AUSLANDER-REITEN COMPONENTS WITH BOUNDED SHORT CYCLES

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Abstract. We study Auslander-Reiten components of an artin algebra with bounded short cycles, namely, there exists a bound for the depths of maps appearing on short cycles of non-zero non-invertible maps between modules in the given component. First, we give a number of combinatorial characterizations of almost acyclic Auslander-Reiten components. Then, we shall show that an Auslander-Reiten component with bounded short cycles is obtained, roughly speaking, by gluing the connecting components of finitely many tilted quotient algebras. In particular, the number of such components is finite and each of them is almost acyclic with only finitely many DTr-orbits. As an application, we show that an artin algebra is representation-finite if and only if its module category has bounded short cycles. This includes a well known result of Ringel's, saying that a representation-directed algebra is representation-finite.

Introduction

Let A be an artin algebra, and let $\operatorname{mod} A$ stand for the category of finitely generated left A-modules. We shall denote by $\operatorname{rad}(\operatorname{mod} A)$ and $\varGamma_A$ the Jacobson radical and the Auslander-Reiten quiver of $\operatorname{mod} A$, respectively. The ultimate objective of the representation theory is to study the full subcategory $\operatorname{ind} A$ of $\operatorname{mod} A$ generated by the indecomposable modules. In [21], Ringel initiated the study of directing modules, that is, indecomposable modules not lying on any cycle of non-zero non-invertible maps in $\operatorname{ind} A$. He showed that directing modules are uniquely determined (up to isomorphism) by their composition factors, and A is representation-finite if $\operatorname{ind} A$ has no cycle; see [21, (2.4)]. In general, the directing modules fall into finitely many DTr-orbits of $\varGamma_A$; see [18, 27]. Later, these results have been generalized to indecomposable modules not lying on any short cycle (that is, cycles of at most two maps) in $\operatorname{ind} A$. For instance, these modules are also uniquely determined (up to isomorphism) by their composition factors; see [1, 20], and A is representation-finite if $\operatorname{ind} A$ has no short cycle; see [3]. Moreover, $\varGamma_A$ has at most finitely connected components $\mathcal{C}$ such that $\operatorname{add}(\mathcal{C})$ has no short cycle, and each of such components contains only finitely many DTr-orbits; see [13, (2.6),(2.8)].

On the other hand, the representation theory of A is determined to certain extend by the Jacobson radical of $\operatorname{mod} A$. Indeed, a well known result of Auslander's says that A is representation-finite if and only if the infinite radical of $\operatorname{mod} A$ vanishes; see [24, (1.1)], or equivalently, every non-zero map in $\operatorname{mod} A$ is of finite depth. For many

2010 *Mathematics Subject Classification.* 16G30, 16G60, 16G70.

Key words and phrases. Representation-finite algebras; tilted algebras; Jacobson radical; cycles in a module category; irreducible maps; Auslander-Reiten sequences; translation quivers; Auslander-Reiten quiver; Auslander-Reiten components.

The first author is supported in part by the Natural Sciences and Engineering Research Council of Canada.

best understood classes of representation-infinite algebras, such as tame concealed algebras, the cycles in their module category contains only maps of finite depth; see [21]. Motivated by this fact, Skowroński studied in [26] cycles of finite depth, which are originally called *finite cycles*. Indeed, Auslander-Reiten components whose non-directing modules lie only on cycles of finite depth are described in [17], and algebras whose module category contains only cycles of finite depth are extensively studied by many authors; see, for example, [16, 17, 23, 26]. More generally, the connected components $\mathcal{C}$ of Γ_A such that $\mathrm{add}(\Gamma)$ contains only short cycles of finite depth are studied in [13].

The main purpose of this paper is to investigate the connected components $\mathcal{C}$ of Γ_A such that $\mathrm{add}(\mathcal{C})$ has bounded short cycles, that is, there exists a bound for the depths of the maps appearing on short cycles in $\mathrm{add}(\mathcal{C})$. To start with, we shall study almost acyclic Auslander-Reiten components, which have played a special role in the study of generalized double tilted algebras introduced by Reiten and Skowroński; see [19]. Our main result says that if $\mathcal{C}$ is a connected component of Γ_A with bounded short cycles then, roughly speaking, $\mathcal{C}$ is obtained by gluing along a finite core the connecting components of finitely many tilted quotient algebras of A; see (3.4). In particular, given a connected component $\mathcal{C}$ of Γ_A, if $\mathcal{C}$ is semi-regular, then $\mathrm{add}(\mathcal{C})$ has bounded short cycles if and only if $A/\mathrm{ann}(\mathcal{C})$ is tilted with $\mathcal{C}$ being its connecting component; see (3.5); and if $\mathcal{C}$ is generalized standard, then $\mathrm{add}(\mathcal{C})$ has bounded short cycles if and only if $A/\mathrm{ann}(\mathcal{C})$ is generalized double tilted algebra with $\mathcal{C}$ being its connecting component; see (3.7). In general, Γ_A has at most finitely many connected components with bounded short cycles, and each of them is almost acyclic with only finitely many DTr-orbits; see (3.6). Finally, we shall show that an artin algebra is representation-finite if and only if $\mathrm{ind}\, A$ has bounded short cycles; see (3.8). This includes the above-mentioned result of Ringel's as a special case.

1. Preliminaries

The objective of this section is to collect some terminology and fix some notation which will be used throughout this paper.

1.1. Quivers. All quivers in this paper are locally finite. Let Q be a quiver. We shall say that Q is *trivial* if it consists of a single vertex, *acyclic* if Q contains no oriented cycle, and *almost acyclic* if Q contains at most finitely many vertices which lie on some oriented cycles. Given two vertices a, b, the *interval* $[a, b]$ is the set of vertices lying on some path from a to b in Q. We shall say that Q is *interval-finite* if all intervals in Q are finite; and *strongly interval-finite* if, given any vertices a, b, the number of paths in Q from a to b is finite. If Q is strongly interval-finite, then it is clearly interval-finite and acyclic. A full subquiver Σ of Q is called *convex in* Q if it contains all the paths with end-points lying in Σ; *predecessor-closed* if it contains all the predecessors of its vertices in Q; and *successor-closed* if it contains all the successors of its vertices in Q. Finally, an infinite path in Q is called *left infinite* if it has no starting point; *right infinite* if it has no ending point; and *double infinite* if it has neither starting point nor ending point.

1.2. Translation Quivers. Let Γ be a translation quiver with translation τ; see, for definition, [21, Page 47]. A path $a_0 \longrightarrow a_1 \longrightarrow \cdots \longrightarrow a_n$ in Γ is called *sectional* if there exists no $0 < i < n$ such that $a_{i-1} = \tau a_{i+1}$. An infinite path is *sectional* if all its finite subpaths are sectional. A vertex a of Γ is called *left stable* (respectively, *right stable*) if $\tau^n a$ is defined for all $n \geq 0$ (respectively, for all $n \leq 0$); *stable* if its left and right stable; and *τ-periodic* if $\tau^n a = a$ for some $n > 0$. Moreover, Γ is called *left stable* (respectively, *right stable*, *stable*, *τ-periodic*) if every vertex of Γ is left stable (respectively, right stable, stable, τ-periodic).

The full subquivers ${}_l\Gamma$, ${}_r\Gamma$ and ${}_s\Gamma$ of Γ generated by the left stable vertices, by the right stable vertices, and by the stable vertices are called the *left stable part*, the *right stable part*, and the *stable part* of Γ respectively. Moreover, the connected components of the quiver ${}_l\Gamma$ (respectively, ${}_r\Gamma$, ${}_s\Gamma$) are called the *left stable components* (respectively, *right stable components, stable components*) of Γ; and a left or right stable component of Γ is simply called a *semi-stable component*. Since Γ is locally finite, a semi-stable component of Γ is τ-periodic if it contains a τ-periodic vertex.

Given an acyclic quiver Δ, one constructs in a canonical way a stable translation quiver $\mathbb{Z}\Delta$ with translation ρ; see, for example, [8, Section 2]. If Δ is of type $\mathbb{A}_\infty$, then $\mathbb{Z}\Delta$ does not depend on the orientation of Δ, and hence, $\mathbb{Z}\Delta$ will be simply written as $\mathbb{Z}\mathbb{A}_\infty$. A translation quiver is called a *stable tube* if it is isomorphic to $\mathbb{Z}\mathbb{A}_\infty/{<}\rho^n{>}$ for some integer $n > 0$; and *quasi-serial* if it is a stable tube or of shape $\mathbb{Z}\mathbb{A}_\infty$. Starting with a quasi-serial translation quiver, one obtains new translations by *ray insertions* or by *co-ray insertions*; see [12, Section 2].

Let Σ be a connected full subquiver of Γ. Recall that Σ is a *section* of Γ if it is acyclic, convex, and contains exactly one vertex of each τ-orbit in Γ; see [8, (2.1)]. In this case, there exists an embedding $\Gamma \to \mathbb{Z}\Sigma$, which sends a vertex $\tau^n x$ with $n \in \mathbb{Z}$ and $x \in \Sigma$ to the vertex $(-n, x)$; see [8, (2.3)]. More generally, Σ is called a *cut* of Γ; see [15, (2.1)] provided, for any arrow $a \to b$ in Γ, that the following two conditions are verified.

(1) If $a \in \Sigma$, then either b or τb, but not both, lies in Σ;
(2) If $b \in \Sigma$, then either a or $\tau^- a$, but not both, lies in Σ.

1.3. Module Category. Throughout this paper, A stands for an artin algebra over a commutative artinian ring R. We shall denote by $\mathrm{mod}A$ the category of finitely generated left A-modules, and by $\mathrm{rad}(\mathrm{mod}A)$ the Jacobson radical of $\mathrm{mod}A$. Recall that $\mathrm{rad}^\infty(\mathrm{mod}A) = \cap_{n\geq 0}\,\mathrm{rad}^n(\mathrm{mod}A)$, where $\mathrm{rad}^n(\mathrm{mod}A)$ stands for the n-th power of $\mathrm{rad}(\mathrm{mod}A)$, is called the *infinite radical* of $\mathrm{mod}A$. Given a map f in $\mathrm{mod}A$, its *depth* $\mathrm{dp}(f)$ is ∞ if $f \in \mathrm{rad}^\infty(X,Y)$; and otherwise, $\mathrm{dp}(f)$ is the minimal integer $n \geq 0$ for which $f \in \mathrm{rad}^n(X,Y)\backslash\mathrm{rad}^{n+1}(X,Y)$; see [15, (1.2)].

We shall denote by $\mathrm{ind}A$ a full subcategory of $\mathrm{mod}A$ generated by a complete set of representatives of the isomorphism classes of the indecomposable modules in $\mathrm{mod}A$. A *path* of length n in $\mathrm{ind}A$ is sequence

$$\sigma:\ X_0 \xrightarrow{f_1} X_1 \longrightarrow \cdots \longrightarrow X_{n-1} \xrightarrow{f_n} X_n$$

of n non-zero maps in $\mathrm{rad}(\mathrm{ind}A)$, whose *depth* is defined to be the supremum of the $\mathrm{dp}(f_i)$ with $1 \leq i \leq n$ and written as $\mathrm{dp}(\sigma)$. Such a path σ is called a *cycle* if $X_n = X_0$; and a *short cycle* if, in addition, $n \leq 2$.

1.4. Auslander-Reiten Quiver. The reader is referred to [2] for the Auslander-Reiten theory of irreducible maps and almost split sequences in mod$A$. For each module X in ind A, we set $D_X = \mathrm{End}(X)/\mathrm{rad}(\mathrm{End}(X))$. The *Auslander-Reiten quiver* Γ_A of A is a valued translation quiver defined as follows; see, for example, [14, Section 2]. The vertices are the objects of ind$A$. Given two vertices X, Y, there exists an arrow $X \to Y$ in Γ_A with valuation (d_{XY}, d'_{XY}) if and only if there exists an irreducible map $f : X \to Y$ in mod$A$, where d_{XY} and d'_{XY} are the dimensions of $\mathrm{rad}(X,Y)/\mathrm{rad}^2(X,Y)$ over D_Y and over D_X, respectively. The translation τ is the Auslander-Reiten translation DTr so that $\tau Z = X$ if and only if there exists an almost split sequence $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ in mod$A$. The connected components of Γ_A are called *Auslander-Reiten components* of Γ_A, and a connected component of Γ_A is called *semi-regular* if it contains no projective module or no injective module; see [9].

Let $\mathcal{C}$ be a connected component of Γ_A with a full subquiver Σ. Given $n \in \mathbb{Z}$, we shall denote by $\tau^n\Sigma$ the full subquiver (possibly empty) of $\mathcal{C}$ generated by the modules $\tau^n X$ with $X \in \Sigma$. The *annihilator* of Σ is $\mathrm{ann}(\Sigma) = \cap_{X\in\Sigma}\, \mathrm{ann}(X)$, where $\mathrm{ann}(X)$ stands for the annihilator of X in A. One says that Σ is *faithful* if $\mathrm{ann}(\Sigma) = 0$; *sincere* if every simple A-module is a composition factor of some module in Σ; *convex in* ind$A$ if every path in ind$A$ with end-points lying in Σ passes only modules in Σ; and *generalized standard* if $\mathrm{rad}^\infty(X,Y) = 0$ for all modules $X, Y \in \Sigma$; compare [25]. Recall that Σ is a *slice* in mod$A$ if Σ is a cut of $\mathcal{C}$ which is sincere and convex in ind$A$; see [21], and in this case, $\mathcal{C}$ is called a *connecting component* of Γ_A. Finally, A is *tilted* if $A = \mathrm{End}_H(T)$, where H is a hereditary artin algebra and T is a tilting module in mod$H$; see [5]. If A is connected, then it is tilted if and only if Γ_A has a connecting component; see [21].

2. Almost acyclic components

The objective of this section is to study some combinatorial properties of the Auslander-Reiten quiver Γ_A. This will allow us to obtain a number of combinatorial characterizations of the almost acyclic components of Γ_A. Let us start with an easy observation.

2.1. Lemma. *If X, Y are modules in Γ_A, then the number of sectional paths in Γ_A from X to Y is finite.*

Proof. Let X, Y be modules in Γ_A. Since $\mathrm{Hom}_A(X,Y)$ is of finite R-length, there exists some integer $r > 0$ such that $\mathrm{rad}^r(X,Y) = \mathrm{rad}^\infty(X,Y)$. Suppose that

$$X = X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_{s-1} \longrightarrow X_s = Y$$

is a sectional path in Γ_A. Choosing irreducible maps $f_i : X_{i-1} \to X_i$ in mod$A$, $i = 1, \ldots, s$, we obtain a map $f_s \cdots f_1$ of depth s; see [7, (13.3)]. In particular, $s < r$. Since Γ_A is locally finite, by König's Lemma, the number of paths from X to Y of length at most r is finite. The proof of the lemma is completed.

Next, we shall study some properties of the semi-stable components of Γ_A.

2.2. Lemma. *Let Γ be a semi-stable component of Γ_A. If Γ is acyclic, then it is strongly interval-finite.*

Proof. We shall consider only the case where Γ is a non-trivial acyclic left stable component. Observe that Γ contains a section Δ; see [12, (3.3)]. Let $X, Y \in \Gamma$ be such that Γ contains some paths from X to Y. Since Γ embeds in $\mathbb{Z}\Delta$, we see that $X \in \tau^s\Delta$ and $Y \in \tau^t\Delta$ with $s \geq t$. We shall proceed by induction on $t - s$. Assume that $s = t$. Since Δ is convex in Γ, so is $\tau^s\Delta$. Thus, all the paths $X \rightsquigarrow Y$ in Γ belong to Δ, and in particular, they are all sectional. By Lemma 2.1, the number of paths in Γ from X to Y is finite.

Suppose that $s > t$ but Γ contains infinitely many paths $\rho_n : X \rightsquigarrow Y$, $n \in \mathbb{N}$. Since Γ is locally finite, we may assume that the lengths $\ell(\rho_n)$ are pairwise different. By Lemma 2.1, we may assume further that none of the ρ_n is sectional. Then, each path ρ_n induces a path $\sigma_n : X \rightsquigarrow \tau Y$ in Γ of length $\ell(\rho_n) - 2$. In particular, the σ_n with $n \in \mathbb{N}$ are pairwise different paths in Γ from X to τY. Since $\tau Y \in \tau^{t+1}\Delta$, we obtain a contradiction to the induction hypothesis. The proof of the lemma is completed.

2.3. LEMMA. *Let Γ be a semi-stable component of Γ_A, not being τ-periodic. If Γ has oriented cycles, then it contains a module M with two infinite sectional paths*

$$\cdots \longrightarrow \tau^{2t}M_1 \longrightarrow \tau^t M_r \longrightarrow \cdots \longrightarrow \tau^t M_1 \longrightarrow M_r \longrightarrow \cdots \longrightarrow M_1 = M$$

and

$$M = N_1 \longrightarrow \cdots \longrightarrow N_s \longrightarrow \tau^t N_1 \longrightarrow \cdots \longrightarrow \tau^t N_s \longrightarrow \tau^{2t} N_1 \longrightarrow \cdots$$

where r, s, t are some positive integers.

Proof. We shall consider only the case where Γ is a left stable component, which contains oriented cycles. Having no τ-periodic module, Γ contains a sectional path

$$\tau^m X_1 \longrightarrow X_n \longrightarrow \cdots \longrightarrow X_1,$$

where $m > n \geq 1$, and $X_1, \ldots, X_n$ lie in pairwise different τ-orbits; see [9, (2.2)]. This yields two sectional paths in Γ as follows:

$$\tau^m X_n \longrightarrow \tau^m X_{n-1} \longrightarrow \cdots \longrightarrow \tau^m X_1 \longrightarrow X_n$$

and

$$X_n \longrightarrow \tau^{m-1} X_1 \longrightarrow \cdots \longrightarrow \tau^{m-(n-1)} X_{n-1} \longrightarrow \tau^{m-n} X_n.$$

Writing $Y_1 = Z_1 = X_n$ and setting $Y_i = \tau^m X_{i-1}$ and $Z_i = \tau^{m-(i-1)} X_{i-1}$, for $i = 2, \ldots, n$, we obtain two sectional paths $\tau^m Y_1 \longrightarrow Y_n \longrightarrow \cdots \longrightarrow Y_1$ and $Z_1 \longrightarrow \cdots \longrightarrow Z_n \longrightarrow \tau^{m-n} Z_1$. Applying repeatedly τ^m and τ^{m-n} to them, we get two infinite sectional paths

$$\cdots \longrightarrow \tau^{2m} Y_1 \longrightarrow \tau^m Y_n \longrightarrow \cdots \longrightarrow \tau^m Y_1 \longrightarrow Y_n \longrightarrow \cdots \longrightarrow Y_1$$

and

$$Z_1 \longrightarrow \cdots \longrightarrow Z_n \longrightarrow \tau^{m-n} Z_1 \longrightarrow \cdots \longrightarrow \tau^{m-n} Z_n \longrightarrow \tau^{2(m-n)} Z_1 \longrightarrow \cdots$$

in Γ. Writing $t = m(m-n)$ and renaming the modules in the above infinite paths so that $M_1 = Y_1$ and $N_1 = Z_1$, we obtain two desired infinite sectional paths

$$\cdots \longrightarrow \tau^{2t} M_1 \longrightarrow \tau^t M_r \longrightarrow \cdots \longrightarrow \tau^t M_1 \longrightarrow M_r \longrightarrow \cdots \longrightarrow M_1$$

and

$$N_1 \longrightarrow \cdots \longrightarrow N_s \longrightarrow \tau^t N_1 \longrightarrow \cdots \longrightarrow \tau^t N_s \longrightarrow \tau^{2t} N_1 \longrightarrow \cdots$$

with $M_1 = N_1$. The proof of the lemma is completed.

We now state some properties of semi-stable but not τ-periodic modules.

2.4. Lemma. *Let X be a module in Γ_A which is not τ-periodic.*

(1) *If X is left stable, then there exists some $s \geq 0$ such that the predecessors of $\tau^s X$ in Γ_A all are left stable.*
(2) *If X is right stable, then there exists some $t \geq 0$ such that the successors of $\tau^{-t} X$ in Γ_A all are right stable.*

Proof. We shall consider only the case where X is left stable. Since X is not τ-periodic, there exists some $r \geq 0$ such that none of the $\tau^i X$ with $i \geq r$ has an immediate projective predecessor in Γ_A. Then, the $\tau^i X$ with $i \geq r$ belong to a non-trivial left stable component Γ of Γ_A. Since X is not τ-periodic, Γ contains no τ-periodic module.

We claim that Γ contains a connected subquiver Σ such that the modules in Σ form a complete set of τ-orbit representatives of Γ and have no projective predecessor in Γ_A. Indeed, if Γ contains no oriented cycle, then it contains a section Σ with the claimed property; see [12, (3.3)]. Otherwise, Γ contains a path

$$\tau^m X_1 \longrightarrow X_n \longrightarrow \cdots \longrightarrow X_1,$$

where $m > n \geq 1$, and $X_1, \ldots, X_n$ form a complete set of representatives of the τ-orbit in Γ; see [12, (3.6)]. For each $1 \leq j \leq n$, since X_j is not τ-periodic, there exists some $s_j \geq 0$ such that none of the $\tau^i X_j$ with $i \geq s_j$ has a projective immediate predecessor in Γ_A. Setting $t = \max\{s_1, \ldots, s_n\}$, we see easily that

$$\Sigma : \tau^t X_n \longrightarrow \cdots \longrightarrow \tau^t X_1$$

has the claimed properties. This establishes our claim. In particular, $\tau^r X = \tau^l Y$, for some $l \in \mathbb{Z}$ and $Y \in \Sigma$. Setting $s = r - l$, we see that the predecessors of $\tau^s X$ in Γ_A are all left stable. The proof of the lemma is completed.

The following statement and its dual exhibit some interesting properties of not semi-stable modules in Γ_A.

2.5. Lemma. *Let M be a module in Γ_A. If M has infinitely many not left stable predecessors in Γ_A, then all of its successors belong to an infinite right stable component of Γ_A which is not left stable and contains oriented cycles.*

Proof. Let M_i, $i \in \mathbb{N}$, be pairwise distinct and not left stable predecessors of M in Γ. Since Γ_A contains only finitely many projective modules, we may assume that there exists a projective module P in Γ_A such that $M_i = \tau^{-n_i} P$ for some integer $n_i \geq 0$. Then, the n_i with $i \geq 1$ are pairwise distinct, and in particular, they are unbounded. As a consequence, P is right stable. By Lemma 2.4(2), we obtain an integer $t > 0$ such that the successors in Γ_A of $\tau^{-t} P$ belong to a right stable component Γ, which is clearly not left stable. Since the n_i are unbounded, there exists no loss of generality in assuming that $n_i \geq t$ for all $i \geq 1$. In particular, M is a successor of $\tau^{-t} P$. Therefore, the successors of M in Γ_A are successors of $\tau^{-t} P$, and hence, they all belong to Γ. Since each of the $\tau^{-n_i} P$ with $i \geq 1$ lies on a path from $\tau^{-t} P$ to M, by Lemma 2.2, Γ contains oriented cycles. The proof of the lemma is completed.

We are ready to give a number of characterizations of almost acyclic components of Γ_A. For this purpose, given a connected component $\mathcal{C}$ of Γ_A, we define its *core* to be the full subquiver generated by the modules lying on some path from a projective module to an injective module. Clearly, the core of $\mathcal{C}$ is convex in $\mathcal{C}$.

2.6. THEOREM. *Let A be an artin algebra with $\mathcal{C}$ a connected component of Γ_A. The following conditions are equivalent.*

(1) *The component $\mathcal{C}$ is almost acyclic.*
(2) *The component $\mathcal{C}$ is interval-finite.*
(3) *Every infinite semi-stable component of $\mathcal{C}$ is acyclic.*
(4) *The core of $\mathcal{C}$ is finite and contains all possible oriented cycles in $\mathcal{C}$.*

Proof. First of all, it is evident that Statement (4) implies Statement (1). Suppose that $\mathcal{C}$ has an infinite semi-stable component Γ with oriented cycles. If Γ contains some τ-periodic modules, then it is τ-periodic. Being infinite, Γ is a stable tube; see, for example, [9, (3.4)]. In this case, it is easy to see that Γ is not interval-finite and every module in Γ lies on an oriented cycle. Assume that Γ contains no τ-periodic module. In view of the second infinite path stated in Lemma 2.3, we obtain a module $M \in \Gamma$ and a positive integer t such that Γ contains infinitely many oriented cycles $M \rightsquigarrow \tau^{tn}M \rightsquigarrow M$, where $n \in \mathbb{Z}$. In particular, Γ is not interval-finite and has infinitely many modules lying on oriented cycles. This shows that each of Statements (1) and (2) implies Statement (3).

It remains to show that Statement (3) implies Statements (2) and (4). Indeed, suppose that Statement (3) holds. Assume on the contrary that there exist some modules M, N in $\mathcal{C}$ such that the interval $[M, N]$ is infinite. That is, the full subquiver $\mathcal{D}$ of $\mathcal{C}$ generated by the modules lying on paths $M \rightsquigarrow N$ is infinite. By König's Lemma, $\mathcal{D}$ has an infinite path

$$(*) \qquad \cdots \longrightarrow N_i \longrightarrow \cdots \longrightarrow N_1 \longrightarrow N_0 = N,$$

where the N_i with $i \geq 0$ are pairwise distinct. In view of Statement (3), we deduce from Lemma 2.5 that there exists an integer r such that N_i is left stable for every $i \geq r$. Let Γ be the left stable component of $\mathcal{C}$ containing the N_i with $i \geq r$. By Statement (3), Γ contains no oriented cycle, and by Lemma 2.2, it is interval-finite. In particular, $M \notin \Gamma$. Then, for each $i \geq r$, there exists a path $\zeta_i : M \rightsquigarrow N_i$ in $\mathcal{D}$, which is the composite of a path $\xi_i : M \rightsquigarrow X_i$, an arrow $\alpha_i : X_i \to Y_i$ with X_i not left stable, and a path $\eta_i : Y_i \rightsquigarrow N_i$ in Γ. By Lemma 2.5, the set $\{X_i \mid i \geq r\}$ is of finite cardinality, and so is $\{Y_i \mid i \geq r\}$. Therefore, we may assume that $Y_i = Y$ for some $Y \in \Gamma$ and all $i \geq r$. This yields infinitely many paths $Y \overset{\eta_i}{\rightsquigarrow} N_i \overset{\omega_i}{\rightsquigarrow} N_r$ in Γ, where $\omega_i : N_i \rightsquigarrow N_r$ is the subpath of the infinite path $(*)$, a contradiction to Lemma 2.2. This establishes Statement (2). As a consequence, the core of $\mathcal{C}$, written as Ω, is finite. Consider an oriented cycle

$$\sigma : \;\; Z_0 \longrightarrow Z_1 \longrightarrow \cdots \longrightarrow Z_{n-1} \longrightarrow Z_n = Z_0$$

in $\mathcal{C}$. Assume first that σ is contained in a semi-stable component Θ of $\mathcal{C}$. By Statement (3), Θ is finite, and hence, τ-periodic. It is well known that $\Theta \neq \mathcal{C}$; see [2, (VII.2.1)]. Thus, $\mathcal{C}$ contains an edge U—V with $U \in \Theta$ and $V \notin \Theta$. Since U is τ-periodic and V is not, the τ-orbit of V contains a projective module P and an injective module I. As a consequence, $\mathcal{C}$ has a path $P \rightsquigarrow U \rightsquigarrow I$. Let $Z \in \Theta$.

Being τ-periodic, Θ contains an oriented cycle $U \rightsquigarrow Z \rightsquigarrow U$, and hence, Z lies in the core Ω. In particular, σ lies entirely in Ω.

Assume next that Z_s is not left stable and Z_t is not right stable for some integers $0 \le s, t \le n$. Then, $\mathcal{C}$ contains a path $P \rightsquigarrow Z_s$ with P projective and a path $Z_t \rightsquigarrow I$ with I injective. Since σ is an oriented cycle, $\mathcal{C}$ contains paths $P \rightsquigarrow Z_i \rightsquigarrow I$ for all $1 \le i \le n$. That is, σ lies in Ω. This proves Statement (4). The proof of the theorem is completed.

Remark. The almost acyclic components have been characterized by the existence of a multisection; see [19, (2.5)]. Note that the core of a multisection of an almost acyclic component seems different from the core of the component defined here.

3. Components with bounded short cycles

The objective of this section is to study the connected components of Γ_A with bounded short cycles. This will yields a new characterization of representation-finite algebras, which includes a well known result of Ringle's saying that a representation-directed algebra is representation-finite; see [21, (2.4)].

3.1. Definition. Let Γ be a full subquiver of Γ_A. A *cycle* in $\mathrm{add}(\Gamma)$ is a cycle in $\mathrm{ind}A$ passing only through modules in Γ. We shall say that Γ is a subquiver with *bounded short cycles* if there exists a bound for the depths of all possibles short cycles in $\mathrm{add}(\Gamma)$, and otherwise, Γ is a subquiver with *unbounded short cycles.*

Given any finite subquiver Γ of Γ_A, it is evident that $\mathrm{add}(\Gamma)$ has bounded short cycle. The following result says that an infinite semi-stable component with bounded short cycles contains no oriented cycle.

3.2. Lemma. *Let Γ be a semi-stable component of Γ_A with bounded short cycles. If Γ is infinite, then it is acyclic.*

Proof. We shall consider only the case where Γ is an infinite left stable component of Γ_A. Suppose that Γ contains some τ-periodic modules. Being infinite, Γ is a stable tube, say of rank r; see [4]. Fix a module $X \in \Gamma$. It is evident that there exists an infinite sectional path

$$X = X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_n \longrightarrow \cdots$$

in Γ. Setting $Y_n = \tau^n X_n$ for all $n \ge 0$, we obtain another infinite sectional path

$$\cdots \longrightarrow Y_n \longrightarrow \cdots \longrightarrow Y_1 \longrightarrow Y_0 = X$$

in Γ. Observe that $Y_{rn} = \tau^{nr} X_{rn} = X_{rn}$ for all $n \ge 0$. Choosing irreducible maps $f_n : X_{n-1} \to X_n$ and $g_n : Y_n \to Y_{n-1}$ in $\mathrm{mod}A$ for all $n \ge 1$, we see that the maps $f_{nr} \cdots f_1 : X_0 \to X_{nr}$ and $g_{nr} \cdots g_1 : X_{nr} \to X_0$ form a short cycle of depth nr; see [7, (13.3)], a contradiction. Suppose now that Γ contains oriented cycles but no τ-periodic module. By Proposition 2.3, Γ contains two infinite sectional paths

$$\cdots \longrightarrow \tau^{2t} M_1 \longrightarrow \tau^t M_r \longrightarrow \cdots \longrightarrow \tau^t M_1 \longrightarrow M_r \longrightarrow \cdots \longrightarrow M_1$$

and

$$N_1 \longrightarrow \cdots \longrightarrow N_s \longrightarrow \tau^t N_1 \longrightarrow \cdots \longrightarrow \tau^t N_s \longrightarrow \tau^{2t} N_1 \longrightarrow \cdots$$

with $r, s, t > 0$ and $M_1 = N_1$. Given an integer n, using a similar argument as above, we can find a map $u_n : \tau^{nt} M_1 \to M_1$ of depth rn and a map $v_n : M_1 \to \tau^{nt} M_1$

of depth sn. In particular, $\mathrm{add}(\Gamma)$ has short cycles of arbitrarily large depth, a contradiction. The proof of the lemma is completed.

We shall regard a full subquiver of Γ_A as a translation quiver with the induced translation. The following result and its dual generalize slightly the result stated in [10, (2.1)].

3.3. Lemma. *Let Γ be a connected full subquiver of Γ_A, and let Δ be a section of Γ containing no right infinite path. If Γ contains all the predecessors of Δ in Γ_A, then* $\mathrm{ann}(\Delta) = \mathrm{ann}(\Omega)$, *where Ω is the full subquiver of Γ_A generated by the predecessors of Δ in Γ_A.*

Proof. Containing no right infinite path, by Krönig's Lemma, Δ contains only finite many paths starting at any given module. For each module $M \in \Delta$, we shall denote by $d(M)$ the maximal length of the paths in Δ starting at M.

Suppose that Ω is contained in Γ. Write $I = \mathrm{ann}(\Delta)$. Fix a module $Y \in \Omega$. Then, $Y \in \tau^n \Delta$ for some integer $n \geq 0$. We shall show by an induction on n that $IY = 0$. Indeed, this is trivial for $n = 0$. Suppose that $n > 0$ and the statement holds for $n - 1$. Then, $Y = \tau^n X$ with $X \in \Delta$, and there exists in $\mathrm{mod} A$ an almost split sequence

$$0 \longrightarrow Y \longrightarrow Y_1 \oplus \cdots \oplus Y_r \longrightarrow \tau^{n-1} X \longrightarrow 0,$$

where $Y_1, \ldots, Y_r \in \Omega$. We claim that $IY_i = 0$ for $i = 1, \ldots, r$. Indeed, for each $1 \leq i \leq r$, there exists some $X_i \in \Delta$ such that $Y_i = \tau^{n-1} X_i$ or $Y_i = \tau^n X_i$, where the second case occurs if and only if Δ contains an arrow $X \to X_i$. We shall establish the claim by an induction on $d(X)$. If $d(X) = 0$, then X is a sink vertex in Δ. Thus, $Y_i = \tau^{n-1} X_i$, and by the induction hypothesis on $n - 1$, we obtain $IY_i = 0$, for $i = 1, \ldots, r$. Suppose now that $d(X) > 0$. Fix an integer i with $1 \leq i \leq r$. If $Y_i = \tau^{n-1} X_i$, then $IY_i = 0$ by the induction hypothesis on $n - 1$. If $Y_i = \tau^n X_i$, then Δ has an arrow $X \to X_i$, and thus, $d(X_i) < d(X)$. Therefore, $IY_i = 0$ by the induction hypothesis on $d(X)$. This establishes our claim. As a consequence, $IY = 0$. The proof of the lemma is completed.

We are ready to describe the shape of the connected components with bounded short cycles of Γ_A.

3.4. Theorem. *Let A be an artin algebra with $\mathcal{C}$ a connected component of Γ_A. If* $\mathrm{add}(\mathcal{C})$ *has bounded short cycles, then $\mathcal{C}$ is almost acyclic and consists of*

(1) *a finite core containing all possible oriented cycles in $\mathcal{C}$;*
(2) *some infinite left stable components $\Gamma_1, \ldots, \Gamma_r$ with $r \geq 0$, where each Γ_i has a finite section Δ_i such that $B_i = A/\mathrm{ann}(\Delta_i)$ is tilted and the predecessors of Δ_i in $\mathcal{C}$ form a predecessor-closed subquiver of the connecting component of Γ_{B_i};*
(3) *some infinite right stable components $\Theta_1, \ldots, \Theta_s$ with $s \geq 0$, where each Θ_i has a finite section Σ_i such that $C_i = A/\mathrm{ann}(\Sigma_i)$ is tilted and the successors of Σ_i in $\mathcal{C}$ form a successor-closed subquiver of the connecting component of Γ_{C_i}.*

Proof. Suppose that $\mathrm{add}(\mathcal{C})$ has bounded short cycles. By Lemma 3.2, every infinite semi-stable component of $\mathcal{C}$ is acyclic, and by Theorem 2.6, $\mathcal{C}$ is almost acyclic with a finite core containing all possible oriented cycles in $\mathcal{C}$. Note that $\mathcal{C}$ has only finitely many semi-stable components; see [12, (3.1)]. Let $\Gamma_1, \ldots, \Gamma_r$ with $r \geq 0$ be the infinite left stable components, and $\Theta_1, \ldots, \Theta_s$ with $s \geq 0$ the infinite right stable components, of $\mathcal{C}$. Let X be a module in $\mathcal{C}$. If all the predecessors of X in $\mathcal{C}$ are left stable, then X belongs to a non-trivial left stable component Γ of $\mathcal{C}$. If Γ is finite, then all the modules in Γ are τ-periodic, and then, it is well known that X has a projective predecessor in $\mathcal{C}$, a contradiction. Thus, $\Gamma = \Gamma_i$ for some $1 \le i \le r$. Dually, if all the successors of X in $\mathcal{C}$ are right stable, then X belongs to one of the Θ_j with $1 \le j \le s$. Therefore, if X does not belong to any of the Γ_i, Θ_j, then X admits a projective predecessor and an injective successor in $\mathcal{C}$. That is, X belongs to the core of $\mathcal{C}$. This shows that $\mathcal{C}$ is the union of its core, $\Gamma_1, \ldots, \Gamma_r$ and $\Theta_1, \ldots, \Theta_s$.

Now, let Γ be an infinite left stable component of $\mathcal{C}$. In particular, $\mathrm{add}(\Gamma)$ has bounded short cycles. By Lemma 3.2, Γ is acyclic, and hence, it contains a section Σ with a unique sink and no projective predecessor in Γ_A; see [12, (3.3)]. Now, $\Delta = \tau\Sigma$ is a section of Γ with a unique sink and no projective immediate successor in Γ_A. Moreover, for any predecessor Y of Δ in Γ_A, both Y and $\tau^{-}Y$ belong to Γ. In particular, the full subquiver Ω of Γ_A generated by the predecessors of Δ in Γ_A is contained in Γ.

Since Δ has no projective immediate successor in Γ_A, we see that Δ is a cut of $\mathcal{C}$. If Δ is infinite, then $\mathrm{add}(\Gamma)$ contains a short cycle passing through modules in Γ; see the proofs of [13, (2.2), (2.3), (2.4)]. Since Γ is acyclic and contains all the predecessors of Δ in Γ_A, this short cycle is of infinite depth, a contradiction. Thus, Δ is finite. We claim that $\mathrm{rad}^{\infty}(M, N) = 0$ for all modules $M, N \in \Delta$. Indeed, suppose on the contrary that there exist modules $M, N \in \Delta$ such that $\mathrm{rad}^{\infty}(M, N)$ has a non-zero map $f_0 : M \to N$. Since Γ contains all the predecessors of Δ in $\mathcal{C}$, we can find an infinite path

$$\cdots \longrightarrow N_i \longrightarrow N_{i-1} \longrightarrow \cdots \longrightarrow N_1 \longrightarrow N_0 = N$$

in Γ such that $\mathrm{rad}^{\infty}(M, N_i)$ has a non-zero map f_i, for every $i \ge 0$; see [15, (1.1)]. Since Δ is a finite section of Γ, there exists some minimal integer $m \ge 0$ such that N_m is a predecessor of M in Γ, say, there exists a path

$$\rho: \qquad N_m = M_t \longrightarrow M_{t-1} \longrightarrow \cdots \longrightarrow M_1 \longrightarrow M_0 = M$$

in Γ. Suppose that ρ is not sectional, that is, $M_{j+1} = \tau M_{j-1}$ for some $0 < j < t$. Then, since Δ is a section, $m > 0$. By our choice of Δ, the $\tau^{-}M_i$ with $t \ge i \ge j+1$ belong to Γ. This yields a path

$$N_{m-1} \longrightarrow \tau^{-}M_t \longrightarrow \cdots \longrightarrow \tau^{-}M_{j+1} \longrightarrow M_{j-2} \longrightarrow \cdots \longrightarrow M_0 = M,$$

a contradiction to the minimality of m. Therefore, ρ is sectional, and consequently, $\mathrm{Hom}_A(N_m, M) \ne 0$. Thus, we obtain a short cycle

$$M \xrightarrow{f_s} N_m \xrightarrow{g} M$$

of infinite depth in $\mathrm{add}(\Gamma)$, a contradiction. This establishes our claim.

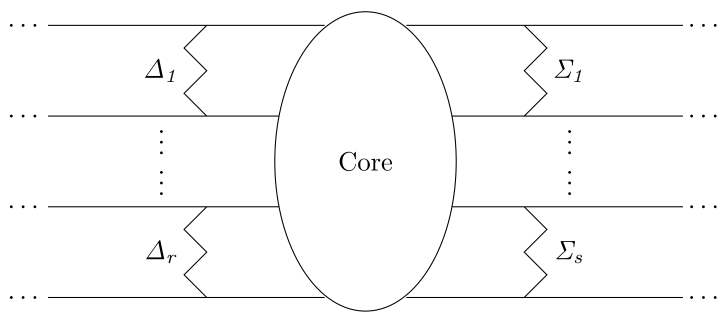
Next, suppose that there exists a non-zero map $g_0 : M \to \tau N_0$ with $M, N_0 \in \Delta$. Observe that M is not a predecessor of τN_0 in Γ. Since all the predecessors of N in Γ_A belong to Γ, we see that M is not a predecessor of τN_0 in Γ_A. In particular, $g_0 \in \mathrm{rad}^{\infty}(M, \tau N_0)$. Considering the minimal left almost split monomorphism for τN_0, we obtain an irreducible map $g_1 : \tau N_0 \to M_1$ with $M_1 \in \Gamma$ such that $0 \ne g_1 g_0 \in \mathrm{rad}^{\infty}(M, M_1)$. By the above claim, $M_1 \notin \Delta$, and thus, $M_1 = \tau N_1$ with

$N_1 \in \mathit{\Delta}$. This yields an arrow $N_0 \to N_1$ in $\mathit{\Delta}$ with $\mathrm{rad}^\infty(M, \tau N_1) \neq 0$. Continuing this process, we obtain an infinite path

$$N_0 \longrightarrow N_1 \longrightarrow \cdots \longrightarrow N_{j-1} \longrightarrow N_j \longrightarrow \cdots$$

in $\mathit{\Delta}$, contrary to the finiteness of $\mathit{\Delta}$. This shows that $\mathrm{Hom}_A(\mathit{\Delta}, \tau\mathit{\Delta}) = 0$. As a consequence, $B = A/\mathrm{ann}(\mathit{\Delta})$ is a tilted algebra with $\mathit{\Delta}$ being a slice of $\mathit{\Gamma}_B$; see [15, (2.8)]. By Lemma 3.3, the predecessors of $\mathit{\Delta}$ in $\mathit{\Gamma}_A$ are B-modules. Therefore, Ω is a subquiver of $\mathit{\Gamma}_B$. Since $\mathit{\Omega}$ is left stable and predecessor-closed in $\mathit{\Gamma}_A$, we see that $\mathit{\Omega}$ is predecessor-closed in $\mathit{\Gamma}_B$. This shows that each of the $\mathit{\Gamma}_i$ satisfies the condition stated in Statement (2). Dually, each of the $\mathit{\Theta}_i$ satisfies the condition stated in Statement (3). The proof of the theorem is completed.

Remark. In view of Theorem 3.4, a connected component with bounded short cycles of $\mathit{\Gamma}_A$ can be pictured as follows.



The following consequence of Theorem 3.4 includes the result stated in [13, (2.7)] on semi-regular components with no short cycles.

3.5. Theorem. *Let A be an artin algebra. If $\mathcal{C}$ is a semi-regular component of $\mathit{\Gamma}_A$, then* $\mathrm{add}(\mathcal{C})$ *has bounded short cycles if and only if $B = A/\mathrm{ann}(\mathcal{C})$ is tilted with $\mathcal{C}$ being a connecting component of $\mathit{\Gamma}_B$.*

Proof. The sufficiency is evident, see, for example, [13, (2.7)]. Let $\mathcal{C}$ be a connected component of $\mathit{\Gamma}_A$ with no projective module. In particular, $\mathcal{C}$ is infinite. Assume that $\mathrm{add}(\mathit{\Gamma})$ has bounded short cycles. In view of Theorem 3.4, we see that $\mathcal{C}$ contains a section $\mathit{\Delta}$ such that $B = A/\mathrm{ann}(\mathit{\Delta})$ is tilted with $\mathcal{C}$ a connecting component of $\mathit{\Gamma}_B$. Since $\mathrm{ann}(\mathit{\Delta}) = \mathrm{ann}(\mathcal{C})$; see [13, (2.1)], we obtain $B = A/\mathrm{ann}(\mathcal{C})$. The proof of the theorem is completed.

The following statement extends some results on Auslander-Reiten components without short cycles, which are stated in [13, (2.6),(2.8)].

3.6. Theorem. *Let A be an artin algebra. The Auslander-Reiten quiver $\mathit{\Gamma}_A$ has at most finitely many connected components with bounded short cycles, and each of them has only finitely many τ-orbits.*

Proof. Let $\mathcal{C}$ be a connected component with bounded short cycles of $\mathit{\Gamma}_A$. The second part is an immediate consequence of Theorem 3.4. If $\mathcal{C}$ is semi-regular then, by Theorem 3.5, $\mathcal{C}$ is a connecting component of a tilted algebra $A/\mathrm{ann}(\mathcal{C})$. In particular, $\mathrm{add}(\mathcal{C})$ contains no short cycle; see [13, (2.7)]. Having at most finitely many connected components which are not semi-regular and at most finitely many connected components with no short cycle; see [13, (2.8)], Γ_A has at most finitely many connected components with bounded short cycles. The proof of the theorem is completed.

A connected artin algebra A is *generalized double tilted* if and only if Γ_A contains a faithful, almost acyclic and generalized standard component, which is called a *connecting component*; see [19, Section 3].

3.7. Proposition. *Let A be an artin algebra. A connected component $\mathcal{C}$ of Γ_A is generalized standard with bounded short cycles if and only if $B = A/\mathrm{ann}(\mathcal{C})$ is generalized double tilted with $\mathcal{C}$ a connecting component of Γ_B.*

Proof. The necessity follows immediately from Theorem 3.6. Assume now that $B = A/\mathrm{ann}(\mathcal{C})$ is generalized double tilted and $\mathcal{C}$ is a connecting component of Γ_B. Being almost acyclic, by Theorem 2.6, $\mathcal{C}$ contains a finite core Ω. Let b be the maximal R-length of the modules in Ω, where R is the center of A. Consider a short cycle σ consisting of two maps $f : M \to N$ and $g : N \to M$ in $\mathrm{rad}(\mathcal{C})$. Since $\mathcal{C}$ is generalized standard, we deduce from Theorem 2.6 that both f and g are sums of composites of irreducible maps in $\mathrm{add}(\Omega)$. In view of the Harada-Sai Lemma; see [6], and also [2, (VI.1.3)], we see that both f and g are of depth less than 2^b. That is, $\mathrm{dp}(\sigma) < 2^b$. The proof of the proposition is completed.

Example. Let $A = kQ/I$, where k is a field, Q is the quiver

$$\begin{tikzcd} 5 \arrow[r] & 4 \arrow[r, shift left] \arrow[r, shift right] & 3 \arrow[r] & 2 \arrow[r, shift left] & 1, \arrow[l, shift left] \end{tikzcd}$$

and I is the ideal in the path algebra kQ generated by the paths of length two. It is easy to see that Γ_A contains a generalized standard component with bounded short cycles as follows:

$$\begin{tikzcd}[column sep=small]
 & & & & & & P_4 \arrow[dr, shift left] \arrow[dr, shift right] & \\
 & & & P_1 \arrow[dr] & & S_3 \arrow[ur, shift left] \arrow[ur, shift right] & & \cdots \\
S_1 \arrow[r] & P_2 \arrow[r] & S_2 \arrow[ur] \arrow[dr] & & I_2 \arrow[ur] \arrow[dr] & & & \\
 & & & P_3 \arrow[ur] & & S_1 & &
\end{tikzcd}$$

It has been shown that the artin algebra A is representation-finite if $\mathrm{mod}\,A$ contains no short cycle; see [3]. In order to improve this result, we shall say that $\mathrm{mod}\,A$ has *bounded short cycles* if there exists a bound for the depths of all possible short cycles in $\mathrm{ind}\,A$.

3.8. Theorem. *An artin algebra A is of finite representation type if and only if $\mathrm{mod}\,A$ has bounded short cycles.*

Proof. Suppose first that A is of finite representation type. Then $\mathrm{rad}^n(\mathrm{mod}\,A) = 0$ for some integer $n > 0$; see [2, (V.7.7)]. That is, every non-zero map in $\mathrm{ind}\,A$ is of depth less than n. In particular, $\mathrm{mod}\,A$ has bounded short cycles.

Suppose now that $\mathrm{mod}\,A$ has bounded short cycles but is of infinite type. Then, Γ_A has an infinite connected component $\mathcal{C}$; see [2, (VI.1.4)]. We may assume with no loss of generality that $\mathcal{C}$ contains an infinite left-stable component Γ; see [12, (3.1)]. By Theorem 3.4, Γ contains a section Δ such that $B = A/\mathrm{ann}(\Delta)$ is a tilted algebra, and the predecessors of Δ in Γ_A generate a predecessor-closed subquiver of a connecting component of Γ_B. In particular, B is representation-infinite. Then, Γ_B has a connected component $\mathscr{C}$ containing non-directing modules; see [22, (2.4)], and also [3]. Observe that $\mathscr{C}$ cannot be a prepropjective component, a preinjective component or a connecting component of Γ_B; see [22, (2.4), (4.2)]. Therefore, $\mathscr{C}$ is either quasi-serial or is obtained from a quasi-serial translation quiver by ray insertions or by co-ray insertions; see [11, (3.7)]. If $\mathscr{C}$ contains oriented cycles, by Lemma 3.2, $\mathrm{add}(\mathscr{C})$ has unbounded short cycles in $\mathrm{ind}B$. Otherwise, $\mathscr{C}$ is obtained from a translation quiver of shape $\mathbb{Z}\mathbb{A}_\infty$ by ray insertions or by co-ray insertions. In particular, $\mathscr{C}$ has infinitely many τ_B-orbits, and by Theorem 3.6, $\mathrm{add}(\mathscr{C})$ has unbounded short cycles. In all cases, $\mathrm{mod}B$ has unbounded short cycles, and so does $\mathrm{mod}A$, a contradiction. The proof of the theorem is completed.

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