The radical nipotency of the module category of a hereditary algebra of Dynkin type

Shiping Liu and Gordana Todorov

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Objective

Understand the representation theory of representation-finite artin algebras in terms of the nilpotency of $\mathrm{rad}(\mathrm{mod}\, A)$.

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In this talk, we calculate the nilpotency of $\mathrm{rad}(\mathrm{mod} A)$ in case A is hereditary.

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   called the Coxeter transformation of $K_0(A)$.

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   - $d' = \dim \mathrm{Ext}^1_A(S, T)_{\mathrm{End}(S)}$.

Dynkin diagram

$$\mathbb{A}_n : \quad 1 \text{ --- } 2 \text{ --- } \cdots \text{ --- } n \qquad (n \geq 1)$$

$$\mathbb{B}_n : \quad 1 \overset{(1,2)}{\text{------}} 2 \text{ --- } \cdots \text{ --- } n \qquad (n \geq 2)$$

$$\mathbb{C}_n : \quad 1 \overset{(2,1)}{\text{------}} 2 \text{ --- } 3 \text{ --- } \cdots \text{ --- } n \qquad (n \geq 3)$$

$$\mathbb{D}_n : \quad \begin{array}{ccccccc} & & 1 & & & & \\ & & | & & & & \\ 2 & \text{---} & 3 & \text{---} & 4 & \text{---} \cdots \text{---} & n \end{array} \qquad (n \geq 4)$$

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$$\mathbb{E}_n : \quad \begin{array}{ccccccccccccc} & & & & 1 & & & & & & & & \\ & & & & | & & & & & & & & \\ 2 & \text{---} & 3 & \text{---} & 4 & \text{---} & 5 & \text{---} & 6 & \text{---} & \cdots & \text{---} & n \end{array} \qquad (n = 6, 7, 8)$$

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$$\mathbb{G}_2 : \quad 1 \overset{(1,3)}{\text{——}} 2$$

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$\overline{Q}_A$	$\mathbb{A}_n$	$\mathbb{B}_n$	$\mathbb{C}_n$	$\mathbb{D}_n$	$\mathbb{E}_6$	$\mathbb{E}_7$	$\mathbb{E}_8$	$\mathbb{F}_4$	$\mathbb{G}_2$
c_A	$n+1$	$2n$	$2n$	$2(n-1)$	12	18	30	12	6

Main Result of this talk

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If A is hereditary artin algebra of finite representation type, then $\mathrm{rad}(\mathrm{mod} A)$ is of nilpotency $c_A - 1$.

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Theorem (Chaio, Liu, 2012)

$$\mathrm{dp}(\mathrm{mod}A) = \max\{\mathrm{dp}(\iota_s \circ \pi_s) \mid S \in \mathrm{mod}A \text{ is simple}\}.$$

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3. If Γ contains a section Δ, then it embeds in $\mathbb{Z}\Delta$ as a convex translation subquiver.

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   - $\Delta \cong Q_A^{\mathrm{op}}$;
   - $\Gamma_A = \Gamma$, a convex subquiver of $\mathbb{Z}Q_A^{\mathrm{op}}$.
   - Given $M, N \in \Gamma_A$, all $M \rightsquigarrow N$ have the same length.

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