(M)\otimes V}(y,x)$ is the morphism

$$\sum_{\alpha\in Q_1(x,y)} P[\bar\alpha^!]\otimes M(\bar\alpha)\otimes \mathrm{id}_V : P^!_x\langle n\rangle\otimes M_n(x)\otimes V\to P^!_y\langle n{+}1\rangle\otimes M_{n+1}(y)\otimes V.$$

Since $(M\otimes V)(\bar\alpha)=M(\bar\alpha)\otimes \mathrm{id}_V$ for any $\alpha\in Q_1(x,y)$ with $(y,x)\in Q_0\times Q_0$, we have $d^n_{\mathcal{F}(M\otimes V)}=d^n_{\mathcal{F}(M)\otimes V}$. This shows that $\mathcal{F}(M\otimes V)^\bullet=\mathcal{F}(M)^\bullet\otimes V$. And similarly, $\mathcal{G}(M\otimes V)^\bullet=\mathcal{G}(M)^\bullet\otimes V$.

(2) Let $\{M_\sigma\}_{\sigma\in\Sigma}$ be a family of modules in GModΛ. By Proposition 3.2.3(1), we see that

$$\begin{aligned}\mathcal{F}(\oplus_{\sigma\in\Sigma}M_\sigma)^n &= \oplus_{x\in Q_0}(P^!_x\langle n\rangle\otimes(\oplus_{\sigma\in\Sigma}M_\sigma)_n(x))\\ &= \oplus_{\sigma\in\Sigma}\oplus_{x\in Q_0}(P^!_x\langle n\rangle\otimes(M_\sigma)_n(x))\\ &= \oplus_{\sigma\in\Sigma}\mathcal{F}(M_\sigma)^n.\end{aligned}$$

Writing $\mathcal{F}(\oplus_{\sigma\in\Sigma}M_\sigma)^{n+1}=\oplus_{y\in Q_0}(P^!_y\langle n+1\rangle\otimes(\oplus_{\sigma\in\Sigma}(M_\sigma)_{n+1}(y)))$, we have

$$d^n_{\mathcal{F}(\oplus_{\sigma\in\Sigma}M_\sigma)}=(d^n_{\mathcal{F}(\oplus_{\sigma\in\Sigma}M_\sigma)}(y,x))_{(y,x)\in Q_0\times Q_0}:\mathcal{F}(\oplus_{\sigma\in\Sigma}M_\sigma)^n\to\mathcal{F}(\oplus_{\sigma\in\Sigma}M_\sigma)^{n+1},$$

where

$$d^n_{\mathcal{F}(\oplus_{\sigma\in\Sigma}M_\sigma)}(y,x):P^!_x\langle n\rangle\otimes(\oplus_{\sigma\in\Sigma}(M_\sigma)_n(x))\to P^!_y\langle n{+}1\rangle\otimes(\oplus_{\sigma\in\Sigma}(M_\sigma)_{n+1}(y))$$

is the morphism defined by

$$d^n_{\mathcal{F}(\oplus_{\sigma\in\Sigma}M_\sigma)}(y,x)=\sum_{\alpha\in Q_1(x,y)}P[\bar\alpha^!]\otimes(\oplus_{\sigma\in\Sigma}M_\sigma)(\bar\alpha).$$

On the other hand, given $\sigma\in\Sigma$, write $\mathcal{F}(M_\sigma)^{n+1}=\oplus_{y\in Q_0}(P^!_y\langle n{+}1\rangle\otimes(M_\sigma)_{n+1}(y))$. Then, we have

$$d^n_{\mathcal{F}(M_\sigma)}=(d^n_{\mathcal{F}(M_\sigma)}(y,x))_{(y,x)\in Q_0\times Q_0}:\mathcal{F}(M_\sigma)^n\to\mathcal{F}(M_\sigma)^{n+1},$$

where $d^n_{\mathcal{F}(M_\sigma)}(y,x)$ is the morphism

$$\sum_{\alpha\in Q_1(x,y)}P[\bar\alpha^!]\otimes M_\sigma(\bar\alpha): P^!_x\langle n\rangle\otimes(M_\sigma)_n(x)\to P^!_y\langle n{+}1\rangle\otimes(M_\sigma)_{n+1}(y).$$

Thus, we see that

$$\begin{aligned}d^n_{\mathcal{F}(\oplus_{\sigma\in\Sigma}M_\sigma)}(y,x) &= \sum_{\alpha\in Q_1(x,y)}P[\bar\alpha^!]\otimes(\oplus_{\sigma\in\Sigma}M_\sigma)(\bar\alpha)\\ &= \sum_{\alpha\in Q_1(x,y)}P[\bar\alpha^!]\otimes(\oplus_{\sigma\in\Sigma}M_\sigma(\bar\alpha))\\ &= \oplus_{\sigma\in\Sigma}(\sum_{\alpha\in Q_1(x,y)}P[\bar\alpha^!]\otimes M_\sigma(\bar\alpha))\\ &= \oplus_{\sigma\in\Sigma}d^n_{\mathcal{F}(M_\sigma)}(y,x)\\ &= d^n_{\oplus_{\sigma\in\Sigma}\mathcal{F}(M_\sigma)}(y,x);\ \text{ for all } (y,x)\in Q_0\times Q_0.\end{aligned}$$

That is, $d^n_{\mathcal{F}(\oplus_{\sigma\in\Sigma}M_\sigma)}=d^n_{\oplus_{\sigma\in\Sigma}\mathcal{F}(M_\sigma)}$. Consequently, $\mathcal{F}(\oplus_{\sigma\in\Sigma}M_\sigma)^\bullet=\oplus_{\sigma\in\Sigma}\mathcal{F}(M_\sigma)^\bullet$. And similarly, we can show that $\mathcal{G}(\oplus_{\sigma\in\Sigma}M_\sigma)^\bullet=\oplus_{\sigma\in\Sigma}\mathcal{G}(M_\sigma)^\bullet$. The proof of the lemma is completed.

6.2 Complex Koszul functors

In this section, we shall extend the Koszul functors to obtain the complex Koszul functors. They are the graded version of those defined in [16, Section 5] under the non-graded setting and the assumption that the quiver is locally finite gradable.

Throughout this section, we always assume that $\Lambda = kQ/R$ is a quadratic algebra, where Q is a locally finite quiver. As described in Section 1.8, the right Koszul functor $\mathcal{F} : \mathrm{GMod}\Lambda \to C(\mathrm{GMod}\Lambda^!)$ and the left Koszul functor $\mathcal{G} : \mathrm{GMod}\Lambda \to C(\mathrm{GMod}\Lambda^!)$ extend to two additive functors

$$\mathcal{F}^{DC} : C(\mathrm{GMod}\Lambda) \to DC(\mathrm{GMod}\Lambda^!); M^\bullet \mapsto \mathcal{F}(M^\bullet)^\bullet\,;\ f^\bullet \mapsto \mathcal{F}(f^\bullet)^\bullet;$$

and

$$\mathcal{G}^{DC} : C(\mathrm{GMod}\Lambda) \to DC(\mathrm{GMod}\Lambda^!); M^\bullet \mapsto \mathcal{G}(M^\bullet)^\bullet\,;\ f^\bullet \mapsto \mathcal{G}(f^\bullet)^\bullet.$$

For convenience, we shall describe these functors explicitly in the following statement, which follows immediately from the definition of the Koszul functors.

6.2.1 **Lemma**. *Let $\Lambda = kQ/R$ be a quadratic algebra, where Q is a locally finite quiver. If $M^\bullet \in C(\mathrm{GMod}\Lambda)$, then*

(1) *the double complex $\mathcal{F}(M^\bullet)^\bullet$ is given by*

$$\mathcal{F}(M^i)^j = \oplus_{x\in Q_0}(P_x^!\langle j\rangle \otimes M_j^i(x));\ \ \textit{for all } i,j\in\mathbb{Z}.$$

(2) *the double complex $\mathcal{G}(M)^\bullet)^\bullet$ is given by*

$$\mathcal{G}(M)^i)^j = \oplus_{x\in Q_0}(I_x^!\langle j\rangle \otimes M_j^i(x));\ \ \textit{for all } i,j\in\mathbb{Z}.$$

Composing these functors with the functor $\mathbb{T} : DC(\mathrm{GMod}\Lambda^!) \to C(\mathrm{GMod}\Lambda^!)$, sending a double complex to its total complex; see (1.8), we obtain two additive functors

$$\mathcal{F}^C = \mathbb{T}\circ\mathcal{F}^{DC} : C(\mathrm{GMod}\Lambda) \to C(\mathrm{GMod}\Lambda^!)$$

and

$$\mathcal{G}^C = \mathbb{T}\circ\mathcal{G}^{DC} : C(\mathrm{GMod}\Lambda) \to C(\mathrm{GMod}\Lambda^!),$$

which are called the **right** and the **left complex Koszul functors**, respectively. They are explicitly described in the following statement.

6.2.2 **Proposition**. *Let $\Lambda = kQ/R$ be a quadratic algebra, where Q is a locally finite quiver.*

(1) *The right complex Koszul functor $\mathcal{F}^C : C(\mathrm{GMod}\Lambda) \to C(\mathrm{GMod}\Lambda^!)$ is faithfully exact such, for $M^\bullet \in C(\mathrm{GMod}\Lambda)$ and $n \in \mathbb{Z}$, that*

$$\mathcal{F}^C(M^\bullet)^n = \oplus_{i\in\mathbb{Z};x\in Q_0}(P_x^!\langle n-i\rangle \otimes M_{n-i}^i(x)).$$

(2) *The left complex Koszul functor $\mathcal{G}^C : C(\mathrm{GMod}\Lambda) \to C(\mathrm{GMod}\Lambda^!)$ is faithfully exact such, for $M^\bullet \in C(\mathrm{GMod}\Lambda)$ and $n \in \mathbb{Z}$, that*

$$\mathcal{G}^C(M^\bullet)^n = \oplus_{i\in\mathbb{Z};x\in Q_0}(I_x^!\langle n-i\rangle \otimes M_{n-i}^i(x)).$$

Proof. We shall only prove Statement (1). By Proposition 6.1.1, $\mathcal{F}$ is faithful and exact, and by Proposition 1.8.5, $\mathcal{F}^C$ is faithful and exact. Let $M^\bullet \in C(\mathrm{GMod}\Lambda)$. By definition, $\mathcal{F}^C(M^\bullet)^\bullet = \mathbb{T}(\mathcal{F}(M^\bullet)^\bullet)$. Fix $n \in \mathbb{Z}$. In view of Lemma 6.2.1, the n-diagonal of the double complex $\mathcal{F}(M^\bullet)^\bullet$ consists of

$$\mathcal{F}(M^i)^{n-i} = \oplus_{x\in Q_0}(P_x^!\langle n-i\rangle \otimes M_{n-i}^i(x));\ \text{ for all } i \in \mathbb{Z}.$$

Therefore,

$$\mathcal{F}^C(M^\bullet)^n = \mathbb{T}(\mathcal{F}(M^\bullet)^\bullet)^n = \oplus_{i\in\mathbb{Z}}\mathcal{F}(M^i)^{n-i} = \oplus_{i\in\mathbb{Z};x\in Q_0}(P_x^!\langle n-i\rangle \otimes M_{n-i}^i(x)).$$

The proof of the proposition is completed.

Now, we shall compose the Koszul functors and the complex Koszul functors. For this purpose, we need the following statement, which follows immediately from the definition of the Koszul functors.

6.2.3 **Lemma**. *Let $\Lambda = kQ/R$ be a quadratic algebra, where Q is a locally finite quiver. If $M \in \mathrm{GMod}\Lambda$, then*

(1) *the double complex $\mathcal{F}(\mathcal{G}(M)^\bullet)^\bullet$ is given by*

$$\begin{aligned}\mathcal{F}(\mathcal{G}(M)^i)^j &= \oplus_{a,x\in Q_0}(P_a\langle j\rangle \otimes (I_x^!)_{i+j}(a) \otimes M_i(x))\\ &= \oplus_{a,x\in Q_0}(P_a\langle j\rangle \otimes D(e_a\hat{\Lambda}_{-i-j}e_x) \otimes M_i(x));\ \text{for all } i,j \in \mathbb{Z}.\end{aligned}$$

(2) *the double complex $\mathcal{G}(\mathcal{F}(M)^\bullet)^\bullet$ is given by*

$$\begin{aligned}\mathcal{G}(\mathcal{F}(M)^i)^j &= \oplus_{a,x\in Q_0}(I_a\langle j\rangle \otimes (P_x^!)_{i+j}(a) \otimes M_i(x))\\ &= \oplus_{a,x\in Q_0}(I_a\langle j\rangle \otimes e_a\,\Lambda_{i+j}^!e_x \otimes M_i(x));\ \text{for all } i,j \in \mathbb{Z}.\end{aligned}$$

6.2.4 **Proposition**. *Let $\Lambda = kQ/R$ be a quadratic algebra, where Q is a locally finite quiver.*

(1) *The functor $\mathcal{F}^C \circ \mathcal{G} : \mathrm{GMod}\Lambda \to C(\mathrm{GMod}\Lambda)$ is faithfully exact such that*

$$(\mathcal{F}^C \circ \mathcal{G})(M)^n = \oplus_{i\in\mathbb{Z};a,x\in Q_0}(P_a\langle n-i\rangle \otimes D(e_a\hat{\Lambda}_{-n}e_x) \otimes M_i(x)),$$

for all $M \in \mathrm{GMod}\Lambda$ and $n \in \mathbb{Z}$.

(2) *The functor $\mathcal{G}^C \circ \mathcal{F} : \mathrm{GMod}\Lambda \to C(\mathrm{GMod}\Lambda)$ is faithfully exact such that*

$$(\mathcal{G}^C \circ \mathcal{F})(M)^n = \oplus_{i\in\mathbb{Z};x,a\in Q_0}(I_a\langle n-i\rangle \otimes e_a\Lambda^!_n e_x \otimes M_i(x)),$$

for all $M \in \mathrm{GMod}\Lambda$ and $n \in \mathbb{Z}$.

Proof. We shall only prove Statement (1), since the proof of Statement (2) is similar. By Propositions 6.1.1 and 6.2.2, $\mathcal{F}^C$ and $\mathcal{G}$ are faithfully exact, and hence, $\mathcal{F}^C \circ \mathcal{G}$ is faithfully exact. Let $M \in \mathrm{GMod}\Lambda$. By definition, $\mathcal{G}(M)^\bullet$ is given by $\mathcal{G}(M)^i = \oplus_{x\in Q_0}(I^!_x\langle i\rangle \otimes M_i(x))$ for all $i \in \mathbb{Z}$. Moreover,

$$(\mathcal{F}^C \circ \mathcal{G})(M)^\bullet = \mathcal{F}^C(\mathcal{G}(M)^\bullet) = \mathbb{T}(\mathcal{F}(\mathcal{G}(M)^\bullet)^\bullet).$$

Given $n \in \mathbb{Z}$, in view of Lemma 6.2.3(1), we see that the n-diagonal of the double complex $\mathcal{F}(\mathcal{G}(M)^\bullet)^\bullet$ consists of

$$\mathcal{F}(\mathcal{G}(M)^i)^{n-i} = \oplus_{a,x\in Q_0}(P_a\langle n-i\rangle \otimes D(e_a\hat{\Lambda}_{-n}e_x) \otimes M_i(x)); \text{ for all } i \in \mathbb{Z}.$$

Thus,

$$(\mathcal{F}^C \circ \mathcal{G})(M)^n = \oplus_{i\in\mathbb{Z}}\, \mathcal{F}(\mathcal{G}(M)^i)^{n-i} = \oplus_{i\in\mathbb{Z};a,x\in Q_0}(P_a\langle n-i\rangle \otimes D(e_a\hat{\Lambda}_{-n}e_x) \otimes M_i(x)).$$

The proof of the proposition is completed.

To extend the above composite functors, we need the following statement, which follows immediately from the definition of the Koszul functors and Proposition 6.2.4.

6.2.5 **Lemma**. *Let $\Lambda = kQ/R$ be a Koszul algebra, where Q is a locally finite quiver. If $M^\bullet \in C(\mathrm{GMod}\Lambda)$, then*

(1) *the double complex $(\mathcal{F}^C \circ \mathcal{G})(M^\bullet)^\bullet$ is given by*

$$(\mathcal{F}^C \circ \mathcal{G})(M^i)^j = \oplus_{s\in\mathbb{Z};a,x\in Q_0}(P_a\langle j-s\rangle \otimes D(e_a\hat{\Lambda}_{-j}e_x) \otimes M^i_s(x)); \text{ for all } i, j \in \mathbb{Z}.$$

(2) *the double complex* $(\mathcal{G}^C \circ \mathcal{F})(M^\bullet)^\bullet$ *is given by*

$$(\mathcal{G}^C \circ \mathcal{F})(M^i)^j = \oplus_{s\in\mathbb{Z};x,a\in Q_0}(I_a\langle j-s\rangle \otimes e_a\Lambda^!_j e_x \otimes M^i_s(x)); \text{ for all } i,j \in \mathbb{Z}.$$

Now, we are able to describe the extension of the composite functors of the Koszul functors and the complex Koszul functors.

6.2.6 **Proposition**. *Let* $\Lambda = kQ/R$ *be a Koszul algebra, where* Q *is a locally finite quiver.*

(1) *The functor* $(\mathcal{F}^C\circ\mathcal{G})^C : C(\mathrm{GMod}\Lambda)\to C(\mathrm{GMod}\Lambda)$ *is faithfully exact such, for* $M^\bullet \in C(\mathrm{GMod}\Lambda)$ *and* $n \in \mathbb{Z}$, *that*

$$(\mathcal{F}^C\circ\mathcal{G})^C(M^\bullet)^n = \oplus_{i,j\in\mathbb{Z};a,x\in Q_0}(P_a\langle n-i-j\rangle\otimes D(e_a\hat{\Lambda}_{i-n}e_x)\otimes M^i_j(x)).$$

(2) *The functor* $(\mathcal{G}^C\circ\mathcal{F})^C : C(\mathrm{GMod}\Lambda)\to C(\mathrm{GMod}\Lambda)$ *is faithfully exact such, for* $M^\bullet \in C(\mathrm{GMod}\Lambda)$ *and* $n \in \mathbb{Z}$, *that*

$$(\mathcal{G}^C\circ\mathcal{F})^C(M^\bullet)^n = \oplus_{i,j\in\mathbb{Z};\,a,x\in Q_0}\,(I_x\langle n-i-j\rangle \otimes e_a\Lambda^!_{n-i}e_x \otimes M^i_j(a)).$$

Proof. We shall only verify Statement (1), since the verification of Statement (2) is similar. By Proposition 6.2.4, $\mathcal{F}^C\circ\mathcal{G}$ is faithfully exact, and by Proposition 1.8.5, so is $(\mathcal{F}^C\circ\mathcal{G})^C$. Let $M^\bullet \in C(\mathrm{GMod}\Lambda)$. Then, $(\mathcal{F}^C\circ\mathcal{G})^C(M^\bullet) = \mathbb{T}((\mathcal{F}^C\circ\mathcal{G})(M^\bullet)^\bullet)$. Fix $n \in \mathbb{Z}$. By Lemma 6.2.5, the n-diagonal of $(\mathcal{F}^C\circ\mathcal{G})(M^\bullet)^\bullet$ consists of

$$(\mathcal{F}^C\circ\mathcal{G})(M^i)^{n-i} = \oplus_{j\in\mathbb{Z};a,x\in Q_0}(P_a\langle n-i-j\rangle\otimes D(e_a\hat{\Lambda}_{i-n}e_x)\otimes M^i_j(x)); \text{ for all } i \in \mathbb{Z}.$$

As a consequence, we have

$$\begin{aligned}(\mathcal{F}^C\circ\mathcal{G})^C(M^\bullet)^n &= \oplus_{i\in\mathbb{Z}}\,(\mathcal{F}^C\circ\mathcal{G})(M^i)^{n-i}\\ &= \oplus_{i,j\in\mathbb{Z};a,x\in Q_0}(P_a\langle n-i-j\rangle\otimes D(e_a\hat{\Lambda}_{i-n}e_x)\otimes M^i_j(x)).\end{aligned}$$

The proof of the proposition is completed.

6.3 Derived Koszul functors

In this section, we shall show that each complex Koszul functor descends to a 2-real parametrized family of derived Kozul functors from categories derived from subcategories of the category of complexes of a quadratic algebra to those derived from those derived from subcategories of the category of complexes of its quadratic dual. These include the two derived functors constructed in [48, Section 5]. The key ingredients of this section are adapted from those in [16, Section 5] which are in the non-graded setting and under the assumption that the quiver is gradable.

Throughout this section, we always assume that $\Lambda = kQ/R$ is a quadratic algebra, where Q is a locally finite quiver. As mentioned in Section 1.8, the right complex Koszul functor $\mathcal{F}^C : C(\mathrm{GMod}\Lambda) \to C(\mathrm{GMod}\Lambda^!)$ and the left complex Koszul functor $\mathcal{G}^C : C(\mathrm{GMod}\Lambda) \to C(\mathrm{GMod}\Lambda^!)$ do not descend to the whole derived category $D(\mathrm{GMod}\Lambda)$. Therefore, we need to consider some derivable subcategories of $C(\mathrm{GMod}\Lambda)$. For this purpose, we shall view a complex $M^\bullet$ of graded modules $M^i = \oplus_{j\in\mathbb{Z}} M^i_j$ as a bigraded k-vector space M^i_j with $i, j \in \mathbb{Z}$.

6.3.1 **Definition**. Let $\Lambda = kQ/R$ be a quadratic algebra, where Q is a locally finite quiver. Given $p, q \in \mathbb{R}$ with $p \geq 1$ and $q \geq 0$, we denote

(1) by $C^{\downarrow}_{p,q}(\mathrm{GMod}\Lambda)$ the full subcategory of $C(\mathrm{GMod}\Lambda)$ of complexes $M^\bullet$ such that $M^i_j = 0$ for $i + pj \gg 0$ or $i - qj \ll 0$; in other words, $M^\bullet$ concentrates in a lower triangle formed by two lines of slopes $-\frac{1}{p}$ and $\frac{1}{q}$, respectively;

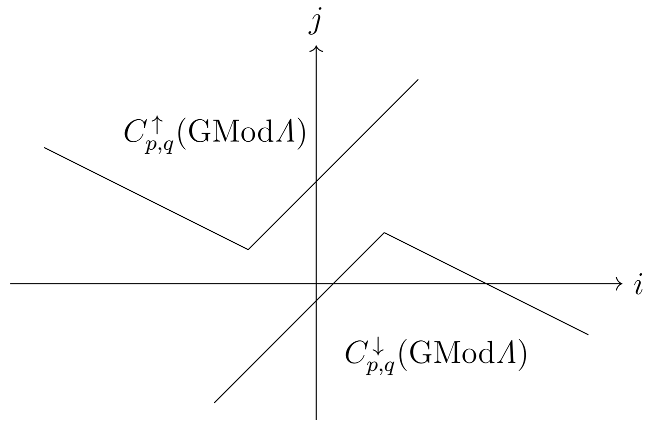
(2) by $C^{\uparrow}_{p,q}(\mathrm{GMod}\Lambda)$ the full subcategory of $C(\mathrm{GMod}\Lambda)$ of complexes $M^\bullet$ such that $M^i_j = 0$ for $i + pj \ll 0$ or $i - qj \gg 0$; in other words, $M^\bullet$ concentrates in an upper triangle formed by two lines of slopes $-\frac{1}{p}$ and $\frac{1}{q}$, respectively.

Remark. (1) Taking $p = 1$ and $q = 0$, we cover the categories $C^{\downarrow}(\Lambda)$ and $C^{\uparrow}(\Lambda)$ as defined in [13, (2.12)]; see also [48, (2.4)].

(2) The categories $C^{\downarrow}_{p,q}(\mathrm{GMod}\Lambda)$ are pairwise distinct derivable subcategories of $C(\mathrm{GMod}^-\Lambda)$ containing $C^b(\mathrm{GMod}^-\Lambda)$.

(3) The categories $C^{\uparrow}_{p,q}(\mathrm{GMod}\Lambda)$ are pairwise distinct derivable subcategories of $C(\mathrm{GMod}^+\Lambda)$ containing $C^b(\mathrm{GMod}^+\Lambda)$.

(4) The triangular regions in which the complexes of these categories are concentrated can be visualized as follows:



Let $\mathfrak{A}$ be an abelian subcategory of GModΛ. We denote by $C^{\downarrow}_{p,q}(\mathfrak{A})$ the full subcategory of $C^{\downarrow}_{p,q}(\mathrm{GMod}\Lambda)$ of complexes of graded modules in $\mathfrak{A}$, by $K^{\downarrow}_{p,q}(\mathfrak{A})$ the quotient category modulo null-homotopic morphisms of $C^{\downarrow}_{p,q}(\mathfrak{A})$, and by $D^{\downarrow}_{p,q}(\mathfrak{A})$ the localization at quasi-isomorphisms of $K^{\downarrow}_{p,q}(\mathfrak{A})$. We define categories $C^{\downarrow}_{p,q}(\mathfrak{A})$, $K^{\uparrow}_{p,q}(\mathfrak{A})$ and $D^{\uparrow}_{p,q}(\mathfrak{A})$ in the same fashion. In other words, $D^{\downarrow}_{p,q}(\mathfrak{A})$ and $D^{\uparrow}_{p,q}(\mathfrak{A})$ are categories derived from $C^{\downarrow}_{p,q}(\mathfrak{A})$ and $C^{\uparrow}_{p,q}(\mathfrak{A})$, respectively.

In order to show that the complex Koszul functors descend to these derived categories, we shall restrict complexes of graded modules to obtain complexes of k-vector spaces. Consider a complex $M^\bullet$ and a morphism $f^\bullet : M^\bullet \to N^\bullet$ in $C(\mathrm{GMod}\Lambda)$. Fix $i \in \mathbb{Z}$. By restricting $M^\bullet$ and $f^\bullet$ to the degree i, we obtain a complex

$$M^\bullet_i : \qquad \cdots \longrightarrow M^{n-1}_i \xrightarrow{(d^{n-1}_M)_i} M^n_i \xrightarrow{(d^n_M)_i} M^{n+1}_i \longrightarrow \cdots$$

and a morphism $f^\bullet_i : M^\bullet_i \to N^\bullet_i$ in $C(\mathrm{Mod}k)$. Similarly, restricting an object $M^{\bullet\bullet}$ and a morphism $f^{\bullet\bullet} : M^{\bullet\bullet} \to N^{\bullet\bullet}$ in $DC(\mathrm{GMod}\Lambda)$ to the degree i, we obtain an object $M^{\bullet\bullet}_i$ and a morphism $f^{\bullet\bullet}_i : M^{\bullet\bullet}_i \to N^{\bullet\bullet}_i$ in $DC(\mathrm{Mod}k)$.

6.3.2 **Lemma**. *Let $\Lambda = kQ/R$ be a quadratic algebra, where Q is a locally finite quiver.*

(1) *If $M^\bullet \in C(\mathrm{GMod}\Lambda)$, then $\mathrm{H}^n(M^\bullet)_i \cong \mathrm{H}^n(M^\bullet_i)$ for all $i \in \mathbb{Z}$.*

(2) *If $f^\bullet : M^\bullet \to N^\bullet$ is a morphism in $C(\mathrm{GMod}\Lambda)$, then $\mathrm{H}^n(f^\bullet)_i \cong \mathrm{H}^n(f^\bullet_i)$, for all $n, i \in \mathbb{Z}$.*

(3) *If* $M^{\bullet\bullet}$ *is an object in* $DC(\mathrm{GMod}\Lambda)$, *then* $\mathbb{T}(M^{\bullet\bullet})_i = \mathbb{T}(M_i^{\bullet\bullet})$, *for all* $i \in \mathbb{Z}$.

(4) *If* $f^{\bullet\bullet}$ *is a morphism in* $DC(\mathrm{GMod}\Lambda)$, *then* $\mathbb{T}(f^{\bullet\bullet})_i = \mathbb{T}(f_i^{\bullet\bullet})$, *for all* $i \in \mathbb{Z}$.

Proof. (1) Consider a complex $(M^\bullet, d^\bullet)$ in $C(\mathrm{GMod}\Lambda)$. In view of Proposition 3.1.8, we see that

$$\mathrm{H}^n(X^\bullet) = \frac{\mathrm{Ker} d^n}{\mathrm{Im} d^{n-1}} = \frac{\oplus_{i\in\mathbb{Z}}\mathrm{Ker}(d_i^n)}{\oplus_{i\in\mathbb{Z}}\mathrm{Im}(d_i^{n-1})} \cong \oplus_{i\in\mathbb{Z}} \frac{\mathrm{Ker}(d_i^n)}{\mathrm{Im}(d_i^{n-1})} = \oplus_{i\in\mathbb{Z}}\mathrm{H}^n(X_i^\bullet).$$

Thus, $\mathrm{H}^n(M^\bullet)_i \cong \mathrm{H}^n(M_i^\bullet)$ for all $i \in \mathbb{Z}$.

(2) Let $f^\bullet : M^\bullet \to N^\bullet$ be a morphism in $C(\mathrm{GMod}\Lambda)$. Fix $n \in \mathbb{Z}$. We have a commutative diagram with exact rows

$$\begin{CD}
0 @>>> \mathrm{Im} d_M^{n-1} @>{q_M^n}>> \mathrm{Ker} d_M^n @>{p_M^n}>> \mathrm{H}^n(M^\bullet) @>>> 0\\
@. @V{\tilde{f}^n}VV @VV{\bar{f}^n}V @VV{\mathrm{H}^n(f^\bullet)}V @.\\
0 @>>> \mathrm{Im} d_N^{n-1} @>{q_N^n}>> \mathrm{Ker} d_N^n @>{p_N^n}>> \mathrm{H}^n(N^\bullet) @>>> 0,
\end{CD}$$

where $\tilde{f}^n$ and $\bar{f}^n$ are induced from f^n. For any $i \in \mathbb{Z}$, we have a commutative diagram with exact rows

$$\begin{CD}
0 @>>> (\mathrm{Im} d_M^{n-1})_i @>{(q_M^n)_i}>> (\mathrm{Ker} d_M^n)_i @>{(p_M^n)_i}>> \mathrm{H}^n(M^\bullet)_i @>>> 0\\
@. @V{\tilde{f}_i^n}VV @VV{\bar{f}_i^n}V @VV{\mathrm{H}^n(f^\bullet)_i}V @.\\
0 @>>> (\mathrm{Im} d_N^{n-1})_i @>{(q_N^n)_i}>> (\mathrm{Ker} d_N^n)_i @>{(p_N^n)_i}>> \mathrm{H}^n(N^\bullet)_i @>>> 0.
\end{CD}$$

By Proposition 3.1.8, this is the commutative diagram with exact rows

$$\begin{CD}
0 @>>> \mathrm{Im}(d_M^{n-1})_i @>{(q_M^n)_i}>> \mathrm{Ker}(d_M^n)_i @>{(p_M^n)_i}>> \mathrm{H}^n(M^\bullet)_i @>>> 0\\
@. @V{\tilde{f}_i^n}VV @VV{\bar{f}_i^n}V @VV{\mathrm{H}^n(f^\bullet)_i}V @.\\
0 @>>> \mathrm{Im}(d_N^{n-1})_i @>{(q_N^n)_i}>> \mathrm{Ker}(d_N^n)_i @>{(p_N^n)_i}>> \mathrm{H}^n(N^\bullet)_i @>>> 0.
\end{CD}$$

Therefore, $\mathrm{H}^n(f^\bullet)_i \cong \mathrm{H}^n(f_i^\bullet)$.

(3) Let $(M^{\bullet\bullet}, v^{\bullet\bullet}, h^{\bullet\bullet})$ be an object in $DC(\mathrm{GMod}\Lambda)$, where $v^{\bullet\bullet}$ is the vertical differential and $h^{\bullet\bullet}$ is the horizontal differential. Fix $i, n \in \mathbb{Z}$. Then, by Proposition 3.2.3(1),

$$\mathbb{T}(M^{\bullet\bullet})_i^n = (\oplus_{j\in\mathbb{Z}} M^{j,n-j})_i = \oplus_{j\in\mathbb{Z}} M_i^{j,n-j} = \mathbb{T}(M_i^{\bullet\bullet}).$$

Moreover,

$$(d^n_{\mathbb{T}(M^{\bullet\bullet})})_i = ((d^n_{\mathbb{T}(M^{\bullet\bullet})}(l,j)_i)_{(l,j)\in\mathbb{Z}\times\mathbb{Z}} : \mathbb{T}(M^{\bullet\bullet})^n_i \to \mathbb{T}(M^{\bullet\bullet})^{n+1}_i = (\oplus_{l\in\mathbb{Z}} M^{l,n-l})_i,$$

where $d^n_{\mathbb{T}(M^{\bullet\bullet})}(l,j)_i : M_i^{j,n-j} \to M_i^{l,n-l}$ is defined such that $d^n_{\mathbb{T}(M^{\bullet\bullet})}(j,j)_i = v_i^{j,n-j}$; $d^n_{\mathbb{T}(M^{\bullet\bullet})}(j+1,j)_i = h_i^{j,n-j}$ and $d^n_{\mathbb{T}(M^{\bullet\bullet})}(l,j)_i = 0$ if $j \neq l$ or $l+1$. On the other hand,

$$d^n_{\mathbb{T}(M_i^{\bullet\bullet})} = (d^n_{\mathbb{T}(M_i^{\bullet\bullet})}(l,j))_{(l,j)\in\mathbb{Z}\times\mathbb{Z}} : \mathbb{T}(M_i^{\bullet\bullet})^n \to \mathbb{T}(M_i^{\bullet\bullet})^{n+1} = (\oplus_{l\in\mathbb{Z}} M_i^{l,n-l}),$$

where $d^n_{\mathbb{T}(M_i^{\bullet\bullet})}(l,j) : M_i^{j,n-j} \to M_i^{l,n-l}$ is defined such that $d^n_{\mathbb{T}(M_i^{\bullet\bullet})}(j,j) = v_i^{j,n-j}$; $d^n_{\mathbb{T}(M_i^{\bullet\bullet})}(j+1,j) = h_i^{j,n-j}$ and $d^n_{\mathbb{T}(M_i^{\bullet\bullet})}(l,j) = 0$ if $j \neq l$ or $l+1$. So, $(d^n_{\mathbb{T}(M^{\bullet\bullet})})_i = d^n_{\mathbb{T}(M_i^{\bullet\bullet})}$. Therefore, $\mathbb{T}(M^{\bullet\bullet})_i = \mathbb{T}(M_i^{\bullet\bullet})$, for all $i \in \mathbb{Z}$.

(4) Let $f^{\bullet\bullet} : M^{\bullet\bullet} \to N^{\bullet\bullet}$ be a morphism in $DC(\mathrm{GMod}\Lambda)$. Fix $i, n \in \mathbb{Z}$. By definition,

$$\mathbb{T}(f^{\bullet\bullet})^n_i = (\mathbb{T}(f^{\bullet\bullet})^n_i(l,j))_{(l,j)\in\mathbb{Z}\times\mathbb{Z}} : (\oplus_{j\in\mathbb{Z}} M^{j,n-j})_i \to (\oplus_{l\in\mathbb{Z}} N^{l,n-l})_i,$$

where $\mathbb{T}(f^{\bullet\bullet})^n_i(l,j) : M_i^{j,n-j} \to N_i^{l,n-l}$ is given by $\mathbb{T}(f^{\bullet\bullet})^n_i(j,j) = f_i^{j,n-j}$ and $\mathbb{T}(f^{\bullet\bullet})^n_i(l,j) = 0$ for all $j \neq l$. On the other hand,

$$\mathbb{T}(f_i^{\bullet\bullet})^n = (\mathbb{T}(f_i^{\bullet\bullet})^n(l,j))_{(l,j)\in\mathbb{Z}\times\mathbb{Z}} : \oplus_{j\in\mathbb{Z}} M_i^{j,n-j} \to \oplus_{l\in\mathbb{Z}} N_i^{l,n-l},$$

where $\mathbb{T}(f_i^{\bullet\bullet})^n(l,j) : M_i^{j,n-j} \to N_i^{l,n-l}$ is given by $\mathbb{T}(f_i^{\bullet\bullet})^n(j,j) = f_i^{j,n-j}$ and $\mathbb{T}(f_i^{\bullet\bullet})^n(l,j) = 0$ for all $j \neq l$. So, $\mathbb{T}(f^{\bullet\bullet})^n_i = \mathbb{T}(f_i^{\bullet\bullet})^n$. Consequently, $\mathbb{T}(f^{\bullet\bullet})_i = \mathbb{T}(f_i^{\bullet\bullet})$, for all $i \in \mathbb{Z}$. The proof of the lemma is completed.

We are ready to have the main result of this section. It is adapted from Theorem 5.3 in [16], which is in the non-graded setting and under the assumption that the quiver is gradable.

6.3.3 **Theorem**. *Let $\Lambda = kQ/R$ be a quadratic algebra, where Q is a locally finite quiver. Consider $p, q \in \mathbb{R}$ with $p \geq 1$ and $q \geq 0$.*

(1) *The right complex Koszul functor $\mathcal{F}^C$ induces a commutative diagram of functors*

$$\begin{array}{ccccc} C^{\downarrow}_{p,q}(\mathrm{GMod}\Lambda) & \longrightarrow & K^{\downarrow}_{p,q}(\mathrm{GMod}\Lambda) & \longrightarrow & D^{\downarrow}_{p,q}(\mathrm{GMod}\Lambda) \\ \downarrow{\scriptstyle \mathcal{F}^C_{p,q}} & & \downarrow{\scriptstyle \mathcal{F}^K_{p,q}} & & \downarrow{\scriptstyle \mathcal{F}^D_{p,q}} \\ C^{\uparrow}_{q+1,p-1}(\mathrm{GMod}\Lambda^!) & \longrightarrow & K^{\uparrow}_{q+1,p-1}(\mathrm{GMod}\Lambda^!) & \longrightarrow & D^{\uparrow}_{q+1,p-1}(\mathrm{GMod}\Lambda^!), \end{array}$$

where $\mathrm{GMod}\Lambda$ *and* $\mathrm{GMod}\Lambda^!$ *can be replaced simultaneously by* $\mathrm{gmod}\Lambda$ *and* $\mathrm{gmod}\Lambda^!$, *respectively.*

(2) *The left complex Koszul functor* $\mathcal{G}^C$ *induces a commutative diagram of functors*

$$\begin{array}{ccccc} C^{\uparrow}_{p,q}(\mathrm{GMod}\Lambda) & \longrightarrow & K^{\uparrow}_{p,q}(\mathrm{GMod}\Lambda) & \longrightarrow & D^{\uparrow}_{p,q}(\mathrm{GMod}\Lambda) \\ \Big\downarrow {\scriptstyle \mathcal{G}^C_{p,q}} & & \Big\downarrow {\scriptstyle \mathcal{G}^K_{p,q}} & & \Big\downarrow {\scriptstyle \mathcal{G}^D_{p,q}} \\ C^{\downarrow}_{q+1,p-1}(\mathrm{GMod}\Lambda^!) & \longrightarrow & K^{\downarrow}_{q+1,p-1}(\mathrm{GMod}\Lambda^!) & \longrightarrow & D^{\downarrow}_{q+1,p-1}(\mathrm{GMod}\Lambda^!), \end{array}$$

where $\mathrm{GMod}\Lambda$ *and* $\mathrm{GMod}\Lambda^!$ *can be replaced simultaneously by* $\mathrm{gmod}\Lambda$ *and* $\mathrm{gmod}\Lambda^!$, *respectively.*

Proof. (1) Let $M^{\bullet} \in C^{\downarrow}_{p,q}(\mathrm{GMod}\Lambda)$, say $M^i_j = 0$ for $i + pj > s$ or $i - qj < t$, where $s, t \in \mathbb{Z}$. Fix $n, r \in \mathbb{Z}$. In view of Proposition 6.2.2(1), we have

$$\mathcal{F}^C(M^{\bullet})^n_r = \oplus_{i \le n+r; x \in Q_0}((P^!_x)_{n+r-i} \otimes M^i_{n-i}(x)).$$

Fix some $i \le n + r$. If $n + (q + 1)r < t$, then $i - q(n - i) < t$; and if $n - (p - 1)r > s$, then $i + p(n - i) > s$. Thus, $\mathcal{F}^C(M^{\bullet})^n_r = 0$ if $n+(q + 1)r < t$ or $n - (p - 1)r > s$. That is, $\mathcal{F}^C(M^{\bullet})^n \in C^{\uparrow}_{q+1,p-1}(\mathrm{GMod}\Lambda^!)$. This yields a functor

$$\mathcal{F}^C_{p,q} : C^{\downarrow}_{p,q}(\mathrm{GMod}\Lambda) \to C^{\uparrow}_{q+1,p-1}(\mathrm{GMod}\Lambda^!).$$

Given $n \in \mathbb{Z}$, by Proposition 6.2.1(1), the n-diagonal of the double complex $\mathcal{F}(M^{\bullet})^{\bullet}$ consists of

$$\mathcal{F}(M^i)^{n-i} = \oplus_{x \in Q_0} (P^!_x\langle n-i \rangle \otimes M^i_{n-i}(x)); \text{ for any } i \in \mathbb{Z}.$$

If $i < (nq + t)(1 + q)^{-1}$, then $M^i_{n-i} = 0$, and hence, $\mathcal{F}(M^i)^{n-i} = 0$. Thus, $\mathcal{F}(M^{\bullet})^{\bullet}$ is diagonally bounded below. By Theorem 1.8.7(1), $\mathcal{F}^C_{p,q}$ sends acyclic complexes to acyclic ones, and by Theorem 1.8.7(2), $\mathcal{F}^C_{p,q}$ induces a commutative diagram as stated in Statement (1). Assume now that the $M_i(x)$ are finite dimensional for all $(i, x) \in \mathbb{Z} \times Q_0$. Given $y \in Q_0$, we have

$$\mathcal{F}^C(M^{\bullet})^n_r(y) = \oplus_{i \le n+r; x \in Q_0}(e_y \Lambda^!_{n+r-i} e_x \otimes M^i_{n-i}(x)).$$

It is easy to see that $\mathcal{F}^C(M^{\bullet})^n_r(y) \ne 0$ only if $(qn + t)(q + 1)^{-1} \le i \le n + r$. So $\mathcal{F}^C(M^{\bullet})^n_r(y)$ is finite dimensional. Thus, $\mathcal{F}^C(M^{\bullet})^{\bullet} \in C(\mathrm{gmod}\Lambda^!)$. That is, $\mathcal{F}^C$ restricts to a functor $\mathcal{F}^C_{p,q} : C^{\downarrow}_{p,q}(\mathrm{gmod}\Lambda) \to C^{\uparrow}_{q+1,p-1}(\mathrm{gmod}\Lambda^!)$. As seen above, it induces a diagram as stated in Statement (1) with $\mathrm{GMod}\Lambda$ and $\mathrm{GMod}\Lambda^!$ replaced simultaneously by $\mathrm{gmod}\Lambda$ and $\mathrm{gmod}\Lambda^!$, respectively.

(2) Similarly, $\mathcal{G}^C$ restricts to functors $\mathcal{G}^C_{p,q} : C^{\uparrow}_{p,q}(\mathrm{GMod}\Lambda) \to C^{\downarrow}_{q+1,p-1}(\mathrm{GMod}\Lambda^!)$ and $\mathcal{G}^C_{p,q} : C^{\uparrow}_{p,q}(\mathrm{gmod}\Lambda) \to C^{\downarrow}_{q+1,p-1}(\mathrm{gmod}\Lambda^!)$. Let $N^\bullet$ be an acyclic complex in $C(\mathrm{GMod}\Lambda)$. Then, $\mathcal{G}^C(N^\bullet) = \mathbb{T}(\mathcal{G}(N^\bullet)^\bullet)$. Fix $r \in \mathbb{Z}$. In view of Proposition 6.3.2(2), $\mathcal{G}^C(N^\bullet)_r = \mathbb{T}(\mathcal{G}(N^\bullet)^\bullet_r)$. Since $\mathcal{G}$ is exact, the double complex $\mathcal{G}(N^\bullet)^\bullet$ has acyclic rows, and so does the double complex $\mathcal{G}(N^\bullet)^\bullet_r$. Given $n \in \mathbb{Z}$, by Lemma 6.2.1(2), the n-diagonal of $\mathcal{G}(N^\bullet)^\bullet_r$ consists of

$$\mathcal{G}(N^i)^{n-i}_r = \oplus_{x \in Q_0}((I^!_x)_{n+r-i} \otimes N^i_{n-i}(x)); \text{ for any } i \in \mathbb{Z}.$$

If $i < n + r$, then $(I^!_x)_{n+r-i} = 0$, and hence, $\mathcal{G}(N^i)^{n-i}_r = 0$. Thus, $\mathcal{G}(N^\bullet)^\bullet{}_r$ is diagonally bounded-below. By Proposition 1.8.2, $\mathbb{T}(\mathcal{G}(N^\bullet)^\bullet_r)$ is acyclic. That is, $\mathcal{G}^C(N^\bullet)_r$ is acyclic, for all $r \in \mathbb{Z}$. By Lemma6.3.2(1), $\mathcal{G}^C(N^\bullet)$ is acyclic. By Theorem 1.8.7(2), $\mathcal{G}^C$ induces a commutative diagram as stated in Statement (2). The proof of the theorem is completed.

REMARK. In case $p = 1$ and $q = 0$, Theorem 6.3.3 has been established for positively graded quadratic categories; see [48, Proposition 20].

(2) By Theorem 1.8.7, the functors $\mathcal{F}^D_{pq}$ and $\mathcal{G}^D_{pq}$ in Theorem 6.3.3 are triangle-exact , which will be called the **right** and the **left derived Koszul functors**, respectively.

Next, we shall show that the complex Koszul functors descend always to the bounded derived category of finitely piece-supported graded modules. For this purpose, we observe that $K^b(\mathrm{GProj}\Lambda)$ and $K^b(\mathrm{GInj}\Lambda)$ are full triangulated subcategories of $D^b(\mathrm{GMod}\Lambda)$, while $K^b(\mathrm{gproj}\Lambda)$ and $K^b(\mathrm{ginj}\Lambda)$ are full triangulated subcategories of $D^b(\mathrm{gmod}\Lambda)$; see (1.7.9).

6.3.4 **Theorem**. *Let $\Lambda = kQ/R$ be a quadratic algebra, where Q is a locally finite quiver.*

(1) *The right Koszul functor $\mathcal{F} : \mathrm{GMod}\Lambda \to C(\mathrm{GMod}\Lambda^!)$ induces a commutative diagram of functors*

$$\begin{array}{ccccc} C^b(\mathrm{GMod}^b\Lambda) & \longrightarrow & K^b(\mathrm{GMod}^b\Lambda) & \longrightarrow & D^b(\mathrm{GMod}^b\Lambda) \\ \downarrow{\scriptstyle \mathcal{F}^C} & & \downarrow{\scriptstyle \mathcal{F}^K} & & \downarrow{\scriptstyle \mathcal{F}^D} \\ C^b(\mathrm{GProj}\Lambda^!) & \longrightarrow & K^b(\mathrm{GProj}\Lambda^!) & \longrightarrow & D^b(\mathrm{GMod}\Lambda^!), \end{array}$$

where $\mathrm{GMod}^b\Lambda$, $\mathrm{GProj}\Lambda^!$ *and* $\mathrm{GMod}\Lambda^!$ *can be replaced simultaneously by* $\mathrm{gmod}^b\Lambda$, $\mathrm{gproj}\Lambda^!$ *and* $\mathrm{gmod}\Lambda^!$, *respectively.*

(2) *The left Koszul functor* $\mathcal{G}\colon \mathrm{GMod}\Lambda \to C(\mathrm{GMod}\Lambda^!)$ *induces a commutative diagram of functors*

$$\begin{array}{ccccc}
C^b(\mathrm{GMod}^b\Lambda) & \longrightarrow & K^b(\mathrm{GMod}^b\Lambda) & \longrightarrow & D^b(\mathrm{GMod}^b\Lambda) \\
{\scriptstyle \mathcal{G}^C}\big\downarrow & & {\scriptstyle \mathcal{G}^K}\big\downarrow & & {\scriptstyle \mathcal{G}^D}\big\downarrow \\
C^b(\mathrm{GInj}\Lambda^!) & \longrightarrow & K^b(\mathrm{GInj}\Lambda^!) & \longrightarrow & D^b(\mathrm{GMod}\Lambda^!),
\end{array}$$

where $\mathrm{GMod}^b\Lambda$, $\mathrm{GInj}\Lambda^!$ *and* $\mathrm{GMod}\Lambda^!$ *can be replaced simultaneously by* $\mathrm{gmod}^b\Lambda$, $\mathrm{ginj}\Lambda^!$ *and* $\mathrm{gmod}\Lambda^!$, *respectively.*

Proof. We shall only prove Statement (1). Let $M^\bullet \in C^b(\mathrm{GMod}^b\Lambda)$. There exist integers $s, t > 0$ such that $M^i_j \neq 0$ only if $-s \le i \le s$ and $-t \le j \le t$. Given $n \in \mathbb{Z}$, by Proposition 6.2.2(1), we have

$$\mathcal{F}^C(M^\bullet)^n = \oplus_{(i,x)\in\mathbb{Z}\times Q_0}(P^!_x\langle n-i\rangle \otimes M^i_{n-i}(x)).$$

Since $M^i_{n-i} \neq 0$ only if $-s-t \le n \le s+t$, the complex $\mathcal{F}^C(M^\bullet)$ is bounded. And since the M^i are finitely piece-supported, $M^i_{n-i}(x) \neq 0$ only for finitely many pairs (i,x) with $-s \le i \le s$ and $x \in Q_0$. Thus, $\mathcal{F}^C(M^\bullet) \in C^b(\mathrm{GProj}\Lambda^!)$. This yields a functor $\mathcal{F}^C\colon C^b(\mathrm{GMod}^b\Lambda) \to C^b(\mathrm{GProj}\Lambda^!)$. As seen in the proof of Theorem 6.3.3, it induces a commutative diagram as stated in Statement (1).

Suppose that $M^\bullet \in C^b(\mathrm{gmod}^b\Lambda)$. Then, the $M^i_{n-i}(x)$ are finite dimensional. So, $\mathcal{F}^C(M^\bullet) \in C^b(\mathrm{gproj}\Lambda)$. This yields a functor $\mathcal{F}^C : C^b(\mathrm{gmod}^b\Lambda) \to C^b(\mathrm{gproj}\Lambda^!)$. As seen above, it induces a commutative diagram as stated in Statement (1) with $\mathrm{GMod}^b\Lambda$, $\mathrm{GProj}\Lambda^!$ and $\mathrm{GMod}\Lambda^!$ replaced by $\mathrm{gmod}^b\Lambda$, $\mathrm{gproj}\Lambda^!$ and $\mathrm{gmod}\Lambda^!$, respectively. The proof of the theorem is completed.

In case $\Lambda^!$ is locally left or locally right bounded, as shown below, the functor $\mathcal{F}^D : D^b(\mathrm{GMod}^b\Lambda) \to D^b(\mathrm{GMod}\Lambda^!)$ or $\mathcal{G}^D : D^b(\mathrm{GMod}^b\Lambda) \to D^b(\mathrm{GMod}\Lambda^!)$ co-restricts to $D^b(\mathrm{GMod}^b\Lambda^!)$, respectively.

6.3.5 **Corollary**. *Let* $\Lambda = kQ/R$ *be a quadratic algebra, where* Q *is a locally finite quiver.*

(1) *In case* $\Lambda^!$ *is locally left bounded, the right Koszul functor* $\mathcal{F} : \mathrm{GMod}\Lambda \to C(\mathrm{GMod}\Lambda^!)$ *induces a commutative diagram of functors*

$$\begin{array}{ccccc}
C^b(\mathrm{GMod}^b\Lambda) & \longrightarrow & K^b(\mathrm{GMod}^b\Lambda) & \longrightarrow & D^b(\mathrm{GMod}^b\Lambda) \\
{\scriptstyle \mathcal{F}^C}\big\downarrow & & {\scriptstyle \mathcal{F}^K}\big\downarrow & & {\scriptstyle \mathcal{F}^D}\big\downarrow \\
C^b(\mathrm{GProj}\Lambda^!) & \longrightarrow & K^b(\mathrm{GProj}\Lambda^!) & \longrightarrow & D^b(\mathrm{GMod}^b\Lambda^!),
\end{array}$$

where $\mathrm{GMod}^b\Lambda$*,* $\mathrm{GProj}\Lambda^!$ *and* $\mathrm{GMod}^b\Lambda^!$ *can be replaced simultaneously by* $\mathrm{gmod}^b\Lambda$*,* $\mathrm{gproj}\Lambda^!$ *and* $\mathrm{gmod}^b\Lambda^!$*, respectively.*

(2) *In case* $\Lambda^!$ *is locally right bounded, the left Koszul functor* $\mathcal{G} : \mathrm{GMod}\Lambda \to C(\mathrm{GMod}\Lambda^!)$ *induces a commutative diagram of functors*

$$\begin{array}{ccccc} C^b(\mathrm{GMod}^b\Lambda) & \longrightarrow & K^b(\mathrm{GMod}^b\Lambda) & \longrightarrow & D^b(\mathrm{GMod}^b\Lambda) \\ \downarrow{\scriptstyle \mathcal{G}^C} & & \downarrow{\scriptstyle \mathcal{G}^K} & & \downarrow{\scriptstyle \mathcal{G}^D} \\ C^b(\mathrm{GInj}\Lambda^!) & \longrightarrow & K^b(\mathrm{GInj}\Lambda^!) & \longrightarrow & D^b(\mathrm{GMod}^b\Lambda^!), \end{array}$$

where $\mathrm{GMod}^b\Lambda$*,* $\mathrm{GInj}\Lambda^!$ *and* $\mathrm{GMod}^b\Lambda^!$ *can be replaced simultaneously by* $\mathrm{gmod}^b\Lambda$*,* $\mathrm{gproj}\Lambda^!$ *and* $\mathrm{gmod}^b\Lambda^!$*, respectively.*

Proof. Suppose that $\Lambda^!$ is left locally bounded, that is, $P_x^!$ is finite dimensional for every $x \in Q_0$. Then, $\mathrm{GProj}\Lambda^! \subseteq \mathrm{GMod}^b\Lambda^!$ and $\mathrm{gproj}\Lambda^! \subseteq \mathrm{gmod}^b\Lambda^!$. Therefore, $K^b(\mathrm{GProj}\Lambda^!)$ and $K^b(\mathrm{gproj}\Lambda^!)$ are full triangulated subcategories of $D^b(\mathrm{GMod}^b\Lambda^!)$ and $D^b(\mathrm{gmod}^b\Lambda^!)$, respectively. Now, Statement (1) follows from Theorem 6.3.4(1). Dually, Statement (2) follows from Theorem 6.3.4(2). The proof of the corollary is completed.

REMARK. By Theorem 1.8.7, the functors $\mathcal{F}^D : D^b(\mathrm{GMod}^b\Lambda) \to D^b(\mathrm{GMod}^b\Lambda^!)$ and $\mathcal{G}^D : D^b(\mathrm{GMod}^b\Lambda) \to D^b(\mathrm{GMod}^b\Lambda^!)$ are triangle-exact, which will be called the **right** and the **left bounded derived Koszul functors**, respectively.

6.4 Koszul dualities

The objective of this section is to show that the derived Koszul functors for a Koszul algebra and those for its Koszul dual form two 2-real parametrized family of dualities, which contain particularly the classical Koszul duality of Beilinson, Ginzburg and Soergel; see [13, (2.12.1)]; also [48, Theorem 30]. The key ingredients of this section are adapted from those in [16, Section 5] which are in the non-graded setting and under the assumption that the quiver is gradable.

Throughout this section, let $\Lambda = kQ/R$ be a Koszul algebra, where Q is a locally finite quiver. We start with the following important property of Koszul functors; see [13, (1.2.6)] and [48, Theorem 30].

6.4.1 **Lemma**. *Let $\Lambda = kQ/R$ be a Koszul algebra, where Q is a locally finite quiver. If $a \in Q_0$, then $S^!_a$ has $\mathcal{F}(I_a)^{\bullet}$ as a truncated linear projective resolution and $\mathcal{G}(P_a)^{\bullet}$ as a truncated colinear injective coresolution.*

Proof. Fix $a \in Q_0$. Since $\Lambda^!$ is Koszul; see (5.4.3), $S^!_a$ has a truncated colinear injective coresolution $\mathcal{I}^{\bullet}_{a^!}$; see (5.4.2). Since $(\Lambda^!)^! = \Lambda$; see (5.3.3), in view of the definition of $\mathcal{G}$, we see that $\mathcal{G}(P_a)^{\bullet} = \mathcal{I}^{\bullet}_{a^!}$. Next, by Lemma 5.4.1, $S^!_a$ has a truncated linear projective resolution $\mathcal{P}^{\bullet}_{a^!}$ as follows:

$$\cdots \longrightarrow \mathcal{P}^{-n}_{a^!} \xrightarrow{\ell^{-n}} \mathcal{P}^{1-n}_{a^!} \longrightarrow \cdots \longrightarrow \mathcal{P}^{-1}_{a^!} \xrightarrow{\ell^{-1}} \mathcal{P}^{0}_{a^!} \longrightarrow 0 \longrightarrow \cdots,$$

where $\mathcal{P}^{-n}_{a^!} = \oplus_{x\in Q_0}(P^!_x\langle -n\rangle \otimes D(e_a\Lambda_n e_x))$ and $\mathcal{P}^{1-n}_{a^!} = \oplus_{y\in Q_0}(P^!_y\langle 1-n\rangle \otimes D(e_a\Lambda_{n-1} e_y))$. Moreover, $\ell^{-n} = (\ell^{-n}(y,x))_{(y,x)\in Q_0\times Q_0}$, where

$$\ell^{-n}(y,x) = \textstyle\sum_{\alpha\in Q_1(x,y)} P[\bar\alpha^!]\otimes DP[\bar\alpha] : P^!_x\langle -n\rangle\otimes D(e_a\Lambda_n e_x) \to P^!_y\langle 1-n\rangle \otimes D(e_a\Lambda_{n-1}e_y).$$

On the other hand, for any $n \in \mathbb{Z}$, we have

$$\mathcal{F}(I_a)^{-n} = \oplus_{x\in Q_0}(P^!_x\langle -n\rangle \otimes (I_a)_{-n}(x)) = \oplus_{x\in Q_0}(P^!_x\langle -n\rangle \otimes D(e_x\Lambda^{\mathrm{o}}_n e_a)).$$

In particular, $\mathcal{F}(I_a)^{-n} = \mathcal{P}^{-n}_{a^!} = 0$, for all $n < 0$. Fix an integer $n \geq 1$. Write $\mathcal{F}(I_a)^{1-n} = \oplus_{y\in Q_0}(P^!_y\langle 1-n\rangle \otimes D(e_y\Lambda^{\mathrm{o}}_{n-1}e_a))$. Then, $d^{-n}_{\mathcal{F}(I_a)} = (d^{-n}_{\mathcal{F}(I_a)}(y,x))_{(y,x)\in Q_0\times Q_0}$, where

$$d^{-n}_{\mathcal{F}(I_a)}(y,x) : P^!_x\langle -n\rangle \otimes D(e_x\Lambda^{\mathrm{o}}_n e_a) \to \oplus_{y\in Q_0} P^!_y\langle 1-n\rangle \otimes D(e_y\Lambda^{\mathrm{o}}_{n-1}e_a)$$

is given by

$$d^{-n}_{\mathcal{F}(I_a)}(y,x) = \textstyle\sum_{\alpha\in Q_1(x,y)} P[\bar\alpha^!]\otimes I_a(\bar\alpha) = \sum_{\alpha\in Q_1(x,y)} P[\bar\alpha^!]\otimes DP^{\mathrm{o}}_a(\bar\alpha^{\mathrm{o}}).$$

Consider the canonical k-linear isomorphism $\theta^n_x : e_a\Lambda_n e_x \to e_x\Lambda^{\mathrm{o}}_n e_a; \bar\gamma \mapsto \bar\gamma^{\mathrm{o}}$. This yields a k-linear isomorphism $D\theta^n_x : D(e_x\Lambda^{\mathrm{o}}_n e_a) \to D(e_a\Lambda_n e_x)$. Given any $\alpha \in Q_1(x,y)$, it is easy to verify that the diagram

$$\begin{array}{ccc} e_a\Lambda_n e_x & \xrightarrow{P[\bar\alpha]} & e_a\Lambda_{n+1}e_y \\ \downarrow{\scriptstyle\theta^n_x} & & \downarrow{\scriptstyle\theta^{n+1}_y} \\ e_x\Lambda^{\mathrm{o}}_n e_a & \xrightarrow{P^{\mathrm{o}}_a(\bar\alpha^{\mathrm{o}})} & e_y\Lambda^{\mathrm{o}}_{n+1}e_a \end{array}$$

commutes. Hence, $DP[\bar\alpha]\circ D\theta^n_x = D\theta^{n-1}_y \circ DP^{\mathrm{o}}_a(\bar\alpha^{\mathrm{o}})$. Therefore,

$$\ell^{-n}(y,x)\circ(\mathrm{id}\otimes D\theta^n_x) = (\mathrm{id}\otimes D\theta^{n-1}_y)\circ d^{-n}_{\mathcal{F}(I_a)}(y,x),$$

for all $(y, x) \in Q_0 \times Q_0$. That is, the graded Λ-linear isomorphisms

$$\oplus_{x\in Q_0}(\mathrm{id} \otimes D\theta^n_x) : \oplus_{x\in Q_0}(P^!_x\langle -n\rangle \otimes D(e_x\Lambda^{\mathrm{o}}_n e_a)) \to \oplus_{x\in Q_0}(P^!_x\langle -n\rangle \otimes D(e_x\Lambda_n e_a))$$

with $n \in \mathbb{Z}$ form a complex isomorphism $\mathcal{F}(I_a)^{\bullet} \cong \mathcal{P}^{\bullet}_{a^!}$. The proof of the lemma is completed.

Remark. Lemma 6.4.1 is adapted from Lemma 5.4 in [16], which is in the non-graded setting and under the assumption that Q is gradable.

More generally, composing the left Koszul functor and the right complex Koszul functor yields graded projective resolution for graded modules.

6.4.2 **Lemma**. *Let $\Lambda = kQ/R$ be a Koszul algebra, where Q is a locally finite quiver. Consider $\mathcal{F}^C \circ \mathcal{G} : \mathrm{GMod}\Lambda \to C(\mathrm{GMod}\Lambda)$. Given $M \in \mathrm{GMod}\Lambda$, there exists a natural quasi-isomorphism $\eta^{\bullet}_M : (\mathcal{F}^C \circ \mathcal{G})(M)^{\bullet} \to M$.*

Proof. Let $M \in \mathrm{GMod}\Lambda$. By definition, $(\mathcal{F}^C \circ \mathcal{G})(M)^{\bullet} = \mathbb{T}(\mathcal{F}(\mathcal{G}(M)^{\bullet})^{\bullet})$. For any $n \in \mathbb{Z}$, by Proposition 6.2.4(1),

$$(*) \qquad (\mathcal{F}^C \circ \mathcal{G})(M)^n = \oplus_{i\in\mathbb{Z};a,x\in Q_0}(P_a\langle n-i\rangle \otimes D(e_a\hat{\Lambda}_{-n}e_x) \otimes M_i(x)).$$

So, $(\mathcal{F}^C \circ \mathcal{G})(M)^n = 0$ for $n > 0$. We shall divide the rest of the proof into two statements.

Statement 1. *If $n < 0$, then* $\mathrm{H}^n((\mathcal{F}^C \circ \mathcal{G})(M)^{\bullet}) = 0$.

Indeed, fix some $n < 0$. Note that $\mathrm{H}^n((\mathcal{F}^C \circ \mathcal{G})(M)^{\bullet}) = \mathrm{H}^n(\mathbb{T}(\mathcal{F}(\mathcal{G}(M)^{\bullet})^{\bullet}))$. For any $i \in \mathbb{Z}$, as described in Section 1.7, the i-th column of $\mathcal{F}(\mathcal{G}(M)^{\bullet})^{\bullet}$ is

$$\mathfrak{t}^i(\mathcal{F}(\mathcal{G}(M)^i)^{\bullet}) = \oplus_{x\in Q_0}\mathfrak{t}^i(\mathcal{F}(I^!_x\langle i\rangle \otimes M_i(x))^{\bullet}) = \oplus_{x\in Q_0}\mathfrak{t}^i(\mathcal{F}(I^!_x\langle i\rangle)^{\bullet}) \otimes M_i(x),$$

where $\mathfrak{t}$ is the twist functor. Since $n < 0$, by Lemma 6.4.1, $\mathrm{H}^n(\mathcal{F}(I^!_x)^{\bullet}) = 0$, for any $x \in Q_0$. Now, for all $i \in \mathbb{Z}$ and $x \in Q_0$, it follows from Corollary 6.1.4 that

$$\mathrm{H}^{n-i}(\mathfrak{t}^i(\mathcal{F}(I^!_x\langle i\rangle)^{\bullet})) = \mathrm{H}^{n-i}(\mathcal{F}(I^!_x\langle i\rangle)^{\bullet}) = \mathrm{H}^n(\mathcal{F}(I^!_x)^{\bullet})\langle -i\rangle = 0.$$

So, $\mathrm{H}^{n-i}(\mathfrak{t}^i(\mathcal{F}(\mathcal{G}(M)^i)^{\bullet})) \cong \oplus_{x\in Q_0}\mathrm{H}^{n-i}(\mathfrak{t}^i(\mathcal{F}(I^!_x\langle i\rangle)^{\bullet}) \otimes M_i(x) = 0$, for all $i \in \mathbb{Z}$.

Fix $p \in \mathbb{Z}$. Consider the double complex $\mathcal{F}(\mathcal{G}(M)^{\bullet})^{\bullet}_p$. Given $i \in \mathbb{Z}$, the i-th column of $\mathcal{F}(\mathcal{G}(M)^{\bullet})^{\bullet}_p$ is the complex $\mathfrak{t}^i(\mathcal{F}(\mathcal{G}(M)^i)^{\bullet})_p$. It follows from Lemma 6.3.2(1) that $\mathrm{H}^{n-i}(\mathfrak{t}^i(\mathcal{F}(\mathcal{G}(M)^i)^{\bullet})_p) = \mathrm{H}^{n-i}(\mathfrak{t}^i(\mathcal{F}(\mathcal{G}(M)^i)^{\bullet}))_p = 0$.

On the other hand, we deduce from Lemma 6.2.3(1) that the n-diagonal of $\mathcal{F}(\mathcal{G}(M)^{\bullet})^{\bullet}_p$ consists of

$$\mathcal{F}(\mathcal{G}(M)^i)^{n-i}_p = \oplus_{a,x\in Q_0}(\Lambda_{p+n-i}e_a \otimes (I^!_x)_n(a) \otimes M_i(x)); \text{ for all } i\in\mathbb{Z}.$$

In particular, $\mathcal{F}(\mathcal{G}(M)^i)_p^{n-i}=0$ for all $i>n+p$. So, $\mathcal{F}(\mathcal{G}(M)^\bullet)_p^\bullet$ is n-diagonally bounded-above. Applying Lemma 1.8.1, we see that $\mathrm{H}^n(\mathbb{T}(\mathcal{F}(\mathcal{G}(M)^\bullet)_p^\bullet))=0$, that is, $\mathrm{H}^n((\mathcal{F}^C\circ\mathcal{G})(M)_p^\bullet)=0$, for any $p\in\mathbb{Z}$. So, $\mathrm{H}^n((\mathcal{F}^C\circ\mathcal{G})(M)^\bullet)=0$. This establishes Statement 1.

To conclude the proof, it suffices to show that $\mathrm{H}^0((\mathcal{F}^C\circ\mathcal{G})(M)^\bullet)$ is naturally isomorphic to M. In view of the equation $(*)$, we see that the 1-diagonal of $\mathcal{F}(\mathcal{G}(M)^\bullet)^\bullet$ is null. Since $(I_x^!\otimes M)(\bar{\alpha}^!)=I_x^!(\bar{\alpha}^!)\otimes \mathrm{id}_M$ for $\alpha\in Q_1(a,x)$, the 0-diagonal and the (-1)-diagonal of $\mathcal{F}(\mathcal{G}(M)^\bullet)^\bullet$ are illustrated as

$$\begin{array}{ccc}\oplus_{b\in Q_0}(P_b\langle -i\rangle\otimes(I_b^!)_0(b)\otimes M_i(b)) & & \\ \Big\uparrow{\scriptstyle v^{i,-i-1}} & & \\ \oplus_{a,x\in Q_0}(P_a\langle -i-1\rangle\otimes(I_x^!)_{-1}(a)\otimes M_i(x)) & \xrightarrow{h^{i,-i-1}} & \oplus_{c\in Q_0}(P_c\langle -i-1\rangle\otimes(I_c^!)_0(c)\otimes M_{i+1}(c)),\end{array}$$

where $v^{i,-i-1}=(v^{i,-i-1}(b,a,x))_{(b,a,x)\in Q_0\times Q_0\times Q_0}$ with

$$v^{i,-i-1}(b,a,x)=\begin{cases}\sum_{\alpha\in Q_1(x,a)}(-1)^iP[\bar{\alpha}]\otimes I_x^!(\bar{\alpha}^!)\otimes\mathrm{id}, & \text{if } b=x;\\ 0, & \text{if } b\neq x,\end{cases}$$

and $h^{i,-i-1}=(h^{i,-i-1}(c,a,x))_{(c,a,x)\in Q_0\times Q_0\times Q_0}$ with

$$h^{i,-i-1}(c,a,x)=\begin{cases}\sum_{\alpha\in Q_1(x,a)}\mathrm{id}\otimes I[\bar{\alpha}^!]\otimes M(\bar{\alpha}), & \text{if } c=a;\\ 0, & \text{if } c\neq a.\end{cases}$$

In particular,

$$(\mathcal{F}^C\circ\mathcal{G})(M)^{-1}=\oplus_{i\in\mathbb{Z};a,x\in Q_0}(P_a\langle -i-1\rangle\otimes(I_x^!)_{-1}(a)\otimes M_i(x)),$$

where $(I_x^!)_{-1}(a)=D(e_a\hat{\Lambda}_1e_x)$ has a k-basis $\{\hat{\beta}^\star\mid\beta\in Q_1(x,a)\}$, that is the dual basis of $\{\hat{\beta}\mid\beta\in Q_1(x,a)\}$. Moreover, by Proposition 6.2.4(1),

$$(\mathcal{F}^C\circ\mathcal{G})(M)^0=\oplus_{(i,b)\in\mathbb{Z}\times Q_0}P_b\langle -i\rangle\otimes(I_b^!)_0(b)\otimes M_i(b),$$

where $(I_b^!)_0(b)=D(e_b\hat{\Lambda}_0e_b)$ with a k-basis $\{\hat{e}_b^\star\}$.

STATEMENT 2. *Let d^{-1} is the differential of degree -1 of $(\mathcal{F}^C\circ\mathcal{G})(M)^\bullet$. We have a natural graded epimorphism*

$$\eta_M:(\mathcal{F}^C\circ\mathcal{G})(M)^0\to M;\ \textstyle\sum_{(i,b)\in\mathbb{Z}\times Q_0}u_{i,b}\otimes\hat{e}_b^\star\otimes m_{i,b}\mapsto\sum_{(i,b)\in\mathbb{Z}\times Q_0}(-1)^{\frac{i(i+1)}{2}}u_{i,b}m_{i,b},$$

such that $\eta_M^0\circ d^{-1}=0$.

Indeed, the existence and the naturality of $\eta_{_M}$ are evident. Consider an element $\omega \in (\mathcal{F}^C\circ \mathcal{G})(M)^{-1}$. We may assume that

$$\omega \in P_a\langle -i-1\rangle \otimes I_x^!\langle i\rangle_{-i-1}(a)\otimes M_i(x)$$

for some $i\in\mathbb{Z}$ and $a,x\in Q_0$. Further, we may assume that $\omega = u_0\otimes \hat{\beta}_0^\star \otimes m_0$, for some $u_0\in P_a\langle -i-1\rangle$, $\beta_0\in Q_1(x,a)$ and $m_0\in M_i(x)$. Write $\hat{P}_x = \hat{\Lambda} e_x$. For $\alpha\in Q_1(x,a)$, since $(\bar{\alpha}^!)^{\rm o} = \hat{\alpha}$, we obtain $I_x^!(\bar{\alpha}^!) = D\hat{P}_x(\hat{\alpha})$ and $I_x^![\bar{\alpha}^!] = DP[\hat{\alpha}]$. Thus, $I_x^!(\bar{\alpha}^!)(\hat{\beta}_0^\star)(e_x) = \hat{\beta}_0^\star(\hat{\alpha})$ and $I[\bar{\alpha}^!](\hat{\beta}_0^\star)(e_a) = \hat{\beta}_0^\star(\hat{\alpha})$. Hence,

$$I_x^!(\bar{\alpha}^!)(\hat{\beta}_0^\star) = I[\bar{\alpha}^!](\hat{\beta}_0^\star) = \begin{cases} \hat{e}_x^\star; & \text{if } \alpha=\beta_0;\\ 0, & \text{otherwise.}\end{cases}$$

This yields

$$\begin{array}{rcl} d^{-1}(\omega) & = & (-1)^i\sum_{\alpha\in Q_1(x,a)}(P[\bar{\alpha}]\otimes I_x^!(\bar{\alpha}^!)\otimes \mathrm{id})(u_0\otimes\hat{\beta}_0^\star\otimes m_0)\\ & & +\sum_{\alpha\in Q_1(x,a)}(\mathrm{id}\otimes I[\bar{\alpha}^!]\otimes M(\bar{\alpha}))(u_0\otimes\hat{\beta}_0^\star\otimes m_0) \qquad (**)\\ & = & (-1)^i(u_0\bar{\beta}_0)\otimes\hat{e}_x^\star\otimes m_0 + u_0\otimes\hat{e}_a^\star\otimes(\bar{\beta}_0 m_0).\end{array}$$

Since $u_0\in P_a\langle -i-1\rangle$ and $u_0\bar{\beta}_0\in P_x\langle -i\rangle$, we obtain

$$(\eta_{_M}\circ d^{-1})(\omega) = (-1)^{\frac{i(i+1)}{2}+i}(u_0\bar{\beta}_0 m_0) + (-1)^{\frac{(i+1)(i+2)}{2}}(u_0\bar{\beta}_0 m_0) = 0.$$

This establishes Statement 2.

It remains to verify that $\mathrm{Ker}(d^{-1})\subseteq \mathrm{Im}(\eta_{_M})$. Fix $\omega\in\mathrm{Ker}(\eta_{_M})$. We may assume that

$$\omega \in (\mathcal{F}^C\circ\mathcal{G})(M)_p^0(a) = \oplus_{i\le p;\, x\in Q_0}(e_aP_x\langle -i\rangle_p\otimes (I_x^!)_0(x)\otimes M_i(x)),$$

where $e_aP_x\langle -i\rangle_p = e_a\Lambda_{p-i}e_x$, for some pair $(p,a)\in\mathbb{Z}\times Q_0$. Thus, we may find some $i_s\le\cdots\le i_2\le i_1=p$ and $x_1,\ldots,x_s\in Q_0$ such that $\omega=\sum_{j=1}^s\bar{\gamma}_j\otimes\hat{e}_{x_j}^\star\otimes m_j$, where the $\bar{\gamma}_j$ with $\gamma_j\in Q_{p-i_j}(x_j,a)$ are pairwise distinct and $m_j\in M_{i_j}(x_j)$. In particular, $\gamma_1=\varepsilon_a$. We shall proceed by induction on the minimal integer n_ω for which ω can be written in this form and $n_\omega=\sum_{j=1}^s(p-i_j)$.

If $n_\omega=0$, then $s=1$ and $m_1=\pm\eta_{_M}(\omega)=0$, and hence, $\omega=0$. Suppose that $n_\omega>0$. Since $\gamma_j\ne\gamma_1=\varepsilon_a$, we may write $\gamma_j=\sigma_j\beta_j$ with $\beta_j\in Q_1(x_j,y_j)$ and $\sigma_j\in Q_{p-i_j-1}(y_j,a)$, for $2\le j\le s$. Set $\sigma=\sum_{j=2}^s(-1)^{i_j}\bar{\sigma}_j\otimes\hat{\beta}_j^\star\otimes m_j$, where $\bar{\sigma}_j\in P_{y_j}\langle -i_j-1\rangle_p$. In view of the equations $(**)$, we obtain

$$\begin{array}{rcl} d^{-1}(\sigma) & = & \sum_{j=2}^s((-1)^{2i_j}(\bar{\sigma}_j\,\bar{\beta}_j)\otimes\hat{e}_{x_j}^\star\otimes m_j + (-1)^{i_j}\bar{\sigma}_j\otimes\hat{e}_{y_j}^\star\otimes(\bar{\beta}_j\,m_j))\\ & = & \omega + e_a\otimes\hat{e}_a^\star\otimes(-m_1) + \sum_{j=2}^s\bar{\sigma}_j\otimes\hat{e}_{y_j}^\star\otimes(-1)^{i_j}(\bar{\beta}_j\,m_j).\end{array}$$

Put $\omega' = d^{-1}(\sigma) - \omega = e_a \otimes \hat{e}_a^\star \otimes (-m_1) + \sum_{j=2}^s \bar{\sigma}_j \otimes \hat{e}_{y_j}^\star \otimes (-1)^{i_j} (\bar{\beta}_j \, m_j)$. Then, $\omega' \in \mathrm{Ker}(\eta_{_M})$ with $n_{\omega'} < n_\omega$. Thus $\omega' \in \mathrm{Im}(d^{-1})$, and hence, $\omega \in \mathrm{Im}(d^{-1})$. The proof of the lemma is completed.

As shown below, composing the right Koszul functor and the left complex Koszul functor yields graded injective coresolutions for bounded above graded modules.

6.4.3 **Lemma**. *Let $\Lambda = kQ/R$ be a Koszul algebra, where Q is a locally finite quiver. Consider $\mathcal{G}^C \circ \mathcal{F} : \mathrm{GMod}\Lambda \to C(\mathrm{GMod}\Lambda)$. Given $M \in \mathrm{GMod}^-\Lambda$, there exists a natural quasi-isomorphism $\zeta_{_M}^\bullet : M \to (\mathcal{G}^C \circ \mathcal{F})(M)^\bullet$.*

Proof. Let $M \in \mathrm{GMod}^-\Lambda$. Assume that r is an integer such that $M_i = 0$ for all $i \geq r$. For any $n \in \mathbb{Z}$, by Proposition 6.2.4(2),

$$(\mathcal{G}^C \circ \mathcal{F})(M)^n = \oplus_{i \in \mathbb{Z}; x, a \in Q_0} (I_a\langle n-i \rangle \otimes e_a \Lambda_n^! e_x \otimes M_i(x)).$$

In particular, $(\mathcal{G}^C \circ \mathcal{F})(M)^n = 0$ for $n < 0$. We shall split the rest of the proof into several statements.

Recall that $(\mathcal{G}^C \circ \mathcal{F})(M)^\bullet = \mathbb{T}(\mathcal{G}(\mathcal{F}(M)^\bullet)^\bullet)$. For any $i \in \mathbb{Z}$, the i-th column of $\mathcal{G}(\mathcal{F}(M)^\bullet)^\bullet$ is the following complex

$$\mathfrak{t}^i(\mathcal{G}(\mathcal{F}(M)^i)^\bullet) = \oplus_{a \in Q_0} \mathfrak{t}^i(\mathcal{G}(P_a^!\langle i \rangle \otimes M_i(a))^\bullet) = \oplus_{a \in Q_0} \mathfrak{t}^i(\mathcal{G}(P_a^!\langle i \rangle)^\bullet) \otimes M_i(a).$$

Statement 1. *Given any integers n, i, we have*

$$\mathrm{H}^{n-i}(\mathfrak{t}^i(\mathcal{G}(\mathcal{F}(M)^i)^\bullet)) \cong \oplus_{a \in Q_0} \mathrm{H}^n(\mathcal{G}(P_a^!)^\bullet)\langle -i \rangle \otimes M_i(a).$$

Indeed, fix some $n, i \in \mathbb{Z}$. We deduce from Lemma 6.4.1 that

$$\mathrm{H}^{n-i}(\mathfrak{t}^i(\mathcal{G}(P_a^!\langle i \rangle)^\bullet)) = \mathrm{H}^{n-i}(\mathcal{G}(P_a^!\langle i \rangle)^\bullet) = \mathrm{H}^n(\mathcal{G}(P_a^!)^\bullet)\langle -i \rangle.$$

Since $\mathcal{F}(M)^i = \oplus_{a \in Q_0} P_a^!\langle i \rangle \otimes M_i(a)$ by definition, it follows from Lemma 6.1.5 and the above equation that

$$\begin{array}{rcl} \mathrm{H}^{n-i}(\mathfrak{t}^i(\mathcal{G}(\mathcal{F}(M)^i)^\bullet)) & = & \oplus_{x \in Q_0} \mathrm{H}^{n-i}(\mathfrak{t}^i(\mathcal{G}(P_a^!\langle i \rangle)^\bullet \otimes M_i(a)) \\ & \cong & \oplus_{x \in Q_0} \mathrm{H}^{n-i}(\mathfrak{t}^i(\mathcal{G}(P_a^!\langle i \rangle)^\bullet) \otimes M_i(a) \\ & \cong & \oplus_{x \in Q_0} \mathrm{H}^n(\mathcal{G}(P_a^!)^\bullet)\langle -i \rangle \otimes M_i(a). \end{array}$$

This establishes Statement 1.

Statement 2. *If $n > 0$, then $\mathrm{H}^n((\mathcal{G}^C \circ \mathcal{F})(M)^\bullet) = 0$.*

Indeed, fix some $n > 0$. Note that $\mathrm{H}^n(\mathcal{G}^C \circ \mathcal{F})(M^\bullet) = \mathrm{H}^n(\mathbb{T}(\mathcal{G}(\mathcal{F}(M^\bullet)^\bullet)))$. It follows from Lemma 6.2.3(2) that the n-diagonal of the double complex $\mathcal{G}(\mathcal{F}(M^\bullet)^\bullet)$ consists of

$$(*)\qquad \mathcal{G}(\mathcal{F}(M)^i)^{n-i} = \oplus_{x,a\in Q_0}(I_x\langle n-i\rangle \otimes e_x\Lambda^!_n e_a \otimes M_i(a));\ \text{ for all } i \in \mathbb{Z}.$$

Since $M_i = 0$ for all $i \geq r$, we see that $\mathcal{G}(\mathcal{F}(M)^i)^{n-i} = 0$ for all $i \geq r$. That is, $\mathcal{G}(\mathcal{F}(M^\bullet)^\bullet)$ is n-diagonally bounded-above.

On the other hand, for any $i \in \mathbb{Z}$, the i-th column of $\mathcal{G}(\mathcal{F}(M^\bullet)^\bullet)$ is the $\mathfrak{t}^i(\mathcal{G}(\mathcal{F}(M)^i)^\bullet)$. Since $n > 0$, it follows from Lemma 6.4.1 that $\mathrm{H}^n(\mathcal{G}(P^!_a)^\bullet) = 0$. Thus, it follows from Statement 1 that

$$\mathrm{H}^{n-i}(\mathfrak{t}^i(\mathcal{G}(\mathcal{F}(M)^i)^\bullet)) \cong \oplus_{a\in Q_0} \mathrm{H}^n(\mathcal{G}(P^!_a)^\bullet)\langle -i\rangle \otimes M_i(a) = 0.$$

Thus, by Lemma 1.8.1, $\mathrm{H}^n(\mathbb{T}(\mathcal{G}(\mathcal{F}(M^\bullet)^\bullet))) = 0$. That is, $\mathrm{H}^n((\mathcal{G}^C \circ \mathcal{F})(M^\bullet)) = 0$. This establishes Statement 2.

To conclude that proof, it suffices to construct a natural graded isomorphism $M \to \mathrm{H}^0((\mathcal{G}^C \circ \mathcal{F})(M^\bullet))$. Indeed, by Proposition 6.2.4(2), we see that

$$(\mathcal{G}^C \circ \mathcal{F})(M)^0 = \oplus_{i\in\mathbb{Z};\, a\in Q_0}\, (I_a\langle -i\rangle \otimes e_a\Lambda^!_0 e_a \otimes M_i(a)).$$

Given $(i, a) \in \mathbb{Z} \times Q_0$, we shall construct a morphism

$$f^i_a : M \to I_a\langle -i\rangle \otimes e_a\Lambda^!_0 e_a \otimes M_i(a)$$

in $\mathrm{GMod}\Lambda$. For this purpose, we define a k-linear map

$$f^i_{a,j} : M_j \to I_a\langle -i\rangle_j \otimes e_a\Lambda^!_0 e_a \otimes M_i(a)$$

for every $j \in \mathbb{Z}$, where $I_a\langle -i\rangle_j = D(\Lambda^{\mathrm{o}}_{i-j} e_a)$. Clearly, $f^i_{a,j} = 0$ in case $j > i$. Fix j with $j \leq i$. We have a k-linear map

$$\psi^i_{a,j} : M_j \to \mathrm{Hom}_k(\Lambda^{\mathrm{o}}_{i-j} e_a,\, e_a\Lambda^!_0 e_a \otimes M_i(a)); w \mapsto \psi^i_{a,j}(w),$$

where $\psi^i_{a,j}(w)$ sends $\bar{\gamma}^{\mathrm{o}}$ to $e_a \otimes \bar{\gamma} w$, for $\gamma \in kQ_{i-j}(-, a)$. Since $\Lambda^{\mathrm{o}}_{i-j} e_a$ is finite dimensional, in view of Corollary 2.1.2(1), we obtain a k-linear isomorphism

$$\theta^i_{a,j} : D(\Lambda^{\mathrm{o}}_{i-j} e_a) \otimes e_a\Lambda^!_0 e_a \otimes M_i(a) \to \mathrm{Hom}_k(\Lambda^{\mathrm{o}}_{i-j} e_a,\, e_a\Lambda^!_0 e_a \otimes M_i(a)).$$

Now, put $f^i_{a,j} = (\theta^i_{a,j})^{-1} \circ \psi^i_{a,j} : M_j \to I_a\langle -i\rangle_j \otimes e_a\Lambda^!_0 e_a \otimes M_i(a)$, which can be computed in the following way.

Statement 3. *Let* $\{\bar{\gamma}_1^{\text{o}}, \ldots, \bar{\gamma}_s^{\text{o}}\}$ *with* $\gamma_p \in kQ_{i-j}(-, a)$ *be a* k*-basis of* $\Lambda^{\text{o}}_{i-j}e_a$ *with dual basis* $\{\bar{\gamma}_1^{\text{o},\star}, \ldots, \bar{\gamma}_s^{\text{o},\star}\}$*. Then* $f^i_{a,j}(w) = \sum_{p=1}^s \bar{\gamma}_p^{\text{o},\star} \otimes e_a \otimes \bar{\gamma}_p w$*, for* $w \in M_j$*.*

Indeed, every $\bar{\gamma}^{\text{o}} \in \Lambda^{\text{o}}_{i-j}e_a$ is written as $\bar{\gamma}^{\text{o}} = \sum_{t=1}^s \lambda_t \bar{\gamma}_t^{\text{o}}$ with $\lambda_t \in k$. Given $w \in M_j$, by the definition given in Corollary 2.1.2(1), we obtain

$$\theta^i_{a,j}(\textstyle\sum_{p=1}^s \bar{\gamma}_p^{\text{o},\star} \otimes e_a \otimes \bar{\gamma}_p w)(\bar{\gamma}^{\text{o}}) = e_a \otimes (\sum_{t=1}^s \lambda_t \bar{\gamma}_t^{\text{o}})w = \psi^i_{a,j}(w)(\bar{\gamma}^{\text{o}}).$$

Thus, $\theta^i_{a,j}(\sum_{p=1}^s \bar{\gamma}_p^{\text{o},\star} \otimes e_a \otimes \bar{\gamma}_p w) = \psi^i_{a,j}(w)$. This establishes Statement 3.

Statement 4. *Given* $(i, a) \in \mathbb{Z} \times Q_0$*, there exists a natural graded* Λ*-linear morphism* $f^i_a : M \to I_a\langle -i\rangle \otimes e_a\Lambda^!_0 e_a \otimes M_i(a)$ *such that* $(f^i_a)_j = f^i_{a,j}$*, for all* $j \in \mathbb{Z}$*.*

Indeed, given $\alpha \in Q_1$ and $j \le i$, we clearly have a commutative diagram

$$\begin{array}{ccccc}
M_j & \xrightarrow{\psi^i_{a,j}} & \operatorname{Hom}((P^{\text{o}}_a)_{i-j},\, e_a\Lambda^!_0 e_a \otimes M_i(a)) & \xleftarrow{\theta^i_{a,j}} & I_a\langle -i\rangle_j \otimes e_a\Lambda^!_0 e_a \otimes M_i(a) \\
\downarrow{\scriptstyle M(\bar{\alpha})} & & \downarrow{\scriptstyle \operatorname{Hom}(P^{\text{o}}_a(\bar{\alpha}^{\text{o}}), e_a\Lambda^!_0 e_a \otimes M_i(a))} & & \downarrow{\scriptstyle I_a\langle -i\rangle(\bar{\alpha})\otimes \mathrm{id}\otimes \mathrm{id}} \\
M_{j+1} & \xrightarrow{\psi^i_{a,j+1}} & \operatorname{Hom}((P^{\text{o}}_a)_{i-j-1}, e_a\Lambda^!_0 e_a \otimes M_i(a)). & \xleftarrow{\theta^i_{a,j+1}} & I_a\langle -i\rangle_{j+1} \otimes e_a\Lambda^!_0 e_a \otimes M_i(a)
\end{array}$$

So, f^i_a is a graded Λ-linear morphism. Similarly, one can verify that f^i_a is natural in M. This establishes Statement 4.

Fix $i \in \mathbb{Z}$. Given $a \in Q_0$, by Statement 4, we obtain a natural graded Λ-linear morphism $g^i_a : M \to I_a\langle -i\rangle \otimes e_a\Lambda^!_0 e_a \otimes M_i(a)$ where

$$(g^i_a)_j = (-1)^{\frac{(i-1)i}{2}} f^i_{a,j},$$

which will be written as $g^i_{a,j}$, for all $j \in \mathbb{Z}$. Let $w = \sum_{x\in Q_0; j\in\mathbb{Z}} w_{x,j} \in M$ with $w_{x,j} \in e_x M_j$. If $g^i_a(w_{j,x}) = g^i_{a,j}(w_{j,x}) \neq 0$ for some $a \in Q_0$, then $f^i_{a,j}(w_{j,x}) \neq 0$. Hence, $j \le i$, and by Statement 3, $kQ_{i-j}(x, a) \neq 0$. Since Q is locally finite, $g^i_a(w) = 0$ for all but finitely many $a \in Q_0$. Therefore, we have a graded Λ-linear morphism

$$g^i = (g^i_a)_{a\in Q_0} : M \to \mathcal{G}(\mathcal{F}(M)^i)^{-i} = \oplus_{a\in Q_0}(I_a\langle -i\rangle \otimes e_a\Lambda^!_0 e_a \otimes M_i(a))$$

where

$$g^i_j = (g^i_{a,j})_{a\in Q_0} : M_j \to \oplus_{a\in Q_0}((I_a\langle -i\rangle)_j \otimes e_a\Lambda^!_0 e_a \otimes M_i(a)),$$

for all $j \in \mathbb{Z}$.

Statement 5. *There exists a natural graded monomorphism*

$$\zeta_{_M} = (g^i)_{i\in\mathbb{Z}} : M \to (\mathcal{G}^C \circ \mathcal{F})(M)^0 = \oplus_{i\in\mathbb{Z}} \mathcal{G}(\mathcal{F}(M)^i)^{-i}.$$

Observe that $\mathcal{G}(\mathcal{F}(M)^i)^{-i} = 0$, for all $i \geq r$. Let $w = \sum_{j\in\mathbb{Z}} w_j \in M$ with $w_j \in M_j$. If $g^i(w_j) = \sum_{a\in Q_0} g^i_a(w_j) = g^i_{a,j}(w_j) \neq 0$ for some i, then $j \leq i$, and hence, $j \leq i < r$. As a consequence, $g^i(w) = 0$ for all but finitely many $i \in \mathbb{Z}$. Thus, we obtain a graded Λ-linear morphism $\zeta_M = (g^i)_{i\in\mathbb{Z}} : M \to (\mathcal{G}^C\circ \mathcal{F})(M)^0$, which is clearly natural in M. Assume that $\zeta_M(w) = 0$, for some $w \in M_j$ with $j \in \mathbb{Z}$. In particular, $g^j(w) = 0$, that is, $g^j_j(w_j) = \sum_{a\in Q_0} g^j_{a,j}(w) = 0$. Thus, $g^j_{a,j}(w) = 0$, and hence, $f^j_{a,j}(w) = 0$, for all $a \in Q_0$. Since $\{e_a\}$ is a basis of $\Lambda^{\rm o}_{j-j}e_a$, by Statement 3, $e_a^{{\rm o},\star} \otimes e_a \otimes e_a w = 0$, and hence, $e_a w = 0$, for all $a \in Q_0$. That is, $w = 0$. So ζ_M is a monomorphism. Statement 5 is established.

In view of the equation $(*)$, we see that the (-1)-diagonal of $\mathcal{G}(\mathcal{F}(M)^\bullet)^\bullet$ is null, while the 0-diagonal and the 1-diagonal can be illustrated as follows:

$$\begin{array}{ccc}
\oplus_{b\in Q_0}(I_b\langle -i\rangle \otimes e_b\Lambda^!_0 e_b \otimes M_i(b)) & \xrightarrow{h^{i,-i}} & \oplus_{a,x\in Q_0}(I_x\langle -i\rangle \otimes e_x\Lambda^!_1 e_a \otimes M_{i+1}(a)) \\
 & & \uparrow {\scriptstyle v^{i+1,-i-1}} \\
 & & \oplus_{c\in Q_0}(I_c\langle -i-1\rangle \otimes e_c\Lambda^!_0 e_c \otimes M_{i+1}(c)),
\end{array}$$

where $h^{i,-i} = (h^{i,-i}(a,x,b))_{a,x,b\in Q_0}$ with

$$h^{i,-i}(a,x,b) = \begin{cases} \sum_{\alpha\in Q_1(x,a)} \mathrm{id}\otimes P[\bar\alpha^!]\otimes M(\bar\alpha), & \text{if } b = x; \\ 0, & \text{if } b \neq x, \end{cases}$$

and $v^{i+1,-i-1} = (v^{i+1,-i-1}(a,x,c))_{a,x,c\in Q_0}$ with

$$v^{i+1,-i-1}(a,x,c) = \begin{cases} \sum_{\alpha\in Q_1(x,a)} (-1)^{i+1} I[\bar\alpha]\otimes P^!_a(\bar\alpha^!)\otimes \mathrm{id}, & \text{if } c = a; \\ 0, & \text{if } c \neq a. \end{cases}$$

Statement 6. *If we denote by d^0 the differential of degree 0 of the complex $(\mathcal{G}^C\circ\mathcal{F})(M)^\bullet$, then $d^0\circ\zeta_M = 0$.*

It amounts to show, for any $i \in \mathbb{Z}$, that the diagram

$$\begin{array}{ccc}
\oplus_{x\in Q_0}(I_x\langle -i\rangle \otimes e_x\Lambda^!_0 e_x \otimes M_i(x)) & \xrightarrow{\oplus_{x\in Q_0} h^{i,-i}(a,x,x)} & \oplus_{a,x\in Q_0}(I_x\langle -i\rangle \otimes e_x\Lambda^!_1 e_a \otimes M_{i+1}(a)) \\
\uparrow {\scriptstyle (g^i_x)_{x\in Q_0}} & & \uparrow {\scriptstyle \oplus_{a\in Q_0} v^{i+1,-i-1}(a,x,a)} \\
M & \xrightarrow{(g^{i+1}_a)_{a\in Q_0}} & \oplus_{a\in Q_0}(I_a\langle -i-1\rangle \otimes e_a\Lambda^!_0 e_a \otimes M_{i+1}(a)),
\end{array}$$

is anti-commutative, or equivalently, the diagram

$$\begin{array}{ccc}
\oplus_{x\in Q_0}(I_x\langle -i\rangle_j \otimes e_x\Lambda^!_0 e_x \otimes M_i(x)) & \xrightarrow{\oplus h^{i,-i}(a,x,x)_j} & \oplus_{a,x\in Q_0}(I_x\langle -i\rangle_j \otimes e_x\Lambda^!_1 e_a \otimes M_{i+1}(a)) \\
\uparrow {\scriptstyle (g^i_{x,j})_{x\in Q_0}} & & \uparrow {\scriptstyle \oplus v^{i+1,-i-1}(a,x,a)_j} \\
M_j & \xrightarrow{(g^{i+1}_{a,j})_{a\in Q_0}} & \oplus_{a\in Q_0}(I_a\langle -i-1\rangle_j \otimes e_a\Lambda^!_0 e_a \otimes M_{i+1}(a))
\end{array}$$

is anti-commutative for all $i, j \in \mathbb{Z}$, where $I_x\langle -i\rangle_j = D(\Lambda^{\rm o}_{i-j}e_x)$. Indeed, this is evident in case $j > i$. Fix $i \geq j$ and $a, x \in Q_0$. Then, we have a k-linear isomorphism $\theta^i_j : D(\Lambda^{\rm o}_{i-j}e_x) \otimes e_x\Lambda^!_1 e_a \otimes M_{i+1}(a) \to \mathrm{Hom}_k(\Lambda^{\rm o}_{i-j}e_x,\, e_x\Lambda^!_1 e_a \otimes M_{i+1}(a))$ as stated in Corollary 2.1.2(1). Consider $\alpha \in Q_1(x, a)$ and $w \in M_j$. We choose a k-basis $\{\bar\delta^{\rm o}_1, \ldots, \bar\delta^{\rm o}_s\}$ for $\Lambda^{\rm o}_{i-j}e_x$. By Statement 3, we obtain

$$(\mathrm{id} \otimes P[\bar\alpha^!] \otimes M(\bar\alpha))(g^i_{x,j}(w)) = (-1)^{\frac{(i-1)i}{2}} \textstyle\sum_{p=1}^s \bar\delta^{{\rm o},\star}_p \otimes \bar\alpha^! \otimes \bar\alpha\bar\delta_p w.$$

As a consequence, we see that

$$\theta^i_j\big[(\mathrm{id} \otimes P[\bar\alpha^!] \otimes M(\bar\alpha))(g^i_{x,j}(w))\big](\bar\delta^{\rm o}_p) = (-1)^{\frac{(i-1)i}{2}}(\bar\alpha^! \otimes \bar\alpha\bar\delta_p w),\ p = 1, \ldots, s.$$

On the other hand, for any k-basis $\{\bar\gamma^{\rm o}_1, \ldots, \bar\gamma^{\rm o}_t\}$ of $\Lambda^{\rm o}_{i+1-j}e_a$, by Statement 3,

$$(I[\bar\alpha] \otimes P_a(\bar\alpha^!) \otimes \mathrm{id})(g^{i+1}_{a,j}(w)) = (-1)^{\frac{i(i+1)}{2}} \textstyle\sum_{q=1}^t (\bar\gamma^{{\rm o},\star}_q \circ P[\bar\alpha^{\rm o}]) \otimes \bar\alpha^! \otimes \bar\gamma_q w.$$

And hence, for any $1 \leq p \leq s$, we obtain

$$\theta^i_j\big[(I[\bar\alpha] \otimes P_a(\bar\alpha^!) \otimes \mathrm{id})(g^{i+1}_{a,j}(w))\big](\bar\delta^{\rm o}_p) = (-1)^{\frac{i(i+1)}{2}} \textstyle\sum_{q=1}^t \bar\gamma^{{\rm o},\star}_q(\bar\delta^{\rm o}_p\bar\alpha^{\rm o}) \cdot (\bar\alpha^! \otimes \bar\gamma_q w).$$

Fix $1 \leq p \leq s$. If $\bar\delta^{\rm o}_p\,\bar\alpha^{\rm o} = 0$, then $\bar\alpha\bar\delta_p = 0$. In this case, we see trivially that

$$\theta^i_j\big[(\mathrm{id}\otimes P[\bar\alpha^!]\otimes M(\bar\alpha))(g^i_{j,x}(w))\big](\bar\delta^{\rm o}_p) = (-1)^i\theta^i_j\big[(I[\bar\alpha]\otimes P_a(\bar\alpha^!)\otimes\mathrm{id})(g^{i+1}_{j,a}(w))\big](\bar\delta^{\rm o}_p).$$

If $\bar\delta^{\rm o}_p\,\bar\alpha^{\rm o} \neq 0$, then $\Lambda^{\rm o}_{i+1-j}e_a$ has a basis $\{\bar\gamma^{\rm o}_1, \ldots, \bar\gamma^{\rm o}_t\}$, where $\bar\gamma^{\rm o}_1 = \bar\delta^{\rm o}_p\,\bar\alpha^{\rm o}$. Noting that $\bar\gamma_1 = \bar\alpha\bar\delta_p$, we obtain

$$\begin{aligned}\theta^i_j\big[(I[\bar\alpha]\otimes P_a(\bar\alpha^!)\otimes\mathrm{id})(g^{i+1}_{j,a}(w))\big](\bar\delta^{\rm o}_p) &= (-1)^{\frac{i(i+1)}{2}}(\bar\alpha^!\otimes\bar\alpha\bar\delta_p\, w)\\ &= (-1)^i\theta^i_j\big[(\mathrm{id}\otimes P[\bar\alpha^!]\otimes M(\bar\alpha))(g^i_{j,x}(w))\big](\bar\delta^{\rm o}_p).\end{aligned}$$

Thus, $(I[\bar\alpha]\otimes P_a(\bar\alpha^!)\otimes\mathrm{id})(g^{i+1}_{j,a}(w)) = (-1)^i(\mathrm{id}\otimes P[\bar\alpha^!]\otimes M(\bar\alpha))(g^i_{j,x}(w))$. It is now is easy to see that

$$(h^{i,-i}_j(a,x,x) \circ g^i_{j,x})(w) + (v^{i+1,-i-1}_j(a,x,a) \circ g^{i+1}_{j,a})(w) = 0.$$

Our claim is established. Thus, Statement 6 holds.

It remains to show that $\mathrm{Ker}(d^0) \subseteq \mathrm{Im}(\zeta_{_M})$. Let $\omega = \sum_{i\in\mathbb{Z}} \omega^i \in \mathrm{Ker}(d^0)$, where

$$\omega^i \in \mathcal{G}(\mathcal{F}(M)^i)^{-i} = \oplus_{c\in Q_0}\,(I_c\langle -i\rangle \otimes e_c\Lambda^!_0 e_c \otimes M_i(c)).$$

Since $M_i = 0$ for all $i \geq r$, we see that $\omega^i = 0$ for all $i \geq r$. We proceed by induction on the maximal integer$n_\omega \leq r$ such that $\omega^i = 0$ for all $i < n_\omega$. If $n_\omega = r$, then $\omega = 0 \in \mathrm{Im}(\zeta_{_M})$. Assume that $n_\omega < r$ and write $n = n_\omega$. Since

$d^0(\omega) = 0$, we have $v^{n,-n}(\omega^n) = -h^{n-1,1-n}(\omega^{n-1}) = 0$. Applying Statement 1, we see that

$$\mathrm{H}^{-n}(\mathfrak{t}^n(\mathcal{G}(\mathcal{F}(M)^n)^\bullet))) \cong \oplus_{c\in Q_0} \mathrm{H}^0(\mathcal{G}(P_a)^\bullet)\langle -n\rangle \otimes M_n(c).$$

It follows from Lemma 6.4.1 that the n-th column of $\mathcal{G}(\mathcal{F}(M)^\bullet)^\bullet$ is a truncated colinear injective coresolution of $\oplus_{c\in Q_0}(S_c \otimes e_c\Lambda_0^! e_c \otimes M_n(c))$, which is graded semisimple. Therefore, $\omega^n \in \mathrm{Ker}(v^{n,-n}) = \mathrm{Soc}(\mathcal{G}(\mathcal{F}(N)^n)^{-n})$. Now, we may write $\omega^n = \sum_{c\in Q_0} e_c^{\mathrm{o},\star} \otimes e_c \otimes u_c$, where $u_c \in M_n(c) = e_c M_n$. Putting

$$u = (-1)^{\frac{(n-1)n}{2}} \sum_{c\in Q_0} u_c \in M_n,$$

we obtain $\zeta_{_M}(u) = \sum_{i\in\mathbb{Z}} u^i$, where $u^i = g^i(u) = g_n^i(u) \in \mathcal{G}(\mathcal{F}(M)^i)^{-i}$. It follows from Statement 3 that

$$\begin{array}{rcl} u^n & = & \sum_{c,a\in Q_0}(-1)^{\frac{(n-1)n}{2}} g_{a,n}^n(w_c) \\ & = & \sum_{c,a\in Q_0} f_{a,n}^n(w_c) \\ & = & \sum_{c,a\in Q_0} e_c^{\mathrm{o},\star} \otimes e_a \otimes e_a w_c \\ & = & \sum_{c\in Q_0} e_c^{\mathrm{o},\star} \otimes e_c \otimes w_c. \end{array}$$

Put $\nu = \omega - \zeta_{_M}(u) = \sum_{i\in\mathbb{Z}}(\omega^i - u^i) \in \mathrm{Ker}(d^0)$. If $i < n$, then $g_n^i(u) = 0$, and hence, $\omega^i - u^i = 0$. Since $u^n = \omega^n$, we have $n_\omega < n_\nu$. Thus, $\nu \in \mathrm{Im}(\zeta_{_M})$, and hence, $\omega \in \mathrm{Im}(\zeta_{_M})$. The proof of the lemma is completed.

The next statement describes a projective resolution for every complex in $C^{\uparrow}_{p,q}(\mathrm{GMod}\Lambda)$ in terms of the extension of the composite of the left Koszul functor and the right complex Koszul functor.

6.4.4 **Proposition**. *Let $\Lambda = kQ/R$ be a Koszul algebra, where Q is a locally finite quiver. Consider $(\mathcal{F}^C \circ \mathcal{G})^C : C(\mathrm{GMod}\Lambda) \to C(\mathrm{GMod}\Lambda)$. Let $M^\bullet \in C^{\uparrow}_{p,q}(\mathrm{GMod}\Lambda)$ for some $p, q \in \mathbb{R}$ with $p \geq 1$ and $q \geq 0$. There exists a natural quasi-isomorphism $\eta^C_{M^\bullet} : (\mathcal{F}^C \circ \mathcal{G})^C(M^\bullet) \to M^\bullet$.*

Proof. Consider $\mathcal{F}^C \circ \mathcal{G} : \mathrm{GMod}\Lambda \to C(\mathrm{GMod}\Lambda)$ and the embedding functor $\kappa : \mathrm{Mod}\Lambda \to C(\mathrm{Mod}\Lambda)$. In view of Lemma 6.4.2, we obtain a functorial morphism $\eta = (\eta^\bullet_M)_{M\in\mathrm{Mod}\Lambda} : \mathcal{F}^C \circ \mathcal{G} \to \kappa$, where $\eta^\bullet_M : (\mathcal{F}^C \circ \mathcal{G})(M) \to M$ is a quasi-isomorphism, for any $M \in \mathrm{Mod}\Lambda$. By Lemma 1.8.8, η extends to a functorial morphism $\eta^C : (\mathcal{F}^C \circ \mathcal{G})^C \to \kappa^C = \mathrm{id}_{C(\mathrm{Mod}\Lambda)}$, where $(\mathcal{F}^C \circ \mathcal{G})^C = \mathbb{T}(\mathcal{F}^C \circ \mathcal{G})$, such that

$$\eta^C_{M^\bullet} = \mathbb{T}(\eta^\bullet_{M^\bullet}) : (\mathcal{F}^C \circ \mathcal{G})^C(M^\bullet) \to M^\bullet,$$

where $\eta^{\bullet}_{M^{\bullet}} : (\mathcal{F}^C \circ \mathcal{G})(M^{\bullet})^{\bullet} \to \kappa(M^{\bullet})^{\bullet}$ is the double complex morphism given by $\eta^j_{M^i} : (\mathcal{F}^C \circ \mathcal{G})(M^i)^j \to \kappa(M^i)^j$ with $i, j \in \mathbb{Z}$.

We claim that $\mathbb{T}(\eta^{\bullet}_{M^{\bullet}})$ is a quasi-isomorphism. By Lemma 6.3.2(2) and (4), this is equivalent to $\mathbb{T}((\eta^{\bullet}_{M^{\bullet}})_s)$ being a quasi-isomorphism for any $s \in \mathbb{Z}$. Fix an integer $s \in \mathbb{Z}$. Consider the following double complex morphism

$$(\eta^{\bullet}_{M^{\bullet}})_s = ((\eta^j_{M^i})_s)_{i,j\in\mathbb{Z}} : (\mathcal{F}^C \circ \mathcal{G})(M^{\bullet})^{\bullet}_s \to \kappa(M^{\bullet})^{\bullet}_s.$$

For any $i \in \mathbb{Z}$, the i-th column of $(\eta^{\bullet}_{M^{\bullet}})_s$ is

$$(\eta^{\bullet}_{M^i})_s : \mathfrak{t}^i((\mathcal{F}^C \circ \mathcal{G})(M^i)^{\bullet})_s \to \mathfrak{t}^i(\kappa(M^i)^{\bullet})_s,$$

which is a quasi-isomorphism by Lemma 6.3.2. Moreover, $\kappa(M^{\bullet})^{\bullet}_s$ is clearly diagonally bounded above. Given $n \in \mathbb{Z}$, we deduce from Lemma 6.2.5(1) that the n-diagonal of $(\mathcal{F}^C \circ \mathcal{G})(M^{\bullet})^{\bullet}_s$ consists of

$$(\mathcal{F}^C \circ \mathcal{G})(M^i)^{n-i}_s = \oplus_{j\in\mathbb{Z};\, a,x\in Q_0} (\Lambda_{n+s-i-j} e_a \otimes D(e_a \hat{\Lambda}_{i-n} e_x) \otimes M^i_j(x)); \text{ for all } i \in \mathbb{Z}.$$

By the assumption on $M^{\bullet}$, there exists some $t \in \mathbb{Z}$ such that $M^i_j = 0$ for all i, j with $i - qj > t$. Fix some $i > (q(n+s)+t)(q+1)^{-1}$. If $j > n+s-i$, then $\Lambda_{n+s-i-j} = 0$; and otherwise, $M^i_j = 0$ since $i - qj \geq i - q(n+s-i) > t$. Thus, $(\mathcal{F}^C \circ \mathcal{G})(M^{\bullet})^{\bullet}_s$ is also diagonally bounded-above. By Lemma 1.8.3, $\mathbb{T}((\eta^{\bullet}_{M^{\bullet}})_s)$ is a quasi-isomorphism, for any $s \in \mathbb{Z}$. This establishes our claim. That is, $\eta^C_{M^{\bullet}}$ is a quasi-isomorphism. The proof of the proposition is completed.

The following statement describes an injective co-resolution for every complex over $\mathrm{GMod}^{-}\Lambda$ in terms of the extension of the composite of the right Koszul functor and the left complex Koszul functor.

6.4.5 **Proposition**. *Let $\Lambda = kQ/R$ be a Koszul algebra, where Q is a locally finite quiver. Consider $(\mathcal{G}^C \circ \mathcal{F})^C : C(\mathrm{GMod}\Lambda) \to C(\mathrm{GMod}\Lambda)$. Let $M^{\bullet} \in C(\mathrm{GMod}^{-}\Lambda)$. There exists a natural quasi-isomorphism $\zeta^C_{M^{\bullet}} : M^{\bullet} \to (\mathcal{G}^C \circ \mathcal{F})^C(M^{\bullet})$.*

Proof. Consider the embedding functor $\kappa : \mathrm{GMod}^{-}\Lambda \to C(\mathrm{GMod}\Lambda)$ and the functor $\mathcal{G}^C \circ \mathcal{F} : \mathrm{GMod}^{-}\Lambda \to C(\mathrm{GMod}\Lambda)$. By Lemma 6.4.3, we have a functorial morphism $\zeta = (\zeta^{\bullet}_M)_{M\in\mathrm{Mod}^{-}\Lambda} : \kappa \to \mathcal{G}^C \circ \mathcal{F}$, where $\zeta^{\bullet}_M : M \to (\mathcal{G}^C \circ \mathcal{F})(M)$ is a quasi-isomorphism for all $M \in \mathrm{GMod}^{-}\Lambda$. By Lemma 1.8.8, ζ extends to a functorial morphism $\zeta^C : \mathrm{id}_{\mathrm{GMod}^{-}\Lambda} = \kappa^C \to (\mathcal{G}^C \circ \mathcal{F})^C = \mathbb{T}(\mathcal{G}^C \circ \mathcal{F})$ such that

$$\zeta^C_{M^{\bullet}} = \mathbb{T}(\zeta^{\bullet}_{M^{\bullet}}) : M^{\bullet} \to (\mathcal{G}^C \circ \mathcal{F})^C(M^{\bullet}),$$

where $\zeta^{\bullet}_{M^{\bullet}} : \kappa(M^{\bullet})^{\bullet} \to (\mathcal{G}^C \circ \mathcal{F})(M^{\bullet})^{\bullet}$ is the double complex morphism given by $\zeta^j_{M^i} : \kappa(M^i)^j \to (\mathcal{G}^C \circ \mathcal{F})(M^i)^j$ with $i, j \in \mathbb{Z}$.

We claim that $\mathbb{T}(\zeta^{\bullet}_{M^{\bullet}})$ is a quasi-isomorphism. Indeed, for any $i \in \mathbb{Z}$, the i-th column of $\zeta^{\bullet}_{M^{\bullet}}$ is $\zeta^{\bullet}_{M^i} : \mathfrak{t}^i(\kappa(M^i)^{\bullet}) \to \mathfrak{t}^i((\mathcal{G}^C \circ \mathcal{F})(M^i)^{\bullet})$, which is clearly a quasi-isomorphism. Moreover, $\kappa(M^{\bullet})^{\bullet}$ is evidently diagonally bounded above. Given any $n \in \mathbb{Z}$, by Lemma 6.2.5(2), the n-diagonal of $(\mathcal{G}^C \circ \mathcal{F})(M^{\bullet})^{\bullet}$ consists of

$$(\mathcal{G}^C \circ \mathcal{F})(M^i)^{n-i} = \oplus_{j \in \mathbb{Z}; x, a \in Q_0} (I_a\langle n-i-j \rangle \otimes e_a \Lambda^!_{n-i} e_x \otimes M^i_j(x)); \text{ for all } i \in \mathbb{Z}.$$

Thus, $(\mathcal{G}^C \circ \mathcal{F})(M^i)^{n-i} = 0$ for any $i > n$. So, $(\mathcal{G}^C \circ \mathcal{F})(M^{\bullet})^{\bullet}$ is n-diagonally bounded above for all $n \in \mathbb{Z}$. That is, $(\mathcal{G}^C \circ \mathcal{F})(M^{\bullet})^{\bullet}$ is diagonally bounded above. By Lemma 1.8.3, $\mathbb{T}(\zeta^{\bullet}_{M^{\bullet}})$ is a quasi-isomorphism, that is, $\zeta^C_{M^{\bullet}}$ is a quasi-isomorphism. The proof of the proposition is completed.

We are ready to prove the main result of this section, which includes the classical Koszul duality of Belinson, Ginzburg and Soergel; see [13].

6.4.6 **Theorem**. *Let $\Lambda = kQ/R$ be a Koszul algebra, where Q is a locally finite quiver. Consider $p, q \in \mathbb{R}$ with $p \geq 1$ and $q \geq 0$.*

(1) *The right derived Koszul functor $\mathcal{F}^D_{p,q} : D^{\downarrow}_{p,q}(\mathrm{GMod}\Lambda) \to D^{\uparrow}_{q+1,p-1}(\mathrm{GMod}\Lambda^!)$ and the left derived Koszul functor*

$$\mathcal{G}^D_{q+1,p-1} : D^{\uparrow}_{q+1,p-1}(\mathrm{GMod}\Lambda^!) \to D^{\downarrow}_{p,q}(\mathrm{GMod}\Lambda)$$

are mutually quasi-inverse, where $\mathrm{GMod}\Lambda$ *and* $\mathrm{GMod}\Lambda^!$ *can be replaced simultaneously by* $\mathrm{gmod}\Lambda$ *and* $\mathrm{gmod}\Lambda^!$*, respectively.*

(2) *The left derived Koszul functor $\mathcal{G}^D_{p,q} : D^{\uparrow}_{p,q}(\mathrm{GMod}\Lambda) \to D^{\downarrow}_{q+1,p-1}(\mathrm{GMod}\Lambda^!)$ and the right derived Koszul functor*

$$\mathcal{F}^D_{q+1,p-1} : D^{\downarrow}_{q+1,p-1}(\mathrm{GMod}\Lambda^!) \to D^{\uparrow}_{p,q}(\mathrm{GMod}\Lambda)$$

are mutually quasi-inverse, where $\mathrm{GMod}\Lambda$ *and* $\mathrm{GMod}\Lambda^!$ *can be replaced simultaneously by* $\mathrm{gmod}\Lambda$ *and* $\mathrm{gmod}\Lambda^!$*, respectively.*

Proof. Note that $C^{\downarrow}_{p,q}(\mathrm{GMod}\Lambda) \subseteq C(\mathrm{GMod}^{-}\Lambda)$. Given $N^{\bullet} \in C^{\downarrow}_{p,q}(\mathrm{GMod}\Lambda)$, by Proposition 1.8.6 and Theorem 6.3.3, $(\mathcal{G}^C \circ \mathcal{F})^C(N^{\bullet}) = (\mathcal{G}^C_{q+1,p-1} \circ \mathcal{F}^C_{p,q})(N^{\bullet})$. Thus, we have a natural quasi-isomorphism $\zeta^C_{N^{\bullet}} : N^{\bullet} \to (\mathcal{G}^C_{q+1,p-1} \circ \mathcal{F}^C_{p,q})(N^{\bullet})$, by Proposition 6.4.5.

Given $M^{\bullet} \in C^{\uparrow}_{q+1,p-1}(\mathrm{GMod}\Lambda^!)$, by Proposition 1.8.6 and Theorem 6.3.3, $(\mathcal{F}^C_{p,q} \circ \mathcal{G}^C_{q+1,p-1})(M^{\bullet}) = (\mathcal{F}^C \circ \mathcal{G})^C(M^{\bullet})$. And by proposition 6.4.4, we have a natural quasi-isomorphism $\eta^C_{M^{\bullet}} : (\mathcal{F}^C_{p,q} \circ \mathcal{G}^C_{q+1,p-1})(M^{\bullet}) = (\mathcal{F}^C \circ \mathcal{G})^C(M^{\bullet}) \to M^{\bullet}$.

This yields a natural isomorphism $\zeta^D_{N^{\bullet}} : N^{\bullet} \to (\mathcal{G}^D_{q+1,p-1} \circ \mathcal{F}^D_{p,q})(N^{\bullet})$ for every $N^{\bullet} \in D^{\downarrow}_{p,q}(\mathrm{GMod}\Lambda)$, and a natural isomorphism $\eta^D_{M^{\bullet}} : (\mathcal{F}^D_{p,q} \circ \mathcal{G}^D_{q+1,p-1})(M^{\bullet}) \to M^{\bullet}$ for every $M^{\bullet} \in D^{\uparrow}_{q+1,p-1}(\mathrm{GMod}\Lambda^!)$. This implies that the derived Koszul functors $\mathcal{F}^D_{p,q} : D^{\downarrow}_{p,q}(\mathrm{GMod}\Lambda) \to D^{\uparrow}_{q+1,p-1}(\mathrm{GMod}\Lambda^!)$ and $\mathcal{G}^D_{q+1,p-1} : D^{\uparrow}_{q+1,p-1}(\mathrm{GMod}\Lambda^!) \to D^{\downarrow}_{p,q}(\mathrm{GMod}\Lambda)$ are also mutually quasi-inverse. Using the same argument, we see that $\mathcal{F}^D_{p,q} : D^{\downarrow}_{p,q}(\mathrm{gmod}\Lambda) \to D^{\uparrow}_{q+1,p-1}(\mathrm{gmod}\Lambda^!)$ and $\mathcal{G}^D_{q+1,p-1} : D^{\uparrow}_{q+1,p-1}(\mathrm{gmod}\Lambda^!) \to D^{\downarrow}_{p,q}(\mathrm{gmod}\Lambda)$ are mutually quasi-inverse. This establishes Statement (1). Similarly, Statement (2) holds. The proof of the theorem is completed.

Remark. In case $p = 1$ and $q = 0$, the first part of Theorem 6.4.6(1) and (2) has been established in [13, (2.12.1)] under the assumption that Λ has an identity, while the second part of Theorem 6.4.6(1) and (2) has been proved for a positively graded Koszul category in [48, Theorem 30].

The following statement says that the bounded derived Koszul functors are also triangle equivalences under some local boundedness conditions.

6.4.7 **Theorem**. *Let $\Lambda = kQ/R$ be a Koszul algebra, where Q is a locally finite quiver.*

(1) *If Λ is locally right bounded and $\Lambda^!$ is locally left bounded, then the bounded derived Koszul functors $\mathcal{F}^D : D^b(\mathrm{GMod}^b\Lambda) \to D^b(\mathrm{GMod}^b\Lambda^!)$ and $\mathcal{G}^D : D^b(\mathrm{GMod}^b\Lambda^!) \to D^b(\mathrm{GMod}^b\Lambda)$ are mutually quasi-inverse, where* $\mathrm{GMod}^b\Lambda$ *and* $\mathrm{GMod}^b\Lambda^!$ *can be replaced simultaneously by* $\mathrm{gmod}^b\Lambda$ *and* $\mathrm{gmod}^b\Lambda^!$, *respectively.*

(2) *If Λ is locally left bounded and $\Lambda^!$ is locally right bounded, then the bounded derived Koszul functors $\mathcal{G}^D : D^b(\mathrm{GMod}^b\Lambda) \to D^b(\mathrm{GMod}^b\Lambda^!)$ and $\mathcal{F}^D : D^b(\mathrm{GMod}^b\Lambda^!) \to D^b(\mathrm{GMod}^b\Lambda)$ are mutually quasi-inverse, where* $\mathrm{GMod}^b\Lambda$ *and* $\mathrm{GMod}^b\Lambda^!$ *can be replaced simultaneously by* $\mathrm{gmod}^b\Lambda$ *and* $\mathrm{gmod}^b\Lambda^!$, *respectively.*

Proof. Suppose that Λ is right locally bounded and $\Lambda^!$ is left locally bounded. By Corollary 6.3.5, we have triangle exact functors $\mathcal{F}^D : D^b(\mathrm{GMod}^b\Lambda) \to D^b(\mathrm{GMod}^b\Lambda^!)$ and $\mathcal{G}^D : D^b(\mathrm{GMod}^b\Lambda^!) \to D^b(\mathrm{GMod}^b\Lambda)$. By the same argument used in the proof of Theorem 6.4.6, we conclude that they are mutual quasi-inverse. The proof of the theorem is completed.

REMARK. (1) In case Λ is of finite dimensional and $\Lambda^!$ is left noetherian, Theorem 6.4.7(3) is established by Beilinson, Ginzburg and Soergel in [13, (2.12.6)].

(2) Assume that Q has no infinite path with an ending point or no infinite path with a starting point. Then Λ is locally right or left bounded and $\Lambda^!$ is locally left or right bounded, respectively. By Theorem 6.4.7, $D^b(\text{GMod}^b\Lambda) \cong D^b(\text{GMod}^b\Lambda^!)$ and $D^b(\text{gmod}^b\Lambda) \cong D^b(\text{gmod}^b\Lambda^!)$.

6.5 Koszul functors, Auslander-Reiten translations and Serre functors

The objective of this section is to show how the derived Koszul functors are related to Auslander-Reiten translations and Serre functors in various derived categories of graded modules over a quadratic algebra and over a Koszul algebra.

Throughout this section, unless otherwise explicitly stated, $\Lambda = kQ/R$ is a quadratic algebra, where Q is a locally finite quiver. First of all, by making use of the bounded derived Koszul functors $\mathcal{F}^D : D^b(\text{gmod}^b\Lambda^!) \to D^b(\text{gmod}\Lambda)$ and $\mathcal{G}^D : D^b(\text{gmod}^b\Lambda^!) \to D^b(\text{gmod}\Lambda)$; see (6.3.4), we may describe some almost split triangles in $D^b(\text{gmod}\Lambda)$ in terms of bounded complexes of finite dimensional graded $\Lambda^!$-modules; compare [10, (5.2)].

6.5.1 **Theorem**. *Let $\Lambda = kQ/R$ be a quadratic algebra, where Q is a locally finite quiver. If $M^\bullet \in C^b(\text{gmod}^b\Lambda^!)$ such that $\mathcal{F}^D(M^\bullet)$ or $\mathcal{G}^D(M^\bullet)$ is indecomposable in $D^b(\text{gmod}\Lambda)$, then there exists an almost split triangle*

$$\mathcal{G}^D(M^\bullet)[-1] \longrightarrow N^\bullet \longrightarrow \mathcal{F}^D(M^\bullet) \longrightarrow \mathcal{G}^D(M^\bullet)$$

in each of $D^b(\text{gmod}\Lambda)$, $D(\text{gmod}\Lambda)$ and $D(\text{GMod}\Lambda)$.

Proof. By Theorem 6.3.5(3), we have functors $\mathcal{F}^C : C^b(\text{gmod}^b\Lambda^!) \to C^b(\text{gproj}\Lambda)$ and $\mathcal{G}^C : C^b(\text{gmod}^b\Lambda^!) \to C^b(\text{ginj}\Lambda)$. Let $M^\bullet \in C^b(\text{gmod}^b\Lambda^!)$. By Proposition 6.2.2(1), $\mathcal{F}^C(M^\bullet)^n = \oplus_{(i,x)\in\mathbb{Z}\times Q_0}(P_x^!\langle n-i\rangle \otimes M_{n-i}^i(x))$, for all $n \in \mathbb{Z}$. Since $\oplus_{i\in\mathbb{Z}}M^i$ is finite dimensional, $\mathcal{F}^C(M^\bullet) \in C^b(\text{gproj}\Lambda^!)$. Applying Theorem 4.1.3 and Proposition 6.2.2(2), we have $\nu\mathcal{F}^C(M^\bullet)^n \cong \oplus_{(i,x)\in\mathbb{Z}\times Q_0}(I_x^!\langle n-i\rangle \otimes M_{n-i}^i(x)) = \mathcal{G}^C(M^\bullet)^n$, for all $n \in \mathbb{Z}$. That is, $\mathcal{G}^C(M^\bullet) \cong \nu\mathcal{F}^C(M^\bullet)$. In particular, $\mathcal{F}^K(M^\bullet)$ is indecomposable in $K^b(\text{gproj}\Lambda)$ if and only if $\mathcal{G}^K(M^\bullet)$ is indecomposable in $K^b(\text{ginj}\Lambda)$.

If $\mathcal{F}^D(M^\bullet)$ or $\mathcal{G}^D(M^\bullet)$ is indecomposable in $D^b(\text{gmod}\Lambda)$, then $\mathcal{F}^K(M^\bullet)$ is indecomposable in $K^b(\text{gproj}\Lambda)$ and $\mathcal{G}^K(M^\bullet)$ is indecomposable in $K^b(\text{ginj}\Lambda)$. And in this case, by Theorem 4.3.2, there exists a desired almost split triangle in each of $D^b(\text{gmod}\Lambda)$, $D(\text{gmod}\Lambda)$ and $D(\text{GMod}\Lambda)$. The proof of the theorem is completed.

Remark. The almost split triangle stated in Theorem 6.5.1 explains our terminology of "left" Koszul functor and "right" Koszul functor.

Example. Let $\Lambda = kQ/R$ be a Koszul algebra with $a \in Q_0$. By Lemma 6.4.1, $\mathcal{F}^D(I^!_a) \cong \mathcal{G}^D(P^!_a) \cong S_a$ in $D^b(\text{gmod}\Lambda)$. It is well known that S is indecomposable in $D^b(\text{gmod}\Lambda)$; see [49, (III.3.4.7)]. If $I^!_a$ or $P^!_a$ is finite dimensional, then S_a is the ending or starting term respectively of an almost split triangle in $D^b(\text{gmod}\Lambda)$.

Next, we shall consider the case where Λ is a Koszul algebra. A complex over gmodΛ is called **derived indecomposable** if it is indecomposable in $D(\text{gmod}\Lambda)$. In case $\Lambda^!$ is locally bounded, we shall establish the existence of almost split triangles in $D^b(\text{gmod}\Lambda)$ for bounded derived indecomposable complexes over $\text{gmod}^b\Lambda$ and describe the Auslander-Reiten translates using bounded derived Koszul functors.

6.5.2 **Theorem**. *Let $\Lambda = kQ/R$ be a Koszul algebra, where Q is a locally finite quiver.*

(1) *Every bounded derived-indecomposable complex $M^\bullet$ over* $\text{gmod}^b\Lambda$ *is the ending term of an almost split triangle in $D^b(\text{gmod}\Lambda)$ if and only if $\Lambda^!$ is locally right bounded; and in this case, $\tau M^\bullet \cong \mathcal{G}^D(\mathcal{G}^D(M^\bullet))[-1]$.*

(2) *Every bounded derived-indecomposable complex $M^\bullet$ over* $\text{gmod}^b\Lambda$ *is the starting term of an almost split triangle in $D^b(\text{gmod}\Lambda)$ if and only if $\Lambda^!$ is locally left bounded; and in this case, $\tau^- M^\bullet \cong \mathcal{F}^D(\mathcal{F}^D(M^\bullet))[1]$.*

(3) *Every bounded derived-indecomposable complex $M^\bullet$ over* $\text{gmod}^b\Lambda$ *is the starting term, as well as the ending term, of an almost split triangle in $D^b(\text{gmod}\Lambda)$ if and only if $\Lambda^!$ is locally bounded.*

Proof. We shall only prove Statement (1). Given any $a \in Q_0$, by Lemma 5.4.1, S_a has a projective resolution $\mathcal{P}^\bullet_a$ in gmodΛ with $\mathcal{P}^{-n}_a = \oplus_{x\in Q_0}(P_x\langle -n\rangle \otimes D(e_a\Lambda^!_n e_x))$, for all $n \in \mathbb{Z}$. Then, $\text{End}_{D^b(\text{gmod}\Lambda)}(S_a) \cong \text{Hom}_{K(\text{gmod}\Lambda)}(P^\bullet_a, S_a) \cong k$; see [61, (10.4.7)]. Thus, S_a is strongly indecomposable in $D(\text{gmod}\Lambda)$. If S_a is the ending term of an almost split triangle in $D^b(\mathrm{gmod}\Lambda)$, then $\mathcal{P}_a^\bullet$ is bounded; see [38, (5.2)]. In particular, $e_a\Lambda^!_n = 0$ for all but finitely many $n \geq 0$, that is, $e_a\Lambda^!$ is finite dimensional. This establishes the necessity of Statement (1).

Suppose that $\Lambda^!$ is locally right bounded. Then $\mathrm{ginj}\Lambda^! \subseteq \mathrm{gmod}^b\Lambda^!$. Consider $M^\bullet \in C^b(\mathrm{gmod}^b\Lambda)$, which is indecomposable in $D^b(\mathrm{gmod}\Lambda)$. By Corollary 6.3.5(3), $\mathcal{G}^C(M^\bullet) \in C^b(\mathrm{ginj}\Lambda^!) \subseteq C^b(\mathrm{gmod}^b\Lambda^!)$. By Theorem 6.3.4(3), $\mathcal{F}^C(\mathcal{G}^C(M^\bullet)) \in C^b(\mathrm{gproj}\Lambda)$. Thus, by Proposition 6.4.4 and Lemma 1.8.6, $\mathcal{F}^D(\mathcal{G}^D(M^\bullet)) \cong M^\bullet$. Now, it follows from Theorem 6.5.1 that there exists an almost split triangle

$$\mathcal{G}^D(\mathcal{G}^D(M^\bullet))[-1] \longrightarrow N^\bullet \longrightarrow M^\bullet \longrightarrow \mathcal{G}^D(\mathcal{G}^D(M^\bullet))$$

in $D^b(\mathrm{gmod}\Lambda)$. The proof of the theorem is completed.

Example. Let $\Lambda = kQ$, where Q is a locally finite quiver. Then $\Lambda^! = kQ^{\mathrm{o}}/R^!$, where $R^!$ is the ideal generated by all paths of length two. Since $\Lambda^!$ is locally bounded, by Theorem 6.5.2, every indecomposable complex in $D^b(\mathrm{gmod}^b\Lambda)$ is the starting term, as well as the ending term, of an almost split triangle in $D^b(\mathrm{gmod}\Lambda)$.

Finally, we shall concentrate on the bounded derived category of finite dimensional graded Λ-modules.

6.5.3 **Lemma**. *Let $\Lambda = kQ/R$ be a Koszul algebra, where Q is a locally finite quiver. Then $D^b(\mathrm{gmod}^b\Lambda)$ is Hom-finite and Krull-Schmidt.*

Proof. Let $M \in \mathrm{gmod}^b\Lambda$. By Lemma 6.4.2, we obtain a quasi-isomorphism $\eta_M^\bullet$: $(\mathcal{F}^C \circ \mathcal{G})^C(M)^\bullet \to M$ with $(\mathcal{F}^C \circ \mathcal{G})^C(M)^n = \oplus_{(i,x)\in\mathbb{Z}\times Q_0}(P_a\langle n-i\rangle \otimes (I^!_x)_n(a) \otimes M_i(x))$. Since $M_i(x) = 0$ for all but finitely many (i,x), we see that $(\mathcal{F}^C \circ \mathcal{G})^C(M)^\bullet$ is a graded projective resolution of M over $\mathrm{gproj}\Lambda$. Given $N \in \mathrm{gmod}^b\Lambda$, we deduce from Lemma 3.4.2 that $\mathrm{GExt}^i_\Lambda(M,N)$ is finite dimensional for all $i \geq 0$. Therefore, $D^b(\mathrm{gmod}^b\Lambda)$ is Hom-finite and Krull-Schmidt; see [32, Corollary B]. The proof of the lemma is completed.

In the locally bounded Koszul case, we shall establish the existence of almost split triangles in $D^b(\mathrm{gmod}^b\Lambda)$ and describe the Serre functors in terms of the bounded derived Koszul functors.

6.5.4 **Theorem**. *Let $\Lambda = kQ/R$ be a locally bounded Koszul algebra, where Q is a locally finite quiver.*

(1) *There exist almost split triangles in* $D^b(\mathrm{gmod}^b\Lambda)$ *on the right if and only if* $\Lambda^!$ *is right locally bounded*; *and in this case,*

$$\mathcal{G}^D\circ\mathcal{G}^D : D^b(\mathrm{gmod}^b\Lambda)\to D^b(\mathrm{gmod}^b\Lambda)$$

is a right Serre functor.

(2) *There exist almost split triangles in* $D^b(\mathrm{gmod}^b\Lambda)$ *on the left if and only if* $\Lambda^!$ *is left locally bounded*; *and in this case,*

$$\mathcal{F}^D\circ\mathcal{F}^D : D^b(\mathrm{gmod}^b\Lambda)\to D^b(\mathrm{gmod}^b\Lambda)$$

is a left Serre functor.

(3) *There exist almost split triangles in* $D^b(\mathrm{gmod}^b\Lambda)$ *if and only if* $\Lambda^!$ *is locally bounded*; *and in this case,* $\mathcal{G}^D\circ\mathcal{G}^D : D^b(\mathrm{gmod}^b\Lambda)\to D^b(\mathrm{gmod}^b\Lambda)$ *is a right Serre equivalence and* $\mathcal{F}^D\circ\mathcal{F}^D : D^b(\mathrm{gmod}^b\Lambda)\to D^b(\mathrm{gmod}^b\Lambda)$ *is a left Serre equivalence.*

Proof. We shall only prove Statement (1). Since Λ is locally bounded, $\mathrm{gproj}\Lambda$ and $\mathrm{ginj}\Lambda$ are subcategories of $\mathrm{gmod}^b\Lambda$. Given $a\in Q_0$, by Lemma 5.4.1, S_a has a linear projective resolution $\mathcal{P}_a^{\bullet}$ over $\mathrm{gproj}\Lambda$ with $\mathcal{P}_a^{-n}=\oplus_{x\in Q_0}(P_x\langle -n\rangle\otimes D(e_a\Lambda^!_ne_x))$. If S_a is the ending term of an almost split triangle in $D^b(\mathrm{gmod}^b\Lambda)$, then it has a finite projective resolution over $\mathrm{gproj}\Lambda$; see [38, (5.2)]. So, $\mathcal{P}_a^{\bullet}$ is a bounded complex over $\mathrm{gproj}\Lambda$. In particular, $e_a\Lambda^!_n=0$ for $n\gg 0$. That is, $e_a\Lambda^!$ is finite dimensional.

Suppose that $\Lambda^!$ is right locally bounded. Let $M^{\bullet}\in D^b(\mathrm{gmod}^b\Lambda)$ be indecomposable. Since Λ is left locally bounded, by Theorem 6.4.7(3), $M^{\bullet}\cong\mathcal{F}^D(\mathcal{G}^D(M^{\bullet}))$. By Theorem 4.1.3 and Proposition 6.2.2, we deduce that $\nu(\mathcal{F}^D(\mathcal{G}^D(M^{\bullet})))\cong\mathcal{G}^D(\mathcal{G}^D(M^{\bullet}))$. Considering the Nakayama functor $\nu:\mathrm{gproj}\Lambda\to\mathrm{gmod}^b\Lambda$, we obtain an almost split triangle

$$\mathcal{G}^D(\mathcal{G}^D(M^{\bullet}))[-1]\longrightarrow N^{\bullet}\longrightarrow M^{\bullet}\longrightarrow\mathcal{G}^D(\mathcal{G}^D(M^{\bullet}))$$

in $D^b(\mathrm{gmod}^b\Lambda)$; see [38, (5.8)]. This implies that

$$\mathcal{G}^D\circ\mathcal{G}^D : D^b(\mathrm{gmod}^b\Lambda)\to D^b(\mathrm{gmod}^b\Lambda)$$

is a right Serre functor; see [55, (I.2.3)]. The proof of the theorem is completed.

Remark. Let $\Lambda=kQ/R$ be a Koszul algebra, where Q is a locally finite quiver with no infinite path. Since Λ and $\Lambda^!$ are locally bounded, $D^b(\mathrm{gmod}^b\Lambda)$ and $D^b(\mathrm{gmod}^b\Lambda^!)$ are equivalent and have almost split triangles; see (6.4.7) and (6.5.4).

In case $R = 0$, one can probably describe the Aulsander-Reiten components of $\mathrm{gmod}^b kQ$, as is done for $\mathrm{mod}^b kQ$; see [11]. Since $\Lambda^! = kQ/(kQ^+)^2$, this will yield a description of the Auslander-Reiten components for $D^b(\mathrm{gmod}^b kQ)$ and $D^b(\mathrm{gmod}^b kQ/(kQ^+)^2)$, as is done for $D^b(\mathrm{mod}^b kQ)$ and $D^b(\mathrm{mod}^b kQ/(kQ^+)^2)$; see [10, 11].

Bibliography

[1] F. W. Anderson and K. R. Fuller, “Rings and categories of modules”, Second edition, Graduate Texts in Mathematics 13 (Springer-Verlag, New York, 1992).

[2] I. Assem, D. Simson and A. Skowronski, “Elements of the Representation Theory of Associative Algebras”, Volume 1: Techniques of Representation Theory, London Mathematical Society Student Texts 65 (Cambridge University Press, Cambridge, 2006).

[3] M. Auslander, “Coherent functors”, Proceedings of the Conference on Categorical Algebra (La Jolla, California, 1965) (Springer, New York, 1966) 189 – 231.

[4] M. Auslander, “Functors and morphisms determined by objects”, Lecture Notes in Pure Appl. Math. 37 (Dekker, New York, 1978) 1 - 244.

[5] M. Auslander and I. Reiten, “Representation theory of artin algebras III”, Comm. Algebra 3 (1975) 239 - 294.

[6] M. Auslander and I. Reiten, “On a generalized version of the Nakayama conjecture,” Proc. Amer. Math. Soc. 52 (1975) 69-74.

[7] M. Auslander and I. Reiten, “Representation theory of artin algebras IV”, Comm. Algebra 5 (1977) 443 - 518.

[8] M. Auslander, I. Reiten and Smalø, “Representation theory of artin algebras,” Cambridge Studies in Advanced Mathematics 36 (Cambridge University Press, Cambridge, 1995).

[9] R. Bautista and S. Liu, “Covering theory for linear categories with application to derived categories”, J. Algebra 406 (2014) 173 - 225.

[10] R. Bautista and S. Liu, "The bounded derived categories of an algebra with radical squared zero", J. Algebra 482 (2017) 303 - 345.

[11] R. Bautista, S. Liu and C. Paquette, "Representation theory of strongly locally finite quivers", Proc. London Math. Soc. 106 (2013) 97 - 162.

[12] R. Bautista, L. Salmeón and R. Zuazua, "Differential Tensor Algebras and Their Module Categories", London Mathematical Society Lecture Note Series 362 (Cambridge University Press, Cambridge, 2009).

[13] A. Beilinson, V. Ginzburg and W. Soergel, "Koszul duality patterns in representation theory", J. Amer. Math. Soc. 9 (1996) 473 - 527.

[14] K. Bongartz and P. Gabriel, "Covering spaces in representation theory", Invent. Math. 65 (1982) 331 - 378.

[15] M. Bouhada, "Catégories singulières et dualité de Koszul", Ph.D thesis (Université de Sherbrooke, 2019).

[16] A. M. Bouhada, M. Huang and S. Liu, "Koszul duality for non-graded derived categories", arXiv:1908.06153.

[17] M. C. R. Butler and C. M. Ringel, "Auslander-Reiten sequences with few middle terms and applications to string algebras," Comm. Algebr. 15 (1987) 145-179.

[18] H. Cartan and S. Eilenberg, "Homological Algebra", Princeton University Press, Princeton, New Jersey, 1956.

[19] R. Gordon and E. L. Green, "Representation theory of graded Artin algebras", J. Algebra 76 (1982) 138 - 152.

[20] P. Gabriel and A. V. Roiter, "Representations of Finite-Dimensional Algebras", Algebra VIII, Encyclopedia Math. Sci. 73 (Springer, Berlin, 1992).

[21] M. Goresky, R. Kottwitz and R. MacPherson, "Equivariant cohomology, Koszul duality, and the localization theorem", Invent. Math. 131 (1998) 25 - 83.

[22] E.L. Green and R. Martinez-Villa, “Koszul and Yoneda algebras,” Canad. Math. Soc. Conf. Proc. 18 (1996) 247 - 298.

[23] E. L. Green, R. Martinez-Villa, I. Reiten, Ø. Solberg and D. Zacharia, “On modules with linear presentations”, J. Algebra 205 (1998) 578–604.

[24] E. L. Green and D. Zacharia, “The cohomology ring of a monomial algebra”, Manuscripta Math. 85 (1994) 11-23.

[25] P.-P. Grivel,“Catégories dérivées et foncteurs dérivés”, in *Algebraic D-modules*, Perspective in Mathematics 2 (Academic Press Inc., Boston, 1987) 1 - 108.

[26] E. Gupta, “Auslander-Reiten theory and string modules under the locally finite-dimensional setting”, Master’s thesis (Université de Sherbrooke, 2023).

[27] D. Happel, “Triangulated categories in the representation theory of finite dimensional algebras”, London Mathematical Society Lecture Note Series 119 (Cambridge University Press, Cambridge, 1988).

[28] D. Happel, “Auslander-Reiten triangles in derived categories of finite dimensional algebras”, Proc. Amer. Math. Soc. 112 (1991) 641 - 648.

[29] K. Igusa, S. Liu and C. Paquette, “A proof of the strong no loop conjecture,” Adv. Math. 228 (2011) 2731 - 2742.

[30] H. Krause, “Krull–Remak–Schmidt categories and projective covers”, Expo. Math. 33 (4) (2015) 535 - 549.

[31] H. Krause, “A short proof for Auslander’s defect formula”, Linear Algebra Appl. 365 (2003), 267–270.

[32] J. Le and X. Chen, “Karoubianness of a triangulated category”, J. Algebra 310 (2007) 452 - 457.

[33] H. Lenzing, “Hereditary Categories”, *Handbook of tilting theory*, London Mathematical Society Lectures Note Series 332 (Cambridge University Press, Cambridge, 2007) 105 - 146.

[34] H. Lenzing, “Nilpotente Elemente in Ringen von endlicher globaler Dimension”, Math. Z. 108 (1969) 313 - 324.

[35] H. Lenzing and R. Zuazua, “Auslander-Reiten duality for abelian categories”, Bol. Soc. Mat. Mexicana 10 (2004) 169 - 177.

[36] S. Liu, “Auslander-Reiten theory in a Krull-Schmidt category”, Sao Paulo J. Math. Sci. 4 (2010) 425 - 472.

[37] S. Liu and J.-P. Morin, “The strong no loop conjecture for special biserial algebras”, Proc. Amer. Math. Soc. 132 (2004) 3513 - 3523.

[38] S. Liu and H. Niu, “Almost split sequences in tri-exact categories”, J. Pure Appl. Algebra 226 (2022) 1 - 31.

[39] S. Liu, P. Ng and C. Paquette, “Almost split sequence and approximation”, Algebr. Represent. Ther. 16 (2013) 1809 - 1827.

[40] S. MacLane, “Homology”, (Springer-Verlag, Berlin, Heidelberg, New York, 1971.)

[41] D. O. Madsen, “On a common generalization of Koszul duality and tilting equivalence”, Advances in Mathematics, 2011, 227(6): 2327-2348.

[42] R. Martinez-Villa, “Introduction to Koszul algebras,” Rev. Un. Mat. Argentina 48 (2007) 67 - 95.

[43] R. Martinez-Villa, “Applications of Koszul algebras: The preprojective algebra,” Canad. Math. Soc. Conf. Proc. 18 (1996) 487 - 504.

[44] R. Martinez-Villa, “Graded, self-injective and Koszul algebras,” J. Algebra 215 (1999) 34 - 72.

[45] R. Martinez-Villa and Ø. Solberg, “Graded and Koszul categories”, Appl. Categ. Struct. 18 (2010) 615 - 652.

[46] R. Martinez-Villa and M. Saorin, “Koszul equivalences and dualities”, Pacific J. Math. 214 (2004) 359 - 378.

[47] J. P. May, “The cohomology of restricted Lie algebras and of Hopf algebras”, J. Algebra 3 (1966) 123 - 146.

[48] V. Mazorchuk, S. Ovsienko, and C. Stroppel, "Quadratic duals, Koszul dual functors, and applications", Trans. Amer. Math. Soc. 361(3) (2009) 1129 - 1172.

[49] D. Milicic, "Lectures on derived categories," www.math.utah.edu/ ˜milicic/Eprints/dercat. pdf (preprint).

[50] I. Mori and K. Ueyama, "Stable categories of graded maximal Cohen–Macaulay modules over noncommutative quotient singularities", Adv. Math. 297 (2016) 54 - 92.

[51] B. Mitchell, "Theory of Categories," (Academic Press, San Diego, 1965).

[52] C. Năstăsescu and F. Van Oystaeyen, "Graded and Filtered Rings and Modules", Lecture Notes in Mathematics 758 (Springer-Verlag, Berlin, 1979).

[53] C. Năstăsescu and F. Van Oystaeyen, "Methods of Graded Rings", (Springer-Verlag, Berlin, Heidelberg, 2004).

[54] S. B. Priddy, "Koszul resolutions", Trans. Amer. Math. Soc. 152 (1970) 39 - 60.

[55] I. Reiten and M. Van den Bergh, "Notherian hereditary categories satisfying Serre duality", J. Amer. Math. Soc. 15 (2002) 295 - 366.

[56] S. Ryom-Hansen, "Koszul duality of translation and Zuckermann functor", J. Lie Theory 14 (2004) 151 - 163.

[57] P. Schapira, "Categories and Homological Algebra", http:// webusers.imj-prg.fr/ pierre.schapira/.

[58] A. Shah, "Krull-Remak-Schmidt decompositions in Hom-finite additive categories", Expo. Math. 41(1):220–237, 2023.

[59] A. Skowroński and J. Waschbüsch, "Representation-finite biserial algebras," J. Reine Angew Math. 345 (1983) 172-181.

[60] B. Wald and J. Waschbüsch, "Tame biserial algebras", J. Algebra, 95(1985), 480-500.

[61] C. A. Weibel, "An Introduction to Homological Algebra", Cambridge Studies in Advanced Mathematics 38 (Cambridge University Press, Cambridge, 1994).