Geometric properties of matrices

Shiping Liu

Université de Sherbrooke

Lecture
at
Shaoxing University

July 4, 2019

Plan

1. Zariski Topology

Plan

1. Zariski Topology
2. Quivers and Representations

Plan

1. Zariski Topology
2. Quivers and Representations
3. Orbits and orbit closures of Representations

Objective

1. We shall illustrate the interaction of linear algebra with topology and geometry.

Objective

1. We shall illustrate the interaction of linear algebra with topology and geometry.
2. More precisely, we shall describe the orbit closures in the Zariski space of $m \times n$ matrices.

Algebraic sets

1. Throughout, let k be an infinite field.

Algebraic sets

1. Throughout, let k be an infinite field.
2. $\mathbb{A}^n := \{(a_1, \ldots, a_n) \mid a_i \in k\}$, called *affine n-space* over k.

Algebraic sets

1. Throughout, let k be an infinite field.
2. $\mathbb{A}^n := \{(a_1, \ldots, a_n) \mid a_i \in k\}$, called *affine n-space* over k.
3. $k[x_1, \ldots, x_n]$: ring of polynomials in variables $x_1, \ldots, x_n$, which is called *coordinate ring* of $\mathbb{A}^n$.

Algebraic sets

1. Throughout, let k be an infinite field.
2. $\mathbb{A}^n := \{(a_1, \ldots, a_n) \mid a_i \in k\}$, called *affine n-space* over k.
3. $k[x_1, \ldots, x_n]$: ring of polynomials in variables $x_1, \ldots, x_n$, which is called *coordinate ring* of $\mathbb{A}^n$.
4. Given $f \in k[x_1, \ldots, x_n]$ and $p = (a_1, \cdots, a_n) \in \mathbb{A}^n$, put
$$f(p) = f(a_, \cdots, a_n) \in k.$$

Algebraic sets

1. Throughout, let k be an infinite field.
2. $\mathbb{A}^n := \{(a_1, \dots, a_n) \mid a_i \in k\}$, called *affine n-space* over k.
3. $k[x_1, \dots, x_n]$: ring of polynomials in variables $x_1, \dots, x_n$, which is called *coordinate ring* of $\mathbb{A}^n$.
4. Given $f \in k[x_1, \dots, x_n]$ and $p = (a_1, \cdots, a_n) \in \mathbb{A}^n$, put
$$f(p) = f(a_, \cdots, a_n) \in k.$$
5. Given $F \subseteq k[x_1, \cdots, x_n]$, we define

Algebraic sets

1. Throughout, let k be an infinite field.
2. $\mathbb{A}^n := \{(a_1, \ldots, a_n) \mid a_i \in k\}$, called *affine n-space* over k.
3. $k[x_1, \ldots, x_n]$: ring of polynomials in variables $x_1, \ldots, x_n$, which is called *coordinate ring* of $\mathbb{A}^n$.
4. Given $f \in k[x_1, \ldots, x_n]$ and $p = (a_1, \cdots, a_n) \in \mathbb{A}^n$, put
$$f(p) = f(a_, \cdots, a_n) \in k.$$
5. Given $F \subseteq k[x_1, \cdots, x_n]$, we define
$$\mathcal{Z}(F) := \{p \in \mathbb{A}^n \mid f(p) = 0, \text{ for all } f \in F\},$$

Algebraic sets

1. Throughout, let k be an infinite field.
2. $\mathbb{A}^n := \{(a_1, \ldots, a_n) \mid a_i \in k\}$, called *affine n-space* over k.
3. $k[x_1, \ldots, x_n]$: ring of polynomials in variables $x_1, \ldots, x_n$, which is called *coordinate ring* of $\mathbb{A}^n$.
4. Given $f \in k[x_1, \ldots, x_n]$ and $p = (a_1, \cdots, a_n) \in \mathbb{A}^n$, put
$$f(p) = f(a_, \cdots, a_n) \in k.$$
5. Given $F \subseteq k[x_1, \cdots, x_n]$, we define
$$\mathcal{Z}(F) := \{p \in \mathbb{A}^n \mid f(p) = 0, \text{ for all } f \in F\},$$
which is called an *algebraic set* of $\mathbb{A}^n$.

Algebraic sets

1. Throughout, let k be an infinite field.
2. $\mathbb{A}^n := \{(a_1, \ldots, a_n) \mid a_i \in k\}$, called *affine n-space* over k.
3. $k[x_1, \ldots, x_n]$: ring of polynomials in variables $x_1, \ldots, x_n$, which is called *coordinate ring* of $\mathbb{A}^n$.
4. Given $f \in k[x_1, \ldots, x_n]$ and $p = (a_1, \cdots, a_n) \in \mathbb{A}^n$, put
$$f(p) = f(a_, \cdots, a_n) \in k.$$
5. Given $F \subseteq k[x_1, \cdots, x_n]$, we define
$$\mathcal{Z}(F) := \{p \in \mathbb{A}^n \mid f(p) = 0, \text{ for all } f \in F\},$$
which is called an *algebraic set* of $\mathbb{A}^n$.

Example

If $p = (a_1, \ldots, a_n) \in \mathbb{A}^n$, then $\{p\} = \mathcal{Z}(x_1 - a_1, \ldots, x_n - a_n)$ is an algebraic set.

Example

1. Take $k = \mathbb{R}$.

Example

1. Take $k = \mathbb{R}$.
2. $\mathbb{A}^2 = \{(a, b) \mid a, b \in \mathbb{R}\}$: the real plane.

Example

1. Take $k = \mathbb{R}$.
2. $\mathbb{A}^2 = \{(a, b) \mid a, b \in \mathbb{R}\}$: the real plane.
3. The coordinate ring is $\mathbb{R}[x, y]$, real polynomials in x, y.

Example

1. Take $k = \mathbb{R}$.
2. $\mathbb{A}^2 = \{(a, b) \mid a, b \in \mathbb{R}\}$: the real plane.
3. The coordinate ring is $\mathbb{R}[x, y]$, real polynomials in x, y.
4. $\mathcal{Z}(x + y - 1)$: straight line passing through $(1, 0)$, $(0, 1)$.

Example

1. Take $k = \mathbb{R}$.
2. $\mathbb{A}^2 = \{(a, b) \mid a, b \in \mathbb{R}\}$: the real plane.
3. The coordinate ring is $\mathbb{R}[x, y]$, real polynomials in x, y.
4. $\mathcal{Z}(x + y - 1)$: straight line passing through $(1, 0)$, $(0, 1)$.
5. $\mathcal{Z}(x^2 + y^2 - 1)$: unit circle centered at $(0, 0)$.

Example

1. Take $k = \mathbb{R}$.
2. $\mathbb{A}^2 = \{(a, b) \mid a, b \in \mathbb{R}\}$: the real plane.
3. The coordinate ring is $\mathbb{R}[x, y]$, real polynomials in x, y.
4. $\mathcal{Z}(x + y - 1)$: straight line passing through $(1, 0)$, $(0, 1)$.
5. $\mathcal{Z}(x^2 + y^2 - 1)$: unit circle centered at $(0, 0)$.
6. $\mathcal{Z}(x + y - 1, x^2 + y^2 - 1) = \{(1, 0), (0, 1)\}$.

Topology

Definition

1. Let X be a non-empty set.

Definition

1. Let X be a non-empty set.
2. A collection $\mathscr{T}$ of subsets of X is called a *topology* on X if

Topology

Definition

1. Let X be a non-empty set.
2. A collection $\mathscr{T}$ of subsets of X is called a *topology* on X if
   - $X, \emptyset \in \mathscr{T}$;

Topology

Definition

1. Let X be a non-empty set.
2. A collection $\mathscr{T}$ of subsets of X is called a *topology* on X if
   - $X, \emptyset \in \mathscr{T}$;
   - If $C_1, \ldots, C_n \in \mathscr{T}$, then $\cup_{i=1}^{n} C_i \in \mathscr{T}$.

Definition

1. Let X be a non-empty set.
2. A collection $\mathscr{T}$ of subsets of X is called a *topology* on X if
   - $X, \emptyset \in \mathscr{T}$;
   - If $C_1, \ldots, C_n \in \mathscr{T}$, then $\cup_{i=1}^{n} C_i \in \mathscr{T}$.
   - If $C_i \in \mathscr{T}$ for all $i \in I$, then $\cap_{i \in I} C_i \in \mathscr{T}$, where I is arbitrary index set.

Topology

Definition

1. Let X be a non-empty set.
2. A collection $\mathscr{T}$ of subsets of X is called a *topology* on X if
   - $X, \emptyset \in \mathscr{T}$;
   - If $C_1, \ldots, C_n \in \mathscr{T}$, then $\cup_{i=1}^{n} C_i \in \mathscr{T}$.
   - If $C_i \in \mathscr{T}$ for all $i \in I$, then $\cap_{i \in I} C_i \in \mathscr{T}$, where I is arbitrary index set.
3. In this case,
   - The pair $(X, \mathscr{T})$ is called a *topological space*;

Definition

1. Let X be a non-empty set.
2. A collection $\mathscr{T}$ of subsets of X is called a *topology* on X if
   - $X, \emptyset \in \mathscr{T}$;
   - If $C_1, \ldots, C_n \in \mathscr{T}$, then $\cup_{i=1}^{n} C_i \in \mathscr{T}$.
   - If $C_i \in \mathscr{T}$ for all $i \in I$, then $\cap_{i \in I} C_i \in \mathscr{T}$, where I is arbitrary index set.
3. In this case,
   - The pair $(X, \mathscr{T})$ is called a *topological space*;
   - The sets in $\mathscr{T}$ are called *closed sets*;

Topology

Definition

1. Let X be a non-empty set.
2. A collection $\mathscr{T}$ of subsets of X is called a *topology* on X if
   - $X, \emptyset \in \mathscr{T}$;
   - If $C_1, \ldots, C_n \in \mathscr{T}$, then $\cup_{i=1}^{n} C_i \in \mathscr{T}$.
   - If $C_i \in \mathscr{T}$ for all $i \in I$, then $\cap_{i \in I} C_i \in \mathscr{T}$, where I is arbitrary index set.
3. In this case,
   - The pair $(X, \mathscr{T})$ is called a *topological space*;
   - The sets in $\mathscr{T}$ are called *closed sets*;
   - A subset Y of X is called an *open set* if $Y = X \setminus C$ for some $C \in \mathscr{T}$.

Topological spaces

Let $(X, \mathscr{T})$, $(Y, \mathscr{S})$ be topological spaces.

Let $(X, \mathscr{T})$, $(Y, \mathscr{S})$ be topological spaces.

Proposition

$X \times Y = \{(x, y) \mid x \in X, y \in Y\}$ *is a topological space*

Topological spaces

Let $(X, \mathscr{T})$, $(Y, \mathscr{S})$ be topological spaces.

Proposition

$X \times Y = \{(x, y) \mid x \in X, y \in Y\}$ *is a topological space equipped with the topology*

$$\mathscr{T} \times \mathscr{S} = \{C \times D \mid C \in \mathscr{T}, D \in \mathscr{S}\}.$$

Let $(X, \mathscr{T})$, $(Y, \mathscr{S})$ be topological spaces.

Proposition

$X \times Y = \{(x, y) \mid x \in X, y \in Y\}$ is a topological space equipped with the topology

$$\mathscr{T} \times \mathscr{S} = \{C \times D \mid C \in \mathscr{T}, D \in \mathscr{S}\}.$$

Definition

1. A map $f : X \to Y$ is called *continuous* provided, for any closed set C of Y, that $f^{-1}(C)$ is closed.

Topological spaces

Let $(X, \mathscr{T})$, $(Y, \mathscr{S})$ be topological spaces.

Proposition

$X \times Y = \{(x, y) \mid x \in X, y \in Y\}$ is a topological space equipped with the topology

$$\mathscr{T} \times \mathscr{S} = \{C \times D \mid C \in \mathscr{T}, D \in \mathscr{S}\}.$$

Definition

1. A map $f : X \to Y$ is called *continuous* provided, for any closed set C of Y, that $f^{-1}(C)$ is closed.
2. Given $U \subseteq X$, its *closure* $\overline{U}$ is the intersection of all closed sets containing U, which is the smallest closed set in X containing U.

Zariski Topology on $\mathbb{A}^n$

Theorem

The algebraic sets in $\mathbb{A}^n$ form a topology, called Zariski topology.

Zariski Topology on $\mathbb{A}^n$

Theorem

The algebraic sets in $\mathbb{A}^n$ form a topology, called Zariski topology.

Proof.

- $\emptyset = \mathcal{Z}(x_1, x_1 - 1)$;

Zariski Topology on $\mathbb{A}^n$

Theorem

The algebraic sets in $\mathbb{A}^n$ form a topology, called Zariski topology.

Proof.

- $\emptyset = \mathcal{Z}(x_1, x_1 - 1)$;
- $\mathbb{A}^n = \mathcal{Z}(0)$;

Zariski Topology on $\mathbb{A}^n$

Theorem

The algebraic sets in $\mathbb{A}^n$ form a topology, called Zariski topology.

Proof.

- $\emptyset = \mathcal{Z}(x_1, x_1 - 1)$;
- $\mathbb{A}^n = \mathcal{Z}(0)$;
- $\cup_{i=1}^{s} \mathcal{Z}(F_s) = \mathcal{Z}(F)$, where $F = \{f_1 \cdots f_s \mid f_i \in F_i\}$;

Zariski Topology on $\mathbb{A}^n$

Theorem

The algebraic sets in $\mathbb{A}^n$ form a topology, called Zariski topology.

Proof.

- $\emptyset = \mathcal{Z}(x_1, x_1 - 1)$;
- $\mathbb{A}^n = \mathcal{Z}(0)$;
- $\cup_{i=1}^{s} \mathcal{Z}(F_s) = \mathcal{Z}(F)$, where $F = \{f_1 \cdots f_s \mid f_i \in F_i\}$;
- $\cap_{i \in I} \mathcal{Z}(F_i) = \mathcal{Z}(\cup_{i \in I} F_i)$, where I arbitrary index set.

Zariski Topology on $\mathbb{A}^n$

Theorem

The algebraic sets in $\mathbb{A}^n$ form a topology, called Zariski topology.

Proof.

- $\emptyset = \mathcal{Z}(x_1, x_1 - 1)$;
- $\mathbb{A}^n = \mathcal{Z}(0)$;
- $\cup_{i=1}^{s} \mathcal{Z}(F_s) = \mathcal{Z}(F)$, where $F = \{f_1 \cdots f_s \mid f_i \in F_i\}$;
- $\cap_{i \in I} \mathcal{Z}(F_i) = \mathcal{Z}(\cup_{i \in I} F_i)$, where I arbitrary index set.

Remark

1. The finite sets in $\mathbb{A}^n$ are closed.

Zariski Topology on $\mathbb{A}^n$

Theorem

The algebraic sets in $\mathbb{A}^n$ form a topology, called Zariski topology.

Proof.

- $\emptyset = \mathcal{Z}(x_1, x_1 - 1)$;
- $\mathbb{A}^n = \mathcal{Z}(0)$;
- $\cup_{i=1}^{s} \mathcal{Z}(F_s) = \mathcal{Z}(F)$, where $F = \{f_1 \cdots f_s \mid f_i \in F_i\}$;
- $\cap_{i \in I} \mathcal{Z}(F_i) = \mathcal{Z}(\cup_{i \in I} F_i)$, where I arbitrary index set.

Remark

1. The finite sets in $\mathbb{A}^n$ are closed.
2. $\mathbb{A}^m \times \mathbb{A}^n = \mathbb{A}^{m+n}$.

Properties of $\mathbb{A}^1$

1. $\mathbb{A}^1 = k$, where k is an infinite field.

Properties of $\mathbb{A}^1$

1. $\mathbb{A}^1 = k$, where k is an infinite field.
2. The coordinate ring is $k[x]$.

Properties of $\mathbb{A}^1$

1. $\mathbb{A}^1 = k$, where k is an infinite field.
2. The coordinate ring is $k[x]$.

Proposition

1. *The closed sets in $\mathbb{A}^1$ are finite sets and $\mathbb{A}^1$.*

Properties of $\mathbb{A}^1$

1. $\mathbb{A}^1 = k$, where k is an infinite field.
2. The coordinate ring is $k[x]$.

Proposition

1. *The closed sets in* $\mathbb{A}^1$ *are finite sets and* $\mathbb{A}^1$.
2. *If* Y *is an infinite subset of* $\mathbb{A}^1$, *then* $\overline{Y} = \mathbb{A}^1$.

Properties of $\mathbb{A}^1$

1. $\mathbb{A}^1 = k$, where k is an infinite field.
2. The coordinate ring is $k[x]$.

Proposition

1. *The closed sets in $\mathbb{A}^1$ are finite sets and $\mathbb{A}^1$.*
2. *If Y is an infinite subset of $\mathbb{A}^1$, then $\overline{Y} = \mathbb{A}^1$.*

Proof. Let $C = \mathcal{Z}(F)$ for some $F \subseteq K[x]$.

Properties of $\mathbb{A}^1$

1. $\mathbb{A}^1 = k$, where k is an infinite field.
2. The coordinate ring is $k[x]$.

Proposition

1. *The closed sets in $\mathbb{A}^1$ are finite sets and $\mathbb{A}^1$.*
2. *If Y is an infinite subset of $\mathbb{A}^1$, then $\overline{Y} = \mathbb{A}^1$.*

Proof. Let $C = \mathcal{Z}(F)$ for some $F \subseteq K[x]$.

Since $k[x]$ is a principal ideal domain, $\mathcal{Z}(F) = \mathcal{Z}(f)$ for some $f \in k[x]$.

Properties of $\mathbb{A}^1$

1. $\mathbb{A}^1 = k$, where k is an infinite field.
2. The coordinate ring is $k[x]$.

Proposition

1. *The closed sets in $\mathbb{A}^1$ are finite sets and $\mathbb{A}^1$.*
2. *If Y is an infinite subset of $\mathbb{A}^1$, then $\overline{Y} = \mathbb{A}^1$.*

Proof. Let $C = \mathcal{Z}(F)$ for some $F \subseteq K[x]$.

Since $k[x]$ is a principal ideal domain, $\mathcal{Z}(F) = \mathcal{Z}(f)$ for some $f \in k[x]$.

If $f \neq 0$, since k is a field, $C = \mathcal{Z}(f)$ is finite.

Properties of $\mathbb{A}^1$

1. $\mathbb{A}^1 = k$, where k is an infinite field.
2. The coordinate ring is $k[x]$.

Proposition

1. *The closed sets in $\mathbb{A}^1$ are finite sets and $\mathbb{A}^1$.*
2. *If Y is an infinite subset of $\mathbb{A}^1$, then $\overline{Y} = \mathbb{A}^1$.*

Proof. Let $C = \mathcal{Z}(F)$ for some $F \subseteq K[x]$.

Since $k[x]$ is a principal ideal domain, $\mathcal{Z}(F) = \mathcal{Z}(f)$ for some $f \in k[x]$.

If $f \neq 0$, since k is a field, $C = \mathcal{Z}(f)$ is finite.

If $f = 0$, then $C = \mathcal{Z}(0) = \mathbb{A}^1$.

Zariski space of matrices

1. $M_{m\times n}(k) := \{(a_{ij})_{m\times n} \mid a_{ij} \in k\}$.

Zariski space of matrices

1. $M_{m\times n}(k) := \{(a_{ij})_{m\times n} \mid a_{ij} \in k\}$.
2. We shall identify $M_{m\times n}(k) = \mathbb{A}^{mn}$, the affine *mn*-space.

Zariski space of matrices

1. $M_{m\times n}(k) := \{(a_{ij})_{m\times n} \mid a_{ij} \in k\}$.
2. We shall identify $M_{m\times n}(k) = \mathbb{A}^{mn}$, the affine *mn*-space.
3. $M_{m\times n}(k)$ has coordinate ring

$$k[x_{ij}; i = 1, \ldots m; j = 1, \ldots, n].$$

Zariski space of matrices

1. $M_{m\times n}(k) := \{(a_{ij})_{m\times n} \mid a_{ij} \in k\}$.
2. We shall identify $M_{m\times n}(k) = \mathbb{A}^{mn}$, the affine *mn*-space.
3. $M_{m\times n}(k)$ has coordinate ring

$$k[x_{ij}; i = 1, \ldots m; j = 1, \ldots, n].$$

Example

1. Consider $M_{2\times 2}(\mathbb{R}) = \mathbb{A}^4$.

Zariski space of matrices

1. $M_{m\times n}(k) := \{(a_{ij})_{m\times n} \mid a_{ij} \in k\}$.
2. We shall identify $M_{m\times n}(k) = \mathbb{A}^{mn}$, the affine *mn*-space.
3. $M_{m\times n}(k)$ has coordinate ring

$$k[x_{ij};\, i = 1, \ldots m;\, j = 1, \ldots, n].$$

Example

1. Consider $M_{2\times 2}(\mathbb{R}) = \mathbb{A}^4$.
2. The coordinate ring is $\mathbb{R}[x_{11}, x_{12}, x_{21}, x_{22}]$ with

Zariski space of matrices

1. $M_{m\times n}(k) := \{(a_{ij})_{m\times n} \mid a_{ij} \in k\}$.
2. We shall identify $M_{m\times n}(k) = \mathbb{A}^{mn}$, the affine *mn*-space.
3. $M_{m\times n}(k)$ has coordinate ring

$$k[x_{ij}; i = 1, \dots m; j = 1, \dots, n].$$

Example

1. Consider $M_{2\times 2}(\mathbb{R}) = \mathbb{A}^4$.
2. The coordinate ring is $\mathbb{R}[x_{11}, x_{12}, x_{21}, x_{22}]$ with

$$\begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix} = x_{11}x_{22} - x_{12}x_{21} \in \mathbb{R}[x_{11}, x_{12}, x_{21}, x_{22}].$$

General linear groups

$$GL(n,k) := \{A \in M_{n\times n}(k) \mid A \text{ invertible}\}, \text{ a group}$$

General linear groups

$GL(n,k) := \{A \in M_{n\times n}(k) \mid A \text{ invertible}\}$, a group called *general linear group* of degree n.

General linear groups

$GL(n,k) := \{A \in M_{n\times n}(k) \mid A \text{ invertible}\}$, a group called *general linear group* of degree n.

Proposition

$GL(n,k)$ is an open set of $M_{n\times n}(k)$.

General linear groups

$GL(n,k) := \{A \in M_{n\times n}(k) \mid A \text{ invertible}\}$, a group called *general linear group* of degree n.

Proposition

$GL(n,k)$ is an open set of $M_{n\times n}(k)$.

1. *Proof.* The coordinate ring of $M_{n\times n}(k)$ contains

General linear groups

$GL(n,k) := \{A \in M_{n\times n}(k) \mid A \text{ invertible}\}$, a group called *general linear group* of degree n.

Proposition

$GL(n,k)$ is an open set of $M_{n\times n}(k)$.

1. *Proof.* The coordinate ring of $M_{n\times n}(k)$ contains

$$D_n := \begin{vmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{vmatrix} = \sum_{\sigma \in S_n} x_{1,\sigma(1)} \cdots x_{n,\sigma(n)}.$$

General linear groups

$GL(n,k) := \{A \in M_{n\times n}(k) \mid A \text{ invertible}\}$, a group called *general linear group* of degree n.

Proposition

$GL(n,k)$ is an open set of $M_{n\times n}(k)$.

1. *Proof.* The coordinate ring of $M_{n\times n}(k)$ contains

$$D_n := \begin{vmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{vmatrix} = \sum_{\sigma \in S_n} x_{1,\sigma(1)} \cdots x_{n,\sigma(n)}.$$

2. Given $A = (a_{ij})_{n\times n} \in M_{n\times n}(k)$, we have

General linear groups

$GL(n,k) := \{A \in M_{n\times n}(k) \mid A \text{ invertible}\}$, a group called *general linear group* of degree n.

Proposition

$GL(n,k)$ is an open set of $M_{n\times n}(k)$.

1. *Proof.* The coordinate ring of $M_{n\times n}(k)$ contains

$$D_n := \begin{vmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{vmatrix} = \sum_{\sigma \in S_n} x_{1,\sigma(1)} \cdots x_{n,\sigma(n)}.$$

2. Given $A = (a_{ij})_{n\times n} \in M_{n\times n}(k)$, we have

$$A \in GL(n,k) \Leftrightarrow \det(A) \neq 0 \Leftrightarrow A \notin \mathcal{Z}(D_n).$$

General linear groups

$GL(n,k) := \{A \in M_{n\times n}(k) \mid A \text{ invertible}\}$, a group called *general linear group* of degree n.

Proposition

$GL(n,k)$ is an open set of $M_{n\times n}(k)$.

1. *Proof.* The coordinate ring of $M_{n\times n}(k)$ contains

$$D_n := \begin{vmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{vmatrix} = \sum_{\sigma\in S_n} x_{1,\sigma(1)} \cdots x_{n,\sigma(n)}.$$

2. Given $A = (a_{ij})_{n\times n} \in M_{n\times n}(k)$, we have

$$A \in GL(n,k) \Leftrightarrow \det(A) \neq 0 \Leftrightarrow A \notin \mathcal{Z}(D_n).$$

3. Thus, $GL(n,k) = M_{n\times n}(k)\backslash\mathcal{Z}(D_n)$, that is an open set.

Some closed sets in $M_{m\times n}(k)$

1. Set $\mathscr{O}_{m\times n}(r) = \{A \in M_{m\times n}(k) \mid \mathrm{rank}(A) \leq r\}$.

Some closed sets in $M_{m\times n}(k)$

1. Set $\mathscr{O}_{m\times n}(r) = \{A \in M_{m\times n}(k) \mid \mathrm{rank}(A) \leq r\}$.

Proposition

$\mathscr{O}_{m\times n}(r)$ is closed in $M_{m\times n}(k)$, for $r = 0, 1, \ldots, \min\{m, n\}$.

Some closed sets in $M_{m\times n}(k)$

1. Set $\mathscr{O}_{m\times n}(r) = \{A \in M_{m\times n}(k) \mid \mathrm{rank}(A) \le r\}$.

Proposition

$\mathscr{O}_{m\times n}(r)$ is closed in $M_{m\times n}(k)$, for $r = 0, 1, \ldots, \min\{m, n\}$.

2. *Proof.* Consider the variable matrix $X = (x_{ij})_{m\times n}$.

Some closed sets in $M_{m\times n}(k)$

1. Set $\mathscr{O}_{m\times n}(r) = \{A \in M_{m\times n}(k) \mid \mathrm{rank}(A) \leq r\}$.

Proposition

$\mathscr{O}_{m\times n}(r)$ is closed in $M_{m\times n}(k)$, for $r = 0, 1, \ldots, \min\{m, n\}$.

2. *Proof.* Consider the variable matrix $X = (x_{ij})_{m\times n}$.
3. Let $X_1, \cdots, X_s$ the square submatrices of X of order $\leq r$.

Some closed sets in $M_{m\times n}(k)$

1. Set $\mathscr{O}_{m\times n}(r) = \{A \in M_{m\times n}(k) \mid \mathrm{rank}(A) \leq r\}$.

Proposition

$\mathscr{O}_{m\times n}(r)$ is closed in $M_{m\times n}(k)$, for $r = 0, 1, \ldots, \min\{m, n\}$.

2. *Proof.* Consider the variable matrix $X = (x_{ij})_{m\times n}$.
3. Let $X_1, \cdots, X_s$ the square submatrices of X of order $\leq r$.
4. $r_i = \det(X_i) \in k[x_{ij}; i = 1, \ldots, m; j = 1, \ldots, n]$.

Some closed sets in $M_{m\times n}(k)$

1. Set $\mathscr{O}_{m\times n}(r) = \{A \in M_{m\times n}(k) \mid \operatorname{rank}(A) \leq r\}$.

Proposition

$\mathscr{O}_{m\times n}(r)$ is closed in $M_{m\times n}(k)$, for $r = 0, 1, \ldots, \min\{m, n\}$.

2. *Proof.* Consider the variable matrix $X = (x_{ij})_{m\times n}$.
3. Let $X_1, \cdots, X_s$ the square submatrices of X of order $\leq r$.
4. $r_i = \det(X_i) \in k[x_{ij}; i = 1, \ldots, m; j = 1, \ldots, n]$.
5. If $A \in M_{m\times n}(k)$, then $\operatorname{rank}(A) \leq r$

Some closed sets in $M_{m\times n}(k)$

1. Set $\mathscr{O}_{m\times n}(r) = \{A \in M_{m\times n}(k) \mid \mathrm{rank}(A) \le r\}$.

Proposition

$\mathscr{O}_{m\times n}(r)$ is closed in $M_{m\times n}(k)$, for $r = 0, 1, \ldots, \min\{m, n\}$.

2. *Proof.* Consider the variable matrix $X = (x_{ij})_{m\times n}$.
3. Let $X_1, \cdots, X_s$ the square submatrices of X of order $\le r$.
4. $r_i = \det(X_i) \in k[x_{ij}; i = 1, \ldots, m; j = 1, \ldots, n]$.
5. If $A \in M_{m\times n}(k)$, then $\mathrm{rank}(A) \le r$

   $\Leftrightarrow$ determinant of any square submatrix of order $\le r$ is 0.

Some closed sets in $M_{m\times n}(k)$

1. Set $\mathscr{O}_{m\times n}(r) = \{A \in M_{m\times n}(k) \mid \mathrm{rank}(A) \leq r\}$.

Proposition

$\mathscr{O}_{m\times n}(r)$ is closed in $M_{m\times n}(k)$, for $r = 0, 1, \ldots, \min\{m, n\}$.

2. *Proof.* Consider the variable matrix $X = (x_{ij})_{m\times n}$.
3. Let $X_1, \cdots, X_s$ the square submatrices of X of order $\leq r$.
4. $r_i = \det(X_i) \in k[x_{ij}; i = 1, \ldots, m; j = 1, \ldots, n]$.
5. If $A \in M_{m\times n}(k)$, then $\mathrm{rank}(A) \leq r$

   $\Leftrightarrow$ determinant of any square submatrix of order $\leq r$ is 0.

   $\Leftrightarrow r_i(A) = 0, i = 1, \ldots, s$.

Some closed sets in $M_{m\times n}(k)$

1. Set $\mathscr{O}_{m\times n}(r) = \{A \in M_{m\times n}(k) \mid \mathrm{rank}(A) \leq r\}$.

Proposition

$\mathscr{O}_{m\times n}(r)$ is closed in $M_{m\times n}(k)$, for $r = 0, 1, \ldots, \min\{m, n\}$.

2. *Proof.* Consider the variable matrix $X = (x_{ij})_{m\times n}$.
3. Let $X_1, \cdots, X_s$ the square submatrices of X of order $\leq r$.
4. $r_i = \det(X_i) \in k[x_{ij}; i = 1, \ldots, m; j = 1, \ldots, n]$.
5. If $A \in M_{m\times n}(k)$, then $\mathrm{rank}(A) \leq r$

   $\Leftrightarrow$ determinant of any square submatrix of order $\leq r$ is 0.

   $\Leftrightarrow r_i(A) = 0, i = 1, \ldots, s$.

   $\Leftrightarrow A \in \mathscr{Z}(r_1, \ldots, r_s)$.

Some closed sets in $M_{m\times n}(k)$

1. Set $\mathscr{O}_{m\times n}(r) = \{A \in M_{m\times n}(k) \mid \mathrm{rank}(A) \leq r\}$.

Proposition

$\mathscr{O}_{m\times n}(r)$ is closed in $M_{m\times n}(k)$, for $r = 0, 1, \ldots, \min\{m, n\}$.

2. *Proof.* Consider the variable matrix $X = (x_{ij})_{m\times n}$.
3. Let $X_1, \cdots, X_s$ the square submatrices of X of order $\leq r$.
4. $r_i = \det(X_i) \in k[x_{ij}; i = 1, \ldots, m; j = 1, \ldots, n]$.
5. If $A \in M_{m\times n}(k)$, then $\mathrm{rank}(A) \leq r$

   $\Leftrightarrow$ determinant of any square submatrix of order $\leq r$ is 0.

   $\Leftrightarrow r_i(A) = 0, i = 1, \ldots, s$.

   $\Leftrightarrow A \in \mathcal{Z}(r_1, \ldots, r_s)$.
6. That is, $\mathscr{O}_{m\times n}(r) = \mathcal{Z}(r_1, \ldots, r_s)$ is a closed set.

Quivers

1. A *quiver* Q consists of

Quivers

1. A *quiver* Q consists of
   - a finite set Q_0 of vertices enumerated as $1, 2, \ldots, n$;

Quivers

1. A *quiver* Q consists of
   - a finite set Q_0 of vertices enumerated as $1, 2, \ldots, n$;
   - a finite set Q_1 of arrows $\alpha : i \to j$, where $i, j \in Q_0$.

Quivers

1. A *quiver* Q consists of
   - a finite set Q_0 of vertices enumerated as $1, 2, \ldots, n$;
   - a finite set Q_1 of arrows $\alpha : i \to j$, where $i, j \in Q_0$.
2. Given an arrow $\alpha : i \to j$, write $i = s(\alpha)$ and $e(\alpha) = j$.

Quivers

1. A *quiver* Q consists of
   - a finite set Q_0 of vertices enumerated as $1, 2, \ldots, n$;
   - a finite set Q_1 of arrows $\alpha : i \to j$, where $i, j \in Q_0$.
2. Given an arrow $\alpha : i \to j$, write $i = s(\alpha)$ and $e(\alpha) = j$.

Example

1. 

Quivers

1. A *quiver* Q consists of
   - a finite set Q_0 of vertices enumerated as $1, 2, \ldots, n$;
   - a finite set Q_1 of arrows $\alpha : i \to j$, where $i, j \in Q_0$.
2. Given an arrow $\alpha : i \to j$, write $i = s(\alpha)$ and $e(\alpha) = j$.

Example

1. $1 \circlearrowright$
2. $\mathcal{A}_2 : \quad 1 \longrightarrow 2$

Quivers

1. A *quiver* Q consists of
   - a finite set Q_0 of vertices enumerated as $1, 2, \ldots, n$;
   - a finite set Q_1 of arrows $\alpha : i \to j$, where $i, j \in Q_0$.
2. Given an arrow $\alpha : i \to j$, write $i = s(\alpha)$ and $e(\alpha) = j$.

Example

1. $1 \circlearrowright$
2. $\mathcal{A}_2 : \quad 1 \longrightarrow 2$
3. $\mathcal{A}_n : \quad 1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n-1 \longrightarrow n$

Representations of a quiver

1. Let $Q = (Q_0, Q_1)$ be a quiver with $Q_0 = \{1, \ldots, n\}$.

Representations of a quiver

1. Let $Q = (Q_0, Q_1)$ be a quiver with $Q_0 = \{1, \ldots, n\}$.
2. A vector $\mathbf{d} = (d_1, \cdots, d_n)$, with d_i positive integer, is called a *dimension vector* for Q.

Representations of a quiver

1. Let $Q = (Q_0, Q_1)$ be a quiver with $Q_0 = \{1, \ldots, n\}$.
2. A vector $\mathbf{d} = (d_1, \cdots, d_n)$, with d_i positive integer, is called a *dimension vector* for Q.
3. For each $\alpha : i \to j \in Q_1$, choose a matrix $A_\alpha \in M_{d_i \times d_j}(k)$.

Representations of a quiver

1. Let $Q = (Q_0, Q_1)$ be a quiver with $Q_0 = \{1, \ldots, n\}$.
2. A vector $\mathbf{d} = (d_1, \cdots, d_n)$, with d_i positive integer, is called a *dimension vector* for Q.
3. For each $\alpha : i \to j \in Q_1$, choose a matrix $A_\alpha \in M_{d_i \times d_j}(k)$. The collection $A = (A_\alpha)_{\alpha \in Q_1}$ is called a *representation* of Q of dimension vector $\mathbf{d}$.

Representations of a quiver

1. Let $Q = (Q_0, Q_1)$ be a quiver with $Q_0 = \{1, \ldots, n\}$.
2. A vector $\mathbf{d} = (d_1, \cdots, d_n)$, with d_i positive integer, is called a *dimension vector* for Q.
3. For each $\alpha : i \to j \in Q_1$, choose a matrix $A_\alpha \in M_{d_i \times d_j}(k)$. The collection $A = (A_\alpha)_{\alpha \in Q_1}$ is called a *representation* of Q of dimension vector $\mathbf{d}$.
4. $\mathrm{Rep}(Q, \mathbf{d})$: the set of all representations of Q of dimension vector $\mathbf{d}$.

Representations of a quiver

1. Let $Q = (Q_0, Q_1)$ be a quiver with $Q_0 = \{1, \dots, n\}$.
2. A vector $\mathbf{d} = (d_1, \cdots, d_n)$, with d_i positive integer, is called a *dimension vector* for Q.
3. For each $\alpha : i \to j \in Q_1$, choose a matrix $A_\alpha \in M_{d_i \times d_j}(k)$. The collection $A = (A_\alpha)_{\alpha \in Q_1}$ is called a *representation* of Q of dimension vector $\mathbf{d}$.
4. $\mathrm{Rep}(Q, \mathbf{d})$: the set of all representations of Q of dimension vector $\mathbf{d}$.
5. We shall identify

$$\mathrm{Rep}(Q, \mathbf{d}) = \mathbb{A}^n,$$

where $n = \sum_{\alpha \in Q_1} d_{s(\alpha)} d_{e(\alpha)}$.

Example

1. $Q:\ 1 \circlearrowright \alpha$ and $\mathbf{d} = (5)$.

Example

1. $Q: \; 1 \circlearrowleft \alpha$ and $\mathbf{d} = (5)$.

   A representation of Q of dimension vector $\mathbf{d}$ is a matrix $A_\alpha \in M_{5\times 5}(k)$. Thus,

Example

1. $Q: \; 1 \circlearrowright \alpha$ and $\mathbf{d} = (5)$.

   A representation of Q of dimension vector $\mathbf{d}$ is a matrix $A_\alpha \in M_{5\times 5}(k)$. Thus,

Example

1. $Q: \; 1 \circlearrowleft \alpha$ and $\mathbf{d} = (5)$.

   A representation of Q of dimension vector $\mathbf{d}$ is a matrix $A_\alpha \in M_{5\times 5}(k)$. Thus,

   $$\mathrm{Rep}(Q, \mathbf{d}) = M_{5\times 5}(k) = \mathbb{A}^{25}.$$

Example

1. $Q: \ 1 \circlearrowright \alpha$ and $\mathbf{d} = (5)$.

   A representation of Q of dimension vector $\mathbf{d}$ is a matrix $A_\alpha \in M_{5\times 5}(k)$. Thus,

   $$\mathrm{Rep}(Q, \mathbf{d}) = M_{5\times 5}(k) = \mathbb{A}^{25}.$$

2. $\mathcal{A}_3: \ 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$ and $\mathbf{d} = (2, 3, 4)$.

Example

1. $Q: 1 \circlearrowright \alpha$ and $\mathbf{d} = (5)$.

   A representation of Q of dimension vector $\mathbf{d}$ is a matrix $A_\alpha \in M_{5\times 5}(k)$. Thus,

   $$\mathrm{Rep}(Q, \mathbf{d}) = M_{5\times 5}(k) = \mathbb{A}^{25}.$$

2. $\mathcal{A}_3 : 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$ and $\mathbf{d} = (2, 3, 4)$.

   A representation of $\mathcal{A}_3$ of dimension vector $\mathbf{d}$ is a pair of matrices $(A_\alpha, A_\beta) \in M_{2\times 3}(k) \times M_{3\times 4}(k)$. Therefore,

Example

1. $Q: 1 \circlearrowleft \alpha$ and $\mathbf{d} = (5)$.

   A representation of Q of dimension vector $\mathbf{d}$ is a matrix $A_\alpha \in M_{5\times 5}(k)$. Thus,

   $$\mathrm{Rep}(Q, \mathbf{d}) = M_{5\times 5}(k) = \mathbb{A}^{25}.$$

2. $\mathcal{A}_3: 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$ and $\mathbf{d} = (2, 3, 4)$.

   A representation of $\mathcal{A}_3$ of dimension vector $\mathbf{d}$ is a pair of matrices $(A_\alpha, A_\beta) \in M_{2\times 3}(k) \times M_{3\times 4}(k)$. Therefore,

   $$\mathrm{Rep}(\mathcal{A}_3, \mathbf{d}) = M_{2\times 3}(k) \times M_{3\times 4}(k) = \mathbb{A}^6 \times \mathbb{A}^{12} = \mathbb{A}^{18}.$$

Group action

1. Let $Q = (Q_0, Q_1)$ be a quiver, where $Q_0 = \{1, \dots, n\}$.
2. Let $\mathbf{d} = (d_1, \cdots, d_n)$ be a dimension vector.

Group action

1. Let $Q = (Q_0, Q_1)$ be a quiver, where $Q_0 = \{1, \ldots, n\}$.
2. Let $\mathbf{d} = (d_1, \cdots, d_n)$ be a dimension vector.
3. $G(\mathbf{d}) = GL(d_1, k) \times \cdots \times GL(d_n, k)$ is a group.

Group action

1. Let $Q = (Q_0, Q_1)$ be a quiver, where $Q_0 = \{1, \ldots, n\}$.
2. Let $\mathbf{d} = (d_1, \cdots, d_n)$ be a dimension vector.
3. $G(\mathbf{d}) = GL(d_1, k) \times \cdots \times GL(d_n, k)$ is a group.

Definition

Given

$$g = (g_1, \cdots, g_n) \in G(\mathbf{d}) \text{ and } (A_\alpha)_{\alpha \in Q_1} \in \mathrm{Rep}(Q, \mathbf{d}),$$

Group action

1. Let $Q = (Q_0, Q_1)$ be a quiver, where $Q_0 = \{1, \dots, n\}$.
2. Let $\mathbf{d} = (d_1, \cdots, d_n)$ be a dimension vector.
3. $G(\mathbf{d}) = GL(d_1, k) \times \cdots \times GL(d_n, k)$ is a group.

Definition

Given

$$g = (g_1, \cdots, g_n) \in G(\mathbf{d}) \quad \text{and} \quad (A_\alpha)_{\alpha \in Q_1} \in \mathrm{Rep}(Q, \mathbf{d}),$$

we define

$$g \cdot (A_\alpha)_{\alpha \in Q_1} = (B_\alpha)_{\alpha \in Q_1} \in \mathrm{Rep}(Q, \mathbf{d}),$$

Group action

1. Let $Q = (Q_0, Q_1)$ be a quiver, where $Q_0 = \{1, \dots, n\}$.
2. Let $\mathbf{d} = (d_1, \cdots, d_n)$ be a dimension vector.
3. $G(\mathbf{d}) = GL(d_1, k) \times \cdots \times GL(d_n, k)$ is a group.

Definition

Given

$$g = (g_1, \cdots, g_n) \in G(\mathbf{d}) \text{ and } (A_\alpha)_{\alpha \in Q_1} \in \mathrm{Rep}(Q, \mathbf{d}),$$

we define

$$g \cdot (A_\alpha)_{\alpha \in Q_1} = (B_\alpha)_{\alpha \in Q_1} \in \mathrm{Rep}(Q, \mathbf{d}),$$

where, for each arrow $\alpha : i \to j$,

$$B_\alpha = g_i^{-1} A_\alpha \, g_j,$$

$\mathcal{A}_2$-case

1. $\mathcal{A}_2 : \ 1 \xrightarrow{\alpha} 2$ and $\mathbf{d} = (m, n)$.

$\mathcal{A}_2$-case

1. $\mathcal{A}_2 : 1 \xrightarrow{\alpha} 2$ and $\mathbf{d} = (m, n)$.
2. A representation of $\mathcal{A}_2$ of dimension vector $\mathbf{d}$ is a matrix $A \in M_{m \times n}(k)$.

$\mathcal{A}_2$-case

1. $\mathcal{A}_2: \ 1 \xrightarrow{\alpha} 2$ and $\mathbf{d} = (m, n)$.
2. A representation of $\mathcal{A}_2$ of dimension vector $\mathbf{d}$ is a matrix $A \in M_{m \times n}(k)$.
3. $\mathrm{Rep}(\mathcal{A}_2, \mathbf{d}) = M_{m \times n}(k) = \mathbb{A}^{mn}$.

$\mathcal{A}_2$-case

1. $\mathcal{A}_2 : \; 1 \xrightarrow{\alpha} 2$ and $\mathbf{d} = (m, n)$.
2. A representation of $\mathcal{A}_2$ of dimension vector $\mathbf{d}$ is a matrix $A \in M_{m \times n}(k)$.
3. $\mathrm{Rep}(\mathcal{A}_2, \mathbf{d}) = M_{m \times n}(k) = \mathbb{A}^{mn}$.
4. Moreover, $G(\mathbf{d}) = GL(m, k) \times GL(n, k)$.

$\mathcal{A}_2$-case

1. $\mathcal{A}_2 : 1 \xrightarrow{\alpha} 2$ and $\mathbf{d} = (m, n)$.
2. A representation of $\mathcal{A}_2$ of dimension vector $\mathbf{d}$ is a matrix $A \in M_{m \times n}(k)$.
3. $\mathrm{Rep}(\mathcal{A}_2, \mathbf{d}) = M_{m \times n}(k) = \mathbb{A}^{mn}$.
4. Moreover, $G(\mathbf{d}) = GL(m, k) \times GL(n, k)$.
5. Given $(g_1, g_2) \in GL(m, k) \times GL(n, k)$ and $A \in M_{m \times n}(k)$,

$\mathcal{A}_2$-case

1. $\mathcal{A}_2 : 1 \xrightarrow{\alpha} 2$ and $\mathbf{d} = (m, n)$.
2. A representation of $\mathcal{A}_2$ of dimension vector $\mathbf{d}$ is a matrix $A \in M_{m\times n}(k)$.
3. $\mathrm{Rep}(\mathcal{A}_2, \mathbf{d}) = M_{m\times n}(k) = \mathbb{A}^{mn}$.
4. Moreover, $G(\mathbf{d}) = GL(m, k) \times GL(n, k)$.
5. Given $(g_1, g_2) \in GL(m, k) \times GL(n, k)$ and $A \in M_{m\times n}(k)$,

$$(g_1, g_2) \cdot A = g_1^{-1} A g_2.$$

Example

$$\left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) \cdot \begin{pmatrix} 2 & 3 & 4 \\ 3 & 1 & 2 \end{pmatrix}$$

Example

$$\left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) \cdot \begin{pmatrix} 2 & 3 & 4 \\ 3 & 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 2 & 3 & 4 \\ 3 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Example

$$\left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) \cdot \begin{pmatrix} 2 & 3 & 4 \\ 3 & 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 2 & 3 & 4 \\ 3 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 & 4 \\ 3 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Example

$$\left(\begin{pmatrix}1&0\\1&1\end{pmatrix},\begin{pmatrix}1&2&0\\0&1&0\\0&0&1\end{pmatrix}\right)\cdot\begin{pmatrix}2&3&4\\3&1&2\end{pmatrix}$$

$$=\begin{pmatrix}1&0\\1&1\end{pmatrix}^{-1}\begin{pmatrix}2&3&4\\3&1&2\end{pmatrix}\begin{pmatrix}1&2&0\\0&1&0\\0&0&1\end{pmatrix}$$

$$=\begin{pmatrix}1&0\\-1&1\end{pmatrix}\begin{pmatrix}2&3&4\\3&1&2\end{pmatrix}\begin{pmatrix}1&2&0\\0&1&0\\0&0&1\end{pmatrix}$$

$$=\begin{pmatrix}2&3&4\\1&-2&-2\end{pmatrix}\begin{pmatrix}1&2&0\\0&1&0\\0&0&1\end{pmatrix}$$

Example

$$\left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) \cdot \begin{pmatrix} 2 & 3 & 4 \\ 3 & 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 2 & 3 & 4 \\ 3 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 & 4 \\ 3 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 3 & 4 \\ 1 & -2 & -2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 7 & 4 \\ 1 & 0 & -2 \end{pmatrix}.$$

$G(\mathbf{d})$-orbits

1. Let $Q = (Q_0, Q_1)$ be a quiver, where $Q_0 = \{1, \ldots, n\}$.
2. Let $\mathbf{d} = (d_1, \cdots, d_n)$ be a dimension vector.
3. $G(\mathbf{d}) = GL(d_1, k) \times \cdots \times GL(d_n, k)$.

$G(\mathbf{d})$-orbits

1. Let $Q = (Q_0, Q_1)$ be a quiver, where $Q_0 = \{1, \dots, n\}$.
2. Let $\mathbf{d} = (d_1, \cdots, d_n)$ be a dimension vector.
3. $G(\mathbf{d}) = GL(d_1, k) \times \cdots \times GL(d_n, k)$.

Definition

Given $A = (A_\alpha)_{\alpha \in Q_1} \in \mathrm{Rep}(Q, \mathbf{d})$, the set

$$\mathscr{O}(A) = \{g \cdot A \mid g \in G(\mathbf{d})\}$$

is called $G(\mathbf{d})$-*orbit* of A.

$\mathcal{A}_2$-case

1. $\mathcal{A}_2 : \ 1 \xrightarrow{\alpha} 2$ and $\mathbf{d} = (m, n)$.
2. $\mathrm{Rep}(\mathcal{A}_2, \mathbf{d}) = M_{m \times n}(k)$ and $G(\mathbf{d}) = GL(m, k) \times GL(n, k)$.

$\mathcal{A}_2$-case

1. $\mathcal{A}_2 : \ 1 \xrightarrow{\alpha} 2$ and $\mathbf{d} = (m, n)$.
2. $\mathrm{Rep}(\mathcal{A}_2, \mathbf{d}) = M_{m \times n}(k)$ and $G(\mathbf{d}) = GL(m, k) \times GL(n, k)$.

Proposition

Given $A \in M_{m \times n}(k)$, we obtain

$$\mathscr{O}(A) = \{B \in M_{m \times n}(k) \mid \mathrm{rank}(B) = \mathrm{rank}(A)\}.$$

$\mathcal{A}_2$-case

1. $\mathcal{A}_2 : 1 \xrightarrow{\alpha} 2$ and $\mathbf{d} = (m, n)$.
2. $\mathrm{Rep}(\mathcal{A}_2, \mathbf{d}) = M_{m \times n}(k)$ and $G(\mathbf{d}) = GL(m, k) \times GL(n, k)$.

Proposition

Given $A \in M_{m \times n}(k)$, we obtain

$$\mathscr{O}(A) = \{B \in M_{m \times n}(k) \mid \mathrm{rank}(B) = \mathrm{rank}(A)\}.$$

Proof. Given $B \in M_{m \times n}(k)$, by definition, $B \in \mathscr{O}(A)$

1. $\mathcal{A}_2: \ 1 \xrightarrow{\alpha} 2$ and $\mathbf{d} = (m, n)$.
2. $\mathrm{Rep}(\mathcal{A}_2, \mathbf{d}) = M_{m\times n}(k)$ and $G(\mathbf{d}) = GL(m, k) \times GL(n, k)$.

Proposition

Given $A \in M_{m\times n}(k)$, we obtain

$$\mathscr{O}(A) = \{B \in M_{m\times n}(k) \mid \mathrm{rank}(B) = \mathrm{rank}(A)\}.$$

Proof. Given $B \in M_{m\times n}(k)$, by definition, $B \in \mathscr{O}(A)$
$\Leftrightarrow B = g_1^{-1} A g_2$, with $g_1 \in GL(m, k)$, $g_2 \in GL(n, k)$;

$\mathcal{A}_2$-case

1. $\mathcal{A}_2: \ 1 \xrightarrow{\alpha} 2$ and $\mathbf{d} = (m, n)$.
2. $\mathrm{Rep}(\mathcal{A}_2, \mathbf{d}) = M_{m\times n}(k)$ and $G(\mathbf{d}) = GL(m,k) \times GL(n,k)$.

Proposition

Given $A \in M_{m\times n}(k)$, we obtain

$$\mathscr{O}(A) = \{B \in M_{m\times n}(k) \mid \mathrm{rank}(B) = \mathrm{rank}(A)\}.$$

Proof. Given $B \in M_{m\times n}(k)$, by definition, $B \in \mathscr{O}(A)$
$\Leftrightarrow B = g_1^{-1} A g_2$, with $g_1 \in GL(m,k)$, $g_2 \in GL(n,k)$;
$\Leftrightarrow B$ obtained from A by performing some elementary row operations and some elementary column operations;

$\mathcal{A}_2$-case

1. $\mathcal{A}_2: \; 1 \xrightarrow{\alpha} 2$ and $\mathbf{d} = (m, n)$.
2. $\mathrm{Rep}(\mathcal{A}_2, \mathbf{d}) = M_{m \times n}(k)$ and $G(\mathbf{d}) = GL(m, k) \times GL(n, k)$.

Proposition

Given $A \in M_{m \times n}(k)$, we obtain

$$\mathscr{O}(A) = \{B \in M_{m \times n}(k) \mid \mathrm{rank}(B) = \mathrm{rank}(A)\}.$$

Proof. Given $B \in M_{m \times n}(k)$, by definition, $B \in \mathscr{O}(A)$

$\Leftrightarrow B = g_1^{-1} A g_2$, with $g_1 \in GL(m, k)$, $g_2 \in GL(n, k)$;

$\Leftrightarrow B$ obtained from A by performing some elementary row operations and some elementary column operations;

$\Leftrightarrow \mathrm{rank}(B) = \mathrm{rank}(A)$.

1. $\mathcal{A}_2: \ 1 \xrightarrow{\alpha} 2$ and $\mathbf{d} = (m, n)$.
2. $\mathrm{Rep}(\mathcal{A}_2, \mathbf{d}) = M_{m\times n}(k)$ and $G(\mathbf{d}) = GL(m, k) \times GL(n, k)$.

Proposition

Given $A \in M_{m\times n}(k)$, we obtain

$$\mathscr{O}(A) = \{B \in M_{m\times n}(k) \mid \mathrm{rank}(B) = \mathrm{rank}(A)\}.$$

Proof. Given $B \in M_{m\times n}(k)$, by definition, $B \in \mathscr{O}(A)$
$\Leftrightarrow B = g_1^{-1} A g_2$, with $g_1 \in GL(m, k)$, $g_2 \in GL(n, k)$;
$\Leftrightarrow B$ obtained from A by performing some elementary row operations and some elementary column operations;
$\Leftrightarrow \mathrm{rank}(B) = \mathrm{rank}(A)$.

Remark

$\mathscr{O}(0_{m\times n}) = \{0_{m\times n}\}$.

Orbit closure

1. Let $Q = (Q_0, Q_1)$ be a quiver, where $Q_0 = \{1, \ldots, n\}$.
2. Let $\mathbf{d} = (d_1, \cdots, d_n)$ be a dimension vector.
3. $G(\mathbf{d}) = GL(m, k) \times \cdots \times GL(n, k)$.

Orbit closure

1. Let $Q = (Q_0, Q_1)$ be a quiver, where $Q_0 = \{1, \ldots, n\}$.
2. Let $\mathbf{d} = (d_1, \cdots, d_n)$ be a dimension vector.
3. $G(\mathbf{d}) = GL(m, k) \times \cdots \times GL(n, k)$.

Definition

The *orbit closure* of a representation $A \in \mathrm{Rep}(Q, \mathbf{d})$ is $\overline{\mathscr{O}(A)}$, the closure of the $G(\mathbf{d})$-orbit of A in $\mathrm{Rep}(Q, \mathbf{d})$.

Orbit closure

1. Let $Q = (Q_0, Q_1)$ be a quiver, where $Q_0 = \{1, \ldots, n\}$.
2. Let $\mathbf{d} = (d_1, \cdots, d_n)$ be a dimension vector.
3. $G(\mathbf{d}) = GL(m, k) \times \cdots \times GL(n, k)$.

Definition

The *orbit closure* of a representation $A \in \mathrm{Rep}(Q, \mathbf{d})$ is $\overline{\mathscr{O}(A)}$, the closure of the $G(\mathbf{d})$-orbit of A in $\mathrm{Rep}(Q, \mathbf{d})$.

Objective

To describe the orbit closures in $\mathrm{Rep}(Q, \mathbf{d})$.

Orbits closures in $M_{m\times n}(k)$: a special case

Lemma

If $A \in M_{m\times n}(k)$ *with* $\mathrm{rank}(A) = 1$, *then*

$$\overline{\mathscr{O}(A)} = \{B \in M_{m\times n}(k) \mid \mathrm{rank}(B) \leq 1\}.$$

Orbits closures in $M_{m\times n}(k)$: a special case

Lemma

If $A \in M_{m\times n}(k)$ *with* $\mathrm{rank}(A) = 1$, *then*

$$\overline{\mathscr{O}(A)} = \{B \in M_{m\times n}(k) \mid \mathrm{rank}(B) \leq 1\}.$$

1. *Proof.* $\mathscr{E} = \{B \in M_{m\times n}(k) \mid \mathrm{rank}(B) \leq 1\}$ is closed.

Orbits closures in $M_{m\times n}(k)$: a special case

Lemma

If $A \in M_{m\times n}(k)$ *with* $\mathrm{rank}(A) = 1$, *then*

$$\overline{\mathscr{O}(A)} = \{B \in M_{m\times n}(k) \mid \mathrm{rank}(B) \leq 1\}.$$

1. *Proof.* $\mathscr{E} = \{B \in M_{m\times n}(k) \mid \mathrm{rank}(B) \leq 1\}$ is closed.
2. Since $\mathscr{O}(A) = \{B \in M_{m\times n}(k) \mid \mathrm{rank}(B) = 1\}$, we have

Orbits closures in $M_{m\times n}(k)$: a special case

Lemma

If $A \in M_{m\times n}(k)$ *with* $\mathrm{rank}(A) = 1$, *then*

$$\overline{\mathscr{O}(A)} = \{B \in M_{m\times n}(k) \mid \mathrm{rank}(B) \leq 1\}.$$

1. *Proof.* $\mathscr{E} = \{B \in M_{m\times n}(k) \mid \mathrm{rank}(B) \leq 1\}$ is closed.
2. Since $\mathscr{O}(A) = \{B \in M_{m\times n}(k) \mid \mathrm{rank}(B) = 1\}$, we have $\mathscr{E} = \mathscr{O}(A) \cup \{0_{m\times n}\} \Rightarrow \overline{\mathscr{O}(A)} \subseteq \mathscr{E}$.

Orbits closures in $M_{m\times n}(k)$: a special case

Lemma

If $A \in M_{m\times n}(k)$ *with* $\operatorname{rank}(A) = 1$, *then*

$$\overline{\mathscr{O}(A)} = \{B \in M_{m\times n}(k) \mid \operatorname{rank}(B) \leq 1\}.$$

1. *Proof.* $\mathscr{E} = \{B \in M_{m\times n}(k) \mid \operatorname{rank}(B) \leq 1\}$ is closed.
2. Since $\mathscr{O}(A) = \{B \in M_{m\times n}(k) \mid \operatorname{rank}(B) = 1\}$, we have $\mathscr{E} = \mathscr{O}(A) \cup \{0_{m\times n}\} \Rightarrow \overline{\mathscr{O}(A)} \subseteq \mathscr{E}$.
3. $\exists$ continuous map $f : \mathbb{A}^1 \to M_{m\times n}(k) : a \mapsto aE_{11}$,

Orbits closures in $M_{m\times n}(k)$: a special case

Lemma

If $A \in M_{m\times n}(k)$ *with* $\mathrm{rank}(A) = 1$, *then*

$$\overline{\mathscr{O}(A)} = \{B \in M_{m\times n}(k) \mid \mathrm{rank}(B) \leq 1\}.$$

1. *Proof.* $\mathscr{E} = \{B \in M_{m\times n}(k) \mid \mathrm{rank}(B) \leq 1\}$ is closed.
2. Since $\mathscr{O}(A) = \{B \in M_{m\times n}(k) \mid \mathrm{rank}(B) = 1\}$, we have $\mathscr{E} = \mathscr{O}(A) \cup \{0_{m\times n}\} \Rightarrow \overline{\mathscr{O}(A)} \subseteq \mathscr{E}$.
3. $\exists$ continuous map $f : \mathbb{A}^1 \to M_{m\times n}(k) : a \mapsto aE_{11}$, where $(1, 1)$-entry of E_{11} is 1, and all other entries are 0.

Orbits closures in $M_{m\times n}(k)$: a special case

Lemma

If $A \in M_{m\times n}(k)$ *with* $\mathrm{rank}(A) = 1$, *then*

$$\overline{\mathscr{O}(A)} = \{B \in M_{m\times n}(k) \mid \mathrm{rank}(B) \leq 1\}.$$

1. *Proof.* $\mathscr{E} = \{B \in M_{m\times n}(k) \mid \mathrm{rank}(B) \leq 1\}$ is closed.
2. Since $\mathscr{O}(A) = \{B \in M_{m\times n}(k) \mid \mathrm{rank}(B) = 1\}$, we have $\mathscr{E} = \mathscr{O}(A) \cup \{0_{m\times n}\} \Rightarrow \overline{\mathscr{O}(A)} \subseteq \mathscr{E}$.
3. $\exists$ continuous map $f : \mathbb{A}^1 \to M_{m\times n}(k) : a \mapsto aE_{11}$, where $(1,1)$-entry of E_{11} is 1, and all other entries are 0.
4. $a \neq 0 \Rightarrow f(a) \in \mathscr{O}(A) \subseteq \overline{\mathscr{O}(A)} \Rightarrow \mathbb{A}^1 \backslash \{0\} \subseteq f^{-1}(\overline{\mathscr{O}(A)})$.

Orbits closures in $M_{m\times n}(k)$: a special case

Lemma

If $A \in M_{m\times n}(k)$ with $\operatorname{rank}(A) = 1$, then
$$\overline{\mathscr{O}(A)} = \{B \in M_{m\times n}(k) \mid \operatorname{rank}(B) \le 1\}.$$

1. *Proof.* $\mathscr{E} = \{B \in M_{m\times n}(k) \mid \operatorname{rank}(B) \le 1\}$ is closed.
2. Since $\mathscr{O}(A) = \{B \in M_{m\times n}(k) \mid \operatorname{rank}(B) = 1\}$, we have $\mathscr{E} = \mathscr{O}(A) \cup \{0_{m\times n}\} \Rightarrow \overline{\mathscr{O}(A)} \subseteq \mathscr{E}$.
3. $\exists$ continuous map $f : \mathbb{A}^1 \to M_{m\times n}(k) : a \mapsto aE_{11}$, where $(1,1)$-entry of E_{11} is 1, and all other entries are 0.
4. $a \neq 0 \Rightarrow f(a) \in \mathscr{O}(A) \subseteq \overline{\mathscr{O}(A)} \Rightarrow \mathbb{A}^1 \backslash \{0\} \subseteq f^{-1}(\overline{\mathscr{O}(A)})$.
5. Since $\mathbb{A}^1 \backslash \{0\}$ is infinite and $f^{-1}(\overline{\mathscr{O}(A)})$ is closed,

Orbits closures in $M_{m\times n}(k)$: a special case

Lemma

If $A \in M_{m\times n}(k)$ *with* $\mathrm{rank}(A) = 1$, *then*

$$\overline{\mathscr{O}(A)} = \{B \in M_{m\times n}(k) \mid \mathrm{rank}(B) \le 1\}.$$

1. *Proof.* $\mathscr{E} = \{B \in M_{m\times n}(k) \mid \mathrm{rank}(B) \le 1\}$ is closed.
2. Since $\mathscr{O}(A) = \{B \in M_{m\times n}(k) \mid \mathrm{rank}(B) = 1\}$, we have $\mathscr{E} = \mathscr{O}(A) \cup \{0_{m\times n}\} \Rightarrow \overline{\mathscr{O}(A)} \subseteq \mathscr{E}$.
3. $\exists$ continuous map $f : \mathbb{A}^1 \to M_{m\times n}(k) : a \mapsto aE_{11}$, where $(1,1)$-entry of E_{11} is 1, and all other entries are 0.
4. $a \ne 0 \Rightarrow f(a) \in \mathscr{O}(A) \subseteq \overline{\mathscr{O}(A)} \Rightarrow \mathbb{A}^1\backslash\{0\} \subseteq f^{-1}(\overline{\mathscr{O}(A)})$.
5. Since $\mathbb{A}^1\backslash\{0\}$ is infinite and $f^{-1}(\overline{\mathscr{O}(A)})$ is closed,

$$\mathbb{A}^1 = \overline{\mathbb{A}^1\backslash\{0\}} \subseteq f^{-1}(\overline{\mathscr{O}(A)}).$$

Orbits closures in $M_{m\times n}(k)$: a special case

Lemma

If $A \in M_{m\times n}(k)$ with $\mathrm{rank}(A) = 1$, then

$$\overline{\mathscr{O}(A)} = \{B \in M_{m\times n}(k) \mid \mathrm{rank}(B) \leq 1\}.$$

1. *Proof.* $\mathscr{E} = \{B \in M_{m\times n}(k) \mid \mathrm{rank}(B) \leq 1\}$ is closed.
2. Since $\mathscr{O}(A) = \{B \in M_{m\times n}(k) \mid \mathrm{rank}(B) = 1\}$, we have $\mathscr{E} = \mathscr{O}(A) \cup \{0_{m\times n}\} \Rightarrow \overline{\mathscr{O}(A)} \subseteq \mathscr{E}$.
3. $\exists$ continuous map $f : \mathbb{A}^1 \to M_{m\times n}(k) : a \mapsto aE_{11}$, where $(1,1)$-entry of E_{11} is 1, and all other entries are 0.
4. $a \neq 0 \Rightarrow f(a) \in \mathscr{O}(A) \subseteq \overline{\mathscr{O}(A)} \Rightarrow \mathbb{A}^1 \backslash \{0\} \subseteq f^{-1}(\overline{\mathscr{O}(A)})$.
5. Since $\mathbb{A}^1 \backslash \{0\}$ is infinite and $f^{-1}(\overline{\mathscr{O}(A)})$ is closed,
$$\mathbb{A}^1 = \overline{\mathbb{A}^1 \backslash \{0\}} \subseteq f^{-1}(\overline{\mathscr{O}(A)}).$$
6. Thus, $f(0) = 0_{m\times n} \in \overline{\mathscr{O}(A)} \Rightarrow \mathscr{E} \subseteq \overline{\mathscr{O}(A)} \Rightarrow \overline{\mathscr{O}(A)} = \mathscr{E}$.

Orbits closures in $M_{m\times n}(k)$

Theorem

1. *Given any* $A \in M_{m\times n}(k)$*, we obtain*

$$\overline{\mathscr{O}(A)} = \{B \in M_{m\times n}(k) \mid \mathrm{rank}(B) \leq \mathrm{rank}(A)\}.$$

Orbits closures in $M_{m\times n}(k)$

Theorem

1. *Given any* $A \in M_{m\times n}(k)$, *we obtain*

$$\overline{\mathscr{O}(A)} = \{B \in M_{m\times n}(k) \mid \mathrm{rank}(B) \leq \mathrm{rank}(A)\}.$$

2. *The orbits closures in* $M_{m\times n}(k)$ *are as follows:*

$$\mathscr{O}_{m\times n}(r) = \{A \in M_{m\times n}(k) \mid \mathrm{rank}(A) \leq r\},$$

where $r = 0, 1, \ldots, \min\{m, n\}$.