Module categories of small radical nipotency

Shiping Liu*, Université de Sherbrooke
Youqi Yin, Shaoxing University

Advances in Representation Theory of Algebras IX

Kingston, Ontario

June 12 - 16, 2023

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Observation

$n_{\mathrm{rad}(\mathrm{mod}A)} = 1 \Longleftrightarrow A$ is simple.

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If b is the maximal length of modules in $\mathrm{ind}A$, then $n_{\mathrm{rad}(\mathrm{mod}A)} \leq 2^b - 1$.

- This estimate depends on a prior knowledge of all indecomposable modules.
- In 2013, Chaio-Liu gave another approach, which seems more efficient and precise.

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Observation

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2. In this case, $n_{\mathrm{rad}(\mathrm{mod}A)} = \mathrm{dp}(\mathrm{mod}A) + 1$.

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Theorem

$n_{\mathrm{rad}(\mathrm{mod} A)} = \ell\ell(A) \iff$ *A is a hereditary algebra of type $\vec{\mathbb{A}}_n$.*

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We shall find all algebras A with $n_{\mathrm{rad}(\mathrm{mod} A)} \leq 4$.

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If $\operatorname{rad}^4(\operatorname{mod} A) = 0$*, then the middle term of any AR-sequence in* $\operatorname{mod} A$ *has at most two indecomposable direct summands. Being representation-finite,* A *is string algebra (by Auslander).*

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Remark

$P \in \mathrm{ind} A$ is wedged projective $\Longleftrightarrow$ $DP \in \mathrm{ind} A^{\mathrm{op}}$ is co-wedged injective.

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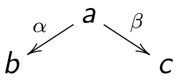

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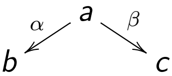

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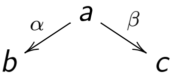

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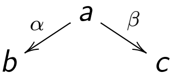

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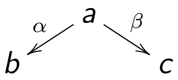

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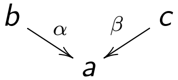

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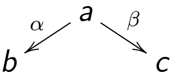

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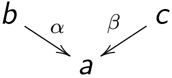

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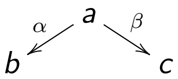

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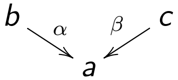

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Wedged string algebras

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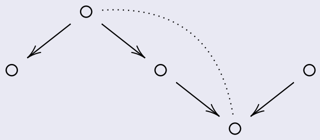

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4. A wedged projective module and a co-wedged injective module have no common composition factor.

Theorem

If A is an artin algebra, then $\mathrm{rad}^4(\mathrm{mod} A) = 0 \Longleftrightarrow$

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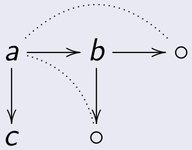

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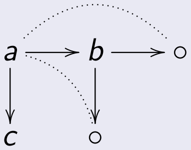

1. $\mathrm{rad}P_a = S_b \oplus S_c$ with $\ell(P_{S_b}) + \ell(I_{S_b}) = 5$.

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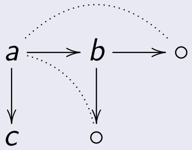

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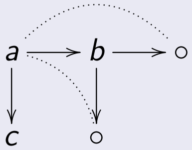

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   - *non-hereditary non-Nakayama tri-string algebras.*

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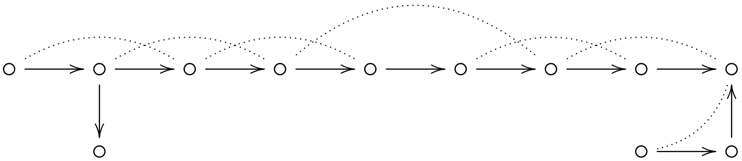

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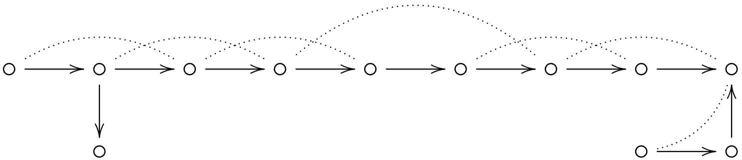

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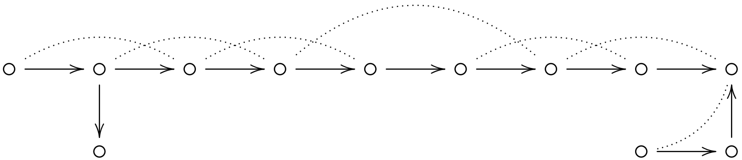

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The representation theory of tri-string algebras

Almost split sequences

Theorem

Let A be tri-string algebra with $M \in \mathrm{ind} A$.

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where $N = I_S$ in case $\ell(I_S) = 2$, and $N = \mathrm{rad} I_S$ in case $\ell(I_S) = 3$.