Triangulated categories with an infinite cluster structure

Shiping Liu (Université de Sherbrooke)

Homological Methods, Representation Theory and Cluster Algebras

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3. This gives some hope to construct non 2-CY cluster categories of infinite rank.

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4. All subcategories of $\mathcal{A}$ are strictly additive.

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2. For $M \in \mathrm{ind}\mathcal{T}$, define

$$\mathcal{T}_M := \mathrm{add}\{N \in \mathrm{ind}\mathcal{T} \mid N \ncong M\}.$$

Cluster structure

Definition (Buan, Iyama, Reiten, Scott)

A collection $(\emptyset \neq)\ \mathfrak{C}$ of subcategories of $\mathcal{A}$ is called a *cluster structure* if, for $\mathcal{T} \in \mathfrak{C}$ and $M \in \mathrm{ind}\mathcal{T}$,

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   where f, u minimal left $\mathcal{T}_M$-approximations;

   g, v minimal right $\mathcal{T}_M$-approximations;
3. $Q_{\mathcal{T}}$ has no cycle of length one or two;
4. $Q_{\mu_{_M}(\mathcal{T})}$ is obtained from $Q_{\mathcal{T}}$ by a mutation at M.

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Theorem (Koenig, Zhu)

If $\mathcal{T}$ is cluster tilting subcategory of $\mathcal{A}$, then

$$\mathrm{mod}\mathcal{T} \cong \mathcal{A}/\mathcal{T}[1]$$

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Canonical orbit categories

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Theorem(Keller)

$\mathscr{C}(\mathcal{H})$ is 2-CY triangulated with exact projection

$$p : D^b(\mathcal{H}) \to \mathscr{C}(Q).$$

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Remark

Holm and Jørgensen constructed a cluster category of type $\mathbb{A}_\infty$ from dg-modules over a polynomial ring.

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3. If Q no infinite paths, then $\mathrm{rep}^+(Q) = \mathrm{rep}(Q)$ has AR-sequences.

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2. $\mathrm{add}\{P_x \mid x \in Q_0\}$ in $\mathscr{C}(Q)$ is cluster tilting.

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Conjecture

The category $\mathscr{C}(Q)$ is 2-CY cluster category.

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   - $r = 0$, if Q of type $\mathbb{A}_\infty$;
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   - $r = 0$, if Q of type $\mathbb{A}_\infty$;
   - $r = 1$, if Q of type $\mathbb{D}_\infty$;
   - $r = 2$, if Q of type $\mathbb{A}_\infty^\infty$; and in this case, the two regular components are orthogonal.

2-CY cluster category of type $\mathbb{A}_\infty$ or $\mathbb{A}_\infty^\infty$

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Proof. Let $\mathcal{T}$ cluster tilting subcategory of $\mathscr{C}(Q)$.

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Objective

Study the bounded derived category $D^b(\mathrm{mod}\Lambda)$.

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Definition

1. Given walk $w = \alpha_r^{d_r} \cdots \alpha_1^{d_1}$, with $d_i = \pm 1$, $\alpha_i \in Q_1$, its *degree* is

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   - $r_Q = \min\{\partial(w) \mid w \text{ closed walks of positive degree }\}$.

Grading for a gradable quiver

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Theorem (Bautista, Liu)

If Q is gradable, then $\hat{\mathcal{F}}$ induces equivalence :

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- *admissible* if it is free and locally bounded.

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3. if $X, Y \in \mathcal{A}$, then
$$\operatorname{Hom}_{\mathcal{B}}(\pi(X), \pi(Y)) \cong \oplus_{g \in G} \operatorname{Hom}_{\mathcal{A}}(X, g \cdot Y)$$

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Proposition

$\exists$ *Galois G-covering* $\pi : \mathcal{A} \to \mathcal{B} \Leftrightarrow \mathcal{B} \cong \mathcal{A}/G$.

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2. The projection functor

$$p : D^b(\mathcal{H}) \to \mathscr{C}(H)$$

is a Galois G-covering, where $G =< F >$.

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     $$\alpha : x \to y \in Q_1.$$

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Theorem (Bautista, Liu)

The covering $\pi : \tilde{\Lambda} \to \Lambda$ induces Galois G-covering

$$\pi_\lambda^D : D^b(\mathrm{mod}\tilde{\Lambda}) \to D^b(\mathrm{mod}\Lambda).$$

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Description of $D^b(\mathrm{mod}\Lambda)$

Theorem (Bautista, Liu)

1. $X^{\cdot} \in D^b(\mathrm{mod}\Lambda)$ has unique decomposition

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- *two orthogonal regular component of shape $\mathbb{Z}\mathbb{A}_\infty$.*