Almost split sequences and approximations

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2. If $\mathcal{A}$ has AR-sequences, when so does an exact subcategory of $\mathcal{A}$?

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$\mathcal{A}$ is *Krull-Schmidt* if every nonzero object is Krull-Schmidt.

History: Existence of AR-sequences

If Λ is artin algebra, then the existence of AR-sequences in $\mathrm{mod}\Lambda$ follows from AR-duality

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   $\mathcal{T}$ has nonzero minimal right $\mathcal{C}$-approximation $f : M \to X$.

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Let $\delta : 0 \longrightarrow X \xrightarrow{u} Y \xrightarrow{v} Z \longrightarrow 0$ be s.e.s. in $\mathcal{A}$.

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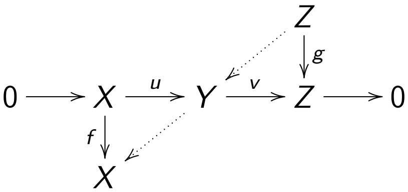

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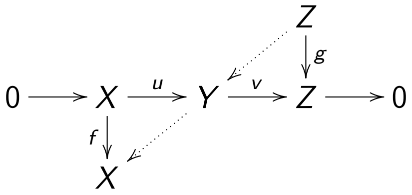

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That is, $\delta \in \mathrm{Soc}_{\mathrm{End}(Z)}\mathrm{Ext}^1(Z, X)$.

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Then there are non-degenerate R-bilinear forms

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Consequently, there are natural R-monomorphisms

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Non-degenerate bilinear form derived from AR-sequence

Proposition (Gabriel-Roiter)

Let $\delta: \ 0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ *be AR-sequence in* $\mathcal{A}$.
Let $\varphi : \mathrm{Ext}^1(Z, X) \to I$ *be R-linear with* $\varphi(\delta) \neq 0$.
Then there are non-degenerate R-bilinear forms

$$< \, , \, >_\varphi : \ \overline{\mathrm{Hom}}(L, X) \times \mathrm{Ext}^1(Z, L) \to I : (\bar{f}, \eta) \mapsto \varphi(f\eta),$$

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Remark

Let Q strongly locally-finite. If $Z \in \mathrm{rep}^+(Q)$ indec non-proj, then $\mathrm{Hom}(Z, L)$, $\mathrm{Hom}(L, \mathrm{DTr}\,Z) \in \mathrm{mod}R$, $\forall\, L \in \mathrm{rep}(Q)$,

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3. *If η or ζ exits, what is the relation between δ and η or ζ?*

Projectively or injectively stable approximations

Definition

A morphism $f : C \to X$ in $\mathcal{A}$ with $C \in \mathcal{C}$ is called

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Let $Z \in \mathcal{C}$ with AR-sequence $\delta: \; 0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ in $\mathcal{A}$. If X has minimal right injectively stable $\mathcal{C}$-approximation $f : M \to X$ with M being nonzero Krull-Schmidt, then $\mathcal{A}$ has pushout diagram

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where η *is AR-sequence in* $\mathcal{C}$*.*

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Corollary

Let $\mathcal{A}$ have AR-sequences. If $\mathcal{C}$ is Krull-Schmidt and functorially-finite in $\mathcal{A}$, then $\mathcal{C}$ has AR-sequences.

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Existence of AR-sequences in subcategories

Theorem

Let R be artinian, $\mathcal{C}$ be stably Hom-finite and Krull-Schmidt.

1. *If $\mathcal{A}$ has right AR-sequences, then $\mathcal{C}$ has right AR-sequences $\Leftrightarrow \forall Z \in \mathrm{ind}\mathcal{C}$ not Ext-projective, $\tau_{\mathcal{A}} Z$ has right injectively stable $\mathcal{C}$-approximation.*
2. *If $\mathcal{A}$ has left AR-sequences, then $\mathcal{C}$ has left AR-sequences $\Leftrightarrow \forall X \in \mathrm{ind}\mathcal{C}$ not Ext-projective, $\tau_{\mathcal{A}}^{-} X$ has left projectively stable $\mathcal{C}$-approximation.*

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1. $\mathrm{Hom}(T, F) = 0$, for $T \in \mathcal{T}$ and $F \in \mathcal{F}$.
2. For any $X \in \mathcal{A}$, $\exists$ short exact sequence

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Remark

$(\mathcal{T}, \mathcal{F})$ torsion theory $\Rightarrow \mathcal{T}$ and $\mathcal{F}$ exact subcategories of $\mathcal{A}$.

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Corollary

Let $\mathcal{A}$ have torsion theory $(\mathcal{T}, \mathcal{F})$.

(1) If $\mathcal{A}$ has right AR-sequences, then so does $\mathcal{T}$.

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Corollary

Let $\mathcal{A}$ have torsion theory $(\mathcal{T}, \mathcal{F})$.

(1) If $\mathcal{A}$ has right AR-sequences, then so does $\mathcal{T}$.

(2) If $\mathcal{A}$ has left AR-sequences, then so does $\mathcal{F}$.

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Theorem (BLP, 2011)

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2. If $M \in \mathrm{rep}^-(Q)$ not injective, then $\mathrm{rep}^-(Q)$ has AR-sequence $0 \to M \to N \to L \to 0 \Leftrightarrow \mathrm{TrD}\, M$ fin dim; and in this case, $L \cong \mathrm{TrD}\, M$.