Representation theory of strongly locally finite quivers

Raymundo Bautista (UNAM)
Shiping Liu (Sherbrooke)*
Charles Paquette (New Brunswick)

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4. representation theory of co-algebras [S].

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There exists a Nakayama equivalence

$$\nu : \ \mathrm{proj}(Q) \ \rightarrow \ \mathrm{inj}(Q) : \ P_x \ \mapsto \ I_x.$$

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Remark. $I_x \in \mathrm{rep}^+(Q) \Leftrightarrow I_x \in \mathrm{rep}^b(Q)$.

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AR-sequences in $\mathrm{rep}^+(Q)$

Theorem

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Corollary

Every AR-sequence in $\mathrm{rep}^+(Q)$ starts with a finite dimensional representation.

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3. Finally, $\Gamma_{\mathrm{rep}^+(Q)}$ is partially valued with all valuations non-symmetric.

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Proposition

If Δ is section of Γ, then there exists embedding

$$\Gamma \to \mathbb{Z}\Delta : \tau^n x \mapsto (-n, x).$$

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Remark

$X \in \Gamma_{\mathrm{rep}^+(Q)}$ regular $\not\Rightarrow$ $\tau^n X$ defined for all $n \in \mathbb{Z}$.

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If one representation in Γ is preprojective (preinjective, regular), then all its representations are preprojective (preinjective, regular). In this case, Γ is called preprojective (preinjective, regular).

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4. Q *has no right infinite path* $\Rightarrow \mathcal{P}_Q \cong \mathbb{N}Q^{\mathrm{op}}$.
5. Q *has right infinite paths* $\Rightarrow \mathcal{P}_Q$ *has right-most section formed by its* ∞*-dim representations. In this case, the* τ*-orbits in* $\mathcal{P}_Q$ *are all finite.*

Example

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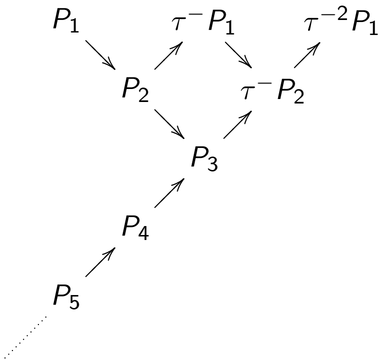
$\mathcal{P}_Q$:



Preinjective components

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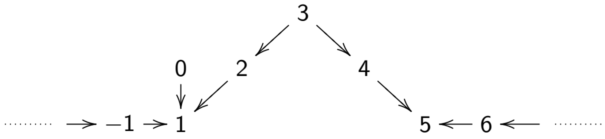

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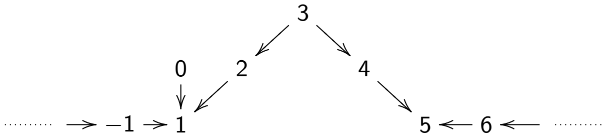

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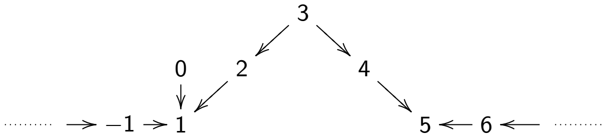

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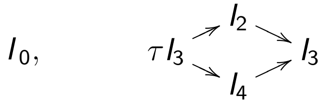
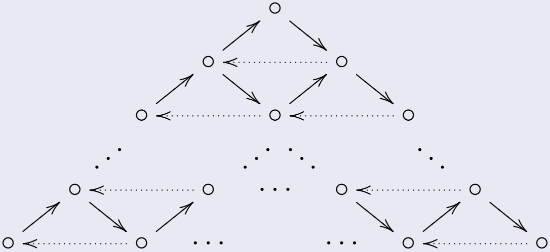


Wings

Definition

A *wing* is a trivially valued translation quiver as follows :



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4. *If Γ has both pseudo-projective representations and ∞-dimensional representations, then Γ is wing.*

Consequence

Corollary

The AR-quiver $\Gamma_{\mathrm{rep}^+(Q)}$ is symmetrically valued.

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$$\begin{array}{l} 1 - \overbrace{2 - 2 - \cdots - 2}^{i\ (>0)\ \text{times}} - 1 - \cdots - 1 - \cdots, \\ \quad\ \ | \\ \quad\ \ 1 \end{array}$$

the last case occurs only when Q has right infinite paths.

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Let Q be infinite Dynkin quiver. Then $\Gamma_{\mathrm{rep}^+(Q)}$ has at most 4 components, all of them are standard, at most one of them is preinjective, and at most 2 of them are regular.

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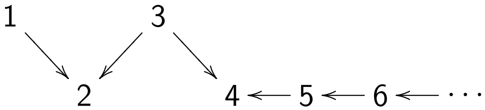

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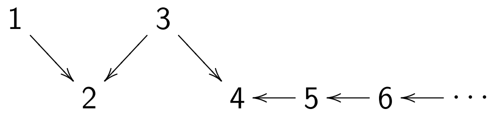

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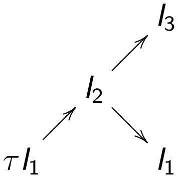

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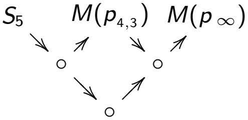

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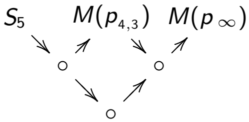

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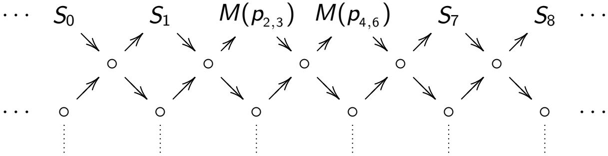

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5. *The AR-triangles in $D^b(\mathrm{rep}^+(Q))$ are the shifts of those as stated above.*

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Corollary

If Q is a infinite Dynkin quiver, then $\Gamma_{D^b(\mathrm{rep}^+(Q))}$ has at most 3 components up to shift, all of them are standard.

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Proposition (RVDB)

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