The bounded derived category of an algebra with radical squared zero

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1. the indecomposable complexes;
2. the AR-components with arbitray $\mathrm{gdim}(A)$.

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1. Galois covering for linear categories;
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3. Representation theory of infinite quivers.

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Definition (Asashiba)

Let $F : \mathcal{A} \to \mathcal{B}$ be a functor of linear categories.

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$$\begin{array}{ccc} F(g \cdot (h \cdot X)) & \xrightarrow{\delta_{g,\, h \cdot X}} & F(h \cdot X) \\ \| & & \downarrow{\scriptstyle \delta_{h,\, X}} \\ F((gh) \cdot X) & \xrightarrow{\delta_{gh,\, X}} & F(X) \end{array}$$

commutes, for $g, h \in G$ and $X \in \mathcal{A}$.

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3. $X, Y \in \mathrm{ind}\,\mathcal{A}$ with $F(X) \cong F(Y) \Rightarrow Y = g \cdot X$, $g \in G$.

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3. *$\Gamma \cong F(\Gamma)$ if $G_{\Gamma} = \{g \in G \mid g(\Gamma) = \Gamma\}$ trivial.*

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2. *The components of $Q^{\mathbb{Z}}$ are pairwise isomorphic, indexed by elements of $\mathbb{Z}_{r(Q)}$.*

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which is an isomorphism $\Leftrightarrow r(Q) = 0$.

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Theorem

π_λ induces commutative diagram

$$\begin{array}{ccc} K^{-,b}(\mathrm{proj}\tilde{A}) & \xrightarrow{\sim} & D^b(\mathrm{mod}\tilde{A}) \\ \pi_\lambda^K \downarrow & & \downarrow \pi_\lambda^D \\ K^{-,b}(\mathrm{proj}A) & \xrightarrow{\sim} & D^b(\mathrm{mod}A), \end{array}$$

where $\pi_\lambda^K, \pi_\lambda^D$ are Galois G-covering.

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5. $G =< \sigma >$ acts on $\mathrm{rep}^-(\tilde{Q}^{\rm op})$.

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2. *If* $X \in \mathrm{ind}(RC^{-,b}(\mathrm{proj}\tilde{A}))$, *then* $X \cong \mathcal{K}(M)[i]$, *for some unique* $i \in \mathbb{Z}$, $M \in \mathrm{rep}^-(\tilde{Q}^{\mathrm{op}})$.

Indecomposable complexes in $D^b(\mathrm{mod} A)$

Let $\mathcal{F}$ be the following composite:

$$\begin{array}{ccc} \mathrm{rep}^-(\tilde{Q}^{\mathrm{op}}) & \xrightarrow{\mathcal{K}} & K^{-,b}(\mathrm{proj}\tilde{A}) & & \\ & & \downarrow \pi_\lambda^K & & \\ & & K^{-,b}(\mathrm{proj}A) & \xrightarrow{\sim} & D^b(\mathrm{mod}A). \end{array}$$

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1. *The non-iso indecomposables in* $D^b(\mathrm{mod} A)$ *are*
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2. $\mathrm{Hom}(\mathcal{F}(M)[i], \mathcal{F}(N)[j]) \neq 0 \Rightarrow j = i, i+1.$

Koszul Equivalence

Theorem

The exact functor $\mathcal{K} : \mathrm{rep}^{-}(\tilde{Q}^{\mathrm{op}}) \to RC^{-,b}(\mathrm{proj}\tilde{A})$ *induces a triangle equivalence*

$$\mathscr{K} : D^b(\mathrm{rep}^{-}(\tilde{Q}^{\mathrm{op}})) \to D^b(\mathrm{mod}\tilde{A}).$$

Last functor

Let $\mathscr{F}$ be the following composite:

$$D^b(\mathrm{rep}^-(\tilde{Q}^{\mathrm{op}})) \xrightarrow[\sim]{\mathscr{K}} D^b(\mathrm{mod}\tilde{A}) \xrightarrow{\pi_\lambda^D} D^b(\mathrm{mod}A).$$

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2. *If $r(Q) = 0$, then $\mathscr{F}$ is a triangle-equivalence.*

AR-components of $D^b(\mathrm{mod} A)$

Ω : the regular AR-cpts of $\mathrm{rep}^-(\tilde{Q}^{\mathrm{op}})$ and the connecting AR-cpt of $D^b(\mathrm{rep}^-(\tilde{Q}^{\mathrm{op}}))$.

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AR-components of $D^b(\mathrm{mod} A)$

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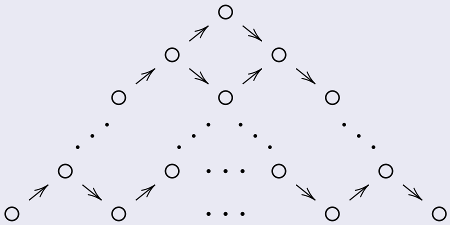

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