Koszul duality for non-graded derived categories

Ales Bouhada, Min Huang, Shiping Liu*
Université de Sherbrooke

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Plan

1. Introduction on Beilinson, Ginzburg and Soergel's work

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   - Q is locally finite gradable quiver;
   - $\Lambda = kQ/(kQ^+)^2$, Koszul with Koszul dual kQ^{op}.

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The application of covering theory requires the study of modules over algebras without identity

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Proposition

If A is a Koszul algebra, then its Koszul dual $A^!$ is also Koszul.

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Remark

Their proof involves spectral sequences.

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5. Given k-space V, write $DV = \mathrm{Hom}_k(V, k)$.

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5. $\mathrm{proj}\,\Lambda$: finite direct sum of the P_x, with $x \in Q_0$.

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Example

If Q is a double infinite path, then kQ is strongly locally finite dimensional but not locally bounded.

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3. For $\gamma = \sum \lambda_i p_i \in kQ_2(x,y)$, writing $\gamma^* = \sum \lambda_i p_i^*$ yields

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7. By definition, $\Lambda^!$ is quadratic with $(\Lambda^!)^! = \Lambda$.

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Theorem

Λ is Koszul $\Leftrightarrow$ $\Lambda^!$ is Koszul; called Koszul dual of Λ.

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5. That is, $\Lambda^! = kQ^{\mathrm{op}}/(kQ^+)^2$.

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Remark

$C^{\downarrow}_{1,0}(\mathrm{Mod}\Lambda) = C^{\downarrow}(\mathrm{Mod}\Lambda)$ and $C^{\uparrow}_{1,0}(\mathrm{Mod}\Lambda) = C^{\uparrow}(\mathrm{Mod}\Lambda)$.

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3. *We prove this by generalizing Acyclic Assembly Lemma on homology of total complexes of double complexes.*

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Corollary

If Q has no right infinite path or no left infinite path, then

$$D^b(\mathrm{Mod}^b\Lambda^!) \cong D^b(\mathrm{Mod}^b\Lambda).$$