Module categories with a null forth power of the radical

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Special session on

Representation Theory of Algebras

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Problem

Can we classify the representation-finite artin algebras in terms of the nilpotency of $\operatorname{rad}(\operatorname{mod} A)$

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2. (Damavandi) *If A is a Nakayama algebra, then* $\mathrm{rad}(\mathrm{mod}\, A)$ *is of nilpotency 3* $\iff A = k\vec{\mathbb{A}}_3$ *or A is non hereditary with* $\mathrm{rad}^2(A) = 0$.

Objective of this talk

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Give a complete list of artin algebras A with $\mathrm{rad}^4(\mathrm{mod}\, A) = 0$.

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Let Δ be a finite valued quiver or valued diagram.

A hereditary algebra A is of type Δ if $Q_A \cong \Delta$ or $\overline{Q_A} \cong \Delta$.

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If A is a non-hereditary Nakayama artin algebra, then

$$\mathrm{rad}^4(\mathrm{mod}\, A) = 0 \Longleftrightarrow \mathrm{rad}^3(\mathrm{mod}\, A) = 0.$$

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Problem

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Remark

$P \in \operatorname{ind} A$ is wedged projective $\Longleftrightarrow$ DP is co-wedged injective.

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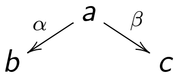

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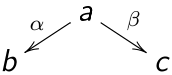

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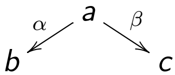

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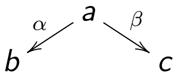

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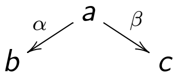

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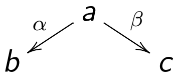

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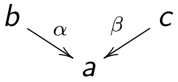

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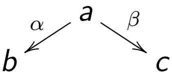

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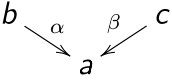

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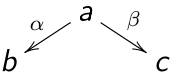

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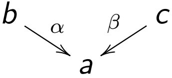

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Local Nakayama algebras of Loewy length 3 satisfy all but (4).

Tri-string case

Proposition

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Main Result

Theorem

If A connected artin algebra, then $\mathrm{rad}^4(\mathrm{mod}\, A) = 0 \iff A$ *is*

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- *a non-hereditary non-Nakayama tri-string algebra.*

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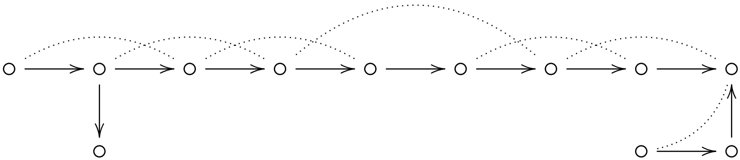

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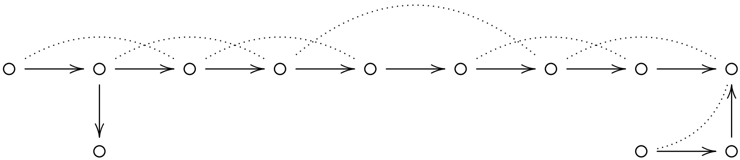

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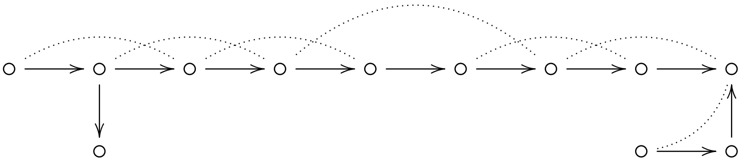

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3. $\mathrm{rad}(\mathrm{mod}\, A)$ is of nilpotency 4.

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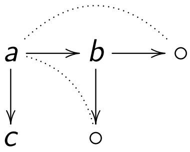

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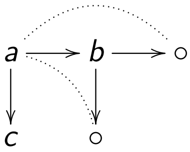

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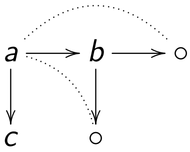

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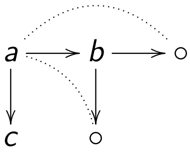

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3. Thus, A is not tri-string algebra.
4. $\mathrm{rad}^4(\mathrm{mod}\, A) \neq 0$.