Covering Theory and Cluster Categories

Shiping Liu (Université de Sherbrooke)

joint with

Fang Li, Jinde Xu and Yichao Yang

International Workshop on Cluster Algebras and Related Topics

Chern Institute of Mathematics

Nankai University (Tianjin, China)

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2. Introduce covering of species and the associated push-down functor.
3. As an application, we shall construct a cluster category of non simply laced type $\mathbb{C}_\infty$.

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2. The unfolding of an exchange matrix is a covering of the corresponding valued quiver.
3. There is a growing interest in cluster categories with a cluster structure of infinite rank, while cluster categories of non simply laced type seems unseen.

Part I

Covering theory
for species and their representations

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Remark

A non-valued quiver without multiple arrows, loops or 2-cycles is regarded as a trivially valued quiver.

Example

A valued quiver of type $\mathbb{C}_\infty$ with a zigzag orientation:

$$0 \xrightarrow{(2,1)} 1 \longleftarrow 2 \longrightarrow 3 \longleftarrow 4 \longrightarrow 5 \longleftarrow \cdots\cdots$$

where trivial valuations are omitted.

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$$(u_{xy}, u_{yx}) \leq (v_{\varphi(x),\varphi(y)}, v_{\varphi(y),\varphi(x)}).$$

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   - for any $y \in \varphi^{-}(b)$, we have
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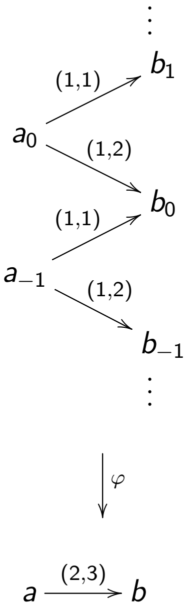
Example of valued quiver covering



Universal cover of a non-valued quiver

Theorem (Bongartz, Gabriel)

Given non-valued quiver Q, one has unique quiver covering

$$\pi : \tilde{Q} \to Q,$$

where $\tilde{Q}$ is a tree, called *universal cover* of Q.

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- $\hat{\Delta}_0 = \Delta_0$;
- For each arrow $\alpha : x \xrightarrow{(v_{xy}, v_{yx})} y$ in Δ, one draws $v_{xy}v_{yx}$ arrows $\alpha_{ij} : x \to y$ in $\hat{\Delta}$, arranged as a $(v_{yx} \times v_{xy})$-matrix

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1,v_{xy}} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2,v_{xy}} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{v_{yx},1} & \alpha_{v_{yx},2} & \cdots & \alpha_{v_{yx},v_{xy}} \end{pmatrix}.$$

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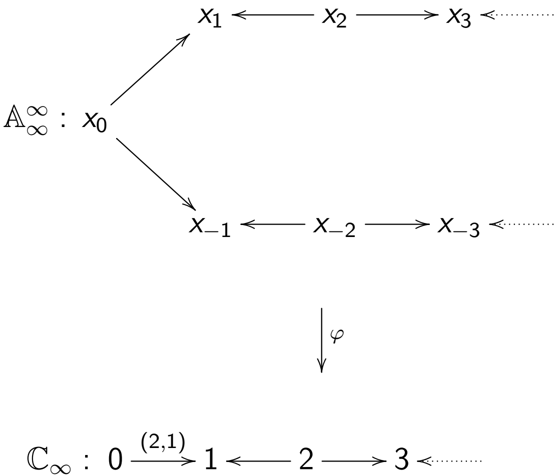
2. *π covers every valued quiver covering $\phi : \Gamma \to \Delta$.*

Example: Universal cover of $\mathbb{C}_\infty$



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Example: $\mathbb{R}$-species of $\mathbb{C}_\infty$

$$\mathbb{C}_\infty : \ 0 \xrightarrow{(2,1)} 1 \longleftarrow 2 \longrightarrow 3 \longleftarrow 4 \cdots\cdots\rightarrow$$

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Proposition (Dlab-Ringel)

The category $\mathrm{rep}(\mathcal{S})$ of finite dimensional representations of $\mathcal{S}$ is a hereditary abelian category.

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   - $\mathcal{S}(a)$-$\mathcal{T}(j)$-bilinear map
   $$\varphi_{\beta} : \mathcal{S}(a) \otimes_{\mathcal{T}(i)} \mathcal{T}(\beta) \to \mathcal{S}(\alpha).$$

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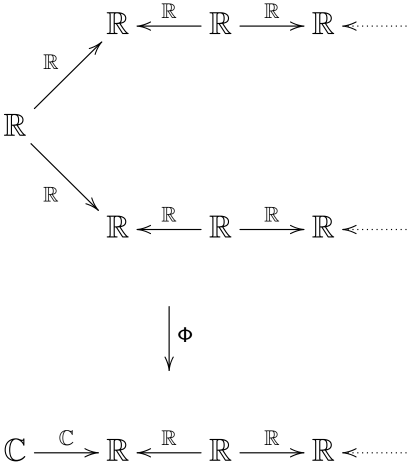
   $$(\varphi_\gamma) : \bigoplus_{\gamma: i \to j \in \varphi^{-}(\alpha)} \mathcal{S}(a) \bigotimes_{\mathcal{T}(i)} \mathcal{T}(\gamma) \to \mathcal{S}(\alpha).$$

Example of species covering



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Adjoint pairs

Theorem

Let $\Phi : \mathcal{T} \to \mathcal{S}$ *be a species covering.*

1. *The push-down functor* $\Phi_\lambda : \mathrm{rep}(\mathcal{T}) \to \mathrm{rep}(\mathcal{S})$ *induces an exact functor* $\Phi_\lambda^D : D^b(\mathrm{rep}(\mathcal{T})) \to D^b(\mathrm{rep}(\mathcal{S}))$.

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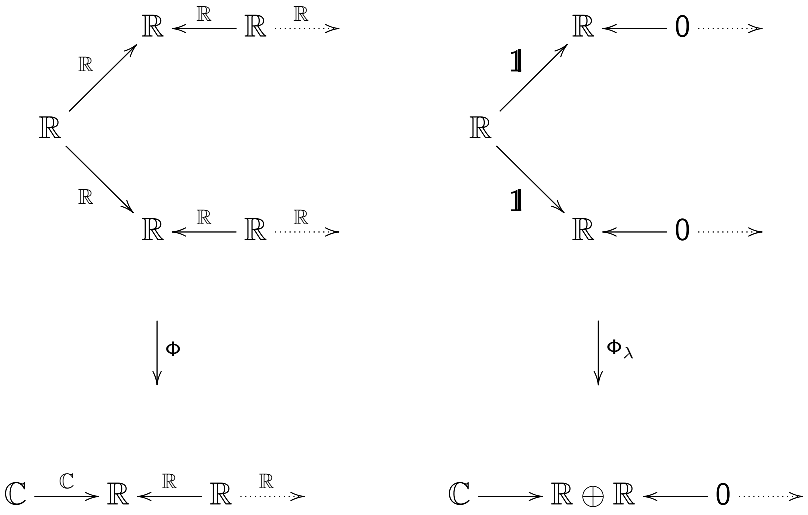
3. *(Φ_λ, Φ_μ) and $(\Phi_\lambda^D, \Phi_\mu^D)$ are adjoint pairs.*

Example of push-down



Part II

Cluster category of type $\mathbb{C}_\infty$

joint with

Jinde Xu and Yichao Yang

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Lemma

$\mathcal{A}/G$ is Hom-finite Krull-Schmidt additive k-category with a canonical embedding
$$\sigma : \mathcal{A} \to \mathcal{A}/G : X \mapsto X; f \mapsto f.$$

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4. There exist in $\mathcal{A}$ two exact triangles :

$$M \xrightarrow{f} N \xrightarrow{g} M^* \longrightarrow M[1];$$

$$M^* \xrightarrow{u} L \xrightarrow{v} M \longrightarrow M^*[1],$$

where f, u are minimal left $\mathcal{C}_M$-approximations;

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Theorem(Keller)

Setting $F = \tau_{\mathscr{D}}^{-1} \circ [1]$, the canonical orbit category

$$\mathscr{C}(\mathcal{H}) = \mathscr{D}^b(\mathcal{H}) / < F >$$

is 2-CY triangulated category.

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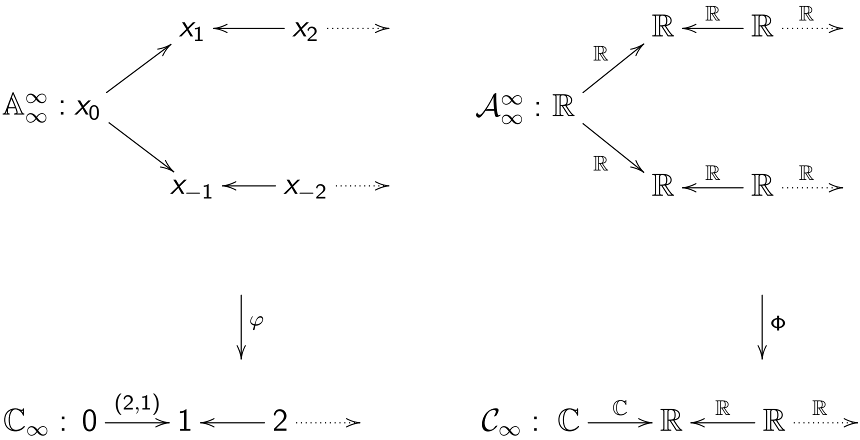
2. We shall show that $\mathscr{C}(\mathrm{rep}(\mathcal{C}_\infty))$ is a cluster category.

Recall the following covering:



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2. The σ-action on $\mathcal{C}_{\mathbb{A}_\infty^\infty}$ is reflection across $\tau_{\mathscr{C}}$-orbit of $P[x_0]$.

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3. *The functor* $\Phi_\lambda^{\mathscr{C}}$ *induces valued translation quiver covering*
$$\Phi_\lambda^{\mathscr{C}} : \Gamma_{\mathscr{C}(\mathrm{rep}(\mathbb{A}_\infty^\infty))} \longrightarrow \Gamma_{\mathscr{C}(\mathrm{rep}(\mathcal{C}_\infty))}.$$

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Remark

In a similar fashion, we can construct a cluster category of type $\mathbb{B}_\infty$.

The infinite strip with marked points

Consider the strip in the plane

$$S[-1,1] = \{(x,y) \in \mathbb{R}^2 \mid -1 \le y \le 1\}$$

with marked points:

$$\mathfrak{l}_n = (n,1), \mathfrak{r}_i = (-n,-1); n \in \mathbb{Z}.$$

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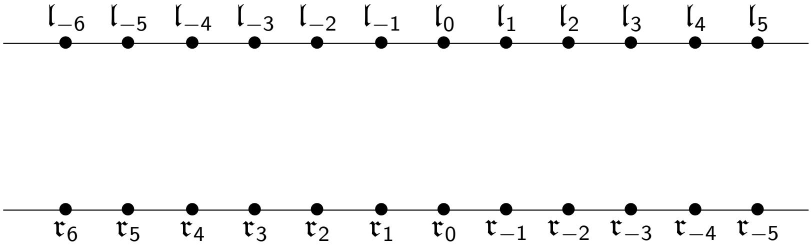
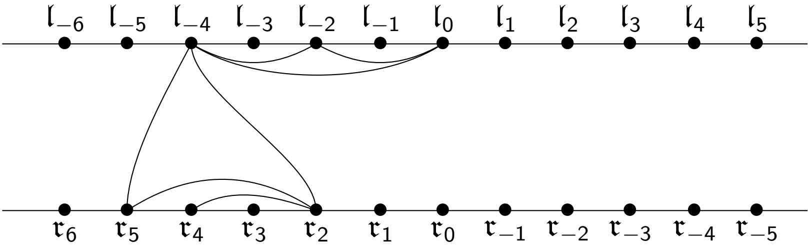


Arcs in $S[-1,1]$

There exist in $S[-1,1]$ three types of arcs:

- *upper arcs* : $[\mathfrak{l}_i, \mathfrak{l}_j]$ with $|i-j| > 1$;
- *lower arcs* : $[\mathfrak{r}_i, \mathfrak{r}_j]$ with $|i-j| > 1$;
- *connecting arcs* : $[\mathfrak{l}_i, \mathfrak{r}_j]$ with $i, j \in \mathbb{Z}$.



Geometric realization of $\mathscr{C}(\mathrm{rep}(\mathbb{A}_\infty^\infty))$

Theorem (Liu, Paquette)

$$\mathscr{C}(\mathrm{rep}(\mathbb{A}_\infty^\infty)) \longleftrightarrow S[-1,1]$$

$$\{\text{objects in } \mathcal{R}_L\} \longleftrightarrow \{\text{upper arcs}\}$$

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$$\{\text{cluster tilting subcategories}\} \longleftrightarrow \{\text{compact triangulations}\}$$

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2. The *twisted strip* is the quotient

$$S[0, 1] = S[-1, 1]/G.$$

Arcs in the twisted strip

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$$\{(x,y) \in \mathbb{R}^2 \mid 0 \leq y \leq 1\} \backslash \{(x,0) \mid x < 0\}$$

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   - *upper arcs* : $[\mathfrak{l}_i, \mathfrak{l}_j]$, with $|i-j| \geq 2$, not passing through the origin O;
   - *connecting arcs* : $[\mathfrak{l}_i, O, \mathfrak{l}_j]$, $i, j \in \mathbb{Z}$, passing through the origin O.

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$$\{\textit{cluster tilting subcategories}\} \longleftrightarrow \{\textit{compact triangulations}\}$$

Match between algebraic covering and topological covering

$$
\begin{array}{ccc}
\mathscr{C}(\mathbb{A}^{\infty}_{\infty}) & \xleftarrow{\text{geometric realization}}\!\!\!\!\rightarrow & S[-1,1] \\
\Big\downarrow{\scriptstyle \Phi^{\mathscr{C}}_{\lambda}} & & \Big\downarrow{\scriptstyle \pi} \\
\mathscr{C}(\mathbb{C}_{\infty}) & \xleftarrow{\text{geometric realization}}\!\!\!\!\rightarrow & S[0,1]
\end{array}
$$