Oriented cycles and the global dimension of an algebra

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Problem

How to determine $\mathrm{gdim}A$ is finite or infinite ?

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$$r \geq 2 \text{ and } Q^+ =< Q_1 >.$$

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If Q has no oriented cycle, then

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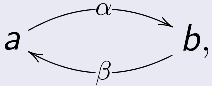

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Then $\mathrm{gdim} A_1 = 2$ and $\mathrm{gdim} A_2 = \infty$.

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Strong No Loop Conjecture (Zacharia, 1980')

If Q has loop at a vertex a, then $\operatorname{pdim} S_a = \infty$.

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2. For the Strong No Loop Conjecture, only partial solutions were obtained until 2010.

Main Result

Theorem (Igusa, Liu, Paquette, 2011)

Let $A = kQ/I$. If Q has a loop at a vertex a, then

$$\mathrm{pdim}\, S_a = \mathrm{idim}\, S_a = \infty.$$

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Let A^{o} denote the opposite algebra of A. Then $\mathrm{HH}_0(A)$ is radical-trivial $\Leftrightarrow$ so is $\mathrm{HH}_0(A^{\mathrm{o}})$.

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Trace of matrices over A

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For $M = (x_{ij})_{n\times n} \in M_n(A)$, one defines

$$\operatorname{tr}(M) = (x_{11} + \cdots + x_{nn}) + [A, A] \in \mathrm{HH}_0(A).$$

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If $M \in M_{m \times n}(A)$ and $N \in M_{n \times m}(A)$, then

$$\mathrm{tr}(MN) = \mathrm{tr}(NM).$$

Trace of endomorphisms of projective modules

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Fix $u \in A$. Consider $\varphi_u : A \mapsto A : x \mapsto ux$. Then

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$\Rightarrow u + [A, A] = \Sigma_{a \in Q_0} e_a u e_a + [A, A] = \mathrm{tr}(\varphi_u)$.

Trace of endomorphisms of modules of fin proj dimension

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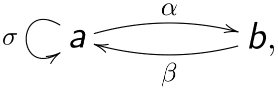

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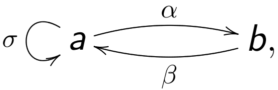

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Consider algebra morphism

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Definition

1. A projective resolution in $\mathrm{mod} A$

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Corollary

$\mathrm{idim}\,S_e < \infty \Rightarrow$ all $M \in \mathrm{mod}A$ are e-bounded.

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Remark

$\mathrm{idim}\, S_e < \infty \Rightarrow \mathrm{tr}_e(\varphi)$ defined for any endomor φ.

Additivity of the e-trace

Lemma

Let $\mathrm{mod} A$ *have exact commutative diagram*

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If two of L, M, N *are e-bounded, then all of them are e-bounded with*

$$\mathrm{tr}_e(\varphi) = \mathrm{tr}_e(\phi) + \mathrm{tr}_e(\psi).$$

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An *e-bounded filtration* of M is a series

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Proof. Let σ be loop at $a \in Q_0$.

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Combinatorial terminology

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1. A *minimal relation* for A is an element $\rho = \lambda_1 p_1 + \cdots + \lambda_r p_r \in I$, $\lambda_i \in k^*$, p_i distinct, such that $\Sigma_{i \in \Omega} \lambda_i p_i \notin I$ for any $\Omega \subset \{1, \ldots, r\}$.

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3. A path p in Q is *free* in A if it is not summand of any minimal relation for A.

Example

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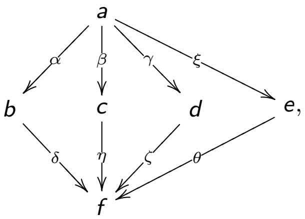

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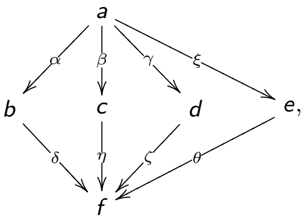

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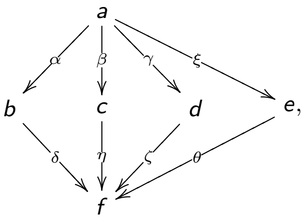

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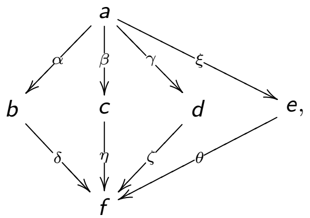

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1. Let $A = kQ/I$, where

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Lemma

Let σ be oriented cycle in Q. If σ is cyclically free in A, then $\bar{\sigma} \notin [A, A]$.

Further result

If σ is oriented cycle in Q passing through distinct vertices $a_1, \ldots, a_s$, put $e_\sigma = e_{a_1} + \cdots + e_{a_s}$.

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Corollary

Let $A = kQ/I$ be monomial. If Q has oriented cycle which is cyclically nonzero in A, then $\mathrm{gdim} A = \infty$.

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Remark

This conjecture holds true for monomial algebras and special biserial algebras.