Almost split sequences in tri-eaxct categories

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Advance in Representation Theory of Algebras

In memory of

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Motivation

Under the Hom-finite setting, many researchers have studied the existence of

- almost split sequences in abelian categories and exact categories (Aulander, Reiten, Bautista, Lenzing, Zuazua, etc);
- almost split triangles in triangulated categories and their extension-closed subcategories (Happel, Reiten, Van den Bergh, Jørgensen, etc).

Auslander's Theorem

Λ: any ring.

$Z \in \mathrm{Mod}\Lambda$ finitely presented, non-projective, $\mathrm{End}(Z)$ local.

$\Sigma := \mathrm{End}(\mathrm{Tr}Z)^{\mathrm{op}}$.

I: the injective envelope of $\mathrm{End}(Z)/\mathrm{rad}(\mathrm{End}(Z))_{\Sigma}$.

$X := \mathrm{Hom}_{\Sigma}(\mathrm{Tr}Z, I) \in \mathrm{Mod}\Lambda$.

Theorem (Auslander)

1. $\mathrm{Ext}^1_{\Lambda}(-, X) \cong \mathrm{Hom}_{\Sigma}(\underline{\mathrm{Hom}}_{\Lambda}(Z, -), I)$.
2. $\mathrm{Mod}\Lambda$ has an almost split sequence

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0.$$

Krause's Theorem

$\mathcal{A}$: triangulated category, generated by the compact objects.

$Z \in \mathcal{A}$ compact with $\Sigma = \mathrm{End}(Z)$ being local.

I: the injective envelope of $\mathrm{End}(Z)/\mathrm{rad}(\mathrm{End}(Z))_\Sigma$.

Theorem (Krause)

1. (Brown) There exists $X \in \mathcal{A}$ such that
$$\mathrm{Hom}_{\mathcal{A}}(-, X) \cong \mathrm{Hom}_{\Sigma}(\mathrm{Hom}_{\mathcal{A}}(Z, -), I).$$
2. $\mathcal{A}$ has an almost split triangle
$$X[-1] \longrightarrow Y \longrightarrow Z \longrightarrow X.$$

Objective

To unify various existence theorems of

- almost split sequences in abelian categories
- almost split triangles in triangulated categories

under setting of tri-exact categories without Hom-finiteness.

Application

Existence of almost split triangles in $D(\mathfrak{A})$ and $D^b(\mathfrak{A})$, where $\mathfrak{A}$ is abelian category without Hom-finiteness.

Tri-exact categories

1. Let $\mathcal{C}$ be a tri-exact category, that is an extension-closed subcategory of a triangulated category $\mathcal{A}$ with shift [1].
2. Given $X, Y \in \mathcal{C}$, we put

$$\mathrm{Ext}^1_{\mathcal{C}}(X, Y) := \mathrm{Hom}_{\mathcal{A}}(X, Y[1]).$$

Definition

A morphism $f : X \to Y$ in $\mathcal{C}$ is called

- *projectively trivial* if, for all $M \in \mathcal{C}$,

$$\mathrm{Ext}^1_{\mathcal{C}}(f, M) : \mathrm{Ext}^1_{\mathcal{C}}(Y, M) \to \mathrm{Ext}^1_{\mathcal{C}}(X, M) : \delta \mapsto \delta \circ f = 0;$$

- *injectively trivial* if, for all $M \in \mathcal{C}$,

$$\mathrm{Ext}^1_{\mathcal{C}}(M, f) : \mathrm{Ext}^1_{\mathcal{C}}(M, X) \to \mathrm{Ext}^1_{\mathcal{C}}(M, Y) : \delta \mapsto f[1] \circ \delta = 0.$$

Stable categories

Given $X, Y \in \mathcal{C}$, we put

1. $\overline{\mathrm{Hom}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y)/\mathscr{I}(X, Y)$,
   where $\mathscr{I}(X, Y)$: injectively trivial morphisms.

2. $\underline{\mathrm{Hom}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y)/\mathscr{P}(X, Y)$,
   where $\mathscr{P}(X, Y)$: projectively trivial morphisms.

Remark

- $\mathrm{Ext}^1_{\mathcal{C}}(X, Y)$ is a $\overline{\mathrm{End}}(Y)$-$\underline{\mathrm{End}}(X)$-bimodule.
- If $\mathcal{C}$ is a triangulated category, then

$$\underline{\mathrm{Hom}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y) = \overline{\mathrm{Hom}}(X, Y).$$

Tri-exact structure and almost split sequences

An extension $\delta \in \mathrm{Ext}^1_{\mathcal{C}}(Z, X[1])$ defines exact triangle in $\mathcal{A}$

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{\delta} X[1].$$

Call $X \xrightarrow{u} Y \xrightarrow{v} Z$ *tri-exact sequence* in $\mathcal{C}$ defined by δ.

Remark

The tri-exact sequences in $\mathcal{C}$ yields a tri-exact structure.

Definition

A tri-exact sequence $X \xrightarrow{u} Y \xrightarrow{v} Z$ in $\mathcal{C}$ is *almost split* if

1. u is minimal left almost split ;
2. v is minimal right almost split.

$\mathscr{C}$: an extension-closed subcategory of abelian category $\mathfrak{A}$.

$\mathscr{C}[0]$: the additive subcategory of $D(\mathfrak{A})$ generated by the complexes isomorphic to $X[0]$ with $X \in \mathscr{C}$.

Proposition

1. *$\mathscr{C}[0]$ is an extension-closed subcategory of $D(\mathfrak{A})$.*
2. $\mathrm{Ext}^1_{\mathscr{C}}(X, Y) \cong \mathrm{Ext}^1_{\mathscr{C}[0]}(X[0], Y[0])$, *for all $X, Y \in \mathscr{C}$.*
3. $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ *is almost split sequence in $\mathscr{C}$*

   $\Leftrightarrow X[0] \longrightarrow Y[0] \longrightarrow Z[0]$ *is almost split sequence in $\mathscr{C}[0]$.*
4. *Every almost split sequence in $\mathscr{C}[0]$ is of the above form.*

1. Let $X, Z \in \mathcal{C}$ with $\mathrm{End}(X)$ and $\mathrm{End}(Z)$ being local.
2. Ring homomorphisms $\Gamma \to \overline{\mathrm{End}}(X)$ and $\Sigma \to \underline{\mathrm{End}}(Z)$.
3. ${}_{\Gamma}I$: injective co-generator of ${}_{\Gamma}\mathrm{End}(X)/\mathrm{rad}(\mathrm{End}(X))$.
4. I_{Σ}: injective co-generator of $\mathrm{End}(Z)/\mathrm{rad}(\mathrm{End}(Z))_{\Sigma}$.

Theorem

$\mathcal{C}$ has an almost split sequence $X \longrightarrow Y \longrightarrow Z$

$\Leftrightarrow$ $\mathrm{Ext}^1_{\mathcal{C}}(-, X)$ *is subfunctor of* $\mathrm{Hom}_{\Sigma}(\underline{\mathrm{Hom}}_{\mathcal{C}}(Z, -), I_{\Sigma})$;
$\mathrm{Soc}\,(\mathrm{Ext}^1_{\mathcal{C}}(Z, X)_{\underline{\mathrm{End}}(Z)}) \neq 0$.

$\Leftrightarrow$ $\mathrm{Ext}^1_{\mathcal{C}}(Z, -)$ *is subfunctor of* $\mathrm{Hom}_{\Gamma}(\overline{\mathrm{Hom}}_{\mathcal{C}}(-, X), {}_{\Gamma}I)$;
$\mathrm{Soc}\,({}_{\overline{\mathrm{End}}(X)}\mathrm{Ext}^1_{\mathcal{C}}(Z, X)) \neq 0$.

If $\mathcal{C}$ is tri-exact R-category, where R commutative ring, we may choose $\Gamma = \Sigma = R$.

Application to almost split triangles in derived categories of abelian categories

Happel's Result

Λ: a finite dimensional algebra over a field k.
$\mathrm{mod}\Lambda$: the category of finite dimensional Λ-modules.

Theorem

(1) *If* $M^{\cdot} \in D^b(\mathrm{mod}\Lambda)$ *is indecomposable, then*

- $D^b(\mathrm{mod}\Lambda)$ *has an almost split triangle in*

$$X^{\cdot} \longrightarrow Y^{\cdot} \longrightarrow M^{\cdot} \longrightarrow X^{\cdot}[1]$$

$\Longleftrightarrow M^{\cdot} \cong P^{\cdot}$, *a bounded complex of projective modules.*

- $D^b(\mathrm{mod}\Lambda)$ *has an almost split triangle*

$$M^{\cdot} \longrightarrow Y^{\cdot} \longrightarrow Z^{\cdot} \longrightarrow M^{\cdot}[1]$$

$\Longleftrightarrow M^{\cdot} \cong I^{\cdot}$, *a bounded complex of injective modules.*

(2) $D^b(\mathrm{mod}\Lambda)$ *has almost split triangles* $\Longleftrightarrow \mathrm{gdim}(\Lambda) < \infty$.

Reflexive modules

1. R: a commutative ring.
2. I_R: a minimal injective co-generator for $\mathrm{Mod}\,R$.
3. $D = \mathrm{Hom}_R(-, I_R) : \mathrm{Mod}\,R \to \mathrm{Mod}\,R$ is exact.
4. An R-module M is *reflexive* if $\exists$ isomorphism

$$\sigma_M : M \to D^2M : x \mapsto [f \mapsto f(x)].$$

Proposition

The category $\mathrm{RMod}R$ of reflexive R-modules

- *is abelian;*
- *contains all R-modules of finite length;*
- *admits duality $D : \mathrm{RMod}R \to \mathrm{RMod}R$.*

5. An R-category $\mathcal{A}$ is called *Hom-reflexive* if

$$\mathrm{Hom}_{\mathcal{A}}(X, Y) \in \mathrm{RMod}R, \text{ for all } X, Y \in \mathcal{A}.$$

Projective resolutions

$\mathfrak{A}$: an abelian R-category.

$\mathcal{P}$: a subcategory of projective objects of $\mathfrak{A}$.

$\mathcal{I}$: a subcategory of injective objects of $\mathfrak{A}$.

Definition

Let $X^\bullet$ be a complex over $\mathfrak{A}$.

1. A *projective resolution over* $\mathcal{P}$ of $X^\bullet$ is quasi-isomorphism
$$p^\bullet : P^\bullet \to X^\bullet$$
where $P^\bullet$ is a bounded-above complex over $\mathcal{P}$.
2. An *injective co-resolution* over $\mathcal{I}$ of $X^\bullet$ is quasi-iso
$$q^\bullet : X^\bullet \to I^\bullet$$
where $I^\bullet$ is a bounded-below complex over $\mathcal{I}$.

Necessity for the existence of an almost split sequence

$\mathfrak{A}$: an abelian R-category.

$\mathcal{P}$: a subcategory of projective objects of $\mathfrak{A}$.

$\mathcal{I}$: a subcategory of injective objects of $\mathfrak{A}$.

Theorem

Consider an almost split triangle

$$X^{\cdot} \longrightarrow Y^{\cdot} \longrightarrow Z^{\cdot} \longrightarrow X^{\cdot}[1]$$

in $D^*(\mathfrak{A})$ *with* $* \in \{\emptyset, b\}$.

1. *If* $Z^{\cdot}$ *has a projective resolution over* $\mathcal{P}$, *then* $Z^{\cdot} \cong P^{\cdot}$, *a bounded complex over* $\mathcal{P}$.
2. *If* $X^{\cdot}$ *has an injective co-resolution over* $\mathcal{I}$, *then* $X^{\cdot} \cong I^{\cdot}$, *a bounded complex over* $\mathcal{I}$.

Application

Let A be any R-algebra.

Theorem

If $D^b(\mathrm{Mod}\, A)$ has an almost split triangle

$$X^{\cdot} \longrightarrow Y^{\cdot} \longrightarrow Z^{\cdot} \longrightarrow X^{\cdot}[1],$$

then

1. $Z^{\cdot} \cong P^{\cdot}$, *a bounded complex of projective A-modules.*
2. $X^{\cdot} \cong I^{\cdot}$, *a bounded complex of injective A-modules.*

We shall study the sufficiency for the existence

Nakayama functor

$\mathfrak{A}$: an abelian R-category.

$\mathcal{P}$: a subcategory of projective objects of $\mathfrak{A}$.

Definition

A functor $\nu : \mathcal{P} \to \mathfrak{A}$ is called *Nakayama functor* if

$$\mathrm{Hom}_{\mathfrak{A}}(-, \nu P) \cong D\mathrm{Hom}_{\mathfrak{A}}(P, -), \text{ for all } P \in \mathcal{P}.$$

In this case,

1. νP is injective in $\mathfrak{A}$, for all $P \in \mathcal{P}$.
2. $K^b(\mathcal{P})$ and $K^b(\nu\mathcal{P})$: triangulated subcategories of $D(\mathfrak{A})$.

Examples

Lemma

Given any R-algebra A, we obtain a Nakayama functor

$$\nu_A = D\mathrm{Hom}_A(-, A) : \mathrm{proj}A \to \mathrm{Mod}A,$$

1. $\mathrm{proj}A$: *category of finitely generated projective A-modules.*
2. $\mathrm{inj}A := \nu(\mathrm{proj}A)$, *a category of injective A-modules.*

Many more abelain categories with Nakayama functor

- $\mathrm{Mod}A$, where $A = kQ/I$ is locally finite dimensional.

Proposition

Let $\nu : \mathcal{P} \to \mathfrak{A}$ be a Nakayama functor.

1. *It induces a triangle functor $\nu : K^b(\mathcal{P}) \to D(\mathfrak{A})$.*
2. *For any $P^\bullet \in K^b(\mathcal{P})$, we have*
$$\mathrm{Hom}_{D(\mathfrak{A})}(-, \nu P^\bullet) \cong D\mathrm{Hom}_{D(\mathfrak{A})}(P^\bullet, -).$$
3. *If $\mathcal{P}$ Hom-reflexive over R, then ν co-restricts to an equiv*
$$\nu : \mathcal{P} \xrightarrow{\cong} \nu\mathcal{P},$$
which induces an equivalence
$$\nu : K^b(\mathcal{P}) \xrightarrow{\cong} K^b(\nu\mathcal{P}).$$

Theorem

Let $\nu : \mathcal{P} \to \mathfrak{A}$ *be a Nakayama functor.*

If $P^{\cdot} \in K^b(\mathcal{P})$ *with* $\mathrm{End}(P^{\cdot})$ *and* $\mathrm{End}(\nu P^{\cdot})$ *local, then* $D^b(\mathfrak{A})$ *has an almost split triangle*

$$\nu P^{\cdot}[-1] \longrightarrow M^{\cdot} \longrightarrow P^{\cdot} \longrightarrow \nu P^{\cdot},$$

which is also an almost split triangle in $D(\mathfrak{A})$*.*

Remark

With $\nu_A : \mathrm{proj}A \to \mathrm{Mod}A$, where A is R-algebra,

the above result applies to $D^b(\mathrm{Mod}A)$ and $D(\mathrm{Mod}A)$.

Let $\nu : \mathcal{P} \to \mathfrak{A}$ be Nakayama functor,

- $\mathcal{P}$ is Hom-reflexive over R;
- $\mathfrak{A}$ has enough projectives in $\mathcal{P}$; enough injectives in $\nu\mathcal{P}$.

Theorem

(1) *If* $M^\cdot \in D^b(\mathfrak{A})$ *such that* $\mathrm{End}(M^\cdot)$ *is local, then*

- $D^b(\mathfrak{A})$ *has almost split triangle* $X^\cdot \longrightarrow Y^\cdot \longrightarrow M^\cdot \longrightarrow X^\cdot[1]$ $\Longleftrightarrow M^\cdot \cong P^\cdot \in K^b(\mathcal{P})$; *in this case,* $X^\cdot \cong \nu P^\cdot$.
- $D^b(\mathfrak{A})$ *has almost split triangle* $M^\cdot \longrightarrow Y^\cdot \longrightarrow Z^\cdot \longrightarrow M^\cdot[1]$ $\Longleftrightarrow M^\cdot \cong I^\cdot \in K^b(\nu\mathcal{P})$; *in this case,* $Z^\cdot \cong \nu^- I^\cdot$.

(2) *If* $\mathfrak{A}$ *Krull-Schmidt, then* $D^b(\mathfrak{A})$ *has almost split triangles* $\Longleftrightarrow$ *every object in* $\mathfrak{A}$ *has*

- *a finite projective resolution over* $\mathcal{P}$;
- *a finite injective co-resolution over* $\nu\mathcal{P}$.

Let A be R-algebra, ${}_AA$ be R-noetherian and R-reflexive.

Then $\mathrm{proj}A$ and $\mathrm{inj}A$ are Hom-reflexive over R.

mod^+A: category of modules finitely generated by $\mathrm{proj}A$.

mod^-A: category of modules finitely co-generated by $\mathrm{inj}A$.

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