Auslander-Reiten components with bounded short cycles

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joint with

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5. As application, give a new characterization of representation-finiteness.

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6. $\mathrm{rad}^\infty(\mathrm{mod}\, A) := \cap_{n\geq 0} \mathrm{rad}^n(\mathrm{mod}\, A)$, infinite radical.
7. Let Γ_A be the AR-quiver of A, with AR-translation τ.

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Proposition (Igusa-Todorov)

If $X_0 \xrightarrow{f_1} X_1 \longrightarrow \cdots \longrightarrow X_{n-1} \xrightarrow{f_n} X_n$ is a sectional path of irreducible maps in $\mathrm{ind}\, A$, then $\mathrm{dp}(f_n \cdots f_1) = n$.

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4. If all the X_i belong to a subquiver Γ of Γ_A, then σ is called *cycle* in $\operatorname{add}(\Gamma)$.

Theorem

An artin algebra A is representation-finite if

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Module categories without cycles

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2. $\operatorname{ind} A$ *contains no short cycle* (Happel-Liu).

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An artin algebra A is generalized double tilted $\Leftrightarrow$ Γ_A has a faithful, generalized standard and short-cycle-bounded component.

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Corollary (Liu)

If Δ is a τ-rigid cut of Γ_A, then the quotient algebra

$$B = A/\mathrm{ann}(\Delta)$$

is tilted with Δ being a slice of Γ_B.

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5. The *core* of $\mathcal{C}$ is the full subquiver generated by the modules lying on $P \rightsquigarrow I$, with P projective and I injective.

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   Thus, $\mathrm{add}(\Gamma)$ has short cycles of arbitrarily large depths.

2. *If Γ contains no oriented cycle, it contains cuts of Γ_A; and if such a cut is not τ-rigid, then $\mathrm{add}(\Gamma)$ contains short cycles of infinite depth.*

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where

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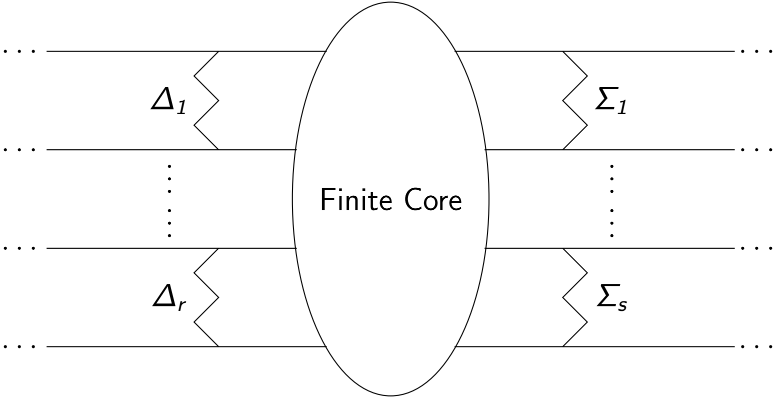
2. *each Θ_i has τ-rigid cut Σ_i such that $C_i = A/\mathrm{ann}(\Sigma_i)$ is tilted and all the successors of Δ_i in $\mathcal{C}$ belong to the connecting component of Γ_{C_i}.*

Illustration of a short-cycle-bounded component



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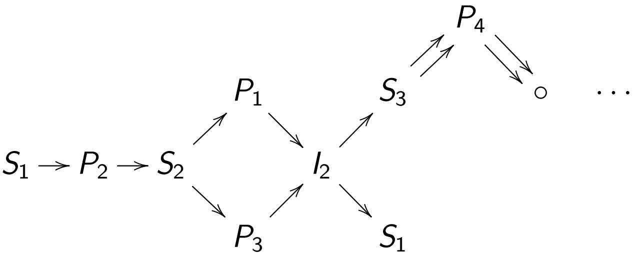
2. We have a short-cycle-bounded AR-component as follows :



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The algebra A is of representation-finite if and only if there exists a bound for the depths of short cycles in $\operatorname{ind} A$