Oriented cycles and global dimension of algebras

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Problem

How to determine $\mathrm{gdim}A$ is finite or infinite ?

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Question

Can one decide $\mathrm{gdim}(A)$ is finite or infinite in terms of the quiver Q?

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$$S_a = P_a / \mathrm{rad} P_a,\ a \in Q_0.$$

The no oriented cycle case

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If Q has no oriented cycle, then

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Remark

For $\mathrm{gdim} A = \infty$, the existence of oriented cycles in Q is necessary but not sufficient.

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1. Consider the following quiver

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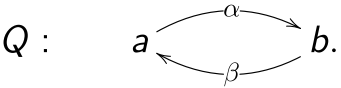

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3. If $B = kQ/<\alpha\beta, \beta\alpha>$, then $\mathrm{gdim} B = \infty$.

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Strong No Loop Conjecture (Zacharia, 1980)

If Q has loop at a vertex a, then $\operatorname{pdim} S_a = \infty$.

Solution of the conjectures

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The Strong No Loop Conjecture was established by Igusa, Liu, Paquette in 1990, by localizing Lenzing's result.

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4) $\mathrm{HH}_0(A)$ is *radical-trivial* if $\mathrm{rad}\, A \subseteq [A, A]$.

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Proposition

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Proof. Let $x, y \in A$. Write

$x = \Sigma_{a \in Q_0} \lambda_a e_a + \Sigma_{\alpha \in Q_1} \lambda_\alpha \bar{\alpha} + \bar{u}, \;\; u \in (kQ^+)^2,$

$y = \Sigma_{b \in Q_0} \mu_b e_b + \Sigma_{\beta \in Q_1} \mu_\beta \bar{\beta} + \bar{v}, \;\; v \in (kQ^+)^2.$

$[x, y] = \Sigma_{\alpha \in Q_1} \lambda_\alpha (\mu_{t(\alpha)} - \mu_{s(\alpha)}) \bar{\alpha}$

$\qquad + \Sigma_{\beta \in Q_1} \mu_\beta (\lambda_{s(\beta)} - \lambda_{t(\beta)}) \bar{\beta} + \bar{w}, \;\; w \in (kQ^+)^2.$

$\bar{\sigma} \in [A, A] \Rightarrow \bar{\sigma} = \Sigma_{\alpha \in \Omega} \nu_\alpha \bar{\alpha} + \bar{u}, \; s(\alpha) \neq t(\alpha), \; u \in (kQ^+)^2,$

$\qquad \Rightarrow \sigma - \Sigma_{\alpha \in \Omega} \nu_\alpha \alpha - u \in I,$

$\qquad \Rightarrow \sigma - \Sigma_{\alpha \in \Omega} \nu_\alpha \alpha \in (kQ^+)^2$, absurd.

Trace of matrices over A

Definition

For $M = (x_{ij})_{n\times n} \in M_n(A)$, one defines

$$\mathrm{tr}(M) = (x_{11} + \cdots + x_{nn}) + [A, A] \in \mathrm{HH}_0(A).$$

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Proposition

If $M \in M_{m\times n}(A)$ and $N \in M_{n\times m}(A)$, then

$$\mathrm{tr}(MN) = \mathrm{tr}(NM).$$

Trace of endomorphisms of projective modules

Definition (Hattori, Stallings)

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Trace of endomorphisms of modules of fin proj dimension

1. Let $M \in \mathrm{mod} A$ have fin proj resolution

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Theorem (Lenzing, 1969)

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Theorem (Igusa, 1990)

If $\mathrm{gdim}(A) < \infty$, then Q has no loop.

Localizing algebra

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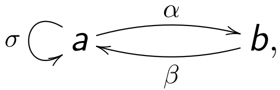
Remark

If σ is a loop at some of the a_i, then it remain a loop in the quiver of A_e.

Example

Let A be given by



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Then A_{e_a} is given by

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This induces group morphism

$$\begin{array}{rrcl} H_e : & \mathrm{HH}_0(A) & \to & \mathrm{HH}_0(A_e) \\ & x + [A, A] & \mapsto & p_e(x) + [A_e, A_e]. \end{array}$$

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Lemma

Let $\varphi \in \mathrm{End}_A(P)$ with P projective. If P, eA have no common summand, then $\mathrm{tr}_e(\varphi) = 0$.

e-bounded modules

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A projective resolution in $\mathrm{mod} A$

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In this case, M is called *e-bounded*.

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M is e-bounded $\Leftrightarrow \mathrm{Ext}^i_A(M, S_e) = 0$, *for $i >> 0$.*

Corollary

$\mathrm{idim}\, S_e < \infty \Rightarrow$ every $M \in \mathrm{mod}A$ is e-bounded.

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$\mathrm{idim}\, S_e < \infty \Rightarrow \mathrm{tr}_e(\varphi)$ defined for any $\varphi : M \to M$.

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Definition

1. A *minimal relation* for A is an element

$$\rho = \lambda_1 p_1 + \cdots + \lambda_r p_r \in I,$$

where $\lambda_i \in k^*$, p_i distinct parallel paths, such that $\sum_{i \in \Omega} \lambda_i p_i \notin I$ for any $\Omega \subset \{1, \dots, r\}$.

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3. A path p in Q is *free* in A if it is not summand of any minimal relation for A.

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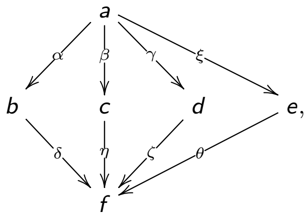

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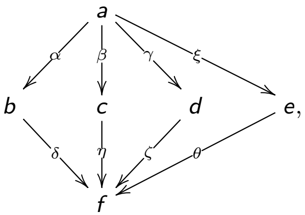

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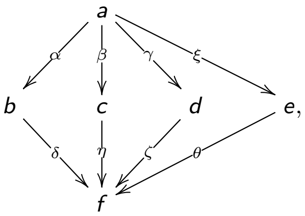

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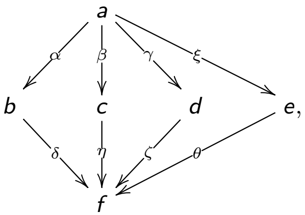

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Oriented cycles

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Remark

A loop in Q is cyclically free in A.

Example

1. Let $A = kQ/I$, where

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4. Cycle $\beta\alpha$ is cyclically free in A.

Lemma

Let σ be oriented cycle in Q. If σ is cyclically free in A, then $\bar{\sigma} \notin [A, A]$.

Further result

If σ is oriented cycle passing through the vertices $a_1, \ldots, a_s$, put $e_\sigma = e_{a_1} + \cdots + e_{a_s}$.

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Corollary

Let $A = kQ/I$ be monomial. If Q has oriented cycle which is cyclically nonzero in A, then $\mathrm{gdim} A = \infty$.

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No Loop Conjecture

Let A be artin algebra. If $\mathrm{gdim}(A) < \infty$, then $\mathrm{Ext}^1(S, S) = 0$ for all simple S.