The strong no loop conjecture for special biserial algebras

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Dedicated to Claus M. Ringel on the occasion of his sixtieth birthday

Introduction

Let A be a finite dimensional algebra over a field given by a quiver with relations. Let S be a simple A-module with a non-split self-extension, that is, the quiver has a loop at the corresponding vertex. The *strong no loop conjecture* claims that S is of infinite projective dimension; see [1, 6]. This conjecture remains open except for monomial algebras; see, for example, [2, 6, 8, 11]. Under certain hypothesis on the loop, Green, Solberg and Zacharia have shown that $\mathrm{Ext}^i_A(S,S)$ does not vanish for every $i \geq 1$; see [5, (4.2)]. In this paper, we shall first present a short proof of this result, because not only is the original proof rather complex, but also our idea possibly works for other cases. Next we observe that this result reduces the conjecture to the case where some power of the loop is a component of a polynomial relation. This reduction works particularly well when A is special biserial, due to a combinatorial description of the syzygies of string modules; see (2.2). Our main result says that $\mathrm{Ext}^i_A(S,S)$ does not vanish for every $i \geq 1$ if the convex support of S is special biserial. We shall also prove that if S has an almost split self-extension, then the block of A supporting S is a local Nakayama algebra, in particular, $\mathrm{Ext}^i_A(S,S)$ does not vanish for every $i \geq 1$. In the course of its proof, we easily get a characterization of Nakayama algebras, strengthening the one stated in [1, (IV.2.10)]. Contrary to what will be seen in this paper, Happel's example stated in [5, Section 4] shows the existence of a simple module S with a loop but $\mathrm{Ext}^i(S,S) = 0$ for infinitely many i. Our motivation for studying special biserial algebras comes from the following two aspects. First of all, since their representations are completely understood, they form naturally a testing class for various well-known conjectures in the representation theory of algebras. Secondly, these algebras play an important role in the modular representation theory of finite groups; see [7], tracing back to the classification of the indecomposable Harish-Chandra modules of the Lorentz group by Gelfand and Ponomarev; see [4].

1. Loops with no power a component of polynomial relations

To begin with, we fix some notation and terminology. Throughout this paper, k stands for a field and Q for a finite quiver. Let kQ be the path algebra of Q over k, and Q^+ the ideal generated by the arrows. Note that we shall compose the paths of Q from the left to the right. If I is an ideal such that $(Q^+)^m \subseteq I \subseteq (Q^+)^2$ for some $m \geq 2$, then the pair (Q, I) is called a *bound quiver*. We shall always assume that I is such an ideal. Let $p_1, \ldots, p_r$ be pairwise distinct paths of Q from a vertex a to a vertex b, and let $\lambda_1, \ldots, \lambda_r \in k$ be nonzero scalars. We call

$$\rho = \lambda_1 p_1 + \cdots + \lambda_r p_r$$

a *relation on* Q if $\rho \in I$ while $\sum_{i\in\Omega} \lambda_i p_i \notin I$ for all $\Omega \subset \{1, \dots, r\}$. In this case, $p_1, \dots, p_r$ are called the *components* of ρ and a the *start-point*. Moreover, ρ is called *monomial, binomial,* or *polynomial* if $r = 1$, $r = 2$, or $r \geq 2$, respectively.

The quotient $A = kQ/I$ is called the *algebra* of the bound quiver (Q, I). If $x \in kQ$, we shall denote by $\bar{x}$ the class $x + I \in A$. An A-module means a right module of finite k-dimension except otherwise stated explicitly. The radical, the top, and the n-th syzygy of an A-module M will be denoted by $\operatorname{rad} M$, $\operatorname{top} M$ and $\Omega^n(M)$, respectively. We call $x \in M$ a *top element* if $x \notin \operatorname{rad} M$. For a vertex a of Q, we shall denote by $S(a)$ and $P(a)$ the simple A-module and the indecomposable projective A-module associated to a, respectively.

The elements of a direct sum of A-modules are written as column matrices. Let $e_1, \dots, e_r$ be idempotents of A. Then an A-linear map $\phi : e_1A \oplus \cdots \oplus e_rA \to M$ is left multiplication by a matrix $(x_1, \dots, x_r)$ with $x_i \in Me_i$. In this case, we say that ϕ is *represented* by $(x_1, \dots, x_r)$. In particular, if $f_1, \dots, f_s$ are also idempotents of A, then an A-linear map from $e_1A \oplus \cdots \oplus e_rA$ to $f_1A \oplus \cdots \oplus f_sA$ is represented by a $(s \times r)$-matrix whose (i, j)-entry is an element of f_iAe_j. For convenience of reference, we state the following well-known result.

1.1. Lemma. *Let A be the algebra of a finite bound quiver with $e_1, ..., e_s$ some primitive idempotents. Let M be a non-zero A-module with $x_i \in Me_i$ for $1 \leq i \leq s$. If the classes of $x_1, ..., x_s$ in $\operatorname{top} M$ are linearly independent over k, then there exist primitive idempotents $e_{s+1}, ..., e_r \in A$ and $x_{s+1} \in Me_{s+1}, \dots, x_r \in Me_r$ such that*

$$(x_1, \dots, x_s, x_{s+1}, \dots, x_r) : \; e_1A \oplus \cdots \oplus e_sA \oplus e_{s+1}A \oplus \cdots \oplus e_rA \longrightarrow M$$

is a projective cover of M.

We now give the promised alternative proof of Proposition 4.2 in [5].

1.2. Proposition (Green-Solberg-Zacharia). *Let A be the algebra of a bound quiver (Q, I), containing a loop α at a vertex a. If for some $n \geq 2$, α^n lies in I but not in $IQ^+ + Q^+I$, then for all $i \geq 1$, $\operatorname{Ext}^i_A(S(a), S(a))$ does not vanish.*

Proof. Suppose that $\alpha^n \in I$ but $\alpha^n \notin IQ^+ + Q^+I$. In particular, $\alpha^{n-1} \notin I$. Let $\alpha_i : a \to c_i$, $i = 1, \dots, t$, be the arrows starting at a with $\alpha_1 = \alpha$. Then $S(a)$ admits a minimal projective resolution

$$\cdots \longrightarrow P_m \xrightarrow{\phi_m} P_{m-1} \longrightarrow \cdots \longrightarrow P_1 \xrightarrow{\phi_1} P_0 \longrightarrow S(a) \longrightarrow 0,$$

where $P_0 = P(a)$, $P_1 = P(c_1) \oplus \cdots \oplus P(c_t)$, and ϕ_1 is represented by $(\bar{\alpha}_1 \dots, \bar{\alpha}_t)$. Assume, for $m \geq 1$, that $P_m \cong P(b_1) \oplus P(b_2) \oplus \cdots \oplus P(b_s)$ with $s \geq 1$ and $P_{m-1} \cong P(a_1) \oplus P(a_2) \oplus \cdots \oplus P(a_r)$ with $r \geq 1$, where the a_i, b_j are vertices with $a_1 = b_1 = a$, while ϕ_m is isomorphic to the map represented by a matrix of the following form:

$$\begin{pmatrix} \bar{x}_{11} & \bar{x}_{12} & \cdots & \bar{x}_{1s} \\ 0 & \bar{x}_{22} & \cdots & \bar{x}_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \bar{x}_{r2} & \cdots & \bar{x}_{rs} \end{pmatrix},$$

where x_{ij} is a linear combination of non-trivial paths from a_i to b_j with x_{11} being α or α^{n-1}. We shall show that ϕ_{m+1} is isomorphic to the map represented by a matrix of this form. By Lemma 1.1, it suffices to prove that $\ker \phi_m$ contains a top element of the form $(\bar{y}_{11}, 0, \dots, 0)^T$ with $y_{11} = \alpha$ or α^{n-1}. Indeed, if $x_{11} = \alpha^{n-1}$, then $(\bar{\alpha}, 0, \ldots, 0)^T$ clearly lies in $\ker \phi_m$ but not in its radical since $\ker \phi_m \subseteq \operatorname{rad} P_m$. If $x_{11} = \alpha$, then $Z = \left(\bar{\alpha}^{n-1}, 0, \ldots, 0\right)^T \in \ker \phi_m$. Assume on the contrary that Z lies in the radical of $\ker \phi_m$. Then $Z = Y\bar{u}$ with $Y \in \ker \phi_m$ and $u \in Q^+$. Since $Y \in \operatorname{rad} P_m$, we may write $Y = (\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_s)^T$ with $y_i \in Q^+$. Now $v = \alpha^{n-1} - y_1 u$, $y_2 u$, $\ldots$, $y_s u \in I$ since $Z = Y\bar{u}$, and $w = \alpha y_1 + x_{12} y_2 + \cdots + x_{1s} y_s \in I$, since $Y \in \ker \phi_m$. Therefore, $\alpha^n = \alpha v + \alpha y_1 u = \alpha v + wu - (x_{12} y_2 u + \cdots + x_{1s} y_s u) \in Q^+ I + I Q^+$, a contradiction. Thus Z is indeed a top element of $\ker \phi_m$. By induction, we have shown that $P(a)$ is a direct summand of P_m for all $m \geq 1$. This completes the proof of the proposition.

We now deduce some useful consequences from the above result.

1.3. Corollary. *Let A be the algebra of a bound quiver (Q, I), containing a loop α at a vertex a. If no power of α is a component of a polynomial relation, then for all $i \geq 1$, $\operatorname{Ext}^i_A(S(a), S(a))$ does not vanish.*

Proof. Let $n \geq 2$ be minimal such that $\alpha^n \in I$, and assume that α^{n-1} is not a component of any polynomial relation. Note that for a relation ρ on Q and $x, y \in kQ$, $x\rho y$ is either zero or a sum of relations on Q. Therefore, if $\alpha^n \in Q^+ I + I Q^+$, then

$$\alpha^n = (\beta_1 \rho_1 + \cdots + \beta_r \rho_r) + (\rho_{r+1} \beta_{r+1} + \cdots + \rho_s \beta_s),$$

where the ρ_i are relations on Q and the β_i are arrows. Hence, α^{n-1} is a component of at least one of the ρ_i, say ρ_1. By the minimality of n, ρ_1 is a polynomial relation, a contradiction. The proof is now completed by applying Proposition 1.2.

The above result implies immediately the following.

1.4. Corollary. *Let A be the algebra of a bound quiver (Q, I), containing a loop α at a vertex a. If $\alpha^2 \in I$, then $\operatorname{Ext}^i_A(S(a), S(a))$ does not vanish for every $i \geq 1$.*

We shall now study simple modules with an almost split self-extension, that is, invariant under the Auslander-Reiten translation $\tau = \mathrm{DTr}$; see [1, (IV.1)]. For this purpose, we need the following result, which is interesting in its own right.

1.5. Proposition. *Let A be the algebra of a bound quiver (Q, I), and let a, b be vertices of Q. Then $\tau S(a) \cong S(b)$ if and only if Q contains an arrow $\alpha : a \to b$ which is the only arrow starting at a and the only one ending at b.*

Proof. If Q satisfies the stated property, then we have a non-split exact sequence :

$$0 \longrightarrow S(b) \longrightarrow M(\alpha) \longrightarrow S(a) \longrightarrow 0.$$

Since α is the only arrow starting at a or ending at b, one verifies easily that the sequence is almost split. Conversely assume that $S(b) \cong \tau S(a)$. Let $\alpha_i : a \to b_i$, $i = 1, \ldots, r$, be the arrows starting at a; and $\beta_j : a_j \to b$, $j = 1, \ldots, s$, be those ending at b. Then $S(a)$ has a minimal projective presentation

$$\delta : \quad P(b_1) \oplus \cdots \oplus P(b_r) \longrightarrow P(a) \longrightarrow S(a) \longrightarrow 0.$$

Applying the duality $\operatorname{Hom}_A(-, A_A)$ to δ, we get a minimal projective presentation

$$\delta^* : \quad Q(a) \longrightarrow Q(b_1) \oplus \cdots \oplus Q(b_r) \longrightarrow \operatorname{Tr} S(a) \longrightarrow 0$$

of the transpose of $S(a)$, where $T(c)$ and $Q(c)$ denote, respectively, the simple and the indecomposable projective left A-module associated to a vertex c. On the other hand, $T(b)$ has a minimal projective presentation

$$\eta : \quad Q(a_1) \oplus \cdots \oplus Q(a_s) \longrightarrow Q(b) \longrightarrow T(b) \longrightarrow 0.$$

Noting $\operatorname{Tr} S(a) \cong \operatorname{D} S(b) \cong T(b)$, we have $\eta \cong \delta^*$. In particular, $r = 1$ with $b_1 = b$ and $s = 1$ with $a = a_1$. The proof of the proposition is completed.

We call a module *homogeneous* if it admits an almost split self-extension.

1.6. COROLLARY. *Let A be the algebra of a connected bound quiver (Q, I). Then A admits a homogeneous simple module if and only if Q consists of one loop. In this case, for any indecomposable non-projective A-module M, $\operatorname{Ext}^i_A(M, M)$ does not vanish for every $i \geq 1$.*

Proof. By Proposition 1.5, Q has a vertex a such that $\tau S(a) \cong S(a)$ if and only if Q contains a loop $\alpha : a \to a$ which is the only arrow starting at or ending at a. This is equivalent to Q consisting of one loop since Q is connected. The rest of the statement is well-known. This completes the proof of the corollary.

An artin algebra is called a *Nakayama algebra* if its indecomposable modules are all uniserial. The algebra of a connected bound quiver is a Nakayama algebra if and only if the quiver is a single path (maybe trivial) or an oriented cycle. The following is another characterization of this class of algebras; compare [1, (IV.2.10)].

1.7. THEOREM. *Let A be the algebra of a connected bound quiver (Q, I). Then A is a Nakayama algebra if and only if there exists a τ-orbit $\mathcal{O}$ consisting of simple A-modules. In this case, $\mathcal{O}$ contains all simple A-modules.*

Proof. If Q is a path $a_1 \to \cdots \to a_n$ with $n \geq 1$ and the a_i pairwise distinct, by Proposition 1.5, $\tau S(a_i) = S(a_{i+1})$ for $1 \leq i < n$. Since $S(a_n)$ is projective and $S(a_1)$ is injective, the $S(a_i)$ form a τ-orbit. If Q is an oriented cycle $a_1 \to \cdots \to a_n \to a_1$ with $n \geq 1$ and the a_i pairwise distinct, then $\tau S(a_i) = S(a_{i+1})$ for $1 \leq i < n$, and $\tau S(a_n) = S(a_1)$. Hence the $S(a_i)$ form a τ-orbit.

Conversely, let $\mathcal{O} = \{S(a_1), \ldots, S(a_n)\}$ be a τ-orbit with the a_i pairwise distinct vertices of Q. Consider first the case $n = 1$. If $S(a_1)$ is projective, then Q consists of the vertex a_1. Otherwise, $\tau S(a_1) = S(a_1)$, and hence Q consists of one loop by Corollary 1.6. Assume now that $n > 1$ and that $\tau S(a_i) = S(a_{i+1})$ for $1 \leq i < n$. For each $1 \leq i < n$, by Proposition 1.5, Q contains an arrow $\alpha_i : a_i \to a_{i+1}$, the only one starting at a_i and the only one ending at a_{i+1}. If $S(a_n)$ is projective, then $S(a_1)$ is injective. Hence Q contains no arrow starting at a_n and the only one ending at a_1. Thus Q consists of the path $a_1 \to \cdots \to a_{n-1} \to a_n$. Otherwise, $\tau S(a_n) = S(a_1)$. Then Q contains an arrow $\alpha_n : a_n \to a_1$, the only one starting at a_n or ending at a_1. Thus Q consists of the oriented cycle $a_1 \to \cdots \to a_n \to a_1$. The proof of the theorem completed.

2. SPECIAL BISERIAL ALGEBRAS

The objective of this section is to establish the conjecture for special biserial algebras. Recall that a finite-dimensional k-algebra is called *special biserial* if it is isomorphic to the algebra of a bound quiver (Q, I) satisfying (1) each vertex is the start-point of at most two arrows and the end-point of at most two arrows; and (2) for each arrow β, there exists at most one arrow α such that $\alpha\beta \notin I$ and at most one arrow γ such that $\beta\gamma \notin I$. We call such a bound quiver *special biserial*.

Let (Q, I) be a special biserial bound quiver. Then a relation on Q is either monomial or binomial; see [9]. Moreover, if a vertex is the start-point of a binomial relation, then such a relation is unique up to a scalar. Thus we may assume, without changing its algebra, that every binomial relation of (Q, I) is a multiple of a binomial relation of the form $p - q$. In this case, we call (p, q) a *binomial pair.*

For a vertex a, the trivial path at a is denoted by ε_a, and for a path p, its start-point and end-point are denoted by $s(p)$ and $e(p)$, respectively. For an arrow α, we introduce a new arrow α^{-1}, the *inverse* of α with $s(\alpha^{-1}) = e(\alpha)$ and $e(\alpha^{-1}) = s(\alpha)$. A *reduced walk* w in Q is either a trivial path or $w = c_1c_2 \cdots c_n$ with $n \geq 1$, where c_i is either an arrow or the inverse of an arrow such that $s(c_{i+1}) = e(c_i)$ and $c_{i+1} \neq c_i^{-1}$ for all $1 \leq i < n$. In the latter case, a path $p = \alpha_1 \cdots \alpha_r$ with α_i arrows is *contained* in w if there exists i with $1 \leq i \leq n$ such that either $c_{i+j} = \alpha_{j+1}$ for all $0 \leq j < r$ or $c_{i+j} = \alpha_{r-j}^{-1}$ for all $0 \leq j < r$. A reduced walk w is a *string* if no path contained in w is a component of a relation on Q. A string p is called *serial* if it is a path and $s(p)$ is not the start-point of any binomial relation. Finally, we say that a string w *starts* or *ends* in a *deep* if there is no arrow γ such that $\gamma^{-1}w$ or $w\gamma$ is a string, respectively.

Let $A = kQ/I$. To each string w, one associates a *string module* $M(w)$, compare [3, 10], as follows: if $w = \varepsilon_a$ for a vertex a, then $M(w) = S(a)$ with $\{a\}$ a k-basis. If $w = c_1c_2 \cdots c_n$ with c_i an arrow or the inverse of an arrow, then $M(w)$ has as a k-basis the ordered family $\{a_0, a_1, \ldots, a_n\}$, where $a_0 = s(c_1)$ and $a_i = e(c_i)$ for $1 \leq i \leq n$. Its multiplication is such that for an arrow α, one has $a_i\bar{\alpha} = a_{i+1}$ if $c_{i+1} = \alpha$ with $0 \leq i < n$, and $a_i\bar{\alpha} = a_{i-1}$ if $c_i = \alpha^{-1}$ with $1 \leq i \leq n$, and $a_i\bar{\alpha} = 0$ otherwise. Here a_i with $0 \leq i \leq n$ is a top element if and only if c_i is the inverse of an arrow whenever $i \geq 1$, and c_{i+1} is an arrow whenever $i < n$.

Furthermore, for a vertex a of Q, we shall fix a k-basis of $P(a)$ with a multiplication. If ε_a is serial, then there exist paths p, q starting at a such that $p^{-1}q$ is a string starting and ending in a deep. In this case, $P(a) = M(p^{-1}q)$. Otherwise, there exists a binomial relation $\alpha_1\alpha_2 \cdots \alpha_r - \beta_1\beta_2 \cdots \beta_s$, where α_i, β_j are arrows with $s(\alpha_1) = s(\beta_1) = a$. Then $P(a)$ has as a k-basis the ordered family $\{a_1, \ldots, a_r, b_1, \ldots, b_s\}$, where $a_i = s(\alpha_i)$ and $b_j = e(\beta_j)$. Its multiplication is such that for each arrow α, one has $b_j\bar{\alpha} = b_{j+1}$ if $\alpha = \beta_{j+1}$ with $1 \leq j < s$ and $b_j\bar{\alpha} = 0$ otherwise; moreover,

$$a_i\bar{\alpha} = \begin{cases} b_1, & i = 1 \text{ and } \alpha = \beta_1, \\ a_{i+1}, & 1 \leq i < r \text{ and } \alpha = \alpha_i, \\ b_s, & i = r \text{ and } \alpha = \alpha_r, \\ 0, & \text{otherwise.} \end{cases}$$

Here a_1 is the only top element. In all cases, we call the fixed k-basis of $M(w)$ or $P(a)$ its *canonical basis.* Clearly, the classes of the top elements of the canonical basis form a basis for the top in each case.

In order to describe the syzygies of string modules, we need the notion of syzygy strings of a string, defined in the following.

2.1. Definition. Let (Q, I) be a special biserial bound quiver. Let w be a string, and write

$$w = p_1^{-1}q_1 \cdots p_r^{-1}q_r,$$

where $r \geq 1$, the p_i, q_i are paths with p_1, q_r the only ones that may be trivial.

(1) In the case that p_1 is non-serial, we let α_1 be an arrow and u_1, v_1 some paths such that $(p_1\alpha_1 u_1, q_1 v_1)$ is a binomial pair. Assume now that p_1 is serial. We first define v_1 to be a path such that $q_1 v_1$ is a string ending in a deep. Moreover, if $p_1^{-1}q_1$ does not start in a deep, then let α_1 be an arrow and u_1 a path such that $u_1^{-1}\alpha_1^{-1}p_1^{-1}q_1$ is a string starting in a deep, and we define neither α_1 nor u_1 otherwise.

(2) Now let i be an integer with $1 < i < r$. If p_i is non-serial, then let u_i, v_i be the paths such that $(p_i u_i, q_i v_i)$ is a binomial pair. Otherwise, let u_i, v_i be paths such that $p_i u_i$ and $q_i v_i$ are strings ending in a deep.

(3) In the case that p_r is non-serial, let α_r be an arrow and u_r, v_r some paths such that $(p_r u_r, q_r \alpha_r v_r)$ is a binomial pair. Assume now that p_r is serial. First let u_r be a path such that $p_r u_r$ is a string ending in a deep. Moreover, if $p_r^{-1}q_r$ does not end in a deep, then let α_r be an arrow and v_r some path such that $p_r^{-1}q_r\alpha_r v_r$ is a string ending in a deep; and we define neither α_r nor v_r otherwise.

For all $1 \leq i < r$, let $w_i = v_i^{-1}u_{i+1}$, which is clearly a string. Moreover, let $w_0 = u_1$ if u_1 is defined, and $w_r = v_r^{-1}$ if v_r is defined. Denote by W the set of defined strings w_i with $0 \leq i \leq r$. We say that w_s, w_t with $0 \leq s < t \leq r$ are *connected* if p_i is non-serial for all $s < i \leq t$. This relation of connectedness generates an equivalence relation on W. It is easy to see that if $\{w_i, w_{i+1}, \ldots, w_j\}$ with $0 \leq i \leq j \leq r$ is an equivalence class of W, then $w_i w_{i+1} \cdots w_j$ is a string, called a *syzygy string* of w. Note that the number of the syzygy strings of w is equal to the number of the equivalence classes of W.

The following is the promised combinatorial description of the first syzygy of a string module.

2.2. Proposition. *Let $A = kQ/I$ with (Q, I) special biserial. Let w be a string and $\Omega(w)$ the set of the syzygy strings of w. If $M(w)$ is the string module associated to w, then $\Omega(M(w)) = \oplus_{\omega \in \Omega(w)} M(\omega)$.*

Proof. We keep all the notation introduced in Definition 2.1. Let $a_i = s(p_i)$, $i = 1, \ldots, r$, be the top elements of the canonical basis of $M(w)$. Then a projective cover of $M(w)$ is given by the map $\phi : P(a_1) \oplus \cdots \oplus P(a_r) \to M(w)$ such that the top element of the canonical basis of $P(a_i)$ maps to that of $M(p_i^{-1}q_i)$ for all $1 \leq i \leq r$. It is easy to see that $M(w)$ is projective if and only if $\Omega(w)$ is empty. Assume that $\Omega(w)$ is non-empty. Since the result is obvious for the case where $r = 1$, we may assume further that $r > 1$.

In order to describe the kernel of ϕ, we need more notations. If u_1 is defined and it is non-trivial whenever p_1 is non-serial, then let b_1 be the element of the canonical basis of $P(a_1)$ corresponding to $s(u_1)$ and let d_0 be the top element of the canonical basis of $M(w_0)$. Otherwise, we define neither b_1 nor d_0. In any case, let c_1 be the element of the canonical basis of $P(a_1)$ corresponding to $e(q_1)$, and let d_1 be the top element of $M(w_1)$. For each $1 < i < r$, let b_i and c_i be the elements of the canonical basis of $P(a_i)$ corresponding to $e(p_i)$ and $e(q_i)$, respectively, and let d_i be the top element of the canonical basis of $M(w_i)$. Finally, if v_r is defined and it is non-trivial whenever q_r is non-serial, then let c_r be the element of the canonical basis of $P(a_r)$ corresponding to $e(\alpha_r)$, and let d_r be the top element of the canonical basis of $M(w_r)$. Otherwise, we define neither c_r nor d_r. In any case, let b_r be the element of the canonical basis of $P(a_r)$ corresponding to $e(p_r)$.

Write $\hat{d}_i = (0, \ldots, 0, d_i, 0, \ldots, 0)^T \in \oplus_{\omega \in \Omega(w)} M(\omega)$ if d_i is defined. Then the classes of the defined $\hat{d}_i$ with $0 \le i \le r$ form a k-basis of the top of $\oplus_{\omega\in\Omega(w)} M(\omega)$. Let $\psi : \oplus_{\omega\in\Omega(w)} M(\omega) \to P(a_1) \oplus \cdots \oplus P(a_r)$ be the map such that

$$\psi(\hat{d}_i) = \begin{cases} (b_1, 0, \ldots, 0)^T, & i = 0 \text{ and } d_0 \text{ is defined,} \\ (0, \ldots, 0, -c_i, b_{i+1}, 0, \ldots, 0)^T, & 1 \le i < r, \\ (0, \ldots, 0, c_r)^T, & i = r \text{ and } d_r \text{ is defined.} \end{cases}$$

One easily verifies that ψ is a monomorphism such that $\psi\phi = 0$. By calculating the dimensions of the modules, we deduce finally that ψ is the kernel of ϕ. This completes the proof of the proposition.

By Corollary 1.3, we need consider only special biserial bound quivers with a loop such that at least one of its powers is a component of a binomial relation. For convenience, we make the following two definitions.

2.3. Definition. Let (Q, I) be a special biserial bound quiver, containing a loop α at a vertex a and a binomial pair $(\alpha^{n+1}, \alpha_1 \cdots \alpha_m)$ with $n \ge 1$ and $m \ge 2$. We shall consider strings of the following forms:

(1) The trivial string ε_a.

(2) The string

$$q_{r+1} q_r^{-1} \cdots q_2 q_1^{-1} \alpha p_1^{-1} p_2 \cdots p_r^{-1} p_{r+1},$$

where $r \ge 1$ is odd, the p_i, q_i are paths that are non-trivial for $1 \le i \le r$ such that $(p_i q_i, p_{i+1} q_{i+1})$ is a binomial pair for each odd i with $1 \le i < r$, and there exists a binomial pair $(p_r q_r, p_{r+1} \beta_{r+1} q_{r+1})$ with β_{r+1} an arrow.

(3) The string

$$q_s^{-1} q_{s-1} \cdots q_2 q_1^{-1} \alpha p_1^{-1} p_2 \cdots p_{s-1} p_s^{-1} \cdots p_r^{-1} p_{r+1},$$

where s, r are odd with $1 \le s \le r$, the p_i, q_j are paths that are non-trivial for $1 \le i \le r$ and $1 \le j < s$ such that $(p_i q_i, p_{i+1} q_{i+1})$ is a binomial pair for each odd i with $1 \le i < s$, and $p_s q_s$ is a serial string ending in a deep.

2.4. Definition. Let (Q, I) be as in Definition 2.3. We shall consider strings of the following forms:

(1) The string

$$p_{r+1} p_r^{-1} \cdots p_2^{-1} p_1 \alpha^{-n} q_1 q_2^{-1} \cdots q_r^{-1} q_{r+1},$$

where $r \ge 0$ is even, the p_i, q_i are paths that are non-trivial for $1 \le i \le r$ such that $(q_i p_i, q_{i+1} p_{i+1})$ is a binomial pair for each even $0 \le i < r$ and there exists a binomial pair $(q_r p_r, q_{r+1} \beta_{r+1} p_{r+1})$ with β_{r+1} an arrow, $q_0 = \alpha^n$ and $p_0 = \alpha$.

(2) The string

$$p_s^{-1} p_{s-1} \cdots p_2^{-1} p_1 \alpha^{-n} q_1 q_2^{-1} \cdots q_{s-1} q_s^{-1} \cdots q_r^{-1} q_{r+1},$$

where s, r are even with $2 \le s \le r$, the p_i, q_j are paths that are non-trivial for $1 \le i < s$ and $1 \le j \le r$, such that $(q_i p_i, q_{i+1} p_{i+1})$ is a binomial pair for each even i with $2 \le i < s$, and $q_s p_s$ is a serial string ending in a deep.

We shall now apply Proposition 2.2 to describe the first syzygy of string modules associated to previously defined strings.

2.5. Lemma. *Let $A = kQ/I$ with (Q, I) being as in Definition 2.3. If M is a string module associated to a string as stated in Definition 2.3, then $\Omega(M)$ has as a direct summand a string module associated to a string as stated in Definition 2.4.*

Proof. Let $M = M(w)$ with w a string. First, if $w = \varepsilon_a$, then $M = S(a)$. Thus $\Omega(M)$ is the string module associated to the string $(\alpha_2 \cdots \alpha_m)\alpha^{-n}$, which is of the form stated in Definition 2.4(1) with $r = 0$ and q_1 trivial. Second, we consider the case

$$w = q_{r+1}q_r^{-1} \cdots q_2 q_1^{-1} \alpha p_1^{-1} p_2 \cdots p_r^{-1} p_{r+1},$$

where $r \geq 1$ is odd, and the p_i, q_i are non-trivial paths such that $(p_i q_i, p_{i+1} q_{i+1})$ is a binomial pair for each odd i with $1 \leq i < r$, and there exists a binomial pair $(p_r q_r, p_{r+1} \beta_{r+1} q_{r+1})$ with β_{r+1} an arrow. Note that $q_1 = \alpha_1 \cdots \alpha_t$ with $1 \leq t < m$.

Suppose first that q_{r+1} is trivial and that there exists no even i with $2 \leq i < r$ such that q_i is serial. If $r = 1$, then let u_1 be the path (maybe trivial) such that $\alpha_{t+1} \cdots \alpha_m = \alpha_{t+1} u_1$. Note that we have binomial pairs $(q_1 \alpha_{t+1} u_1, \alpha\alpha^n)$ and $(p_1 q_1, p_2 \beta_2)$. By Proposition 2.2, $\Omega(M)$ is the string module associated to the string $u_1 \alpha^{-n} q_1$, which is of the form stated in Definition 2.4(1). If $r \geq 3$, let $u_1 = \alpha_{t+1} \cdots \alpha_m$, and for each even i with $2 \leq i \leq r-3$, let u_i, u_{i+1} be non-trivial paths such that $(q_i u_i, q_{i+1} u_{i+1})$ is a binomial pair, and finally, let u_r, u_{r-1} be paths with u_{r-1} non-trivial and δ_r an arrow such that $(q_{r-1} u_{r-1}, q_r \delta_r u_r)$ is a binomial pair. Since none of the paths contained in w is a serial string, $\Omega(M)$ is the string module associated to the string $u_r u_{r-1}^{-1} \cdots u_3 u_2^{-1} u_1 \alpha^{-n} q_1 q_2^{-1} q_3 \cdots q_{r-1}^{-1} q_r$, which is of the form stated in Definition 2.4(1) for $r-1$.

Suppose now that q_{r+1} is non-trivial and that there exists no even i with $2 \leq i \leq r+1$ such that q_i is serial. Let $u_1 = \alpha_{t+1} \cdots \alpha_m$ and let, for each even i with $2 \leq i \leq r-1$, u_i, u_{i+1} be non-trivial paths such that $(q_i u_i, q_{i+1} u_{i+1})$ is a binomial pair, and finally, let u_{r+2}, u_{r+1} be non-trivial paths and δ_{r+2} an arrow such that $(q_{r+1} u_{r+1}, \delta_{r+2} u_{r+2})$ is a binomial pair. For the same reason, $\Omega(M)$ is the string module associated to the string $u_{r+2} u_{r+1}^{-1} \cdots u_2^{-1} u_1 \alpha^{-n} q_1 q_2^{-1} \cdots q_r q_{r+1}^{-1}$, which is of the form stated in Definition 2.4(1) for $r+1$ with q_{r+2} trivial.

Suppose that neither of the above two situations occurs. Then there exists some minimal even integer s with $2 \leq s \leq r+1$ such that q_s is non-trivial and serial. Let $u_1 = \alpha_{t+1} \cdots \alpha_m$ and let, for each even i with $2 \leq i < s$, u_i, u_{i+1} be non-trivial paths such that $(q_i u_i, q_{i+1} u_{i+1})$ is a binomial pair, and finally, let u_s be a path (maybe trivial) such that $q_s u_s$ is a string ending in a deep. Since the p_i, q_j with $1 \leq i \leq r$ and $1 \leq j < s$ are non-serial, $\Omega(M)$ has as a direct summand the string module associated to the string $u_s^{-1} u_{s-1} \cdots u_2^{-1} u_1 \alpha^{-n} q_1 q_2^{-1} \cdots q_{s-1} q_s^{-1} \cdots q_r q_{r+1}^{-1}$, which is of the form stated in Definition 2.4.(2) for s and $r-1$ if $s \leq r-1$ and p_{r+1} is trivial; and for s and $r+1$ otherwise.

We conclude the proof with the final case:

$$w = q_s^{-1} q_{s-1} \cdots q_2 q_1^{-1} \alpha p_1^{-1} p_2 \cdots p_{s-1} p_s^{-1} \cdots p_r^{-1} p_{r+1},$$

where s, r are odd with $1 \leq s \leq r$, and the p_i, q_j are non-trivial paths for $1 \leq i \leq r$ and $1 \leq j < s$ such that $(p_i q_i, p_{i+1} q_{i+1})$ is a binomial pair for each odd i with $1 \leq i < s$, and $p_s q_s$ is a serial string ending in a deep. Suppose first that q_i is non-serial for each odd i with $1 \leq i \leq s$. Let u_s, u_{s-1} be paths with u_{s-1} non-trivial and δ_s an arrow such that $(q_s \delta_s u_s, q_{s-1} u_{s-1})$ is a binomial pair, where $q_0 = \alpha$ and $u_0 = \alpha^{-n}$ in case $s = 1$; and let $u_1 = \alpha_{t+1} \cdots \alpha_m$ if $s > 1$. Finally for each odd i with $3 \leq i < s$, let u_i, u_{i-1} be non-trivial paths such that $(q_{i-1} u_{i-1}, q_i u_i)$ is a binomial pair. Since p_s is serial while the p_i, q_j with $1 \leq i < s$ and $1 \leq j \leq s$ are non-serial, $\Omega(M)$ has as a direct summand the string module associated to the string $u_s u_{s-1}^{-1} \cdots u_1 \alpha^{-n} q_1 q_2^{-1} \cdots q_{s-1}^{-1} q_s$, which is of the form stated in Definition

2.4(1) for $s-1$. Otherwise $s \geq 3$ and there exists a minimal even integer d with $2 \leq d \leq s-1$ such that q_d is serial. Let u_d be a path (maybe trivial) such that $q_d u_d$ is a string ending in a deep. Being non-trivial, $q_1 = \alpha_1 \cdots \alpha_t$ for some $1 \leq t < m$. Let $u_1 = \alpha_{t+1} \cdots \alpha_m$, and for each even i with $2 \leq i < d$, let u_i, u_{i+1} be non-trivial paths such that $(q_i u_i, q_{i+1} u_{i+1})$ is a binomial pair. Since q_d and p_s are serial while the others between them are non-serial, $\Omega(M)$ has as a direct summand the string module associated to the string $u_d^{-1} u_{d-1} \cdots u_2^{-1} u_1 \alpha^{-n} q_1 q_2^{-1} \cdots q_{s-1}^{-1} q_s$, which is of the form stated in Definition 2.4(2) for d and $s-1$. This completes the proof.

2.6. Lemma. *Let $A = kQ/I$ with (Q, I) being as in Definition 2.3. If M is a string module associated to a string as stated in Definition 2.4, then $\Omega(M)$ has as a direct summand a string module associated to a string as stated in Definition 2.3.*

Proof. Let $M = M(w)$ with w a string. Let us begin with the case

$$w = p_{r+1} p_r^{-1} \cdots p_2^{-1} p_1 \alpha^{-n} q_1 q_2^{-1} \cdots q_r^{-1} q_{r+1},$$

where $r \geq 0$ is even, the p_i, q_i are non-trivial for $1 \leq i \leq r$ such that for each even i with $0 \leq i < r$, $(q_i p_i, q_{i+1} p_{i+1})$ is a binomial pair with $q_0 = \alpha^n$ and $p_0 = \alpha$, and there exists a binomial pair $(q_r p_r, q_{r+1} \beta_{r+1} p_{r+1})$ with β_{r+1} some arrow.

Suppose first that p_{r+1} is trivial and that there exists no odd i with $1 \leq i < r$ such that p_i is serial. If $r = 0$, then $q_1 = \alpha_1 \cdots \alpha_{m-1}$ by hypothesis. Thus $\Omega(M) = M(\varepsilon_a)$. If $r \geq 2$, then $p_1 = \alpha_t \cdots \alpha_m$ with $1 < t \leq m$. For each odd i with $1 \leq i \leq r-3$, let u_i, u_{i+1} be non-trivial paths such that $(p_i u_i, p_{i+1} u_{i+1})$ is a binomial pair, and let δ_r be an arrow and u_r, u_{r-1} some paths with u_{r-1} non-trivial such that $(q_{r-1} u_{r-1}, q_r \delta_r u_r)$ is a binomial pair. Since none of the paths contained in w is a serial string, $\Omega(M)$ is the string module associated to the string $u_r u_{r-1}^{-1} \cdots u_2 u_1^{-1} \alpha p_1^{-1} p_2 \cdots p_{r-1}^{-1} p_r$, which is of the form stated in Definition 2.3.(1) for $r-1$.

Suppose secondly that p_{r+1} is non-trivial and that there exists no odd i with $1 \leq i \leq r+1$ such that p_i is serial. For each odd i with $1 \leq i \leq r-1$, let u_i, u_{i+1} be non-trivial paths such that $(q_i u_i, q_{i+1} u_{i+1})$ is a binomial pair. Moreover, let δ_{r+2} be an arrow and u_{r+2}, u_{r+1} some non-trivial paths such that $(q_{r+1} u_{r+1}, \delta_{r+2} u_{r+2})$ is a binomial pair. For the same reason, $\Omega(M)$ is the string module associated to the string $u_{r+2} u_{r+1}^{-1} \cdots u_2 u_1^{-1} \alpha p_1^{-1} p_2 \cdots p_r p_{r+1}^{-1}$, which is of the form stated in Definition 2.3(1) for $r+1$ with p_{r+2} trivial.

Suppose now that neither of the above two situations occurs. Then there exists a minimal odd integer s with $1 \leq s \leq r+1$ such that p_s is non-trivial and serial. Let u_s be a path (maybe trivial) such that $p_s u_s$ is a string ending in a deep, and for each odd i with $1 \leq i < s$, let u_i, u_{i+1} be non-trivial paths such that $(q_i u_i, q_{i+1} u_{i+1})$ is a binomial pair. Since the p_i, q_j with $1 \leq i < s$ and $1 \leq j \leq r$ are all non-serial, $\Omega(M)$ has as a direct summand the string module associated to the string $u_s^{-1} u_{s-1} \cdots u_2 u_1^{-1} \alpha p_1^{-1} p_2 \cdots p_{s-1} p_s^{-1} \cdots p_r p_{r+1}^{-1}$, which is of the form stated in Definition 2.3(2) for s and $r-1$ if p_{r+1} is trivial with $s \leq r-1$; and otherwise, for s and $r+1$ with p_{r+2} trivial. We shall conclude the proof with the case:

$$w = p_s^{-1} p_{s-1} \cdots p_2^{-1} p_1 \alpha^{-n} q_1 q_2^{-1} \cdots q_{s-1} q_s^{-1} \cdots q_r^{-1} q_{r+1},$$

where s, r are even with $2 \leq s \leq r$, and the p_i, q_j are non-trivial paths for $1 \leq i < s$ and $1 \leq j \leq r$ such that for each even i with $0 \leq i < s$, $(q_i p_i, q_{i+1} p_{i+1})$ is a binomial pair with $q_0 = \alpha^n$, $p_0 = \alpha$, whereas $q_s p_s$ is a serial string ending in a deep.

Suppose first that p_i is non-serial for every odd i with $1 \le i < s$. Let δ_s be an arrow and u_s, u_{s-1} some paths with u_{s-1} non-trivial such that $(p_s\delta_s u_s, p_{s-1}u_{s-1})$ is a binomial pair; and for each even i with $2 \le i < s$, let u_i and u_{i-1} be non-trivial paths such that $(q_{i-1}u_{i-1}, q_i u_i)$ is a binomial pair. Since q_s is serial while the p_i, q_j with $1 \le i \le s$ and $1 \le j < s$ are all non-serial, $\Omega(M)$ has as a direct summand the string module associated to the string $u_s u_{s-1}^{-1} \cdots u_1^{-1} \alpha p_1^{-1} p_2 \cdots p_{s-1}^{-1} p_s$, which is of the form stated in Definition 2.3(1) for $s-1$.

Otherwise, there exists a minimal odd integer d with $1 \le d \le s-1$ such that p_d is serial. Let u_d be a path (maybe trivial) such that $q_d u_d$ is a string ending in a deep, and for each odd i with $1 \le i < d$, let u_i, u_{i+1} be non-trivial paths such that $(q_i u_i, q_{i+1}u_{i+1})$ is a binomial pair. Since p_d and q_s are serial while the others between them are all non-serial, $\Omega(M)$ has as a direct summand the string module associated to the string $u_d^{-1} u_{d-1} \cdots u_2 u_1^{-1} \alpha p_1^{-1} p_2 \cdots p_{s-1}^{-1} p_s$, which is of the form stated in Definition 2.3(2) for d and $s-1$. This completes the proof of the lemma.

Let A be the algebra of a bound quiver (Q, I). The *convex support* of an A-module M is the algebra of the bound quiver (Q_M, I_M), where Q_M is the convex hull in Q of the vertices a with $\mathrm{Hom}_A(P(a), M) \ne 0$, and $I_M = I \cap (kQ_M)$.

2.7. Theorem. *Let A be the algebra of a bound quiver (Q, I), and let S be a simple A-module with a non-split self-extension. If the convex support of S is special biserial, then for all $i \ge 1$, $\mathrm{Ext}^i_A(S, S)$ does not vanish.*

Proof. Let B be the convex support of S in A. It is well-known that $\mathrm{Ext}^i_A(S,S) = \mathrm{Ext}^i_B(S,S)$ for all $i \ge 1$. Hence we may assume, without loss of generality, that (Q, I) is special biserial. Let α be a loop at the vertex a such that $S = S(a)$. If no power of α is a component of a binomial relation on Q, then the theorem follows from Corollary 1.3. Assume now that there exists a binomial pair $(\alpha^{n+1}, \alpha_1 \cdots \alpha_m)$ with $n \ge 1$ and $m \ge 2$. Note that $\Omega^0(S) = M(\varepsilon_a)$. Assume, for $i \ge 0$, that $\Omega^i(S)$ has as a direct summand a string module associated to a string stated in Definitions 2.3 or 2.4, and in particular, $S(a)$ is a summand of the top of $\Omega^i(S)$. By Lemmas 2.5 and 2.6, the same holds true for $\Omega^{i+1}(S)$. This completes the proof of the theorem.

We conclude with an even stronger version of the strong no loop conjecture.

2.8. Conjecture. If S is a simple module over an artin algebra A with $\mathrm{Ext}^1_A(S,S)$ not vanishing, then $\mathrm{Ext}^i_A(S,S)$ does not vanish for infinitely many i.

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