Standard Auslander-Reiten components and cluster structure of infinite Dynkin type

Shiping Liu (Université de Sherbrooke)
Charles Paquette (University of New Brunswick)

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- For describing morphisms in $\mathcal{A}$, we apply Auslander-Reiten theory,
  1. irreducible morphisms,
  2. almost split sequences,
  3. AR-quiver.

Pseudo-exact sequences

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Example

Any sequence $X \longrightarrow 0 \longrightarrow Z$ is pseudo-exact.

Almost split sequences

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REMARK. This unifies notions of almost split sequences in abelian categories and almost split triangles in triangulated categories.

Auslander-Reiten quiver

Definition

AR-quiver $\Gamma_{\mathcal{A}}$ of $\mathcal{A}$ is translation quiver as follows:

- The vertices are the non-iso objects in $\mathrm{ind}\mathcal{A}$.
- For vertices X, Y, draw $d_{X,Y}$ arrows $X \to Y$, where
$$d_{X,Y} = \dim_k \mathrm{rad}(X,Y)/\mathrm{rad}^2(X,Y).$$
- If $X \longrightarrow Y \longrightarrow Z$ is almost split, then $\tau Z = X$.

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- $k(\Gamma)$: the mesh category of Γ over k.
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Definition (Ringel)

Γ is *standard* if $\mathcal{A}(\Gamma) \cong k(\Gamma)$.

The module category case

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1) (R, BG) *A is rep-finite with* $\mathrm{char} k \neq 2$.

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Theorem

A component Γ of $\Gamma_{\mathrm{mod} A}$ is standard provided

1) (R, BG) *A is rep-finite with* $\mathrm{char}\, k \neq 2$.
2) (Ringel) *Γ is preprojective or preinjective.*

A general description

Theorem (Skowronski)

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Let Γ be standard component of $\Gamma_{\mathrm{mod} A}$.

1) *All but finitely many τ-orbits in Γ are periodic.*

2) *If Γ is regular, then Γ is stable tube or $\Gamma \cong \mathbb{Z}\Delta$, where Δ finite acyclic quiver.*

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Let $\mathcal{A}$ be Hom-finite Krull-Schmidt k-category.

1. How to decide a component of $\Gamma_{\mathcal{A}}$ is standard?
2. Are there new types of standard components?
3. We consider these questions for AR-components with a section.

Sections of a translation quiver

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A full subquiver Δ of Γ is *section* if

1. Δ is connected, acyclic, and convex in Γ.
2. Δ meets each τ-orbit in Γ exactly once.

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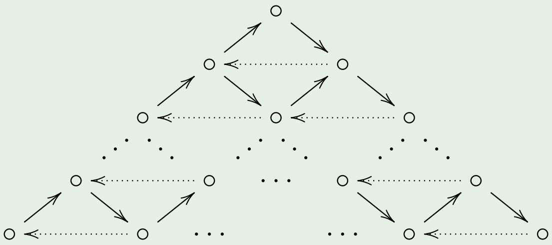

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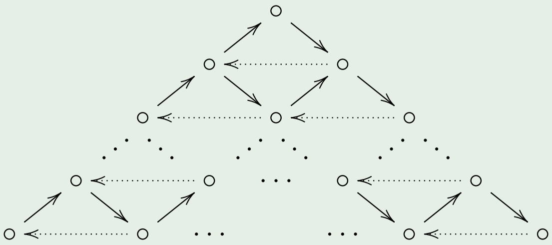

The two exterior paths are sections.

Construction of translation quivers with sections

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Notation

- $\mathbb{N}\Delta =< (x, i) \mid x \in \Delta_0, i \in \mathbb{N} > \subseteq \mathbb{Z}\Delta$.

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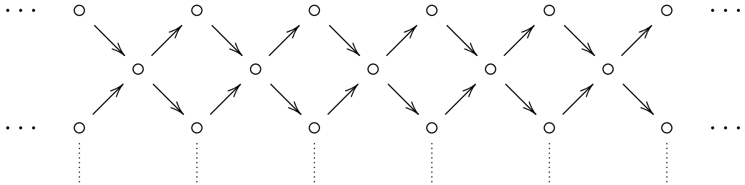
- $\mathbb{N}^{-}\Delta = <(x, -i) \mid x \in \Delta_0, i \in \mathbb{N} > \subseteq \mathbb{Z}\Delta$.

Example

The translation quiver $\mathbb{Z}\mathbb{A}_\infty$ is as follows:



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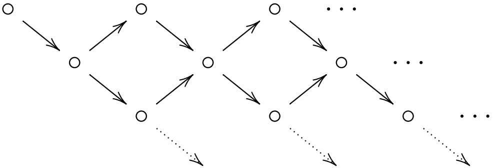

Example

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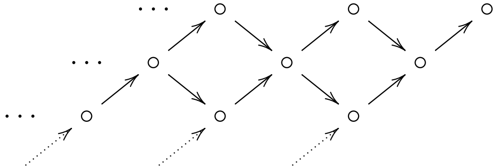
then $\mathbb{N}^-\mathbb{A}_\infty^-$ is as follows:



Stable components

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If Γ is not τ-periodic, then $\Gamma \cong \mathbb{Z}\Delta$, where Δ is locally finite acyclic quiver.

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This yields embedding $\varGamma \to \mathbb{Z}\varDelta : \tau^n X \mapsto (-n, X)$.

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Γ is standard $\Leftrightarrow$ the quasi-simple objects are orthogonal bricks.

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The following are equivalent.

1. *Γ is standard.*
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3. *Γ is a connecting component of $\Gamma_{\mathrm{mod}B}$, where B is a tilted factor algebra of A.*

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Proposition

$\mathrm{rep}^+(Q)$, *cat. of finitely presented representations, is Hom-finite, hereditary, abelian.*

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2. *preinjective* if Γ has some indecomposable injective representation I_x, $x \in Q_0$.
3. *regular* if Γ has no P_x, I_x, $x \in Q_0$.

Preprojective and preinjective components

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1. *The unique preprojective component, and the preinjective components of* $\Gamma_{\mathrm{rep}^+(Q)}$ *are all standard.*
2. *If* Q *has no infinite path, then* $\Gamma_{\mathrm{rep}^+(Q)}$ *has unique preprojective component of shape* $\mathbb{N}Q^{\mathrm{op}}$*, unique preinjective component of shape* $\mathbb{N}^{-}Q^{\mathrm{op}}$*.*

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1. *The regular components of $\Gamma_{\mathrm{rep}^+(Q)}$ are wings or $\mathbb{Z}\mathbb{A}_\infty, \mathbb{N}\mathbb{A}_\infty^+, \mathbb{N}^-\mathbb{A}_\infty^-$.*

Regular components

Theorem

Suppose that Q is infinite.

1. *The regular components of $\Gamma_{\mathrm{rep}^+(Q)}$ are wings or $\mathbb{ZA}_\infty, \mathbb{NA}_\infty^+, \mathbb{N}^-\mathbb{A}_\infty^-$.*
2. *The regular components are all standard $\Leftrightarrow$ Q of infinite Dynkin types $\mathbb{A}_\infty, \mathbb{A}_\infty^\infty, \mathbb{D}_\infty$.*

The bounded derived category $D^b(\mathrm{rep}^+(Q))$

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The bounded derived category $D^b(\mathrm{rep}^+(Q))$

1. $D^b(\mathrm{rep}^+(Q))$ is Hom-finite, Krull-Schmidt.
2. $\Gamma_{D^b(\mathrm{rep}^+(Q))}$ has a *connecting component* $\mathcal{C}_Q$, obtained by gluing the preprojective component and the shift by -1 of all preinjective components of $\Gamma_{\mathrm{rep}^+(Q)}$.

Standard components in $D^b(\mathrm{rep}^+(Q))$

Theorem

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2. $\mathcal{C}_Q \cong \mathbb{Z}Q^{\mathrm{op}} \Leftrightarrow Q$ *has no infinite path.*
3. *The components of* $\Gamma_{D^b(\mathrm{rep}^+(Q))}$ *are all standard* $\Leftrightarrow Q$ *is infinite Dynkin quiver.*

Cluster category of infinite Dynkin type

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The cluster category

$$C(Q) = D^b(\mathrm{rep}^+(Q))/\tau[-1]$$

is Hom-finite, triangulated, 2-CY.

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1. *$r = 0$ if Q of type $\mathbb{A}_\infty$.*
2. *$r = 1$ if Q of type $\mathbb{D}_\infty$.*
3. *$r = 2$ if Q of type $\mathbb{A}_\infty^\infty$. In this case, the regular components are orthogonal.*

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$\mathcal{T}_M$: additive subcategory of $\mathcal{T}$ obtained by deleting M, where $M \in \mathrm{ind}\mathcal{T}$.

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3. The quiver of $\mathrm{add}(\mathcal{T}_M \cup M^*)$ is obtained from $Q_{\mathcal{T}}$ by mutation at M.
4. $\mathcal{A}$ has exact triangles

$$M \xrightarrow{f} N \xrightarrow{g} M^* \to M[1]; \; M^* \xrightarrow{s} L \xrightarrow{t} M \to M^*[1],$$

where f, s are minimal left $\mathcal{T}_M$-approximations,
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Main Results

Theorem (BMRRT, HJ)

If Q is finite or of type $\mathbb{A}_\infty$, then $C(Q)$ admits a cluster structure formed by the cluster tilting subcategories.

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