Cluster category of type $\mathbb{A}_{\infty}^{\infty}$ and triangulations of the infinite strip

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3. In 2006, Caldero, Chapoton and Schiffler described cluster category of type $\mathbb{A}_n$ in terms of triangulations of an $(n+3)$-gon.
4. In 2012, Holm-Jørgensen constructed cluster category of type $\mathbb{A}_\infty$ as finite derived category of dg-modules over the polynomial ring, and described cluster structure in terms of triangulations of the infinity-gon.

Objective of this talk

1. To construct cluster categories of type $\mathbb{A}_\infty^\infty$ using the canonical orbit category of $D^b(\mathrm{rep}Q)$, where Q is a quiver with no infinite path of type $\mathbb{A}_\infty^\infty$.

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2. To given a geometrical realization of the cluster structure of this cluster category in terms of triangulations of the infinite strip with marked points.

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3. A subcategory of $\mathcal{A}$ is called *strictly additive* if it is full, closed under isomorphisms, finite direct sums, and direct summands.

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Theorem (Koenig, Zhu)

If $\mathcal{T}$ is a cluster-tilting subcategory of $\mathcal{A}$, then

$$\mathrm{mod}\mathcal{T} \cong \mathcal{A}/\mathcal{T}[1].$$

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where f, u minimal left $\mathcal{T}_M$-approximations;

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1. it has some cluster-tilting subcategories;
2. the quiver of each cluster-tilting subcategory has no oriented cycle of length one or two.

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4. Γ is called *standard* if $\operatorname{add}(\Gamma) \cong k(\Gamma)$.

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   - *connecting* components $\mathcal{C}_Q[n](\cong \mathbb{Z}\mathbb{A}_{\infty}^{\infty})$, obtained by gluing $\mathcal{I}_Q[n-1]$ with $\mathcal{P}_Q[n]$, where $n \in \mathbb{Z}$.

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5. $\mathscr{C}(Q)$ is 2-CY with triangle-exact projection functor
$$p : D^b(\mathrm{rep}(Q)) \to \mathscr{C}(Q) : X \mapsto X; f \mapsto f.$$

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2. *A subcategory of $\mathscr{C}(Q)$ is weakly cluster-tilting $\Leftrightarrow$ it is maximal rigid.*

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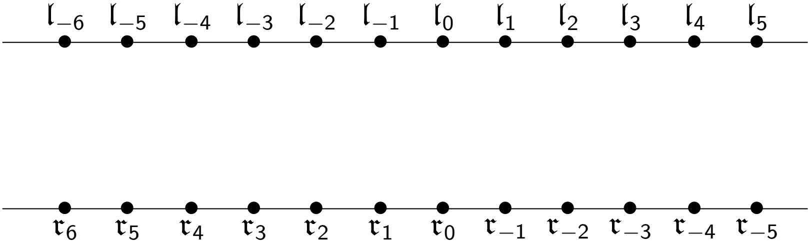

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   - written as $\sigma = [\mathfrak{p}, \mathfrak{q}] = [\mathfrak{q}, \mathfrak{p}]$.

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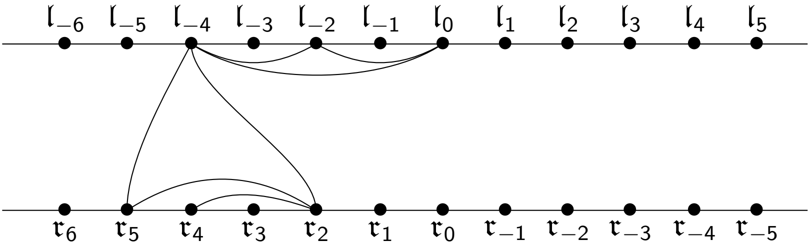

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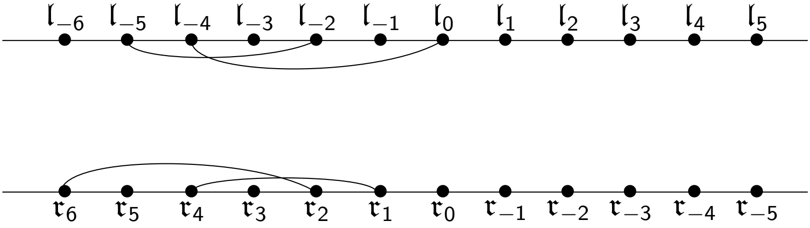
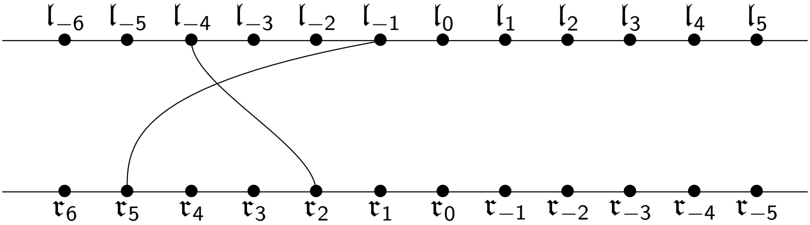


Illustrations



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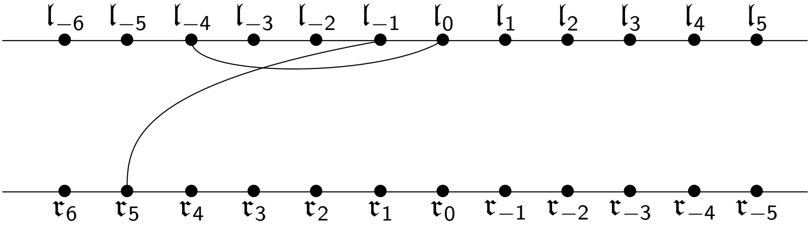

Triangulations of $\mathcal{B}_\infty$

Definition

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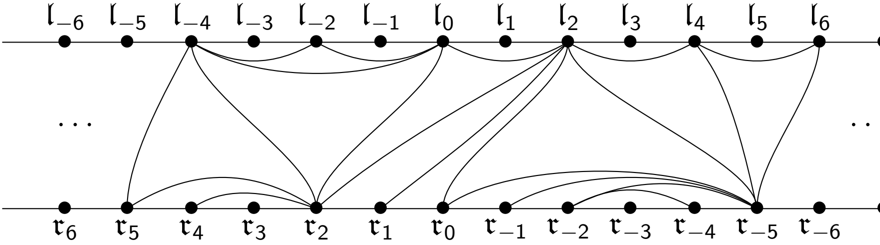

Parametrization of indecomposable objects by arcs

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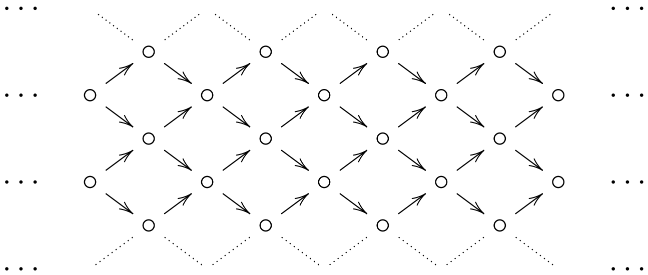
3. We shall construct a bijection

$$\varphi : \mathrm{ind}\mathscr{C}(Q) \to \mathrm{arc}(\mathcal{B}_\infty),$$

where $\mathrm{arc}(\mathcal{B}_\infty)$ is the set of arcs in $\mathcal{B}_\infty$.

Parametrization of objects in $\mathcal{C}_Q \cong \mathbb{Z}\mathbb{A}_\infty^\infty$



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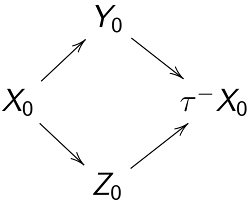

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Parametrization of objects in regular components ($\cong \mathbb{Z}\mathbb{A}_\infty$)

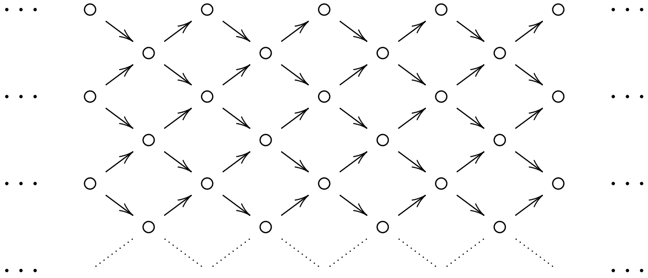

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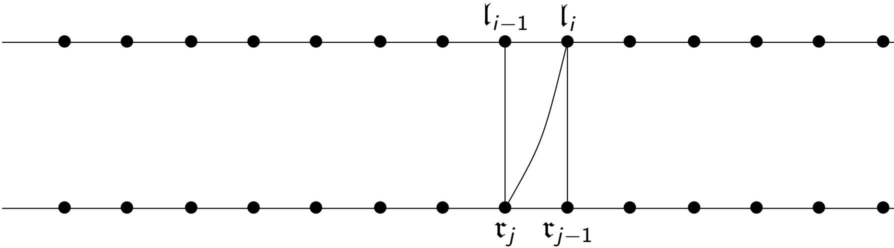
Illustration of arrows



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Theorem

$\mathcal{T}$ is a weakly cluster-tilting subcategory $\Leftrightarrow \operatorname{arc}(\mathcal{T})$ is a triangulation of $\mathcal{B}_\infty$.

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   - *left $\mathbb{T}$-unbounded* if $[\mathfrak{l}_i, \mathfrak{l}_p], [\mathfrak{l}_p, \mathfrak{r}_j] \in \mathbb{T}$ for infinitely many $i < p$ and infinitely many $j > -p$.
   - *right $\mathbb{T}$-bounded* if $[\mathfrak{l}_p, \mathfrak{l}_i], [\mathfrak{l}_p, \mathfrak{r}_j] \in \mathbb{T}$ for only finitely many $i > p$ and only finitely many $j < -p$;
   - *right $\mathbb{T}$-unbounded* if $[\mathfrak{l}_p, \mathfrak{l}_i], [\mathfrak{l}_p, \mathfrak{r}_j] \in \mathbb{T}$ for infinitely many $i > p$ and infinitely many $j < -p$.
2. In a similarly, one defines a lower marked point to be *left $\mathbb{T}$-bounded*, *left $\mathbb{T}$-unbounded*, *right $\mathbb{T}$-bounded*, and *right $\mathbb{T}$-unbounded*.

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3. *full $\mathbb{T}$-fountain base* if $\mathfrak{p}$ is left and right $\mathbb{T}$-unbounded.

Main Result

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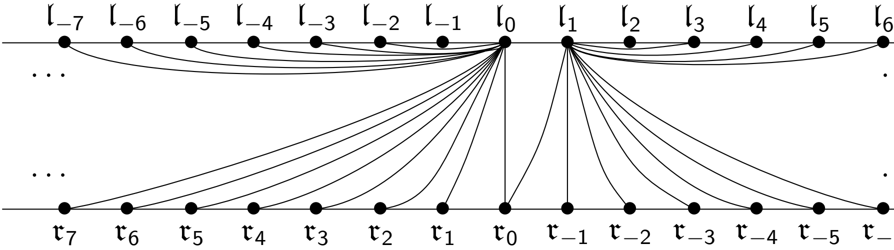
In this case, $\mathcal{B}_{\infty}$ has at most two $\mathrm{arc}(\mathcal{T})$-fountain bases, and if it has two, then one is a left fountain base and the other one is a right fountain base.

A cluster-tilting subcategory with two fountains



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3. *If* v *is the other diagonal of* Σ, *then* $M_u^* = M_v$.