Auslander-Reiten Theory in Tri-eaxct Categories

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Motivation

There have been two parallel Auslander-Reiten theories

- in abelian categories and their extension-closed subcategories (by Aulander, Reiten, Bautista, Lenzing, Zuazua, etc);
- in triangulated categories and their extension-closed subcategories (by Happel, Reiten, Van den Bergh, Jørgensen, etc).

Objective

To unify these two theories under the setting of tri-exact categories without Hom-finiteness.

Preliminaries

1. R: a commutative ring.
2. I_R: a minimal injective co-generator for $\mathrm{Mod}\, R$.
3. $D = \mathrm{Hom}_R(-, I_R) : \mathrm{Mod}\, R \to \mathrm{Mod}\, R$ is exact.
4. An R-module M is *reflexive* if $\exists$ isomorphism

$$\sigma_M : M \to D^2 M : x \mapsto [f \mapsto f(x)].$$

5. An R-category $\mathcal{C}$ is called *Hom-reflexive* provided that

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \text{ is reflexive over } R, \text{ for all } X, Y \in \mathcal{C}.$$

Proposition

The category $\mathrm{RMod}\, R$ of reflexive R-modules

- *is abelian ;*
- *contains all R-modules of finite length ;*
- *admits a duality $D : \mathrm{RMod}\, R \to \mathrm{RMod}\, R$.*

Tri-exact structure

1. Let $\mathcal{C}$ be *tri-exact R-category*, that is an extension-closed subcategory of a triangulated R-category $\mathcal{A}$ with shift [1].
2. Given $X, Z \in \mathcal{C}$, we put
$$\mathrm{Ext}^1_{\mathcal{C}}(Z, X) := \mathrm{Hom}_{\mathcal{A}}(Z, X[1]).$$
3. An *extension* $\delta \in \mathrm{Ext}^1_{\mathcal{C}}(Z, X)$ defines an exact triangle
$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{\delta} X[1] \ \in \mathcal{A},$$
where $X \xrightarrow{u} Y \xrightarrow{v} Z$ is called a *tri-exact sequence* in $\mathcal{C}$.
4. The tri-exact sequences in $\mathcal{C}$ form a *tri-exact structure.*

Pull-back and push-out

Fix a morphism $f : X \to Y$ in $\mathcal{C}$.

Given $M \in \mathcal{C}$, we obtain

1. $\mathrm{Ext}^1_{\mathcal{C}}(f, M) : \mathrm{Ext}^1_{\mathcal{C}}(Y, M) \to \mathrm{Ext}^1_{\mathcal{C}}(X, M) : \delta \mapsto \delta \circ f.$

$$
\begin{array}{ccccccc}
M & \longrightarrow & N' & \longrightarrow & X & \xrightarrow{\mathrm{Ext}^1(f,M)(\delta)} & M[1] \\
\| & & \downarrow & & \downarrow{\scriptstyle f} & & \| \\
M & \longrightarrow & N & \xrightarrow{v} & Y & \xrightarrow{\delta} & M[1].
\end{array}
$$

2. $\mathrm{Ext}^1_{\mathcal{C}}(M, f) : \mathrm{Ext}^1_{\mathcal{C}}(M, X) \to \mathrm{Ext}^1_{\mathcal{C}}(M, Y) : \delta \mapsto f[1] \circ \delta.$

$$
\begin{array}{ccccccc}
X & \xrightarrow{u} & L & \longrightarrow & Z & \xrightarrow{\delta} & X[1] \\
{\scriptstyle f}\downarrow & & \downarrow & & \| & & \downarrow{\scriptstyle f[1]} \\
Y & \longrightarrow & L' & \longrightarrow & Z & \xrightarrow{\mathrm{Ext}^1(f,Z)(\delta)} & Y[1].
\end{array}
$$

Definition

A morphism $f : X \to Y$ in $\mathcal{C}$ is called

1. *projectively trivial* if $\operatorname{Ext}^1_{\mathcal{C}}(f, M) = 0$, for all $M \in \mathcal{C}$,

$$\begin{array}{ccccccc}
M & \longrightarrow & N' & \longrightarrow & X & \xrightarrow{0} & M[1] \\
\| & & \downarrow & & \downarrow{\scriptstyle f} & & \| \\
M & \longrightarrow & N & \xrightarrow{\forall\, v} & Y & \xrightarrow{\delta} & M[1].
\end{array}$$

2. *injectively trivial* if $\operatorname{Ext}^1_{\mathcal{C}}(M, f) = 0$, for all $M \in \mathcal{C}$,

$$\begin{array}{ccccccc}
X & \xrightarrow{\forall\, u} & L & \longrightarrow & Z & \xrightarrow{\delta} & X[1] \\
{\scriptstyle f}\downarrow & & \downarrow & & \| & & \downarrow{\scriptstyle f[1]} \\
Y & \longrightarrow & L' & \longrightarrow & Z & \xrightarrow{0} & Y[1].
\end{array}$$

Stable categories of tri-exact categories

Given $X, Y \in \mathcal{C}$, we put

1. $\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y)/\mathscr{P}(X, Y)$,
   where $\mathscr{P}(X, Y)$: projectively trivial morphisms.
2. $\overline{\mathrm{Hom}}_{\mathcal{C}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y)/\mathscr{I}(X, Y)$,
   where $\mathscr{I}(X, Y)$: injectively trivial morphisms.

Remark

- $\mathrm{Ext}^1_{\mathcal{C}}(X, Y)$ is a $\overline{\mathrm{End}}(Y)$-$\underline{\mathrm{End}}(X)$-bimodule.
- If $\mathcal{C}$ is a triangulated category, then

$$\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y) = \overline{\mathrm{Hom}}_{\mathcal{C}}(X, Y).$$

Almost split sequences in tri-exact categories

Definition

A tri-exact sequence $X \xrightarrow{u} Y \xrightarrow{v} Z$ defined by $\delta \in \mathrm{Ext}^1_{\mathcal{C}}(Z, X)$ is called *almost split sequence* if

- u is minimal left almost split in $\mathcal{C}$;
- v is minimal right almost split in $\mathcal{C}$.

or equivalently,

- $\delta \in \mathrm{Soc}\,(\mathrm{Ext}^1_{\mathcal{C}}(Z, X)_{\underline{\mathrm{End}}(Z)})$;
- $\delta \in \mathrm{Soc}\,({}_{\overline{\mathrm{End}}(X)}\mathrm{Ext}^1_{\mathcal{C}}(Z, X))$.

Unification of AR theory

$\mathscr{C}$: an extension-closed subcategory of abelian category $\mathfrak{A}$.

$\widehat{\mathscr{C}} := \mathrm{add}(X[0] \mid X \in \mathscr{C})$ in the derived category $D(\mathfrak{A})$.

Proposition

1. $\mathscr{C} \cong \widehat{\mathscr{C}}$, *an extension-closed subcategory of* $D(\mathfrak{A})$.
2. *The exact structure on* $\mathscr{C}$ *is equivalent to the tri-exact structure on* $\widehat{\mathscr{C}}$.
3. $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ *is almost split sequence in* $\mathscr{C}$ $\Leftrightarrow X[0] \longrightarrow Y[0] \longrightarrow Z[0]$ *is almost split sequence in* $\widehat{\mathscr{C}}$.
4. *Every almost split sequence in* $\widehat{\mathscr{C}}$ *is of the above form.*

Existence of almost split sequence

1. Given $X, Z \in \mathcal{C}$ with $\mathrm{End}(X)$ and $\mathrm{End}(Z)$ being local,
2. When does $\mathcal{C}$ have an almost split sequence

$$X \longrightarrow Y \longrightarrow Z$$

Existence of almost split sequence

1. Ring homomorphisms $\Gamma \to \overline{\mathrm{End}}(X)$ and $\Sigma \to \underline{\mathrm{End}}(Z)$.
2. ${}_\Gamma I$: injective co-generator of ${}_\Gamma\mathrm{End}(X)/\mathrm{rad}(\mathrm{End}(X))$.
3. I_Σ: injective co-generator of $\mathrm{End}(Z)/\mathrm{rad}(\mathrm{End}(Z))_\Sigma$.

Theorem

$\mathcal{C}$ has an almost split sequence $X \longrightarrow Y \longrightarrow Z$

$\Leftrightarrow$ $\mathrm{Ext}^1_{\mathcal{C}}(-, X)$ *is a subfunctor of* $\mathrm{Hom}_\Sigma(\underline{\mathrm{Hom}}_{\mathcal{C}}(Z, -), I_\Sigma)$;
$\mathrm{Soc}\,(\mathrm{Ext}^1_{\mathcal{C}}(Z, X)_{\underline{\mathrm{End}}(Z)}) \neq 0$.

$\Leftrightarrow$ $\mathrm{Ext}^1_{\mathcal{C}}(Z, -)$ *is a subfunctor of* $\mathrm{Hom}_\Gamma(\overline{\mathrm{Hom}}_{\mathcal{C}}(-, X), {}_\Gamma I)$;
$\mathrm{Soc}\,({}_{\overline{\mathrm{End}}(X)}\mathrm{Ext}^1_{\mathcal{C}}(Z, X)) \neq 0$.

Stable Hom-reflexive cases

Theorem

If $\overline{\mathrm{Hom}}_{\mathcal{C}}(M, X), \underline{\mathrm{Hom}}_{\mathcal{C}}(Z, M) \in \mathrm{RMod} R$, *for all* $M \in \mathcal{C}$, *then* $\mathcal{C}$ *has almost split sequence* $X \longrightarrow Y \longrightarrow Z$

$\Longleftrightarrow\ 0 \neq \mathrm{Ext}^1_{\mathcal{C}}(-, X) \cong D\underline{\mathrm{Hom}}_{\mathcal{C}}(Z, -).$

$\Longleftrightarrow\ 0 \neq \mathrm{Ext}^1_{\mathcal{C}}(Z, -) \cong D\overline{\mathrm{Hom}}_{\mathcal{C}}(-, X).$

Generalizing Happel's result

1. Let $\mathfrak{A}$ an abelian R-category.
2. We shall study the existence of almost split triangles
   - in the derived category $D(\mathfrak{A})$;
   - in the bounded derived category $D^b(\mathfrak{A})$.
3. $D^b(\mathfrak{A})$ is a triangulated subcategory of $D(\mathfrak{A})$.

Nakayama functor

Let $\mathcal{P}$ be a subcategory of projective objects of $\mathfrak{A}$.

Definition

A functor $\nu : \mathcal{P} \to \mathfrak{A}$ is called *Nakayama functor* if

$$\mathrm{Hom}_{\mathfrak{A}}(-, \nu P) \cong D\mathrm{Hom}_{\mathfrak{A}}(P, -), \text{ for all } P \in \mathcal{P}.$$

In this case,

1. νP is injective in $\mathfrak{A}$, for all $P \in \mathcal{P}$.
2. If $\mathcal{P}$ is Hom-reflexive, then ν co-restricts to an equivalence
$$\nu : \mathcal{P} \xrightarrow{\cong} \nu\mathcal{P},$$
$\nu\mathcal{P}$ is Hom-reflexive subcategory of injective objects of $\mathfrak{A}$.

Example

Lemma

Let A be any R-algebra.

$\mathrm{proj}A$: *finitely generated projective left A-modules.*

We obtain a Nakayama functor

$$\nu_A = D\mathrm{Hom}_A(-, A) : \mathrm{proj}A \to \mathrm{Mod}A.$$

1. Let $\nu : \mathcal{P} \to \mathfrak{A}$ Nakayama functor, with $\mathcal{P}$ Hom-reflexive.
2. The bounded homotopy categories $K^b(\mathcal{P})$ and $K^b(\nu\mathcal{P})$ are Hom-reflexive triangulated subcategories of $D(\mathfrak{A})$.

Proposition

1. $\exists$ *an induced triangle exact functor* $\nu : K^b(\mathcal{P}) \to D(\mathfrak{A})$ *such that, for all* $P^{\cdot} \in K^b(\mathcal{P})$,

$$\mathrm{Hom}_{D(\mathfrak{A})}(-, \nu P^{\cdot}) \cong D\mathrm{Hom}_{D(\mathfrak{A})}(P^{\cdot}, -).$$

2. $\exists$ *an induced triangle equivalence*

$$\nu : K^b(\mathcal{P}) \xrightarrow{\cong} K^b(\nu\mathcal{P}).$$

Theorem

Let $\nu : \mathcal{P} \to \mathfrak{A}$ *be Nakayama functor, where* $\mathcal{P}$ *Hom-reflexive.*

1. *If* $P^\cdot \in K^b(\mathcal{P})$ *with* $\mathrm{End}(P^\cdot)$ *local, then* $D^b(\mathfrak{A})$ *has an almost split triangle*

$$\nu P^\cdot[-1] \longrightarrow M^\cdot \longrightarrow P^\cdot \longrightarrow \nu P^\cdot,$$

   which is also almost split in $D(\mathfrak{A})$.

2. *If* $I^\cdot \in K^b(\nu\mathcal{P})$ *with* $\mathrm{End}(I^\cdot)$ *local, then* $D^b(\mathfrak{A})$ *has an almost split triangle*

$$I^\cdot \longrightarrow M^\cdot \longrightarrow \nu^- I^\cdot[1] \longrightarrow I^\cdot[1],$$

   which is also almost split in $D(\mathfrak{A})$.

Theorem

Let $\nu : \mathcal{P} \to \mathfrak{A}$ *be Nakayama functor, where* $\mathcal{P}$ *is Hom-reflexive.*

(1) *Consider* $M^{\cdot} \in D^b(\mathfrak{A})$ *such that* $\mathrm{End}(M^{\cdot})$ *is local.*

a) *If* $M^{\cdot}$ *has projective resolution over* $\mathcal{P}$*, then* $D^b(\mathfrak{A})$ *has almost split triangle* $X^{\cdot} \longrightarrow Y^{\cdot} \longrightarrow M^{\cdot} \longrightarrow X^{\cdot}[1] \iff M^{\cdot} \cong P^{\cdot} \in K^b(\mathcal{P})$*. If so, it is almost split in* $D(\mathfrak{A})$*.*

b) *If* $M^{\cdot}$ *has injective co-resolution over* $\nu\mathcal{P}$*, then* $D^b(\mathfrak{A})$ *has almost split triangle* $M^{\cdot} \longrightarrow Y^{\cdot} \longrightarrow Z^{\cdot} \longrightarrow M^{\cdot}[1] \iff M^{\cdot} \cong I^{\cdot} \in K^b(\nu\mathcal{P})$*. If so, it is almost split in* $D(\mathfrak{A})$*.*

(2) *If* $\mathfrak{A}$ *is Krull-Schmidt with enough projectives in* $\mathcal{P}$*, enough injectives in* $\nu\mathcal{P}$*, then* $D^b(\mathfrak{A})$ *has almost split triangles*

$\iff$ *every object in* $\mathfrak{A}$ *has*

- *a finite projective resolution over* $\mathcal{P}$*;*
- *a finite injective co-resolution over* $\nu\mathcal{P}$*.*

Application

1. Let A be a noetherian R-algebra, where R is complete, noetherian, and local.
2. Then $\mathrm{proj}A$ and $\mathrm{inj}A$ are Hom-reflexive.
3. $\mathrm{mod}^{+}A$: finitely generated A-modules
4. $\mathrm{mod}^{-}A$: finitely co-generated A-modules.

Theorem

(1) *If $M^{\boldsymbol{\cdot}} \in D^b(\mathrm{mod}^{+}A)$ is indecomposable, then $D^b(\mathrm{Mod}A)$ has almost split triangle $X^{\boldsymbol{\cdot}} \longrightarrow Y^{\boldsymbol{\cdot}} \longrightarrow M^{\boldsymbol{\cdot}} \longrightarrow X^{\boldsymbol{\cdot}}[1]$ $\Leftrightarrow$ $M^{\boldsymbol{\cdot}} \cong P^{\boldsymbol{\cdot}} \in K^b(\mathrm{proj}A)$; If so, $X^{\boldsymbol{\cdot}} \in D^b(\mathrm{mod}^{-}A)$.*

(2) *If $M^{\boldsymbol{\cdot}} \in D^b(\mathrm{mod}^{-}A)$ is indecomposable, then $D^b(\mathrm{Mod}A)$ has almost split triangle $M^{\boldsymbol{\cdot}} \longrightarrow Y^{\boldsymbol{\cdot}} \longrightarrow Z^{\boldsymbol{\cdot}} \longrightarrow M^{\boldsymbol{\cdot}}[1]$ $\Leftrightarrow$ $M^{\boldsymbol{\cdot}} \cong I^{\boldsymbol{\cdot}} \in K^b(\mathrm{inj}A)$. If so, $Z^{\boldsymbol{\cdot}} \in D^b(\mathrm{mod}^{+}A)$.*

Application

1. $\Lambda = kQ/(kQ^+)^2$, with k a field, Q a locally finite quiver.
2. Write $P_x = \Lambda e_x$ and $I_x = D(e_x\Lambda)$, for $x \in Q_0$.
3. Set $\mathrm{proj}\Lambda = \mathrm{add}(P_x \mid x \in Q_0)$, which is Hom-finite.
4. We have a Nakayama functor

$$\nu_{\Lambda} : \mathrm{proj}\Lambda \to \mathrm{Mod}\Lambda : P_x \mapsto I_x.$$

Theorem

(1) *Every almost split triangle in $D^b(\mathrm{mod}\Lambda)$ is an almost split triangle in $D(\mathrm{Mod}\Lambda)$.*

(2) *$D^b(\mathrm{mod}\Lambda)$ has (left, right) almost split triangles $\Longleftrightarrow$ Q has no (left, right) infinite path.*