Auslander-Reiten Theory through Triangulated Categories

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5. In extension-closed subcategory of abelian category over arbitrary commutative ring k, Liu, Ng and Paquette (2013) obtained a criterion for the existence of an almost split sequence, using I_k.

Objective

To unify the previously mentioned existence theorems of an almost split sequence or an almost split triangle.

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If $\mathrm{Hom}_{\mathrm{End}(Z)}(\mathrm{Hom}_{\mathcal{C}}(Z,-), I) \cong \mathrm{Ext}^1_{\mathcal{C}}(-, X)$ *for some* $X \in \mathcal{C}$*, then* $\mathcal{C}$ *has almost split sequence* $X \longrightarrow Y \longrightarrow Z$.

left or right triangulated case

1. Let $\mathcal{C}$ be a left triangulated subcategory of $\mathcal{T}$.
2. Let $Z \in \mathcal{C}$ with $\mathrm{End}(Z)$ being local.
3. $S = \mathrm{End}(Z)/\mathrm{rad}(\mathrm{End}(Z))$ is right $\mathrm{End}(Z)$-simple.
4. I the injective envelope of S.

Theorem

If $\mathrm{Hom}_{\mathrm{End}(Z)}(\mathrm{Hom}_{\mathcal{C}}(Z, -), I) \cong \mathrm{Ext}^1_{\mathcal{C}}(-, X)$ *for some* $X \in \mathcal{C}$, *then* $\mathcal{C}$ *has almost split sequence* $X \longrightarrow Y \longrightarrow Z$.

Remark

In case $\mathcal{C}$ is a right triangulated subcategory of $\mathcal{T}$, then the dual version of the above theorem holds.