Tilted Special Biserial Algberas

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Introduction

Tilted algebras, that is endomorphism algebras of tilting modules over a hereditary algebra, have been one of the main objects of study in representation theory of algebras since their introduction by Happel and Ringel [**9**]. As a generalization, Happel, Reiten and Smalø studied endomorphism algebras of tilting objects of a hereditary abelian category which they call quasi-tilted algebras [**8**]. We are concerned with the problem of characterizing these algebras in terms of bound quivers. In our previous paper [**12**], we have found some simple combinatorial criteria to determine if a string algebra is quasi-tilted or tilted or neither. In this paper, we shall consider the same problem for special biserial algebras which are not string algebras. Note that such an algebra is tilted if and only if it is quasi-tilted since there are some indecomposable projective-injective modules [**3**]. Our strategy is to study the combinatorial interpretation of some behavious of the homological dimensions of the indecomposable modules. This enables us to find first a combinatorial characterization of the special biserial algebras of global dimension at most two, and then some simple necessary and sufficient conditions for a special biserial algebra to be tilted. As one of the applications, this allows one to construct a large class of new examples of tilted algebras.

1. Preliminaries

Throughout this paper, denote by A a basic finite dimension algebra over an algebraically closed field k. It is then well-known that $A \cong kQ/I$ with (Q, I) a finite bound quiver [**6**]. In this paper we shall identify the category $\operatorname{mod} A$ of the finite-dimensional (over k) right A-modules with the category of the finite-dimensional representations of (Q, I) over k. We shall consider only the special biserial algebras. This class of algebras have attracted much of attention of current research [**1, 2, 4,13, 17, 18**].

1.1. Definition [2, 7]. *An algebra A is called* special biserial *if $A \cong kQ/I$ with (Q,I) a bound quiver satisfying the following*:

(1) *Each vertex of Q is start-point or end-point of at most two arrows.*

(2) *For an arrow α, there is at most one arrow β such that $\alpha\beta \notin I$ and at most one arrow γ such that $\gamma\alpha \notin I$.*

Moreover A is called a string algebra *if, in addition, I is generated by a set of paths of Q.*

Let (Q,I) be a bound quiver. A path p of Q is called a *zero path* if $p \in I$. A zero path is called a *zero-relation* of (Q,I) if none of its proper subpaths is a zero path. Moreover a pair (p,q) of non-zero paths p,q from a vertex a to a vertex b is called a *binomial relation* of (Q,I) if $\lambda p + \mu q \in I$ for some non-zero coefficients λ, μ. We call a the *start-point*, b the *end-point* and p,q the *maximal subpaths* of (p,q).

In our later use, by saying that $A \cong kQ/I$ is a special biserial algebra we mean that the bound quiver (Q,I) satisfies the conditions as stated in the above definition. In this case, any minimal set of generators of I consists of zero-relations and binomial relations. However in our terminology, some zero-relations may not belong to any minimal set of generators of I.

We now fix more notation and terminology which will be used throughout this paper. Let Q be a finite quiver. For an arrow α of Q, denote by $s(\alpha)$ its start-point, by $e(\alpha)$ its end-point and by α^{-1} its formal inverse with start-point $s(\alpha^{-1}) = e(\alpha)$ and end-point $e(\alpha^{-1}) = s(\alpha)$, and write $(\alpha^{-1})^{-1} = \alpha$. A *walk* of positive length n is a sequence $w = c_1 \cdots c_n$ with c_i an arrow or the inverse of an arrow such that $s(c_{i+1}) = e(c_i)$ for $1 \leq i < n$. We call c_1 the *initial edge* of w and c_n the *terminal edge*. Moreover, we define $s(w) = s(c_1)$ and $e(w) = e(c_n)$. Finally we define $w^{-1} = c_n^{-1} \cdots c_1^{-1}$. A *trivial walk* at a vertex a is the trivial path ε_a with $e(\varepsilon_a) = s(\varepsilon_a)$, its inverse is itself. A walk w is called *reduced* if either w is trivial or $w = c_1 \cdots c_n$ such that $c_{i+1} \neq c_i^{-1}$ for all $1 \leq i < n$. For convenience, we shall allow ourself to add some appropriate trivial paths in the expression of a walk. For example if $\alpha : a \to b$ is an arrow, we may write $\alpha = \varepsilon_a \alpha = \alpha \varepsilon_b^{-1}$. However $\alpha\alpha^{-1}$ and ε_a are two distinct walks.

A non-trivial reduced walk $w = c_1 \cdots c_n$ is called a *reduced cycle* if $s(w) = e(w)$ and $c_n \neq c_1^{-1}$, and a *simple cycle* if in addition $s(c_1), \ldots, s(c_n)$ are pairwise distinct. Another reduced cycle w_1 is said to be equivalent to w if $w_1 = c_i \cdots c_n c_1 \cdots c_{i-1}$ or $w_1^{-1} c_i \cdots c_n c_1 \cdots c_{i-1}$ for some $1 \leq i \leq n$.

Let $w = c_1 \cdots c_n$ be a non-trivial reduced walk in Q. We say that a non-trivial path p of Q is *contained* in w if there are some $1 \leq i \leq j \leq n$ such that $p = c_i \cdots c_j$ or $p^{-1} = c_i \cdots c_j$. Note that a path of Q may appear many times in a reduced walk. Let q be another path contained in w such that q or q^{-1} is equal to $c_r \cdots c_s$ for some $1 \leq r \leq s \leq n$. We say that p, q *point to the same direction* in w if $p = c_i \cdots c_j,\ q = c_r \cdots c_s$ or $p^{-1} = c_i \cdots c_j,\ q^{-1} = c_r \cdots c_s$; otherwise p, q *point to opposite directions* in w.

1.2. Definition. *Let $A = kQ/I$ be a special biserial algebra. A reduced walk w in Q is called a* string *if each path contained in w is neither a zero-relation nor a maximal subpath of a binomial relation of (Q, I).*

Example. Consider the special biserial algebra given by the quiver

$$\begin{tikzcd}
 & \circ \arrow[dl, "\beta"'] \arrow[dr, "\delta"] & & \circ \arrow[dl, "\rho"'] \arrow[dr, "\phi"] & \\
\circ \arrow[dr, "\alpha"'] & & \circ \arrow[dl, "\gamma"] \arrow[dr, "\mu"'] & & \circ \arrow[dl, "\psi"] \\
 & \circ & & \circ &
\end{tikzcd}$$

bound by the relations $\beta\alpha - \delta\gamma$, $\delta\mu$ and $\rho\gamma$. Then $\rho\mu\psi^{-1}\phi^{-1}\rho\delta^{-1}\beta$ is a string while $\beta^{-1}\delta\mu\psi^{-1}$ and $\beta\alpha\gamma^{-1}\mu$ are not strings.

Let $A = kQ/I$ be a special biserial algebra. Let w be a string in (Q, I). Throughout this paper we shall denote by $M(w)$ the string module determined by w. Recall that if w is the trivial path at a vertex a, then $M(w)$ is the simple module at a. Otherwise $w = c_1 c_2 \cdots c_n$, where $n \geq 1$ and c_i or c_i^{-1} is an arrow. For $1 \leq i \leq n+1$, let $U_i = k$; and for $1 \leq i \leq n$, denote by U_{c_i} the identity map sending $x \in U_i$ to $x \in U_{i+1}$ if c_i is an arrow and otherwise the identity map sending $x \in U_{i+1}$ to $x \in U_i$. For a vertex a, if a appears in w, then $M(w)_a$ is the direct sum of the spaces U_i with i such that $s(c_i) = a$ or $i = n+1$ and $e(c_n) = a$; otherwise $M(w)_a = 0$. For an arrow α, if α appears in w, then $M(w)_\alpha$ is the direct sum of the maps U_{c_i} such that $c_i = \alpha$ or $c_i^{-1} = \alpha$; otherwise $M(w)_\alpha$ is the zero map.

One says that a string w in (Q, I) *starts* or *ends* in a *deep* if there is no arrow γ such that $\gamma^{-1} w$ or $w\gamma$ is a string, respectively; and it *starts* or *ends* on a *peak* if there is no arrow δ such that δw or $w\delta^{-1}$ is a string, respectively. For a vertex a of Q, we denote throughout this paper by $S(a), P(a)$ and $I(a)$ the simple module, the indecomposable projective module and the indecomposable injective module at a, respectively. Then $S(a) = M(\varepsilon_a)$ with ε_a the trivial path at a. If a is not the start-point of a binomial relation, then $P(a) = M(u^{-1}v)$, where u, v are paths of non-negative length starting with a such that $u^{-1}v$ is a string starting and ending in a deep. Dually, if a is not the end-point of a binomial relation, then $I(a) = M(pq^{-1})$, where p, q are paths of non-negative length ending with a such that pq^{-1} is a string starting and ending on a peak. Finally if a is the start-point of a binomial relation whose end-point is b, then $P(a)$ is isomorphic to $I(b)$, and hence it is projective-injective.

1.3. Definition. *Let $A = kQ/I$ be a special biserial algebra. A non-trivial reduced cycle of Q is called a* band *if each of its powers is a string and it is not a power of a string of less length.*

Let $A = kQ/I$ be a special biserial algebra. Let $w = c_1c_2\cdots c_n$ be a band in (Q, I), where $n \geq 1$ and c_i or c_i^{-1} is an arrow such that $s(c_1) = e(c_n)$. Let ϕ be an indecomposable automorphism of a k-vector space V. One defines a band module $N = N(w, \phi)$ determined by w and ϕ as follows:

For $1 \leq i \leq n$, define $V_i = V$. For $1 \leq i < n$, let f_{c_i} be the identity map from V_i to V_{i+1} if c_i is an arrow; and otherwise the identity map from V_{i+1} to V_i, and let f_{c_n} be the map sending $x \in V_n$ to $\phi(x) \in V_1$ if c_n is an arrow; and otherwise the map sending $x \in V_1$ to $\phi^{-1}(x) \in V_n$. For each vertex a of Q, if a appears in w, then N_a is the direct sum of the spaces V_i such that $s(c_i) = a$, and otherwise N_a is the zero-space. For each arrow α, if α appears in w, then N_α is the direct sum of the maps f_{c_i} such that $c_i = \alpha$ or $c_i = \alpha^{-1}$; and otherwise N_α is the zero-map.

The indecomposable modules and the almost split sequences over a special biserial algebra are completely described by Wald and Waschbüsch [**18**]. We quote what is needed for our purpose as follows.

1.4. Theorem [18]. *Let $A = kQ/I$ be a special biserial algebra. Then*

(1) *An indecomposable module in* $\operatorname{mod} A$ *is a string module, a band module or a projective-injective module corresponding to a binomial relation.*

(2) *Each band module over A is invariant under the Auslander-Reiten translation.*

It follows from a result of Skowroński [**16**] that a special biserial algebra is of directed representation type if and only if there is no special family of local modules. However his combinatorial interpretation of the existence of a special family of local modules does not seem completely correct. We shall give a combinatorial reformulation of this result. In order to do so, we need the following concept.

1.5. Definition. *Let $A = kQ/I$ be a special biserial algebra. Let $w = c_1 \cdots c_n$ be a reduced cycle of Q, where c_i or c_i^{-1} is an arrow α_i for $1 \le i \le n$. Let $p_1, \ldots, p_s$ ($s \ge 0$) the paths contained in w which are either zero-relations or maximal subpaths of binomial relations of (Q, I). We say that w (and any reduced walk equivalent to w) is clockwise in (Q, I) if $\alpha_1, \alpha_n, p_1, \ldots, p_s$ all point to the same direction in w.*

1.6. Theorem. *Let $A = kQ/I$ be a special biserial algebra. Then A is of directed representation type if and only if (Q, I) admits no clockwise cycle.*

Proof. Let $\bar{I}$ be the ideal of kQ generated by the zero-relations and the maximal subpaths of the binomial relations of (Q, I). Then $\bar{A} = kQ/\bar{I}$ is a string algebra. It is easy to see that a reduced walk of Q is a string or a band in (Q, I) if and only if it is a string or a band in $(Q, \bar{I})$, respectively. Therefore an indecomposable module M in $\operatorname{mod} A$ is a string or a band module over A if and only if it is a string or a band module over $\bar{A}$, respectively. This implies that A finite representation type if and only if so is $\bar{A}$. It now follows from a result of de la Peña [**14**] that $\bar{A}$ is of directed representation type if and only if there is no clockwise cycle in $(Q, \bar{I})$. Clearly a reduced cyle of Q is clockwise in $(Q, \bar{I})$ if and only if it is clockwise in (Q, I). Therefore it suffices to show that A is of directed representation type if and only if so is $\bar{A}$. First that $\bar{A}$ is not of directed representation type implies trivially that A is not either. Assume now that A is not of directed representation type. If A is of infinite representation type, then so is $\bar{A}$. Thus $\bar{A}$ is not of directed representation type [**15**]. Suppose that A is of finite representation type. Then there is no band module in $\operatorname{mod} A$ and there is a cycle

$$(*) \qquad M_0 \xrightarrow{f_1} M_1 \to \cdots \to M_{r-1} \xrightarrow{f_r} M_r = M_0$$

in $\operatorname{mod} A$, where the M_i are indecomposable modules and the f_i are irreducible maps. Then $r > 1$ and $M_{i-1} \ncong M_i$ for all $1 \le i \le r$. Note that each

M_i with $0 \leq i \leq r$ is either a string module or a projective-injective module corresponding to a binomial relation of (Q, I). Let s with $0 \leq s \leq r$ be such that M_s is not a string module. We may assume that $0 < s < r$. Thus M_{s-1} is the radical of M_s and M_{s+1} is the socle-factor of M_s. By computing the dimensions, we conclude that there are irreducibles maps $g_{s-1} : M_{s-1} \longrightarrow N_s$ and $g_s : N_s \longrightarrow M_{s+1}$ with N_s indecomposable and $N_s \not\cong M_s$. Note then that N_s is not projective-injective, and hence N_s is a string module. Replacing f_i by g_i for $s - 1 \leq i \leq s$, we obtain a cycle in $\operatorname{mod} A$ containing fewer number of non-string modules. Therefore we may assume that $(*)$ contains only string modules. This implies that $(*)$ is a cycle in $\operatorname{mod} \bar{A}$. The proof is completed.

2. Projective and injective dimensions

In this section, we shall study the combinatorial interpretation of some behavior of the projective and injective dimensions of the indecomposable modules over a special biserial algebra. To begin with, we state some elementary properties of special biserial algebras, which will be used extensively in our later proofs.

2.1. Lemma. *Let $A = kQ/I$ be a special biserial algebra. Then*

(1) *Any proper subpath of a non-zero path of (Q, I) is a string.*

(2) *Let u, v and p be non-trivial paths with $e(u) = e(v) = s(p)$. If vp is a non-zero path, then up contains a zero-relation which is not contained in u.*

(3) *If p is a non-zero path such that $s(p)$ is the start-point of a binomial relation of (Q, I), then p is contained in this binomial relation.*

(4) *Let (p, q) be a binomial relation of (Q, I). If u is a non-trivial path with $e(u) = s(p)$, then up and uq are both zero paths. If v is a non-trivial path with $s(v) = e(p)$, then pv and qv are both zero paths.*

We skip the proof of the above lemma since it is simply a routine verification of the definition of a special biserial algebra.

We shall now find some sufficient conditions for a string module to be of projective dimension greater than one. Note that each reduced walk w in a quiver can be uniquely written as $w = p_1^{-1} q_1 \cdots p_n^{-1} q_n$, where $n \geq 1$, the p_i and the q_j are paths which are non-trivial for $1 < i \leq n$, $1 \leq j < n$.

2.2. Lemma. *Let $A = kQ/I$ be a special biserial algebra. Let*

$$w = p_1^{-1}q_1 \cdots p_n^{-1}q_n$$

be a string in (Q, I), where $n \geq 1$, the p_i and the q_j are paths which are non-trivial for $1 < i \leq n$, $1 \leq j < n$. Assume that some $s(p_r)$ $(1 \leq r \leq n)$ is the start-point of a binomial relation $(p_r u_r, q_r v_r)$, where u_r, v_r are non-trivial paths and u_r is of length greater than one if $r = 1$ and v_r is of length greater than one if $r = n$. Then the projective dimension of the string module $M(w)$ is greater than one.

Proof. Let $a_i = s(p_i), b_i = e(p_i)$ for $1 \leq i \leq n$ and let $b_{n+1} = e(q_n)$. Clearly the projective cover of $M(w)$ is $\oplus_{i=1}^n P(a_i)$. Let K be the kernel of the canonical epimorphism from $\oplus_{i=1}^n P(a_i)$ to $M(w)$. It suffices to show that K is not projective.

Let $s \leq r \leq t$ be such that a_i is the start-point of a binomial relation for all $s \leq i \leq t$ and such that $t - s$ is maximal with respect to this property. For each $s \leq i \leq t$, let u_i and v_i be paths such that $(p_i u_i, q_i v_i)$ is a binomial relation. Note that the u_i and the v_i are non-trivial since w is a string. If $s > 1$, then let v_{s-1} be the path such that $q_{s-1}v_{s-1}$ is a string ending in a deep and otherwise let v_0 be the trivial path at b_1. Similarly if $t < n$, let u_{t+1} be the path such that $p_{t+1}u_{t+1}$ is a string ending in a deep and otherwise let u_{n+1} be the trivial path at b_{n+1}.

Suppose first that $r = 1$. Then u_1 is of length greater than one by hypothesis. Let $u_1 = \alpha u$ with α an arrow and u a non-trivial path. If $n = 1$, then v_1 is also of length greater than one by hypothesis, and hence $v_1 = \beta v$ with β an arrow v a non-trivial path. Now uv^{-1} is a string such that $K = M(uv^{-1})$, which is not projective since u, v are non-trivial. If $n > 1$, then

$$w_1 = uv_1^{-1}u_2 \cdots u_t v_t^{-1}u_{t+1}$$

is a string such that $M(w_1)$ is a direct summand of K. Note that $M(w_1)$ is not projective since u, v_1 are non-trivial. Hence K is not either. A symmetric argument shows that K is not projective if $r = n$.

Suppose now that $1 < r < n$. Then it is easy to see that

$$w_2 = v_{s-1}^{-1}u_s v_s^{-1} \cdots u_r v_r^{-1} \cdots u_t v_t^{-1} u_{t+1}$$

is a string such that $M(w_2)$ is a direct summand of K. Note that $M(w_2)$ is not projective since u_r, v_r are non-trivial, and hence K is not either. This completes the proof.

2.3. Lemma. *Let $A = kQ/I$ be a special biserial algebra. Let*

$$w = p_1^{-1}q_1 \cdots p_n^{-1}q_n$$

be a string in (Q, I), where $n \geq 1$, the p_i and the q_j are paths which are non-trivial for $1 < i \leq n$, $1 \leq j < n$. Assume that there is a path q with initial arrow α such that $w\alpha$ is a string and $q_r q$ admits exactly one zero-relation, and the zero-relation contains q. Then the projective dimension of $M(w)$ is greater than one.

Proof. First note that q is of length greater than one. Write $q = \alpha u \beta$ with β an arrow and u a path. Then $q_n \alpha u$ and $u\beta$ are non-zero by hypothesis. Let $a_i = s(q_i)$ for $1 \leq i \leq n$. Then $\oplus_{i=1}^n P(a_i)$ is the projective cover of $M(w)$. Let K be the kernel of the canonical epimorphism from $\oplus_{i=1}^n P(a_i)$ to $M(w)$. If a_n is not the start-point of a binomial relation, then it is easy to see that the string module $M(u)$ is a direct summand of K, which is not projective since $u\beta$ is non-zero. Thus the projective dimension of $M(w)$ is greater than one.

Assume now that a_n is the start-point of a binomial relation. It is necessarily of the form $(p_n u_n, q_n \alpha u v_n)$, where u_n is a non-trivial path and v_n is a path such that uv_n is non-trivial, since $q_n \alpha u$ is non-zero and $q_n \alpha$ is a string by hypothesis. If $n > 1$ or $n = 1$ with u_1 of length greater than one, then $M(w)$ is of projective dimension greater than one by Lemma 2.2. Suppose that $n = 1$ and u_1 is an arrow. Then $K = M(uv_1)$. If u is trivial, then $K = M(v_1)$, which is not projective since $v_1^{-1}\beta$ is a string. If u is non-trivial, then v_1 is trivial since $u\beta, uv_1$ are non-zero. Thus $K = M(u)$, which is not projective since $u\beta$ is non-zero. Therefore the projective dimension of $M(w)$ is greater than one. The proof is completed.

2.4. Lemma. *Let $A = kQ/I$ be a special biserial algebra. Let*

$$w = p_1^{-1}q_1 \cdots p_n^{-1}q_n$$

be a string in (Q, I), where $n > 1$, the p_i and the q_j are paths which are non-trivial for $1 < i \leq n$, $1 \leq j < n$. If there is a non-zero path p such that for some $1 < r \leq n$, both $q_{r-1}p$ and $p_r p$ are zero paths, then the projective dimension of $M(w)$ is greater than one.

Proof. For each $1 \leq i \leq n$, let $a_i = s(p_i)$. Then $\oplus_{i=1}^n P(a_i)$ is the projective cover of $M(w)$. Let K be the kernel of the canonical epimorphism from $\oplus_{i=1}^n P(a_i)$ to $M(w)$. Let p be a non-zero path such that $q_{r-1}p$ and $p_r p$

are zero paths for some $1 < r \leq n$. We may assume that p is of minimal length with respect to this property. Write $p = u\alpha$, where α is an arrow and u is a path. Then either $q_{r-1}u$ or $p_r u$ is non-zero by the minimality of p.

Suppose that a_{r-1} is the start-point of a binomial relation. It is necessarily of the form $(p_{r-1}u_{r-1}, q_{r-1}v_{r-1})$ with u_{r-1}, v_{r-1} some non-trivial paths. If $r > 2$ or $r = 2$ with u_{r-1} of length greater than one, then $M(w)$ is of projective dimension greater than one by Lemma 2.2. Assume now that $r = 2$ and u_1 is an arrow. Since p is a non-zero path and $q_1 p$ is a zero path, it is easy to see that $v_1^{-1}p$ is a reduced walk. Moreover a_2 is not the start-point of a binomial relation since $p_2 v_1$ and $p_2 p$ are both zero-paths. If $p_2 u$ is a zero path, then $q_1 u$ is non-zero with u non-trivial, which is impossible. Therefore $p_2 u$ is non-zero, and hence $p_2 u$ is a string ending in a deep. It is now easy to see that $M(v_1^{-1}u)$ is a direct summand of K. Note that $M(v_1^{-1}u)$ is not projective since $u\alpha = p$ is non-zero. Therefore $M(w)$ is of projective dimension greater than one. Using a symmetric argument, one can show that $M(w)$ is of projective dimension greater than one if a_s is the start-point of a binomial relation.

Suppose now that neither a_{s-1} nor a_s is the start-point of a binomial relation. We may assume that $p_s u$ is non-zero. Let v be the path such that $q_{s-1}v$ is a string ending in a deep. Then $M(v^{-1}u)$ is a direct summand of K. Note that $M(v^{-1}u)$ is not projective since $u\alpha$ is non-zero. Therefore $M(w)$ is of projective dimension greater than one. The proof is completed.

We shall now find some necessary conditions for a string module to be of projective dimension greater than one.

2.5. Lemma. *Let $A = kQ/I$ be a special biserial algebra. Let*

$$w = p_1^{-1}q_1 \cdots p_n^{-1}q_n$$

be a string in (Q, I), where $n \geq 1$, the p_i and the q_j are paths which are non-trivial for $1 < i \leq n$ and $1 \leq j < n$. If the projective dimension of $M(w)$ is greater than one, then one of the following cases occurs:

(PD1) *There is a path p with initial arrow α such that $\alpha^{-1}w$ is a reduced walk without zero-relations and $p_1 p$ admits a zero-relation containing p.*

(PD2) *There is a path q with initial arrow β such that $w\beta$ is a reduced walk without zero-relations and $q_n q$ admits a zero-relation containing q.*

(PD3) *There is a non-zero path u such that for some $1 < s \leq n$, both $q_{s-1}u$ and $p_s u$ are zero paths.*

(PD4) *Some $s(p_r)$ with $1 \le r \le n$ is the start-point of a binomial relation.*

Proof. Assume that none of the stated cases occurs. We shall show that the projective dimension of $M(w)$ is less than two. Let $a_i = s(p_i), b_i = e(p_i)$ for $1 \le i \le n$, and let $b_{n+1} = e(q_n)$. Then the projective cover of $M(w)$ is $\oplus_{i=1}^n P(a_i)$. Let K be the kernel of the canonical epimorphism from $\oplus_{i=1}^n P(a_i)$ to $M(w)$. It suffices to show that K is projective.

We fix some more notations. Since (PD4) does not occur, a_i is not the start-point of a binomial relation for all $1 \le i \le n$. Denote by u_i, v_i the paths such that $u_i^{-1}p_i^{-1}q_iv_i$ is a string starting and ending in a deep. Then $P(a_i) = M(u_i^{-1}p_i^{-1}q_iv_i)$ for all $1 \le i \le r$. For each $1 < i \le r$, b_i is not the start-point of a binomial relation since (PD3) does not occur. Thus $v_{i-1}^{-1}u_i$ is a string, which starts and ends in a deep since (PD3) does not occur. Therefore $M(v_{i-1}^{-1}u_i) = P(b_i)$ for all $1 < i \le n$.

Define $K_1 = 0$ if u_1 is trivial. Otherwise write $u_1 = \alpha_1 u_0$ with α_1 an arrow and u_0 a path, and define $K_1 = M(u_0)$. Note that $p_1\alpha_1$ is non-zero. Since (PD1) does not occur, $s(u_0)$ is not the start-point of a binomial relation and u_0 is a string starting and ending in a deep. Thus $K_1 = P(s(u_0))$ is projective. Similarly define $K_{n+1} = 0$ if v_n is trivial. Otherwise write $v_n = \alpha_{n+1}v_{n+1}$ with α_{n+1} an arrow and v_{n+1} a path. Define $K_{n+1} = M(v_{n+1})$, which is in fact $P(s(v_{n+1}))$ since (PD2) does not occur. Now by computing the dimension of modules, we see that

$$0 \to K_1 \oplus_{i=2}^n M(v_{i-1}^{-1}u_i) \oplus K_{n+1} \to \oplus_{i=1}^n P(a_i) \to M(w) \to 0$$

is a short exact sequence in $\operatorname{mod} A$. Thus $K = K_1 \oplus_{i=2}^r M(v_{i-1}^{-1}u_i) \oplus K_{r+1}$ is projective. The proof is completed.

For the convenience of the reader, we state explicitly the dual result concerning the injective dimension of string modules.

2.6. Lemma. *Let $A = kQ/I$ be a special biserial algebra. Let*

$$w = q_1^{-1}p_1^{-1}\cdots q_np_n^{-1}$$

be a string, where $n \ge 1$, the q_i and the p_j are paths which are non-trivial for $1 < i \le n$ and $1 \le j < n$. If the injective dimension of $M(w)$ is greater than one, then one of the following cases occurs:

(ID1) *There is a path q with terminal arrow α such that αw is a reduced walk without zero-relations and qq_1 admits a zero-relation containing q.*

(ID2) *There is a path p with terminal arrow β such that $w\beta^{-1}$ is a reduced walk without zero-relations and pp_n admits a zero-relation containing p.*

(ID3) *There is a non-zero path u such that for some $1 < s \leq n$, both up_{s-1} and uq_s are zero paths.*

(ID4) *Some $e(q_r)$ with $1 \leq r \leq n$ is the end-point of a binomial relation.*

We conclude this section by considering the projective dimension of the band modules.

2.7. Lemma. *Let $A = kQ/I$ be a special biserial algebra. Let*

$$w = p_1 q_1^{-1} \cdots p_n q_n^{-1}$$

be a band, where $n \geq 1$, the p_i and the q_i are non-trivial paths such that $s(p_1) = s(q_n)$. Let N be a band module of support w. If N has projective dimension greater than one, then one of the following cases occurs:

(1) *Some $s(p_r)$ with $1 \leq r \leq n$ is the start-point of a binomial relation.*

(2) *There is a non-zero path u such that both $q_r u$ and $p_r u$ are zero paths for some $1 \leq r \leq n$.*

Proof. Assume that the projective dimension of N is greater than one. Then $\mathrm{Hom}(D({}_AA), N) \neq 0$ since N is invariant under the Auslander-Reiten translation [**15**, (2.4)]. Let a be a vertex such that there is a non-zero map g from $I(a)$ to N. Suppose first that a is the end-point of a binomial relation (p, q). Then $I(a) = P(b)$ is projective, where $b = s(p)$. Thus b appears in w, say in some q_r with $1 \leq r \leq n$. If $b = s(q_r)$, then the first case occurs. Otherwise $q_r = u_r v_r$, where u_r is a non-trivial path and v_r is a path such that $e(u_r) = s(v_r) = b$. Then we may assume that v_r is a proper subpath of p. Write $p = v_r v$ with v a non-trivial path such that $s(v) = e(v_r) = e(p_r)$. Then $p_r v$ is a zero path since p_r, v are both non-trivial. Moreover $q_r v = u_r p$ is a zero path since u_r is non-trivial, that is the second case occurs.

Suppose now that a is not the end-point of a binomial relation, then $I(a) = M(pq^{-1})$, where p, q are paths such that pq^{-1} is a string starting and ending on a peak. Note that g factors through the socle-factor of $I(a)$. Using the same argument in the proof of [**12**, (2.4)], one can show that there is a proper subpath v of p (or q) which is also a proper subpath of some p_s (or q_s) with $1 \leq s \leq n$ so that $p = vu_1$ and $p_s = u_2 v$, where u_1, u_2 are some non-trivial paths. Then $p_s u_1 = u_2 p$ is a zero path since u_2 is non-trivial. Moreover $q_s u_1$ is also a zero path. In fact if v is non-trivial, then $q_s u_1$ is a zero path since $vu_1 = p$ is non-zero; and otherwise $q_s u_1 = q_s p$ is a zero path since p is a string starting on a peak. This completes the proof.

3. Global dimension

In this section, we shall apply the results of the previous section to study the combinatorial interpretation of some behavior of the global dimension of a special biserial algebra.

We shall first find the necessary conditions for a special biserial algebra to be of global dimension at most two.

3.1. Lemma. *Let $A = kQ/I$ be a special biserial algebra of global dimension at most two. Let $(\alpha p\beta,\ \gamma q\delta)$ be a binomial relation of (Q, I), where $\alpha, \beta, \gamma, \delta$ are some arrows and p, q are some paths. If u is a non-zero path with $e(u) = s(p)$, then either $u\alpha p$ or $u\gamma q$ is non-zero. Dually if v is a non-zero path with $s(v) = e(p)$, then either $p\beta v$ or $q\delta v$ is non-zero.*

Proof. Note that $p\beta\delta^{-1}q^{-1}$ is a string such that $M(p\beta\delta^{-1}q^{-1})$ is the radical of the indecomposable projective module at $s(\alpha)$. If there is a non-zero path v such that $p\beta v$ and $q\delta v$ are both zero paths. Then by Lemma 2.4, the projective dimension of $M(p\beta\delta^{-1}q^{-1})$ is greater than one. Hence the projective dimension of the simple module at $s(p)$ is greater than two, which is a contradiction. Dually one can show that there is no non-zero path u such that $u\alpha p$ and $u\gamma q$ are both zero paths. The proof is completed.

3.2. Lemma. *Let $A = kQ/I$ be a special biserial algebra of global dimension at most two. Then the start-point of a binomial relation of (Q, I) does not lie in another different binomial relation.*

Proof. Assume on the contrary that there are two distinct binomial relations (p_1, q_1) and (p_2, q_2) such that $a_2 = s(p_2) \in q_1$. Let $a_1 = s(p_1)$. Then $a_2 \neq a_1$. Write $p_1 = \alpha_1 u_1,\ q_1 = \beta_1 v v_1$, where α_1, β_1 are arrows and u_1, v, v_1 are paths with $e(v) = s(v_1) = a_2$. Then $\mathrm{rad} P(a_1) = M(u_1 v_1^{-1} v^{-1})$. We shall show that $M(u_1 v_1^{-1} v^{-1})$ is of projective dimension greater than one, which will lead to a desired contradiction.

We claim that v_1 is a proper subpath of p_2 or q_2. In fact, suppose that v_1 is non-trivial. The we may assume that the initial arrow α_2 of p_2 is contained in v_1. Now v_1 and p_2 are two non-zero paths having the same initial arrow α_2. Thus one of v_1, p_2 is contained in the other. Note that v_1 is a string while p_2 is not. Thus v_1 is a proper subpath of p_2. Write $p_2 = v_1 u$ with u a non-trivial path. Then $u_1 u$ is a zero path. If v is non-trivial, then $(vv_1)u = vp_2$ is a zero path. It follows now from Lemma 2.4 that the projective dimension of $M(u_1 v_1^{-1} v^{-1})$ is greater than one. If v is trivial, then $M(u_1 v_1^{-1} v^{-1}) = M(u_1 v_1^{-1})$. By Lemma 2.2, the projective dimension of $M(u_1 v_1^{-1})$ is greater than one. This completes the proof.

3.3. Lemma. *Let $A = kQ/I$ be a special biserial algebra of global dimension at most two. Then there is no path in Q of the form $p_1p_2p_3$, where p_1, p_2, p_3 are non-trivial paths such that p_2 is a string and p_1p_2, p_2p_3 are the only zero-relations contained in the path.*

Proof. Assume that the lemma was false. Then there is a path $\alpha_1\alpha_2\cdots\alpha_n$, where $\alpha_i : a_{i-1} \to a_i$ is an arrow, containing exactly two zero-relations $\alpha_1\cdots\alpha_s$, $\alpha_r\cdots\alpha_n$ with $1 < r \le s < n$ and a string $\alpha_r\cdots\alpha_s$.

Suppose first that a_0 is the start-point of a binomial relation (p, q). Let q_1 be the path such that $\alpha_1\cdots\alpha_s = \alpha_1 q_1 \alpha_s$. Then $p = \gamma u, q = \alpha_1 q_1 v$, where γ is an arrow and u, v are some paths. Now $M(uv^{-1}q_1^{-1}) = \mathrm{rad}P(a_0)$. We shall show that the projective dimension $M(uv^{-1}q_1^{-1})$ is greater than one, which will lead to a desired contradiction. In fact, if v is non-trivial, then q_1 is trivial since $q_1\alpha_s$ and $q_1 v$ are both non-zero. Hence $s = r = 2$, and $M(uv^{-1}q_1^{-1}) = M(uv^{-1})$. Note that $\alpha_2\cdots\alpha_t$ is a zero-relation and $uv^{-1}\alpha_2$ is a string. By Lemma 2.3, the projective dimension of $M(uv^{-1})$ is greater than one. If v is trivial, then q_1 is non-trivial and $M(uv^{-1}q_1^{-1}) = M(uq_1^{-1})$. Note that $u\alpha_s$ is a zero path since $q_1\alpha_s$ is non-zero. Therefore $u\alpha_s\cdots\alpha_n$ and $q_1\alpha_s\cdots\alpha_n = \alpha_2\cdots\alpha_n$ are both zero paths. By Lemma 2.4, the projective dimension of $M(uq^{-1})$ is greater than one.

Suppose now that a_0 is not the start-point of a binomial relation. Let q_2 be the path such that $\alpha_1\cdots\alpha_{r-1} = q_2\alpha_{r-1}$. Then q_2 is a string, and $P(a_0)$ is the projective cover of $M(q_2)$. Let q_3 be the path such that $\alpha_r\cdots\alpha_s = q_3\alpha_s$. Then $M(q_3)$ is a direct summand of the first syzygy of $M(q_2)$. Now $\alpha_s\cdots\alpha_n$ is a path such that $q_3\alpha_s\cdots\alpha_n = \alpha_r\cdots\alpha_n$ is a zero-relation and $q_3\alpha_s = \alpha_r\cdots\alpha_s$ is a string by hypothesis. By Lemma 2.3, the projective dimension of $M(q_3)$ is greater than one, and hence that of $M(q_2)$ is greater than two, which is again a contradiction. This completes the proof.

We are now able to get a combinatorial characterization of the special biserial algebras of global dimension at most two.

3.4. Theorem. *Let $A = kQ/I$ be a special biserial algebra. Then the global dimension of A is at most two if and only if (Q, I) satisfies the following properties*:

(GD1) *The start-point of a binomial relation does not lie in another different binomial relation.*

(GD2) *There is no path of the form $p_1p_2p_3$, where p_1, p_2, p_3 are non-trivial paths such that p_2 is a string and p_1p_2, p_2p_3 are the only zero-relations contained in the path.*

(GD3) *Let $(\alpha p\beta, \gamma q\delta)$ be a binomial relation, where $\alpha, \beta, \gamma, \delta$ are some arrows and p, q are some paths. If u is a non-zero path with $e(u) = s(\alpha)$, then either $u\alpha p$ or $u\gamma q$ is non-zero. Dually if v is a non-zero path with $s(v) = e(\beta)$, then either $p\beta v$ or $q\delta v$ is non-zero.*

Proof. We need only to prove the sufficiency. Assume that (Q, I) satisfies (GD1), (GD2) and (GD3). Let a be a vertex of Q, and let $K = \mathrm{rad}P(a)$. We shall show that the projective dimension of K is at most one. This will complete the proof.

Assume first that a is the start-point of a binomial relation $(\alpha q_1, \beta q_2)$, where α, β are some arrows and q_1, q_2 are some non-trivial paths. Then $K = M(q_1q_2^{-1})$. We shall verify that the string $q_1q_2^{-1}$ satisfies none of (PD1), (PD2), (PD3) and (PD4) as stated in Lemma 2.5. In fact it follows easily from (GD2) that (PD1) or (PD2) does not occur. Moreover (PD3) does not occur by (GD3). Finally (PD4) does not occur by (GD1). Therefore the projective dimension of K is at most one in this case.

Assume now that a is not the start-point of a binomial relation. Suppose that K is of projective dimension greater than one. Then there is a non-trivial path $u_1 = \alpha_1 \cdots \alpha_r = \alpha_1 u_2$, where $\alpha_i : a_{i-1} \to a_i$ is an arrow and u_2 is a path, such that $a = a_0$ and $M(u_2)$ is a direct summand of K. Hence $M(u_2)$ is of projective dimension greater than one. By Lemma 2.5, one of the cases (PD1), (PD2) and (PD4) occurs. Now (PD1) does not occur by (GD2). If (PD4) occurs, that is a_1 is the start-point of a binomial relation $(\beta_1 v_1, u_2 v_2)$, where β_1 is an arrow and v_1, v_2 are non-trivial paths since u_2 is a string. Note that $\alpha_1\beta_1$ is a zero-relation, and $\alpha_1 \cdots \alpha_r \beta_2$ with β_2 the initial arrow of v_2 is a zero path. By (GD3), v_2 is an arrow. This implies that $M(v_1)$ is the first syzygy of $M(u_2)$, and hence not projective. So there is an arrow ρ such that either $\rho^{-1}v_1$ is a reduced walk or $v_1\rho$ is a non-zero path. In the first case, $\beta_1\rho$ is a zero-relation, which is contrary to (GD2) shown by the path $\alpha_1\beta_1\rho$. In the second case $v_2\rho$ is a zero-relation. Note that u_1v_2 is also a zero-relation.

This leads also to a contradiction to (GD2) since v_2 is an arrow.

Otherwise a_1 is not the start-point of a binomial relation, and there is a path $\alpha_{r+1}u_3$, where α_{r+1} is an arrow and u_3 is a non-trivial path, such that $u_2\alpha_{r+1}$ is non-zero and $u_2\alpha_{r+1}u_3$ contains a zero-relation $\alpha_s \cdots \alpha_{r+1}u_3$ with $1 < s \le r+1$. Note that $\alpha_1 \cdots \alpha_{s-1}\alpha_s \cdots \alpha_{r+1}$ is also a zero-relation. By (GD2), the non-zero path $\alpha_s \cdots \alpha_{r+1}$ is not string. Thus $\alpha_s \cdots \alpha_{r+1}$ is a maximal subpath of a binomial relation. Hence $s > 2$ since a_1 is not the start-point of a binomial relation. Therefore $\alpha_2 \cdots \alpha_s \cdots \alpha_{r+1} = u_2\alpha_{r+1}$ is a zero path, which is a contradiction. The proof is completed.

4. Main result

In this final section, we shall obtain our promised combinatorial criteria for deciding if a special biserial algebra which is not a string algebra is tilted or not. Note that such a special biserial algebra admits a indecomposable projective-injective module. Hence it is tilted if and only if it is quasi-tilted [**3**]. Recall that an algebra A is *quasi-tilted* if its global dimension is at most two and each indecomposable module is either of projective dimension at most one or of injective dimension at most one [**8**].

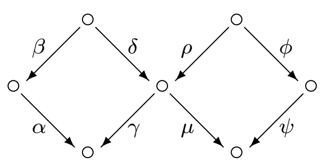
In order to formulate our main result, we need the following concept.

4.1. Definition. *Let $A = kQ/I$ be a special biserial algebra. A reduced walk w in Q is called a* sequential pair of zero-relations *if*

(1) *$w = p_1p_2p_3$, where the p_i are non-trivial paths such that p_1p_2 and p_2p_3 are the only zero-relations contained in w; or*

(2) *$w = pw_1q$, where w_1 is a string and p, q are paths which are the only zero-relations contained in w.*

Example. Consider the special biserial algebra given by the quiver



bound by the relations $\beta\alpha - \delta\gamma$, $\delta\mu$ and $\rho\gamma$. Then $\delta\mu\psi^{-1}\phi^{-1}\rho\gamma$ is a sequential pair of zero-relations, while $\rho\gamma\alpha^{-1}\beta^{-1}\delta\mu$ is not.

Remark. Let $A = kQ/I$ be a special biserial algebra. It is easy to verify that each path with more than one zero-relations contains a sequential pair of zero-relation, and so does a reduced walk of the form pwq, where w is a string and p, q are some zero paths.

4.2. Lemma. *Let $A = kQ/I$ be a special biserial algebra of global dimension two. If there is a string module of projective and injective dimensions both equal to two, then (Q, I) admits a sequential pair of zero-relations.*

Proof. Note first that (Q, I) satisfies (GD1), (GD2) and (GD3) as stated in Theorem 3.4. Let now w be a string such that $M(w)$ has projective and injective dimensions both equal to two. We shall find a sequential pair of zero-relations in (Q, I).

We first consider the case where $w = \varepsilon$ is a trivial path at a vertex a. Then $w = \varepsilon^{-1}\varepsilon = \varepsilon\varepsilon^{-1}$. By Lemma 2.5, either (PD1) or (PD4) occurs, and by Lemma 2.6, either (ID1) or (ID4) occurs. Assume that (PD1) occurs, that is there is a zero-relation p with initial arrow such that $s(\alpha) = a$. If (ID1) occurs, then there is a zero-relation q with $e(q) = a$. Thus qp is a sequential pair of zero-relations. If (ID4) occurs, then there is a binomial relation $(u\beta, v\gamma)$, where u, v are some paths and β, γ are some arrows such that $e(\alpha) = a$. Then either $\beta\alpha$ or $\gamma\alpha$ is a zero-relation, which is a contradiction to (GD2). Dually (PD4) and (ID1) can not both occur. Finally (PD4) and (ID4) can not both occur by (GD1).

Suppose now that w is non-trivial. Write $w = p_1^{-1}q_1 \cdots q_{r-1}p_n^{-1}q_n$, where $n \geq 1$, the p_i and the q_j are paths which are non-trivial for $1 < i \leq n$ and $1 \leq j < n$. We shall consider only the case where p_1 is non-trivial and q_n is trivial, since the other cases can be treated similarly. In this situation, we have $w = q_0p_1^{-1}q_1 \cdots q_{n-1}p_n^{-1}$ with q_0 the trivial path at $e(p_1)$. By Lemma 2.5, one of the cases (PDi) with $1 \leq i \leq 4$ occurs, and by Lemma 2.6, one of the cases (IDi) with $1 \leq i \leq 4$ occurs. We shall complete the proof by considering separately the following cases.

(1) Case (PD1) occurs, that is there is a path p_0 with initial arrow α_1 such that p_1p_0 is a zero path while $p_1\alpha_1$ is non-zero. If (ID1) occurs, then there is a zero-relation with terminal arrow β_0 such that $e(\beta_0) = e(p_1) = s(\alpha_1)$. Now $\beta_0\alpha_1$ is a zero-relation since p_1 is non-trivial and $p_1\alpha_1$ is non-zero. Thus $\beta_0 p$ gives rise to a contradiction to (GD2). If $e(p_1)$ is the end-point of a binomial relation, then it is of the form (u_0, v_0p_1) with u_0, v_0 some non-trivial paths. Note that $v_0p_1\alpha_1$ is a zero path. Thus $v_0p_1p_0$ contains two distinct zero-

relation since $p_1\alpha_1$ is non-zero, and hence a sequential pair of zero-relations. In all other cases, observing that p_i is non-trivial for all $1 \le i \le n$, it is easy to verify that there is a path u_r such that $u_r p_r$ is a zero path for some $1 \le r \le n$. Thus $(u_r p_r) q_{r-1}^{-1} \cdots q_1^{-1}(p_1 p_0)$ contains a sequential pair of zero-relations.

(2) Case (PD2) occurs, that is there is a zero-relation q_{n+1} with initial arrow α_{n+1} such that $p_n^{-1}\alpha_{n+1}$ is a reduced walk. If (ID2) occurs or $e(p_n)$ is the end-point of a binomial relations, then there is an arrow β_{n+1} such that $\beta_{n+1}p_n$ is a non-zero path. Thus $\beta_{n+1}\alpha_{n+1}$ is a zero-relation. Then $\beta_{n+1}q_{n+1}$ gives rise to a contradiction to (GD2). In all other cases, observing that the q_i with $0 < i < n$ are non-trivial, it is easy to verify that there is a path v_s such that $v_s q_s$ is a zero path and $v_s q_s p_{s+1}^{-1}$ is a reduced walk for some $0 \le s < n$. Thus $(v_s q_s) p_{s+1}^{-1} \cdots p_n^{-1}(q_{n+1})$ contains a sequential pair of zero-relations.

(3) Case (PD3) occurs, that is for some $1 < s \le n$, there is a non-zero path u_s such that $p_s u_s$ and $q_{s-1}u_s$ are both zero paths. Note that $e(p_s)$ is not the end-point of a binomial relation by (GD3), moreover, the q_i with $0 < i < s$ and the p_i with $s \le i \le n$ are non-trivial. It is easy to verify that each of (IDi) with $1 \le i \le 4$ implies that either for some $0 \le r < s$, there is a path v_r such that $v_r q_r$ is a zero path and $v_r q_r p_{r+1}^{-1}$ is a reduced walk; or for some $s \le t \le n$, there is a path v_t such that $v_t p_t$ is a zero path. In the first case $(v_r q_r) p_{r+1}^{-1} \cdots p_{s-1}^{-1}(q_{s-1}u_s)$ contains a sequential pair of zero-relations. In the second case, $(v_t p_t) q_{t-1}^{-1} \cdots q_s^{-1}(p_s u_s)$ contains a sequential pair of zero-relations.

(4) The vertex $s(p_1)$ is the start-point of a binomial relation (p_1u_1, q_1v_1), where u_1, v_1 are some non-trivial paths. If (ID1) occurs, then there is a zero path v_0 with terminal arrow β_0 such that $\beta_0 p_1^{-1}$ is a reduced walk. Then $\beta_0\gamma_1$ is a zero-relation, where γ_1 is the initial arrow of u_1. Now $y_0\gamma_1$ gives rise to a contradiction (GD2). If (ID2) occurs, then there is path y_n with terminal arrow β_n such that $y_n p_n$ is a zero path while $\beta_n p_n$ is non-zero. If $n = 1$, then $\beta_1 p_1 u_1$ is a zero path. Therefore $y_1 p_1 u_1$ contains two distinct zero-relations since $\beta_1 p_1$ is a non-zero path, and hence a sequential pair of zero-relations. If $n > 1$, then $p_2 v_1$ is a zero-path. Thus $(y_n p_n) q_{n-1}^{-1} \cdots q_3^{-1}(p_2 v_1)$ contains a sequential pair of zero-relations. Note now that $e(p_1)$ and $e(q_1)$ are not end-points of binomial relations by (GD1) and there is no non-zero path y such that $e(y) = s(p_1)$ and yp_1 and yq_1 are both zero paths. If (ID3) or (ID4) occurs, then it is easy to see that $n > 1$ and for some $2 \le r \le n$, there is a path v_r such that $v_r p_r$ is a zero path. Therefore $(v_r p_r) q_{r-1}^{-1} \cdots q_3^{-1}(p_2 v_1)$ contains a sequential pair of zero-relations.

(5) Assume that $n > 1$ and $s(p_n)$ is the start-point of a binomial relation $(p_n u_n, v_n)$, where u_n, v_n are some non-trivial paths. If (ID2) occurs, then there is a non-trivial path y_n with terminal arrow β_n such that $y_n p_n$ is a zero path while $\beta_n p_n$ is non-zero. Note that $\beta_n p_n u_n$ is a zero path. Thus $y_n p_n u_n$ contains two distinct zero-relations since $\beta_n p_n$ is non-zero, and hence a sequential paire of zero-relations. Note that $e(p_n)$ is not the end-point of a binomial relation. Thus in all other cases, for some $0 \le s < n$, there is a non-zero path y_s such that $y_s q_s$ is a zero-path and $y_s q_s p_{s+1}^{-1}$ is reduced walk. Note that $q_{n-1} u_n$ is a zero path. Thus $(y_s q_s) p_{s+1}^{-1} \cdots p_{n-1}^{-1} (q_{n-1} u_n)$ contains a sequential pair of zero-relations.

(6) Assume that $n > 1$ and some $s(p_s)$ with $1 < s < n$ is the start-point of a binomial relation $(p_s u_s, q_s v_s)$, where u_s, v_s are some non-trivial paths. Note that $e(q_{s-1})$ and $e(q_s)$ are not end-points of binomial relations by (GD1). It is then easy to check that each of the (IDi) with $1 \le i \le 4$ implies that either for some $0 \le r < s$ there is a non-trivial path y_r such that $y_r q_r$ is a zero path and $y_r q_r p_{r+1}^{-1}$ is a reduced walk; or for some $s < t \le n$, there is a non-trivial path y_t such that $y_t p_t$ is a zero path. Note that $q_{s-1} u_s, p_{s+1} v_s$ are both zero paths. Thus in the first case, $(y_r q_r) p_{r+1}^{-1} \cdots p_{s-1}^{-1} (q_{s-1} u_s)$ contains a sequential pair of zero-relation. Similarly in the second case, $(y_t p_t) q_{t-1}^{-1} \cdots q_{s+1}^{-1} (p_{s+1} v_s)$ contains a sequential pair of zero-relation. The proof is now completed.

4.3. Lemma. *Let $A = kQ/I$ be a special biserial algebra of global dimension two. If there is a band module of projective and injective dimensions both equal to two, then (Q, I) admits a sequential pair of zero-relations.*

Proof. First (Q, I) satisfies (GD1), (GD2) and (GD3) as stated in Theorem 3.4. Let $w = p_1 q_1^{-1} \cdots p_n q_n^{-1}$ be a band, where $n \ge 1$, the p_i and the q_i are non-trivial paths such that $s(p_1) = s(q_n)$. Let N be a band module of support w such that N is of projective and injective dimensions both equal to two. Define $p_0 = p_n, p_{n+1} = 1$ and $q_0 = q_n, q_{n+1} = 1$. By Lemma 2.7, either some $s(p_r)$ with $1 \le r \le n$ is the start-point of a binomial relation $(q_{r-1} q, p_r p)$ or there is a path u_s such that $p_s u_s$ and $q_s u_s$ are both zero paths for some $1 \le s \le n$. By the dual of Lemma 2.7, either some $e(q_m)$ with $1 \le m \le n$ is the end-point of a binomial relation $(u p_m, v q_m)$ or there is a path v_t such that $v_t p_t$ and $v_t q_{t-1}$ are both zero paths for some $1 \le t \le n$.

Suppose first that the binomial relation $(q_{r-1} q, p_r p)$ with $1 \le r \le n$ exists. Then $p_{r-1} q, q_r p$ are both zero paths. If the binomial relation $(u p_m, v q_m)$ with $1 \le m \le n$ exists, then $m \ne r - 1$ and $m \ne r$ by (GD1). Note that $v p_{m+1}$ and $u q_{m-1}$ are both zero paths. Thus $(v p_{m+1}) q_{m+1}^{-1} \cdots q_{r-2}^{-1} (p_{r-1} q)$

contains a sequential pair of zero-relations provided that $m < r-1$ and $(uq_{m-1})p_{m-1}^{-1}\cdots p_{r+1}^{-1}(q_r p)$ contains a sequential pair of zero-relations provided that $m > r$. If the path v_t exists such that $v_t p_t$ and $v_t q_{t-1}$ are both zero paths for some $1 \le t \le n$, then $t \neq r$ by (GD3). Thus $(v_t p_t)q_{t+1}^{-1}\cdots q_{r-1}^{-1}(p_{r-1}q)$ contains a sequential pair of zero-relations provided that $t < r$ and $(v_t q_{t-1})p_{t-1}^{-1}\cdots p_{r+1}^{-1}(q_r p)$ contains a sequential pair of zero-relations provided that $t > r$.

Suppose now that the path u_s exists such that $p_s u_s$ and $q_s u_s$ are both zero paths for some $1 \le s \le n$. If the binomial relation (up_m, vq_m) with $1 \le m \le n$ exists, then $m \neq s$ by (GD3). Note that vp_{m+1} and uq_{m-1} are both zero paths. Thus $(vp_{m+1})q_{m+1}^{-1}\cdots q_{s-1}^{-1}(p_s u_s)$ contains a sequential pair of zero-relations provided that $m < s$ and $(uq_{m-1})p_{m-1}^{-1}\cdots p_{s+1}^{-1}(q_s u_s)$ contains a sequential pair of zero-relations provided that $m > s$. If the path v_t exists such that $v_t p_t$ and $v_t q_{t-1}$ are both zero paths for some $1 \le t \le n$, then $(v_t p_t)q_t^{-1}\cdots q_{s-1}^{-1}(p_s u_s)$ contains a sequential pair of zero-relations if $t \le s$; and otherwise $(v_t q_{t-1})p_{t-2}^{-1}\cdots p_{s+1}^{-1}(q_s u_s)$ contains a sequential pair of zero-relations. This completes the proof.

We are now able to obtain our main result as follows.

4.4. Theorem. *Let $A = kQ/I$ be a special biserial algebra which is not a string algebra. Then A is tilted if and only if (Q, I) satisfies the following*:

(1) *There is no sequential pair of zero-relations.*

(2) *The start-point of a binomial relation does not lie in another different binomial relation.*

(3) *Let $(\alpha p\beta, \gamma q\delta)$ be a binomial relation, where $\alpha, \beta, \gamma, \delta$ are some arrows and p, q are some paths. If u is a non-zero path with $e(u) = s(\alpha)$, then either $u\alpha p$ or $u\gamma q$ is non-zero. Dually if v is a non-zero path with $s(v) = e(\beta)$, then either $p\beta v$ or $q\delta v$ is non-zero.*

Proof. Since A is not a string algebra, there is at least one indecomposable projective-injective module. Thus it follows from a result of Coelho and Skowroński [**3**] that A is tilted if and only if A is quasi-tilted. We shall show that A is quasi-tilted if and only if (Q, I) satisfies the conditions as stated in the theorem.

Assume first that (Q, I) satisfies (1), (2) and (3). By theorem 3.4, the global dimension of A is at most two. Let M be an indecomposable module in $\operatorname{mod} A$ which is not projective-injective. Then M is either a string module or a band module by Theorem 1.4. It follows now from Lemmas 4.2 and 4.3

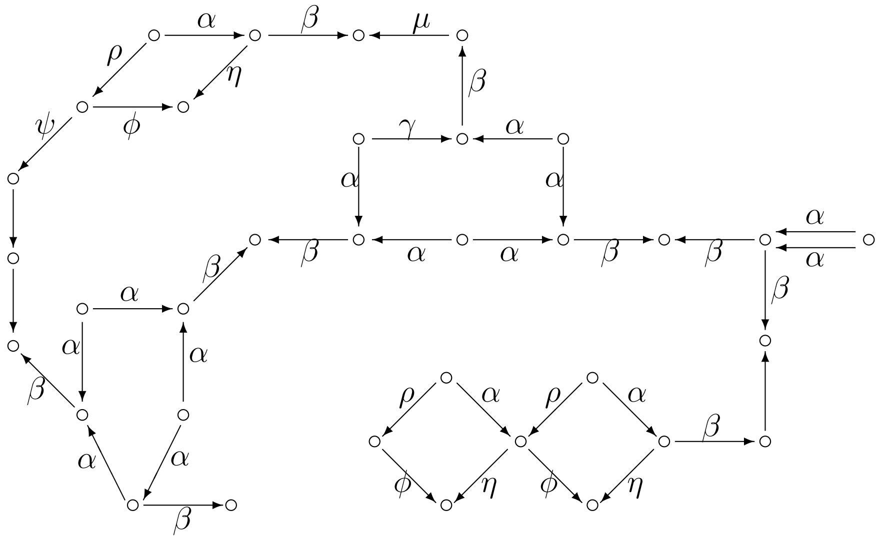
that M is either of projective dimension at most one or of injective dimension at most one. Therefore A is quasi-tilted.

Conversely assume that A is quasi-tilted. In particular the global dimension of A is at most two. By Theorem 3.4, (Q, I) satisfies (2) and (3). Assume on the contrary that (Q, I) admits a sequential pair of zero-relation $\tilde{w}$. If $\tilde{w} = p_1p_2p_3$, where p_1, p_2, p_3 are non-trivial paths and p_1p_2 and p_2p_3 are the only zero-relations contained $\tilde{w}$. By Theorem 3.4, p_2 is not a string. Then there is a binomial relation $(\alpha_1 u \alpha_2, p_2)$, where α_1, α_2 are arrows and u_1 is a path. Let β_1 be the terminal arrow of p_1 and β_3 the initial arrow of p_3. Then $\beta_1\alpha_1, \alpha_2\beta_3$ are zero-relations. Therefore $\beta_1\alpha_1 u \alpha_2\beta_3$ is a seuqential pair of zero-relations with u_1 a string.

Thus we may assume that $\tilde{w}$ is of the form $\tilde{w} = pwq$, where w is a string and p, q are paths which are the only zero-relations contained in $\tilde{w}$. We may further assume that $\tilde{w}$ is such that w is of minimal length. Let α be the terminal arrow of p and β the initial arrows of q. We claim that αw and $w\beta$ are strings. In fact, we write $w = p_1^{-1}q_1 \cdots p_n^{-1}q_n$, where the p_i and the q_j are paths which are non-trivial for $1 < i \le n$ and $1 \le j \le n$. Note that $w\beta = p_1^{-1}q_1 \cdots p_n^{-1}q_n\beta$ is a reduced walk without zero-relations. Suppose that $w\beta$ is not a string. Then $q_n\beta$ is a maximal subpath of a binomial relation, which is necessarily of the form $(p_n\gamma u_n, q_n\beta)$, where γ is an arrow and u_n is a path. If $n = 1$, then $\alpha\gamma$ is a zero-relation. The path $p_1\gamma$ gives rise to a contradiction to Lemma 3.3. Thus $n > 1$, hence p_n and q_{n-1} are non-trivial. Write $q_{n-1} = v_{n-1}\delta$, where v_{n-1} is a path and δ is an arrow. Then $\delta\gamma$ is a zero-relation. Let $w_1 = p_1^{-1}q_1 \cdots p_{n-1}^{-1}v_{n-1}$. Then w_1 is a proper substring of w such that $p_1w_1\delta\gamma$ is a sequential pair of zero-relation. This contradicts the minimality of the length of w. Thus $w\beta$ is a string, and so is αw by duality. It follows now from Lemma 2.3 and its dual that the string module $M(w)$ is of projective and injective dimensions both greater than one, which is a desired contradiction. The proof is completed.

Combining our main result in [**12**] with the above theorem, we obtain a complete characterization of tilted special biserial algebras in terms of bound quivers.

Example. Consider the algebra defined by the bound quiver



where the relations are $\rho\psi$, $\alpha\phi$, $\rho\eta$, $\gamma\beta\mu$ and all possible paths $\alpha\beta$ as well as all possible differences $\rho\phi - \alpha\eta$. This is a special biserial algebra satisfying the conditions (1), (2) and (3) as stated in the above theorem. Thus it is a tilted algebra.

We conclude the paper with some remarks. The module category of a tilted special biserial algebra is well-understood. In fact one easily read off its Auslander-Reiten quiver from its bound quiver. To be more precise, let $A = kQ/I$ be a special biserial algebra which is not hereditary of type $\tilde{A}_n$ and let Γ_A the Auslander-Reiten quiver of A. Then a component of Γ is either of shape $\mathbf{N}\tilde{A}_n$ or $(-\mathbf{N})\tilde{A}_n$, or a standard tube or the connecting component. Assume that (Q, I) admits $r(\geq 0)$ full bound subquivers of type $\tilde{A}_n$ (of which r_1 is not of type $\tilde{A}_2$) having an arrow entering them; and $s(\geq 0)$ full bound subquivers of type $\tilde{A}_n$ (of which s_1 is not of type $\tilde{A}_2$) having an arrow leaving them. Then Γ_A contains exactly $r + s$ standard orthogonal tubular families, r components of shape $\mathbf{N}\tilde{A}_n$, s components of shape $(-\mathbf{N})\tilde{A}_n$ and $2(r_1 + s_1)$ non-homogeneous tubes.

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References

[1] F. M. Bleher, *Special biserial algebras and their automorphisms*, preprint.

[2] M. C. R. Butler and C. M. Ringel, *Auslander-Reiten sequences with few middle terms and applications to string algebras*, Comm. Algebra, **15** (1987), 145 - 179.

[3] F. Coehlo and A. Skowroński, *On Auslander-Reiten components for quasi-tilted algebras*, Fund. Math., **149** (1996), 67 - 82.

[4] W. W. Crawley-Boevey, *Maps between representations of zero-relation algebras*, J. Algebra, **126** (1989), 259 - 263.

[5] K. R. Fuller, *Biserial Rings*, Lecture Notes in Mathematics, **734** (Springer, Berlin, 1979), 64 - 90.

[6] P. Gabriel, *Auslander-Reiten sequences and representation-finite algebras*, Lecture Notes in Mathematics, **831** (Springer, Berlin, 1980), 1 - 71.

[7] E. L. Green, D. Happel and D. Zacharia, *Projective resolutions over algebras with zero relations*, Illinois J. Math., **29** (1985), 180 - 190.

[8] D. Happel, I. Reiten and S. O. Smalø, *Tilting in abelian categories and quasitilted algebras*, Memoirs Amer. Math. Soc., **575** (1996).

[9] D. Happel and C. M. Ringel, *Tilted Algebras*, Trans. Amer. Math. Soc., **274** (1982), 399 - 443.

[10] D. Happel and D. Vossieck, *Minimal algebras of infinite representation type with preprojective component*, Manucripta Math., **42** (1983), 221 - 243.

[11] F. Huard, *Tilted gentle algebras*, Comm. Algebra, (1) **26** (1998) 63 - 72.

[12] F. Huard and Shiping Liu, *Tilted string algebras*, preprint.

[13] H. Krause, *Maps between tree and band modules*, J. Algebra, **126** (1991), 186 - 194.

[14] J. A. de la Peña, *Representation-finite algebras whose Auslander-Reiten quiver is planar*, J. London Math. Soc., (2) **32** (1985), 62 - 74.

[15] C. M. Ringel, *Tame algebras and integral quadratic forms*, Lecture Notes in Mathematics, **1099** (Springer, Berlin, 1984).

[16] A. Skowroński, *On nonexistence of oriented cycles in Auslander-Reiten quivers*, Acta Universitatis Carolinae — Mathematica et Physica, **25** (1984), 45 - 52.

[17] A. Skowroński and J. Waschbüsch, *Representation-finite biserial algebras*, J. Reine Angew Math., **345** (1983) 172 - 181.

[18] B. Wald and J. Waschbüsch, *Tame biserial algberas*, J. Algebra, **95** (1985), 480 - 500.

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