MODULE CATEGORIES OF SMALL RADICAL NILPOTENCY

SHIPING LIU AND YOUQI YIN

Abstract. This paper aims to initiate a study of the representation theory of representation-finite artin algebras in terms of the nilpotency of the radical of their module category. Firstly, we shall calculate this nilpotency explicitly for hereditary algebras of type $\mathbb{A}_n$ and for Nakayama algebras. Surprisingly, this nilpotency for an artin algebra coincides with its Loewy length if and only if the algebra is a hereditary Nakayama algebra. Secondly, given a positive integer m up to four, we shall find all artin algebras for which this nilpotency is equal to m and provide a complete description of their module category.

Introduction

Let A be a connected artin algebra of finite representation type. The central objective of the representation theory is to study the category mod$A$ of finitely generated left A-modules, that is to classify its indecomposable modules and describe the maps between them. For instance, the representation theory of Nakayama algebras is well understood; see, for example, [2, Section V.2]. An important result says that rad(mod$A$), that is the radical of mod$A$, is nilpotent; see [2, (V.7.6)], and also [12, 14]. Observe that A is simple if and only if rad(mod$A$) vanishes. It is natural to ask whether the nilpotency of rad(mod$A$) determines completely, or to what extent, the representation theory of A. One may approach this question from two aspects. Firstly, given a class of representation-finite algebras A, one may calculate or estimate the nilpotency of rad(mod$A$); and secondly, given an integer n, one may find a complete list of representation-finite algebras A such that rad(mod$A$) is of nipotency n and describe their representation theory if possible.

As to the first problem under the most general setting, the Harada-Sai Lemma says that the nipotency of rad(mod$A$) is bounded by $2^b - 1$, where b is the maximal length of all indecomposable modules in mod$A$; see [12]. A sharper bound is given in [10] which depends also on the maximal length of all indecomposable modules. By a completely different approach, this nilpotency is shown to be the maximal depth of the composite of the projective cover and the injective envelope of simple modules in mod$A$; see [9]. In this paper, we shall use the latter result to show that the nilpotency of rad(mod$A$) is equal to n in case A is a hereditary algebra of type $\mathbb{A}_n$; see (2.5), and it is equal to $m - 1$, where m is the maximal sum of the composition length of the projective cover and that of the injective envelope of simple modules in case A is a Nakayama algebra; see (2.7). It is evident that the nilpotency of rad(mod$A$) is greater than or equal to the nilpotency of the radical

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of A. Surprisingly, these two numbers coincide if and only if A is hereditary of type $\mathbb{A}_n$ with a linear orientation, that is, a hereditary Nakayama algebra; see (2.6).

This paper is mainly devoted to the second problem for small nilpotencies up to four. The result for nilpotency two is straightforward: the only artin algebras are hereditary algebras of type $\mathbb{A}_2$; see (5.1). The list for nilpotency three is short and nice, consisting of two well understood classes, namely, the hereditary algebras of type $\mathbb{B}_2$ or $\mathbb{A}_3$ and the non-hereditary Nakayama algebras with radical squared zero; see (5.2). However, it is quite long to obtain the list for nilpotency four. First, we shall show that they are all *string algebras*; see (3.1) and (4.1), an artin version of Butler and Ringel's string algebras defined by a quiver with relations; see [7]. Then, we shall divide them into three classes: the hereditary algebras of type $\mathbb{A}_4$, the non-hereditary Nakayama algebras of Loewy length three, and the non-hereditary non-Nakayama *tri-string algebras*; see (5.3), where the last class of algebras are string algebras with radical cubed zero plus some other additional conditions; see (3.5) and (4.4). The representation theory of all above-mentioned algebras will be explicitly described. In particular, the module category of a tri-string algebra is similar to that of Nakayama algebra; see (5.4) and (5.5).

Our tool consists of the Auslander-Reiten theory of irreducible maps and almost split sequences and the theory of degrees of irreducible maps, the latter appears to be particularly applicable in this topic. Our results provide some evidences that the nilpotency of rad(mod$A$) depends only on the composition lengths of the indecomposable projective or injective modules and determines to certain extent the shape of the Auslander-Reiten quiver of A, and we believe that they will stimulate future research in this direction. We are grateful to Gordana Todorov for pointing out a mistake in a previous version of the paper.

1. Preliminaries

The main objective of this section is to fix the notation and the terminology, which will be used throughout the paper, and collect some known results which will be needed for our investigation. Beside this, we shall also obtain some new results.

1) Radical of module categories. Throughout this paper, A stands for a connected artin algebra and rad$A$ for the radical of A. The *Loewy length* of A, that is the minimal integer s such that $\mathrm{rad}^s A = 0$, will be written as $\ell\ell(A)$. We shall denote by mod$A$ the category of finitely generated left A-modules in which the morphisms are composed from the right to the left, and by ind$A$ the full subcategory of mod$A$ generated by the indecomposable modules. By a *projective* or *injective* module in ind$A$ we mean a module in ind$A$ which is projective or injective in mod$A$, respectively. The *radical* of mod$A$ is the two-sided ideal rad(mod$A$) in mod$A$ generated by the non invertible maps in ind$A$. A map in mod$A$ is called *radical* if it lies in rad(mod$A$). We shall write $\mathrm{rad}^m(\mathrm{mod}A)$ for the m-th power of rad(mod$A$) for each integer $m \geq 0$ and $\mathrm{rad}^\infty(\mathrm{mod}A)$ for the intersection of all $\mathrm{rad}^m(\mathrm{mod}A)$ with $m \geq 0$. In case A is representation-finite, there exists a minimal integer m such that $\mathrm{rad}^m(\mathrm{mod}A) = 0$; see [2, (V.7.6)], which will be called the *nilpotency* of rad(mod$A$), and also, the *radical nipotency* of mod$A$.

Let M be a module in mod$A$. The composition length of M will be simply called the *length* and written as $\ell(M)$. And the top, the radical and the socle of M

will be written as top$M$, rad$M$ and soc$M$, respectively. Moreover, for each simple module S in mod$A$, we shall fix a projective cover $\pi_S : P_S \to S$ and an injective envelope $\iota_S : S \to I_S$, and put $\theta_S = \iota_S \circ \pi_S$. For convenience of reference, we state the following well-known fact ; see, for example, [2, (III.1.15)].

1.1. Lemma. *Let A be an artin algebra. If S, T are simple modules in* mod$A$*, then S is a direct summand of the top of the radical of P_T if and only if T is a direct summand of the socle of the socle-factor of I_S.*

2) Auslander-Reiten theory. A comprehensive account of the Auslander-Reiten theory of irreducible maps and almost split sequences can be found in [2]. By a *sink map* and a *source map* in mod$A$, we mean a minimal left almost split map and a minimal right almost split map respectively. For convenience of reference, we state the following well known fact; see [2, (V.5.5)].

1.2. Lemma. *Let A be an artin algebra. If P is a projective injective module in* mod$A$ *of length at least two, then there exists in* mod$A$ *an almost split sequence*

$$0 \longrightarrow \mathrm{rad}P \xrightarrow{(q,p_1)^T} P \oplus \mathrm{rad}P/\mathrm{soc}P \xrightarrow{(p,q_1)} P/\mathrm{soc}P \longrightarrow 0.$$

The following statement strengthens slightly a result stated in [13].

1.3. Lemma. *Let A be an artin algebra. Consider almost split sequences*

$$0 \longrightarrow X_1 \xrightarrow{(f_1,u_1)^T} Y \oplus M \xrightarrow{(g_1,v_1)} Z_1 \longrightarrow 0$$

and

$$0 \longrightarrow X_2 \xrightarrow{(f_2,u_2)^T} Y \oplus N \xrightarrow{(g_2,v_2)} Z_2 \longrightarrow 0$$

in mod$A$*, where Y is indecomposable. If Y is not a direct summand of M, then $(f_1, f_2) : X_1 \oplus X_2 \to Y$ is irreducible if and only if so is $(g_1, g_2)^T : Y \to Z_1 \oplus Z_2$.*

Proof. We shall prove only the necessity. Assume that $(f_1, f_2) : X_1 \oplus X_2 \to Y$ is irreducible. If $X_1 \not\cong X_2$, then $Z_1 \not\cong Z_2$, and hence, $(g_1, g_2)^T$ is irreducible; see [4, page 92]. Otherwise, there exists an isomorphism $f : X_1 \to X_2$, which induces an isomorphism $g : Z_2 \to X_1$. This gives rise to an almost split sequence

$$0 \longrightarrow X_1 \xrightarrow{(f_2 f,u_2 f)^T} Y \oplus Y' \xrightarrow{(gg_2,gv_2)} Z_1 \longrightarrow 0.$$

On the other hand, $(f_1, f_2 f) : X_1 \oplus X_1 \to Y$ is irreducible. By Lemma 1.10 in [15] and the dual of Corollary 3.4 in [4], $(g_1, gg_2)^T : Y \to Z_1 \oplus Z_1$ is irreducible. As a consequence, $(g_1, g_2)^T$ is irreducible. The proof of the lemma is completed.

Throughout, $\varGamma_A$ stands for the Auslander-Reiten quiver of A, which is a valued translation quiver having as vertex set the set of isomorphism classes of modules in ind$A$ and as translation the Auslander-Reiten translation $\tau = \mathrm{DTr}$ with quasi-inverse $\tau^- = \mathrm{TrD}$, where $D : \mathrm{mod}A \to \mathrm{mod}A^{\mathrm{op}}$ denotes the standard duality and $\mathrm{Tr} : \underline{\mathrm{mod}}A \to \underline{\mathrm{mod}}A^{\mathrm{op}}$ denotes the transpose. For brevity, a module in ind$A$ will be identified with its isomorphism class in $\varGamma_A$. We shall say that $\varGamma_A$ is *planar* if the middle term of any almost split sequence in mod$A$ is either indecomposable or a direct sum of two indecomposable modules.

A path $X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_n$ in $\varGamma_A$ with $n \geq 1$ is called *sectional* provided that $\tau X_{i+1} \not\cong X_{i-1}$ for all $0 < i < n$ and *pre-sectional* provided that $\tau X_{i+1} \cong X_{i-1}$

with $0 < i < n$ occurs only if there exists an irreducible map $f : X_{i-1} \oplus X_{i-1} \to X_i$ or $g : X_i \to X_{i+1} \oplus X_{i+1}$; see [15, (1.4)].

By an *irreducible map* in ind$A$ we mean a map in ind$A$ which is irreducible in mod$A$. A path of irreducible maps in ind$A$ is called *sectional* or *pre-sectional* if the corresponding path in $\varGamma_A$ is sectional or pre-sectional respectively. In a diagram of irreducible maps in ind$A$, a left dotted arrow $X \leftarrow\!\cdots\; Y$ indicates that $X = \tau Y$.

1.4. Definition. A diagram $\varOmega$ of irreducible maps in ind$A$ is called *fitting* provided, for all modules M and $\tau^- M$ in $\varOmega$, that the maps $f_i : M \to M_i$ of domain M and the maps $g_i : M_i \to \tau^- M$ of co-domain $\tau^- M$ in $\varOmega$, for $i = 1, \dots, r$, fit in an almost split sequence

$$0 \longrightarrow M \xrightarrow{(f_1,\cdots,f_r,f)^T} M_1 \oplus \cdots \oplus M_r \oplus X_M \xrightarrow{(g_1,\cdots,g_r,g)} \tau^- M \longrightarrow 0$$

in mod$A$. Such a fitting diagram $\varOmega$ is called *mesh-complete* if $X_M = 0$ for all modules M and $\tau^- M$ in $\varOmega$; and in this case, the sub-diagram formed by f_i and g_i with $1 \le i \le r$ is called a *mesh* in ind$A$.

We shall use frequently the following well known statement to construct fitting diagrams of irreducible maps in ind$A$.

1.5. Lemma. *Let A be an artin algebra. Consider an almost split sequence*

$$0 \longrightarrow X \xrightarrow{(f_1,\cdots,f_r)^T} Y_1 \oplus \cdots \oplus Y_r \xrightarrow{(g_1,\dots,g_r)} Z \longrightarrow 0$$

in mod$A$, *where $Y_1, \dots, Y_r$ are indecomposable.*

(1) *If f_i is a monomorphism for some $1 \le i \le r$, then g_j is a monomorphism, and hence, X_j is not injective for every $j \ne i$.*
(2) *If g_i is an epimorphism for some $1 \le i \le r$, then f_j is an epimorphism, and hence, X_j is not projective for every $j \ne i$.*

3) Degrees of irreducible maps. Given a map $f : X \to Y$ in mod$A$, its *depth* dp(f) is defined to be s if $f \in \text{rad}^s(X,Y) \backslash \text{rad}^{s+1}(X,Y)$; and to be infinity if $f \in \text{rad}^\infty(X,Y)$; see [9]. This terminology allows us to reformulate the notion of degrees of irreducible maps as follows; compare [15, (1.1)].

1.6. Definition. Let $f : X \to Y$ be an irreducible map in mod$A$ with X or Y in ind$A$. The *left degree* $d_l(f)$ of f is defined to be the minimal integer n such that there exists a map $g : M \to X$ of depth n with $fg \in \text{rad}^{n+2}(M,Y)$; and $d_l(f) = \infty$ if such an integer n does not exist. The *right degree* $d_r(f)$ of f is defined dually.

Remark. It is handy to view the degrees of an irreducible map $f : X \to Y$ in the following way. If $g : M \to X$ is a map of depth $s < d_l(f)$, then fg is of depth $s+1$; and if $h : Y \to N$ is a map of depth $t < d_r(f)$, then hf is of depth $t+1$.

We shall use frequently the following statement, which combines the corollaries to Lemmas 1.2 and 1.3 stated in [15].

1.7. Lemma. *Let A be an artin algebra. Consider an irreducible map $f : X \to Y$ in* mod$A$, *where X or Y is indecomposable.*

(1) *The left degree of f is equal to one if and only if f is a sink epimorphism. Moreover, $d_l(f) = \infty$ in case Y is projective.*

(2) *The right degree of f is equal to one if and only if f is a source epimorphism. Moreover, $d_r(f) = \infty$ in case X is injective.*

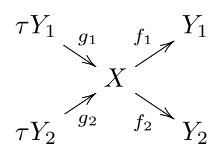
The key ingredient in the application of degrees of irreducible maps is the reduction of finite degrees illustrated in the following two statements, which are quoted or reformulated from results stated in [15, (1.2), (1.3), (1.11)].

1.8. LEMMA. *Let A be an artin algebra. Consider an almost split sequence*

$$0 \longrightarrow X \xrightarrow{(f_1,f_2)^T} Y_1 \oplus Y_2 \xrightarrow{(g_1,g_2)} Z \longrightarrow 0$$

in $\mathrm{mod}A$, *where Y_1, Y_2 are non-zero. If $d_l(g_1) < \infty$, then $d_l(f_2) < d_l(g_1)$; and if $d_r(f_1) < \infty$, then $d_r(g_2) < d_r(f_1)$.*

1.9. LEMMA. *Let A be an artin algebra. If $(f_1, f_2)^T : X \to Y_1 \oplus Y_2$ is an irreducible map of left degree n, where $X, Y_1, Y_2 \in \mathrm{ind}A$, then there exists a fitting diagram*



in $\mathrm{ind}A$ *such that (g_1, g_2) is an irreducible map of left degree $< n$.*

We quote the following statement from [15, (1.6), (1.15)].

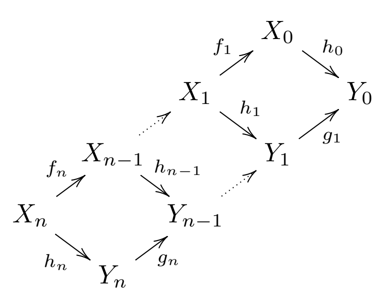
1.10. PROPOSITION. *Let A be an artin algebra, and let Γ_A have a pre-sectional path*

$$Y_n \longrightarrow Y_{n-1} \longrightarrow \cdots \longrightarrow Y_1 \longrightarrow Y_0.$$

(1) *There exist irreducible maps $f_i : Y_i \to Y_{i-1}$ such that* $\mathrm{dp}(f_1 \cdots f_n) = n$.
(2) *If the path is sectional and $f_i : Y_i \to Y_{i-1}$ is irreducible, then* $\mathrm{dp}(f_1 \cdots f_n) = n$.
(3) *If $(f, f_1) : X \oplus Y_1 \to Y_0$ is an irreducible map with $f : X \to Y_0$ non-zero, then $d_l(f) > n$, and $d_l(f) = \infty$ in case Y_i is projective for some $0 \le i \le n$.*
(4) *If $(f_n, g)^T : Y_n \to Y_{n-1} \oplus Y$ is an irreducible map with $g : Y_n \to Y$ non-zero, then $d_r(g) > n$, and $d_r(g) = n$ in case Y_i is injective for some $0 \le i \le n$.*

The following statement will be useful for our investigation.

1.11. LEMMA. *Let A be an artin algebra with a fitting diagram*



in $\mathrm{ind}A$, *where $X_i = \tau Y_{i-1}$ for $i = 1, \ldots, n$. Then, there exists a map $f : X_n \to Y_0$ of depth $n + 1$.*

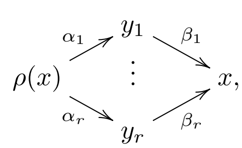
Proof. By the assumption, $X_n \to \cdots \to X_1 \to X_0$ and $Y_n \to \cdots \to Y_1 \to Y_0$ are pre-sectional paths in Γ_A. By Proposition 1.10(1), we can find a map $g : X_n \to X_0$ of depth n, and by Lemma 1.10(3), $d_l(h_0) > n$. Thus, $\mathrm{dp}(h_0 g) = n + 1$. The proof of the lemma is completed.

Remark. In the sequel, a fitting diagram as described in Lemma 1.11 will be called a *ladder* of *height* n from X_n to Y_0.

4) Valued translation quivers. Let Δ be a valued quiver. The valuation (d_{xy}, d'_{xy}) for an arrow $x \to y$ in Δ is *trivial* if $d_{xy} = d'_{xy} = 1$; and in this case, the valuation will be omitted. If Σ is a valued subquiver of Δ, then the valuation for each arrow in Σ will be the same as in Δ. Given an integer $n \geq 1$, we shall denote by $\vec{\mathbb{A}}_n$ the trivially valued quiver, whose vertices are the integers $1, \ldots, n$ and whose arrows are $i \to i+1$, for $i = 1, \ldots, n-1$. Moreover, we shall denote by $\hat{\mathbb{A}}_n$ a trivially valued quiver of type $\mathbb{A}_n$ with a non-linear orientation.

Given a valued quiver Δ with no oriented cycle, one defines a valued translation quiver $\mathbb{Z}\Delta$ as follows. The vertices are the pairs (n, x), where $n \in \mathbb{Z}$ and $x \in \Delta_0$. Each arrow $x \to y$ in Δ with valuation (v_{xy}, v'_{xy}) induces arrows $(n, x) \to (n, y)$ with valuation (v_{xy}, v'_{xy}) and $(n-1, y) \to (n, x)$ with valuation (v'_{xy}, v_{xy}), where $n \in \mathbb{Z}$, in $\mathbb{Z}\Delta$. The translation of $\mathbb{Z}\Delta$ is defined so that it sends (n, x) to $(n-1, x)$, for all $n \in \mathbb{Z}$ and $x \in \Delta_0$; see [6, (1.7)].

Let Γ be a valued translation quiver with translation ρ. If x is a vertex x in Γ such that $\rho(x)$ is defined, then the *mesh* starting with $\rho(x)$ and ending with x is the valued subquiver



where $\alpha_1, \ldots, \alpha_r$ are the arrows in Γ starting with $\rho(x)$, and $\beta_1, \ldots, \beta_r$ are the arrows ending with x. In this case, $\beta_1\alpha_1, \ldots, \beta_r\alpha_r$ are called the *components* of the mesh. Such a mesh is called *monomial* if $r = 1$ and *binomial* if $r = 2$. A full valued translation subquiver Ω of Γ is called *mesh-complete* if every mesh in Γ, whose starting point and ending point lie in Ω, lies entirely in Ω.

Two paths ξ and ζ in Γ are called *homotopic* if there exists a sequence of paths $\xi = \xi_1, \xi_2, \ldots, \xi_m = \zeta$ such that for each $1 \leq i < m$, either $\xi_i = \xi_{i+1}$ or else, $\xi_i = \eta_i\beta_i\alpha_i\theta_i$ and $\xi_{i+1} = \eta_i\delta_i\gamma_i\theta_i$, where η_i, θ_i are paths in Γ, while $\beta_i\alpha_i$ and $\delta_i\gamma_i$ are components of the same mesh; compare [6, (1.2)]. This is an equivalence relation on the set of paths in Γ, which is compatible with the concatenation of paths. Observe that two homotopic paths are parallel (that is, they star with the same vertex and end with the same vertex) of the same length.

1.12. Lemma. *The following statements hold in* $\mathbb{Z}\vec{\mathbb{A}}_n$ *with* $n \geq 2$.

(1) *Any two parallel paths are homotopic, and hence, of the same length.*

(2) *Every path of length* n *is homotopic to a path passing through a monomial mesh.*

Proof. Let ξ and ζ be paths in $\mathbb{Z}\vec{\mathbb{A}}_n$ from a vertex x to another vertex y. It is clear that $\xi = \zeta$ in case ξ or ζ is sectional. In particular, Statement (1) holds if ξ or ζ is of length ≤ 1. Suppose that ξ and ζ are of length > 1. Write $\xi = \xi_1\alpha$ and $\zeta = \zeta_1\gamma$, where $\alpha : x \to x_1$ and $\gamma : x \to y_1$ are arrows. If $\alpha = \gamma$, then ξ is homotopic to ζ by the induction hypothesis. Otherwise, neither ξ nor ζ is sectional. Then, we obtain a path $\eta: \rho^- x \rightsquigarrow y$, where ρ denotes the translation of $\mathbb{Z}\vec{\mathbb{A}}_n$. Consider the arrows $\beta : x_1 \to \rho^- x$ and $\delta : y_1 \to \rho^- x$. By the induction hypothesis, we deduce that ξ is homotopic to $\eta\beta\alpha$, while ζ is homotopic to $\eta\delta\gamma$. Since $\eta\beta\alpha$ and $\eta\delta\gamma$ are homotopic by definition, ξ is homotopic to ζ. This establishes Statement (1).

Assume now that $\xi : x \rightsquigarrow y$ is a path of length n. By Statement (1), it suffices to find a path from x to y, which passes through a monomial mesh. With no loss of generality, we may assume that $x = (0, i)$ for some $1 \le i \le n$. Then, $y = (s, j)$ for some $s \ge 0$ and $1 \le j \le n$. Let Σ be the convex hull generated by $(0, i)$ and $(n - i, i)$, which has two boundary paths

$$x = (0, i) \to (0, i + 1) \to \cdots \to (0, n - 1) \to (0, n) \to (1, n - 1) \to \cdots \to (n - i, i)$$

and

$$x = (0, i) \to (1, i - 1) \to \cdots \to (i - 2, 2) \to (i - 1, 1) \to (i - 1, 2) \to \cdots \to (n - i, i).$$

Since these paths are of length $n - 1$, any path from x to a vertex in Σ is of length $< n$. Hence, $y \notin \Sigma$. That is, either $i \le j \le n$ and $s > n - j$, or else, $1 \le j < i$ and $s > i$. This yields a path

$$x = (0, i) \to \cdots \to (i - 1, 1) \to (i\text{-}1, 2) \to (i, 1) \to \cdots \to (i, j) \to \cdots \to (s, j) = y$$

or

$$x = (0, i) \to \cdots \to (0, n) \to (1, n-1) \to (1, n) \to \cdots \to (n-j+1, j) \to \cdots \to (s, j) = y,$$

which passes through a monomial mesh. The proof of the lemma is completed.

Suppose that Γ is connected. A connected full valued subquiver Δ of Γ is called a *section* if it is convex, contains no oriented cycle, and meets exactly once every ρ-orbit in Γ. In this case, there exists a *canonical embedding* $\Gamma \to \mathbb{Z}\Delta$ sending $\rho^n x$ to $(-n, x)$; see [17, (2.1),(2.3)].

5) Hereditary algebras. Let A be a connected hereditary artin algebra. It is well known that the projective modules in Γ_A generate a section in the preprojective component; see [2, (VIII.1.15)], and dually, the injective modules generate a section in the preinjective component. We shall say that A is *hereditary* of *type* Δ, where Δ is a valued quiver, provided that the section of the preprojective component of Γ_A generated by the projective modules is isomorphic to Δ.

1.13. Proposition. *Let A be a connected non-simple artin algebra. Then A is hereditary of finite representation type if and only if Γ_A has a connected non-trivial mesh-complete valued translation subquiver Γ in which the projective modules generate a section Δ and the injective modules also generate a section. In this case, Γ_A coincides with Γ and embeds in $\mathbb{Z}\Delta$ as a convex translation subquiver.*

Proof. Suppose that A is hereditary and representation-finite. Since A is not simple, Γ_A is non-trivial finite and connected; see [2, (VII.2.1)]. Hence, the projective modules in Γ_A generate a section Δ; see [2, (VIII.1.15)], and dually, the injective modules in Γ_A also generate a section; see [2, (VIII.5.4)]. Consider the canonical embedding of Γ_A into $\mathbb{Z}\Delta$. Let $X_0 \to \cdots \to X_{n-1} \to X_n$ be a path in $\mathbb{Z}\Delta$. We may assume that $X_i = (s_i, P_i)$, where $s_i \ge 0$ and $P_i \in \Delta$ with $\tau^{-s_0}P_0, \tau^{-s_n}P_n \in \Gamma_A$. We shall show that $\tau^{-s_i}P_i \in \Gamma_A$ for every $0 \le i \le n$. Otherwise, there exists some $0 < t \le n$ such that $\tau^{-s_t}P_t \in \Gamma_A$ and $\tau^{-s_{t-1}}P_{t-1} \notin \Gamma_A$. Then, $\tau^{-s}P_{t-1}$ is an injective module, where $0 \le s < s_{t-1}$. Now $X_{t-1} \to X_t$ is induced from an arrow $P_{t-1} \to P_t$ or $P_t \to P_{t-1}$ in Δ. In the first case, $s_t = s_{t-1}$ and $s < s_t$. This yields an arrow $\tau^{-s}P_{t-1} \to \tau^{-s}P_t$ in Γ_A. Since A is hereditary, $\tau^{-s}P_t$ is injective, and hence, $\tau^{-s_t}P_t \notin \Gamma_A$, absurd. In the second case, $s_t = s_{t-1}+1$ and $s+1 < s_t$. This yields an arrow $\tau^{-s}P_{t-1} \to \tau^{-s-1}P_t$ in Γ_A. Then $\tau^{-s-1}P_t$ is injective, and hence, $\tau^{-s_t}P_t \notin \Gamma_A$, absurd. Thus, Γ_A is convex in $\mathbb{Z}\Delta$.

Conversely, assume that Γ is a connected non-trivial mesh-complete valued translation subquiver of Γ_A, having a section Δ generated by the projective modules in Γ and a section Σ generated by the injective modules in Γ. Then, every module M in Γ is uniquely written as $M = \tau^{-s}P = \tau^t I$, where $s, t \ge 0$ and $P \in \Delta$ and $I \in \Omega$. In particular, Γ is finite. Moreover, it is easy to see that $\tau M \in \Gamma$ if M is not projective and $\tau^- M$ is in $\in \Gamma$ if M is injective.

Fix a module P in Δ. We claim that every indecomposable direct summand M of rad$P$ lies in Δ. Suppose that $M \notin \Delta$. Since P is projective, M is not injective. If P is injective, then $M = \text{rad}P$ and $M \to P$ and $P \to \tau^- M$ are the only arrows in Γ_A starting or ending in P; see (1.2). Not being projective, $\tau^- M \notin \Delta$. Thus, $\Delta = \{P\}$ and hence, $\Gamma = \{P\}$, absurd. Thus, P is not injective. As shown above, the path $P \to \tau^- M \to \tau^- P$ lies in Γ, and so does $M \to P \to \tau^- M$, absurd. This establishes our claim. In particular, Γ contains all arrows $X \to P$ in Γ_A. Let $P \to Y$ be an arrow in Γ_A. If P is not injective, then Γ contains the path $P \to Y \to \tau^- P$. Otherwise, $Y = P/\text{soc}P$ with an arrow $\tau Y \to P$. As has been shown, $\tau Y \in \Delta$, and consequently, $Y \in \Gamma$.

Dually, if I is a module in Σ, then Γ contains all arrows $I \to Y$ and $X \to I$ in Γ_A. Furthermore, if $X \to M$ and $M \to Y$ are arrows in Γ_A, where $M \in \Gamma$ and $M \notin \Delta \cup \Sigma$, then Γ contains the paths $\tau M \to X \to M$ and $M \to Y \to \tau^- M$. This shows that Γ is a finite connected component of Γ_A. As a consequence, $\Gamma = \Gamma_A$; see [2, (VII.2.1)]. In particular, Δ contains essentially all projective modules in ind$A$. It follows from our claim that A is hereditary of type Δ. The proof of the proposition is completed.

Remark. An hereditary algebra of type Δ, where Δ is a valued quiver, will also be called *hereditary* of *type* $\bar{\Delta}$, where $\bar{\Delta}$ is the underlying valued graph of Δ.

2. The $\mathbb{A}$-hereditary case and the Nakayama case

The main objective of this section is to calculate the nilpotency of rad(mod$A$) in case A is a hereditary of type $\mathbb{A}_n$ or a Nakayama algebra. We start with some general properties of ind$A$ in case $\text{rad}^n(\text{mod}A)$ vanishes, which confirm that the Auslander-Reiten quiver of mod$A$ is controlled somehow by its radical nilpotency.

2.1. Lemma. *Let A be an artin algebra with* $\text{rad}^n(\text{mod}A) = 0$ *for some $n > 1$. If $f : X \to Y$ is a non-zero map in* ind$A$, *then* $\text{dp}(f) \le n-1$, *where the equality occurs only if X is projective and Y is injective.*

Proof. Let $f : X \to Y$ be a non-zero map in ind$A$. Then, $f \notin \text{rad}^n(X, Y)$, and hence, $\text{dp}(f) \le n-1$. Suppose that $\text{dp}(f) = n-1$. If Y is not injective, then the injective envelope $q : Y \to I$ is a radical map such that $0 \ne qf \in \text{rad}^n(X, I)$, a contradiction. Dually, X is projective. The proof of the lemma is completed.

2.2. Lemma. *Let A be an artin algebra with* $\text{rad}^n(\text{mod}A) = 0$ *for some $n > 1$. Consider non-zero radical maps $f_i : X_i \to X_{i+1}$ in* ind$A$, *for $i = 1, \ldots, n-1$.*
(1) *If the f_i are monomorphisms, then X_1 is simple projective and X_n is injective.*
(2) *If the f_i are epimorphisms, then X_1 is projective and X_n is simple injective.*

Proof. We consider only the case where $f_1, \ldots, f_{n-1}$ are monomorphisms. In view of Lemma 2.1, we see easily that $\mathrm{dp}(f_{n-1} \cdots f_1) = n-1$. Thus, X_1 is projective and X_n is injective. If X_1 is not simple, then there exists a radical monomorphism $f : S \to X_1$, where S is simple. This yields that $0 \neq f f_{n-1} \cdots f_1 \in \mathrm{rad}^n(S, X_n)$, a contradiction. The proof of the lemma is completed.

The following statement says that the degrees of irreducible maps are bounded by the radical nilpotency of $\mathrm{mod}A$.

2.3. Lemma. *Let A be an artin algebra with $\mathrm{rad}^n(\mathrm{mod}A) = 0$ for some $n > 1$.*
(1) *If $f : X \to Y$ is an irreducible epimorphism in $\mathrm{ind}A$, then $d_l(f) \leq n-1$.*
(2) *If $f : X \to Y$ is an irreducible monomorphism in $\mathrm{ind}A$, then $d_r(f) \leq n-1$.*

Proof. Let $f : X \to Y$ be an irreducible epimorphism in $\mathrm{ind}A$. By Lemma 2.1, its kernel $q : L \to X$ is of depth $\leq n-1$. Since $fq = 0$, by definition, $d_l(f) \leq n-1$. The proof of the lemma is completed.

The following statement says that the lengths of pre-sectional paths in Γ_A are bounded by the radical nilpotency of $\mathrm{mod}A$.

2.4. Lemma. *Let A be an artin algebra with $\mathrm{rad}^n(\mathrm{mod}A) = 0$ for some $n > 1$. If Γ_A contains a pre-sectional path $X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_{m-1} \longrightarrow X_m$, then $m \leq n-1$, where the equality occurs only if X_0 is projective and X_{n-1} is injective.*

We are ready to calculate the radical nilpotency of the module category of a hereditary algebra of type $\mathbb{A}_n$.

2.5. Theorem. *Let A be a hereditary artin algebra of type $\mathbb{A}_n$ for some $n \geq 1$. Then the radical of $\mathrm{mod}A$ is nilpotent of nilpotency n.*

Proof. We may assume that $n \geq 2$. Let Δ be the section of Γ_A generated by the projective modules, which is a quiver of type $\mathbb{A}_n$. Since $\mathbb{Z}\Delta \cong \mathbb{Z}\vec{\mathbb{A}}_n$; see [11, (5.6)], we may regard Γ_A as a convex translation subquiver of $\mathbb{Z}\vec{\mathbb{A}}_n$; see (1.13). In particular, every mesh in Γ_A is monomial or binomial. Since Γ_A is finite and contains no oriented cycle, $\mathrm{End}_A(X)$ is a division algebra for any module X in Γ_A. Given an arrow $X \to Y$ in Γ_A, since it has a trivial valuation and is the only path from X to Y, $\mathrm{Hom}_A(X, Y)$ is one dimensional over each of $\mathrm{End}_A(X)$ and $\mathrm{End}_A(Y)$. Combining these two facts with Lemma 1.7, we obtain the following statement.

Sublemma. *Let $f : \tau X \to Y$ and $g : Y \to X$ be irreducible maps in $\mathrm{ind}A$. If the mesh ending with X is monomial, then $gf = 0$; and otherwise, gf forms a basis of $\mathrm{Hom}_A(\tau X, X)$ over each of $\mathrm{End}_A(\tau X)$ and $\mathrm{End}_A(X)$.*

For each arrow $\alpha : X \to Y$ in Γ_A, we choose an irreducible map $f_\alpha : X \to Y$ in $\mathrm{mod}A$. Given a path $\xi = \alpha_1 \cdots \alpha_m$ in Γ_A, where the α_i are arrows, we put $f_\xi = f_{\alpha_1} \cdots f_{\alpha_m}$. Since Γ_A is convex in $\mathbb{Z}\vec{\mathbb{A}}_n$, two paths ξ, ζ in Γ_A are homotopic in Γ_A if and only if they are homotopic in $\mathbb{Z}\mathbb{A}_n$; and in this case, we deduce from the sublemma that $f_\xi = 0$ if and only $f_\zeta = 0$. Moreover, if ξ is a path of length n, then it is homotopic to a path in Γ_A passing through a monomial relation, and by the sublemma, $f_\xi = 0$. This implies that $\mathrm{rad}^n(\mathrm{mod}A) = 0$.

It remains to show that $\mathrm{rad}^{n-1}(\mathrm{mod}A) \neq 0$. Let $M_1 \to \cdots \to M_{t-1} \to M_t$ be a sectional path in Γ_A of maximal length. Then, $M_i = \tau^{-s_i} P_i$, where $s_i \geq 0$ and $P_i \in \Delta$, for $i = 1, \ldots, t$. Since the M_i are pairwise different, so are the P_i, and hence, $t \le n$. Suppose that $t < n$. Since Δ is of type $\mathbb{A}_n$, it contains an edge P—P_1 or P—P_t, where $P \notin \{P_1, \ldots, P_t\}$. Assume that the first case occurs. If M_1 is not projective, then none of the M_i is projective. Applying τ if necessary, we may assume that $M_1 = P_1$. By the maximality of t, we see that $M_1 \to P$ is an arrow in Δ. Hence, there exists an irreducible monomorphism $f_1 : M_1 \to P$. Then, M_1 is not injective and there exists an irreducible monomorphism $f_2 : M_2 \to \tau^- M_1$; see (1.5). By induction, $M_1, \ldots, M_t$ are not injective. This yields a sectional path $P \to \tau^- M_1 \to \cdots \to \tau^- M_t$ in Γ_A, contrary to the maximality of t. Thus, $t = n$. By Lemma 1.10, $\mathrm{rad}^{n-1}(M_1, M_n) \neq 0$. The proof of the theorem is completed.

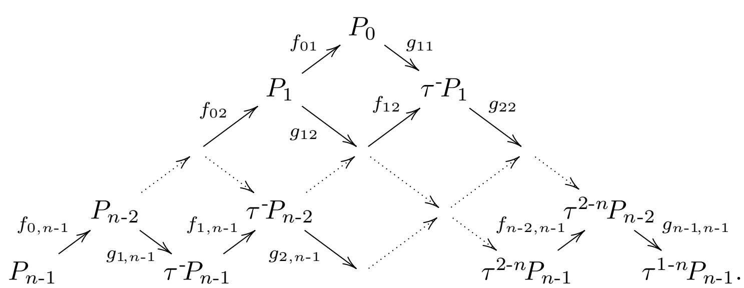
The following statement describes in particular the Auslander-Reiten quiver of a hereditary artin algebra of type $\vec{\mathbb{A}}_n$.

2.6. Theorem. *Let A be a connected artin algebra of finite representation type. If $\ell\ell(A) = n$, then* $\mathrm{rad}(\mathrm{mod}A)$ *is of nilpotency $\ge n$, where the equality occurs if and only if A is hereditary of type $\vec{\mathbb{A}}_n$; and in this case, Γ_A is a wing of rank n.*

Proof. Suppose that $\mathrm{rad}^n A = 0$ and $\mathrm{rad}^{n-1} A \neq 0$ for some integer $n \ge 1$. It is clear that $\mathrm{rad}^{n-1}(\mathrm{mod}A) \neq 0$, and hence, $\mathrm{rad}(\mathrm{mod}A)$ is of nilpotency $\ge n$. If A is hereditary of type $\vec{\mathbb{A}}_n$, then $\mathrm{rad}(\mathrm{mod}A)$ is of nilpotency n; see (2.5). Suppose conversely that $\mathrm{rad}^n(\mathrm{mod}A) = 0$. If $n = 1$, then A is hereditary of type $\vec{\mathbb{A}}_1$. Otherwise, there exist orthogonal primitive idempotents $e_0, e_1, \ldots, e_{n-1}$ in A such that $e_{n-1}(\mathrm{rad}A)e_{n-2} \cdots e_1(\mathrm{rad}A)e_0 \neq 0$. This yields a path of radical maps

$$P_{n-1} \xrightarrow{f_{0,n-1}} P_{n-2} \xrightarrow{f_{0,n-2}} \cdots \xrightarrow{f_{02}} P_1 \xrightarrow{f_{01}} P_0$$

in $\mathrm{ind}\,A$, where $P_j = Ae_j$, such that $f_{01} \cdots f_{0,n-2} f_{0,n-1} \neq 0$. We observe that $f_{0j} \notin \mathrm{rad}^2(P_j, P_{j-1})$, that is, f_{0j} is irreducible, for $j = 1, \ldots, n-1$. Since the P_j are projective, the path is a sectional. Starting with it and applying repeatedly Lemma 1.5, we can construct a fitting diagram



in $\mathrm{ind}A$, where f_{ij} with $0 \le i < j \le n-1$ are irreducible monomorphisms. We shall show that the diagram is mesh-complete.

(1) *The modules $\tau^{-i} P_i$ with $0 \le i \le n-1$ are injective and g_{ij} with $1 \le i \le j < n$ are irreducible epimorphisms.*

Since $f_{01}, \ldots, f_{0,n-1}$ are monomorphisms, by Lemma 2.2, P_0 is injective. Fix $0 < i < n-1$. We obtain two sectional paths of irreducible maps

$$\tau^{-i} P_{n-1} \xrightarrow{f_{i,n-1}} \tau^{-i} P_{n-2} \longrightarrow \cdots \longrightarrow \tau^{-i} P_{i+1} \xrightarrow{f_{i,i+1}} \tau^{-i} P_i$$

and

$$P_{n-1-i} \xrightarrow{g_{1,n-i}} \tau^{-} P_{n-i} \longrightarrow \cdots \longrightarrow \tau^{1-i} P_{n-1} \xrightarrow{g_{i,n-1}} \tau^{-i} P_{n-1}$$

in ind$A$. By Lemma 1.10, $\mathrm{dp}(g_{i,n-1}\cdots g_{1,n-i}) = i$. For each j with $i < j \le n-1$, we have an irreducible map $(f_{ij}, g_{i,j-1}) : \tau^{-i}P_j \oplus \tau^{1-i}P_{j-2} \to \tau^{-i}P_{j-1}$ and a sectional path $P_{j-i-1} \to \tau^{-}P_{j-i-1} \to \cdots \to \tau^{-i}P_{j-1}$ in Γ_A. Since P_{j-i-1} is projective, by Lemma 1.10(3), $d_l(f_{ij}) = \infty$. Thus, $\mathrm{dp}(f_{i,i+1}\cdots f_{i,n-1}g_{i,n-1}\cdots g_{1,n-i}) = n-1$. and by Lemma 2.1, $\tau^{-i}P_i$ is injective. This shows that $P_0, \tau^{-}P_1, \ldots, \tau^{2-n}P_{n-2}$ are injective, and consequently, $g_{11}, \ldots, g_{n-1,n-1}$ are irreducible epimorphisms. Then, by Lemma 2.2(2), $\tau^{1-n}P_{n-1}$ is injective. Finally, we deduce from Lemma 1.5(2) that g_{ij} is an irreducible epimorphism for every $1 \le i \le j < n$.

(2) *The maps* $(g_{ij}, f_{i,j+1}) : \tau^{1-i}P_{j-1} \oplus \tau^{-i}P_{j+1} \to \tau^{-i}P_j$ *with* $0 < i \le j < n-1$ *are sink maps.*

Suppose that this is not true, say $(g_{st}, f_{s,t+1})$ with $0 < s < n-1$ is not a sink map. We may assume that s is minimal. Then, there exists an irreducible map $(g_{st}, f_t, f_{s,t+1}) : \tau^{1-s}P_{t-1} \oplus M_t \oplus \tau^{-s}P_{t+1} \to \tau^{-s}P_t$, where M_t is indecomposable. If $s = 1$, then we obtain a pre-sectional path $P_{n-1} \to \cdots \to P_{t+1} \to P_t \to M_t$ in Γ_A, and by Lemma 1.10(1), we have a map $\theta : P_{n-1} \to M_t$ of depth $n-t$. Since P_j is projective, $d_l(f_t) = d_l(f_{1j}) = \infty$ for $2 \le j \le t-1$; see (1.10). Therefore, $d_l(f_{12}\cdots f_{1t}f_t\theta) = n$, a contradiction. In case $s > 1$, by the minimality of s, we see that M_t is projective. Then, g_{st} is a monomorphism; see (1.5), a contradiction.

(3) *The maps* $g_{i,n-1} : \tau^{1-i}P_{n-2} \to \tau^{-i}P_{n-1}$ *with* $0 < i < n$ *are sink maps.*

Suppose that this does not hold, say $g_{s,n-1}$ with $0 < s < n$ is not a sink map. This yields an irreducible map $(g_{s,n-1}, f_{n-1}) : \tau^{1-s}P_{n-2} \oplus M_{n-1} \to \tau^{-s}P_{n-1}$, where M_{n-1} is indecomposable. We may assume that s is minimal. In case $s = 1$, we obtain pre-sectional path $M_t \to \tau^{-}P_{n-1} \to \cdots \to \tau^{-}P_1$, and then, a ladder of height $n-1$ from P_{n-1} to $\tau^{-}P_1$. By Lemma 1.11, $\mathrm{rad}^n(P_{n-1}, \tau^{-}P_1) \ne 0$, absurd. In case $s > 1$, it follows from the minimality of s that M_{n-1} is projective. Then, $g_{s,n-1}$ is a monomorphism; see (1.5), a contradiction.

By the above statements, the diagram is mesh-complete in ind$A$. Forgetting its irreducible maps, we obtain a mesh-complete translation subquiver Γ of Γ_A, which is a wing of rank n and has all the properties stated in Proposition 1.13. Thus A is hereditary of type $\vec{\mathbb{A}}_n$ with $\Gamma_A = \Gamma$. The proof of the theorem is completed.

We conclude this section with the case where A is a Nakayama algebra. Since Γ_A is planar; see [2, page 197], an irreducible monomorphism or epimorphism in ind$A$ is of infinite left or right degree respectively; see [8, (6.2)].

2.7. Theorem. *Let A be a Nakayama algebra. Then the nilpotency of* $\mathrm{rad}(\mathrm{mod}A)$ *is the maximal number of* $\ell(P_S)+\ell(I_S)-1$, *where S ranges over the simple modules.*

Proof. It amounts to show that $\mathrm{dp}(\theta_S) = \ell(P_S)+\ell(I_S)-2$, for every simple module S in mod$A$; see [9, (2.7)]. Write $n = \ell(I_S)$. If $n = 1$, then ι_S is an isomorphism, and hence, $\mathrm{dp}(\iota_S) = 0$. Otherwise, it is well known; see, for example, [2, Page 197] that there exists a sequence of canonical irreducible monomorphisms

$$S \xrightarrow{q_{n-1}} \mathrm{rad}^{n-2}I_S \xrightarrow{q_{n-2}} \cdots \xrightarrow{q_2} \mathrm{rad}I_S \xrightarrow{q_1} I_S.$$

Thus, $\iota_S = fq_1\cdots q_{n-1}$, where $f : I_S \to I_S$ is an isomorphism. Since $d_l(q_i) = \infty$; see [8, (6.2)], $\mathrm{dp}(\iota_S) = \mathrm{dp}(q_1\cdots q_{n-1}) = n-1$. Thus, $\mathrm{dp}(\iota_S) = \ell(I_S)-1$ in any case. Dually, $\mathrm{dp}(\pi_S) = \ell(P_S)-1$. Now, if $n = 1$, then $\mathrm{dp}(\theta_S) = \mathrm{dp}(\pi_S) = \ell(P_S)+\ell(I_S)-2$. Otherwise, since $d_r(q_i) = \infty$ for $i = 1, \ldots, n-1$, we see that

$$\mathrm{dp}(\theta_S) = \mathrm{dp}(q_1\cdots q_{n-1}\pi_S) = \ell(P_S) - 1 + n - 1 = \ell(P_S) + \ell(I_S) - 2.$$

The proof of the theorem is completed.

As an immediate consequence, we obtain the following interesting statement.

2.8. Corollary. *Let A be a connected Nakayama algebra, and let m be the nilpotency of* rad(mod$A$).

(1) *If A is hereditary, then $m = \ell\ell(A)$; and otherwise, $\ell\ell(A)+1 \le m \le 2\cdot\ell\ell(A)-1$.*
(2) *If the projective modules in* ind$A$ *are of the same length, then $m = 2\cdot\ell\ell(A)-1$.*

Proof. If A is hereditary, then it is not hard to see that A is of type $\vec{\mathbb{A}}_n$; see [2, (VIII.5.4)], and by Theorem 2.6, $m = \ell\ell(A)$. Suppose that A is not hereditary. By Theorem 2.6, $m > \ell\ell(A)$. Moreover, $\ell(P_S) \le \ell\ell(A)$ and $\ell(I_S) \le \ell\ell(A)$, for every simple module S in mod$A$. By Theorem 2.7, $m \le 2\cdot\ell\ell(A)-1$.

Suppose that the projective modules in ind$A$ are of the same length n. Then, $\ell\ell(A) = n$. Given a projective module P in ind$A$, considering the projective cover of the injective envelope of P, we see that P is injective. Thus, by Theorem 2.7, $m = 2n-1$. The proof of the corollary is completed.

Example. Let $A = kQ/(kQ^+)^n$, where k is a field, Q is a quiver consisting of a single oriented cycle and kQ^+ is the ideal in kQ generated by the arrows. Then rad(mod$A$) is nilpotency $2n-1$.

3. String algebras

The objective of this preparatory section is to study the depth of the projective cover and the injective envelope of simple modules over some special classes of algebras. For this purpose, we first generalize the Butler and Ringel's notion of string algebras given by a quiver with relations; see [7] and then introduce a subclass of *wedged string algebras*. When the radical of a wedged string algebra is cubed zero, we shall be able to describe the almost split sequences involving the indecomposable projective or injective modules, which yields in particular a description of the depths of projective covers and injective envelopes of simple modules.

3.1. Definition. An artin algebra A is called a *string algebra* provided that the radical of any projective module, as well as the socle-factor of any injective module, in ind$A$ is either uniserial or a direct sum of two uniserial modules.

Remark. For finite dimensional algebras given by a quiver with relations, our notion of a string algebra coincides with the one defined in [7, Section 3].

As a special case of the theorem stated in [7, Section 1], the following statement gives an explicit description of the almost split sequence with a short proof.

3.2. Proposition. *Let A be an artin algebra with P a projective module in* ind$A$. *If S is a simple direct summand of* rad$P$*, then the canonical short exact sequence*

$$0 \longrightarrow S \xrightarrow{\ q\ } P \xrightarrow{\ p\ } P/S \longrightarrow 0$$

is an almost split sequence if and only if the socle-factor of I_S has a simple socle.

Proof. Let S be a direct summand of rad$P$. The inclusion map $q : S \to P$ is irreducible. Write $J = \mathrm{rad}A$. We may assume that $P = Ae_0$ and $S \cong Ae_1/Je_1$, where e_0, e_1 are primitive idempotents in A. It is well known that $S = Au$, for some

$u \in e_1 J e_0 \backslash e_1 J^2 e_0$. Putting $N = P/S$, we obtain a minimal projective presentation

$$Ae_1 \xrightarrow{R_u} Ae_0 \longrightarrow N \longrightarrow 0,$$

where R_u denotes the right multiplication by u. Applying $\mathrm{Hom}_A(-, A)$ to this sequence yields a minimal projective presentation

$$e_0 A \xrightarrow{L_u} e_1 A \longrightarrow \mathrm{Tr} N \longrightarrow 0$$

in $\mathrm{mod} A^{\mathrm{op}}$, where L_u denotes the left multiplication by u.

Suppose that the socle-factor of $D(e_1 A)$ has a simple socle, that is, $e_1 J$ has a simple top. Since $u \notin e_1 J^2$, we see that $e_1 J / e_1 J^2 = (u + e_1 J^2) A$, and since J is nilpotent, $e_1 J = uA$. Thus, $\mathrm{Tr} N \cong e_1 A / uA = e_1 A / e_1 J \cong S$, and consequently, $\mathrm{DTr} N \cong D(e_1 A / e_1 J) \cong Ae_1 / J e_1 \cong S$. Now, it is not hard to see that the canonical short exact sequence stated in the proposition is an almost split sequence.

Suppose that the canonical short exact sequence is an almost split sequence. In particular, $\mathrm{DTr} N \cong S$, and consequently, S admits a minimal injective co-presentation $0 \longrightarrow S \longrightarrow D(e_1 A) \longrightarrow D(e_0 A)$. In particular, the socle of the socle-factor of $D(e_1 A)$ is isomorphic to the simple socle of $D(e_0 A)$. The proof of the proposition is completed.

Remark. The dual of Proposition 3.2 is left for the reader to formulate.

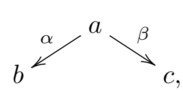
Motivated by the previous statement, we introduce the following notion.

3.3. Definition. Let A be an artin algebra.

(1) A projective module P in $\mathrm{ind} A$ is called *wedged* if $\mathrm{rad} P = S_1 \oplus S_2$, where S_1, S_2 are simple such that the socle-factor of I_{S_i} has a simple socle, for $i = 1, 2$.
(2) An injective module I in $\mathrm{ind} A$ is called *co-wedged* if $I / \mathrm{soc} I = S_1 \oplus S_2$, where S_1, S_2 are simple such that the radical of P_{S_i} has a simple top, for $i = 1, 2$.

Remark. It is easy to see that a projective module P in $\mathrm{ind} A$ is wedged if and only if the injective module DP in $\mathrm{ind} A^{\mathrm{op}}$ is a co-wedged.

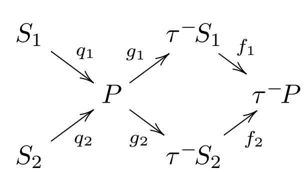
Example. Let A be an algebra over a field given by a quiver Q with relations. A projective module in $\mathrm{ind} A$ is wedged if and only if its support has a wedge shape



where α is the only arrow in Q ending in b and β is the only arrow ending in c.

The following statement plays an important role in our investigation.

3.4. Lemma. *Let A be an artin algebra with P a projective module in* $\mathrm{ind} A$. *If A is a string algebra and P is wedged, then there exists a mesh-complete diagram*



in $\mathrm{ind} A$, *where S_1, S_2 are simple*; *and in this case, I_{S_1}, I_{S_2} are uniserial of length at least two. Conversely, if such a mesh-complete diagram exists in* $\mathrm{ind} A$, *then*

(1) *P is wedged with* $\mathrm{rad}P \cong S_1 \oplus S_2$ *and* $\mathrm{top}P \cong \tau^- P$;
(2) $\mathrm{dp}(\pi_S) = \mathrm{dp}(f_1 g_1) = \mathrm{dp}(g_2 q_1) = \mathrm{dp}(g_1 q_2) = 2$;
(3) *$I_{S_1} \cong \tau^- S_2$ in case $\ell(I_{S_1}) = 2$; and $I_{S_2} \cong \tau^- S_1$ in case $\ell(I_{S_2}) = 2$.*

Proof. Suppose that A is a string algebra and $\mathrm{rad}P = S_1 \oplus S_2$, where S_i is simple such that $\mathrm{soc}(I_{S_i}/S_i)$ is simple, for $i = 1, 2$. Being indecomposable, I_{S_i}/S_i is uniserial, and hence, I_{S_i} is uniserial of length ≥ 2, for $i = 1, 2$. Considering the inclusion map $q_i : S_i \to P$, by Proposition 3.2, we obtain an almost split sequence $0 \longrightarrow S_i \xrightarrow{q_i} P \xrightarrow{g_i} \tau^- S_i \longrightarrow 0$, for $i = 1, 2$. Since $(q_1, q_2) : P_1 \oplus P_2 \to P$ is a sink map, by Lemma 1.3, $(g_1, g_2)^T : P \to \tau^- S_1 \oplus \tau^- S_2$ is irreducible. Being non-uniserial, P is not a direct summand of the radical of any projective module in $\mathrm{ind}A$. This implies that $(g_1, g_2)^T$ is a source map. Since P is not injective; see (1.2), we obtain a mesh diagram in $\mathrm{ind}A$ as stated in the lemma.

Suppose that $\mathrm{ind}A$ contains such a mesh-complete diagram. By Lemma 1.3, (q_1, q_2) is an irreducible map. If it is not a sink map, then we have an irreducible map $(q_1, q_2, f) : S_1 \oplus S_2 \oplus M \to P$, where M is indecomposable. Since P is projective, M is not injective. This yields an irreducible map $g : P \to \tau^- S_1 \oplus \tau^- S_2 \oplus \tau^- M$, absurd. Thus, (q_1, q_2) is a sink monomorphism. In particular, $\mathrm{rad}P \cong S_1 \oplus S_2$. Since q_i is a source map, $\mathrm{soc}(I_{S_i}/S_i)$ is simple; see (3.2), for $i = 1, 2$. That is, P is wedged. Write $S = \tau^- P$. Since $\ell(P) = 3$, we see that S is simple. On the other hand, $\mathrm{dp}(f_1 g_1) = 2$; see (1.11). Thus, $\pi_S = f_1 g_1 h$, where $h : P_S \to P$ is an isomorphism. Therefore, $\mathrm{top}P \cong \tau^- P$ and $\mathrm{dp}(\pi_S) = 2$.

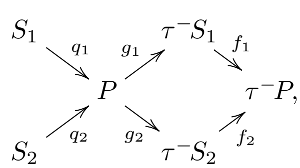
If $g_2 q_1 \in \mathrm{rad}^3(S_1, \tau^- S_2)$, then $q_1 + u q_2 \in \mathrm{rad}^2(S_1, \tau^- S_2)$ for some $u : S_1 \to S_2$; see [15, Lemma 1.2]. Since q_1 is irreducible, u is an isomorphism. Hence, (q_1, q_2) is not irreducible; see [4, Proposition 1], absurd. Thus, $\mathrm{dp}(g_2 q_1) = 2$. And similarly, $\mathrm{dp}(g_1 q_2) = 2$. Since $\ell(\tau^- S_1) = 2$, we see that $S_2 \cong \mathrm{soc}(\tau^- S_1)$. Thus, $I_{S_2} \cong \tau^- S_1$ in case $\ell(I_{S_2}) = 2$. Similarly, $I_{S_2} \cong \tau^- S_1$ in case $\ell(I_{S_2}) = 2$. The proof of the lemma is completed.

Remark. The dual statement of Lemma 3.4 is left for the reader to formulate.

3.5. Definition. A string artin algebra A is called a *wedged string algebra* if every projective module in $\mathrm{ind}A$ is uniserial or wedged, and every injective module in $\mathrm{ind}A$ is uniserial or co-wedged.

By definition, Nakayama algebras are wedged string algebras. On the other hand, as shown below, hereditary non-Nakayama wedged string algebras are rare.

3.6. Proposition. *Let A be a connected artin algebra. Then the following statements are equivalent:*
(1) *A is a hereditary algebra of type $\hat{\mathbb{A}}_3$ or $\mathbb{B}_2$;*
(2) *A is a hereditary non-Nakayama wedged string algebra;*
(3) *A is a hereditary algebra with a wedged projective module in* $\mathrm{ind}A$ *or* $\mathrm{ind}A^{\mathrm{op}}$;
(4) *there exists a mesh-complete diagram*



in $\operatorname{ind} A$ *or* $\operatorname{ind} A^{\rm op}$*, where* S_1, S_2 *are simple projective, and* $\tau^- P$ *is simple injective. In this case,* $\operatorname{rad}^3(\operatorname{mod} A) = 0$.

Proof. It is evident that Statement (2) implies Statement (3). Suppose that Statement (1) holds. Then, the projective modules in Γ_A or those in $\Gamma_{A^{\rm op}}$ generate a section Δ, which is either a trivially valued quiver $S_1 \longrightarrow P \longleftarrow S_2$, or a single valued arrow $S_1 \longrightarrow P$ with valuation $(1, 2)$. Since A is hereditary, S_1, S_2 are simple such that $\operatorname{rad} P = S_1 \oplus S_2$, where $S_2 = S_1$ if the second case occurs. Moreover, the inclusion map $q_i : S_i \to P$ is a source map, and hence, $\operatorname{soc}(I_{S_i}/S_i)$ is simple; see (3.2), for $i = 1, 2$. That is, P is wedged. Thus, Statement (3) holds.

Suppose that Statement (3) holds, say $\operatorname{ind} A$ contains a wedged projective module P with $\operatorname{rad} P = S_1 \oplus S_2$, where S_1, S_2 are simple. By Lemma 3.4, there exists a mesh-complete diagram in $\operatorname{ind} A$ as stated in Statement (4), where $\tau^- P \cong \operatorname{top} P$ and f_1, f_2 are epimorphisms; see (1.5). Since A is hereditary, S_1, S_2 are projective. If f_1 is not a source map, then there exists an irreducible map $(g, f_1) : \tau^- S_1 \to N \oplus \tau^- P$, where N is indecomposable. Since g_1 is a sink map, N is projective, and so is $\tau^- S_1$, absurd. Similarly, f_2 is a source map. As a consequence, $\tau^- S_1$ and $\tau^- S_2$ are injective, and since A is hereditary, so is $\tau^- P$. Thus, Statement (4) holds.

Suppose that Statement (4) holds, say $\operatorname{ind} A$ contains a mesh-complete diagram as stated in Statement (4). Since S_1 is simple projective, P is projective, and since $\tau^- P$ is simple injective, $\tau^- S_1$ and $\tau^- S_2$ are injective. Forgetting the irreducible maps and identifying the isomorphic modules, we obtain a translation subquiver Γ of Γ_A with all the properties stated in Proposition 1.13, in which the projective modules generate a section of type $\hat{\mathbb{A}}_3$ or $\mathbb{B}_2$ in case $S_1 \ncong S_2$ or $S_1 \cong S_2$, respectively. Thus, A is hereditary of type $\hat{\mathbb{A}}_3$ or $\mathbb{B}_2$ with $\Gamma_A = \Gamma$. In particular, S_1, S_2, P are essentially the only projective modules in $\operatorname{ind} A$, while $\tau^- S_1, \tau^- S_2, \tau^- P$ are essentially the only injective modules in $\operatorname{ind} A$. By Lemma 3.4, P is wedged. Being of length two, $\tau^- S_1$ and $\tau^- S_2$ are uniserial. That is, A is a non-Nakayama wedged string algebra. Therefore, Statements (1) and (2) hold. Finally, since every path of three irreducible maps in the diagram has a zero composite, $\operatorname{rad}^3(\operatorname{mod} A) = 0$. The proof of the proposition is completed.

We shall concentrate on a smaller class of wedged string algebras.

3.7. Lemma. *Let* A *be a wedged string algebra with radical cubed zero.*
(1) *Every projective or injective module in* $\operatorname{ind} A$ *is of length* ≤ 3.
(2) *Every uniserial module of length* 3 *in* $\operatorname{ind} A$ *is projective injective.*

Proof. Since $\operatorname{rad}^3 A = 0$, every uniserial module in $\operatorname{mod} A$ is of length ≤ 3. Since A is a wedged string algebra, every projective or injective module in $\operatorname{ind} A$ is of length ≤ 3. Let L be a uniserial module of length 3. Consider its projective cover $f : P \to L$ and injective envelope $g : L \to I$. Since L has a simple top and a simple socle, P and I are indecomposable. Since $\ell(P) \leq 3$ and $\ell(I) \leq 3$, both f and g are isomorphisms. The proof of the lemma is completed.

Example. The path algebra over a field of the quiver $\circ \rightrightarrows \circ$ is a string algebra with radical squared zero. However, it is not a wedged string algebra.

We shall describe almost split sequences involving indecomposable projective or injective modules over a wedged string algebra with radical cubed zero. By Lemma 3.4 and its dual, it suffices to consider uniserial projective or injective module.

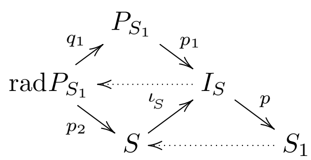
3.8. LEMMA. *Let A be a wedged string algebra with radical cubed zero. Let S be a simple module in* $\mathrm{mod}A$ *such that I_S is uniserial of length 2 with $I_S/S = S_1$.*

(1) *If P_{S_1} is uniserial of length 2, then there exists an almost split sequence*

$$0 \longrightarrow S \xrightarrow{\iota_S} I_S \xrightarrow{p} S_1 \longrightarrow 0$$

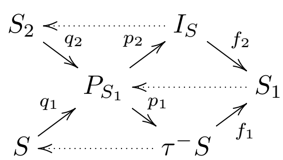
in $\mathrm{mod}A$*, where $I_S \cong P_{S_1}$. In particular,* $\mathrm{dp}(\iota_S) = 1$.

(2) *If P_{S_1} is uniserial of length 3, then there exists a mesh-complete diagram*



in $\mathrm{ind}A$*. In particular,* $\mathrm{dp}(\iota_S) = 1$.

(3) *If P_{S_1} is wedged, then there exists a mesh-complete diagram*



in $\mathrm{ind}A$ *such that $\iota_S = p_2 q_1$. In particular,* $\mathrm{dp}(\iota_S) = 2$.

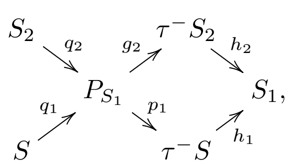
Proof. If P_{S_1} is uniserial, then $\mathrm{rad}P_{S_1}$ has a simple top. By the dual of Lemma 3.2, we obtain an almost split sequence as stated in Statement (1). If $\ell(P_{S_1}) = 2 = \ell(I_S)$, then it is clear that $P_{S_1} \cong I_S$. This establishes Statement (1).

Suppose that P_{S_1} is uniserial of length 3. Then, P_{S_1} is projective-injective; see (3.7), and we obtain an almost split sequence as stated in Statement (1). Since $S_1 = \mathrm{soc}(I_S/S)$, by Lemma 1.1, $S = \mathrm{top}(\mathrm{rad}P_{S_1}) = \mathrm{rad}P_{S_1}/\mathrm{soc}P_{S_1}$. By Lemma 1.2, we obtain an almost split sequence

$$0 \longrightarrow \mathrm{rad}P_{S_1} \xrightarrow{(q_1,p_2)^T} P_{S_1} \oplus S \xrightarrow{(p_1,q_2)} P_{S_1}/\mathrm{soc}P_{S_1} \longrightarrow 0$$

with $S = \mathrm{soc}(P_{S_1}/\mathrm{soc}P_{S_1})$. Since $\ell(P_{S_1}/\mathrm{soc}P_{S_1}) = 2 = \ell(I_S)$, we have an isomorphism $u : P_{S_1}/\mathrm{soc}P_{S_1} \to I_S$ such that $\iota_S = uq_2$. Replacing q_2 by ι_S, we obtain a mesh-complete diagram as stated in Statement (2). This establishes Statement (2).

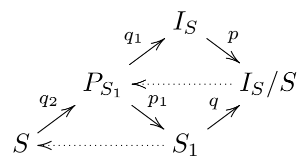
Suppose that P_{S_1} is wedged. Since $S_1 = \mathrm{soc}(I_S/S)$, we see that $\mathrm{rad}P_{S_1} = S \oplus S_2$, where S_2 is simple; see (1.1). By Lemma 3.4, we obtain a mesh-complete diagram



in $\mathrm{ind}A$ such that $\mathrm{dp}(g_2 q_1) = 2$. Since $\ell(I_S) = 2$, we see that $\iota_S = v g_2 q_1$, for some isomorphism $v : \tau^- S_2 \to I_S$; see (3.4). Putting $p_2 = v g_2$, we obtain a mesh-complete diagram as stated in Statement (3). The proof of the lemma is completed.

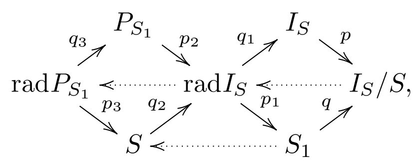
3.9. LEMMA. *Let A be a wedged string algebra with radical cubed zero. Let S be a simple module in* $\mathrm{mod}A$ *such that I_S is uniserial of length 3 with* $\mathrm{soc}(I_S/S) = S_1$.

(1) *If P_{S_1} is uniserial of length 2, then there exists a mesh-complete diagram*



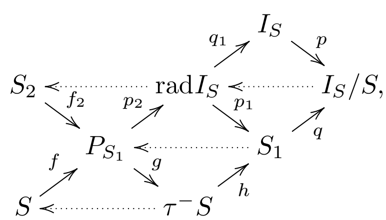
in ind$A$ *such that* $\iota_S = q_1 q_2$. *In particular,* $\mathrm{dp}(\iota_S) = 2$.

(2) *If P_{S_1} is uniserial of length 3, then there exists a mesh-complete diagram*



in ind$A$ *such that* $\iota_S = q_1 q_2$. *In particular,* $\mathrm{dp}(\iota_S) = 2$.

(3) *If P_{S_1} is wedged, then there exists a mesh-complete diagram*



in ind$A$ *such that* $\iota_S = q_1 p_2 f$. *In particular,* $\mathrm{dp}(\iota_S) = 3$.

Proof. Since $\ell(I_S) = 3$, we see that $S_1 = \mathrm{top}(\mathrm{rad}I_S)$ with a canonical projection $p_1 : \mathrm{rad}I_S \to S_1$. Therefore, we have a projective cover $p_2 : P_{S_1} \to \mathrm{rad}I_S$. This yields two short exact sequences

$$(\star) \qquad 0 \longrightarrow S \xrightarrow{q_2} \mathrm{rad}I_S \xrightarrow{p_1} S_1 \longrightarrow 0$$

and

$$(\dagger) \qquad 0 \longrightarrow S_2 \xrightarrow{f_2} P_{S_1} \xrightarrow{p_2} \mathrm{rad}I_S \longrightarrow 0,$$

where S_2 is a (possibly zero) submodule of $\mathrm{rad}P_{S_1}$. Since I_S is projective-injective; see (3.7), by Lemma 1.2, we obtain an almost split sequence

$$(*) \qquad 0 \longrightarrow \mathrm{rad}I_S \xrightarrow{(q_1,p_1)^T} I_S \oplus S_1 \xrightarrow{(p,q)} I_S/S \longrightarrow 0.$$

Suppose first that P_{S_1} is uniserial of length 2. Then p_2 is an isomorphism. Hence, we may assume that $P_{S_1} = \mathrm{rad}I_S$. Then $S = \mathrm{rad}P_{S_1}$. In particular, $q_2 : S \to P_{S_1}$ is irreducible. Since p_1 is irreducible, $(\star)$ is an almost split sequence; see [2, (V.5.9)]. This yields a mesh-complete diagram in ind$A$ as stated in Statement (1). Since $S \to P_{S_1} \to I_S$ is a sectional path, $\mathrm{dp}(q_1q_2) = 2$; see (1.10). Since $q_1q_2 \neq 0$, we may assume that $\iota_S = q_1q_2$. This establishes Statement (1).

Suppose that P_{S_1} is uniserial of length 3. Then, P_{S_1} is projective-injective; see (3.7). Since $\ell(\mathrm{rad}I_S) = 2$, we see that S_2 is simple, and hence, $S_2 = \mathrm{soc}P_{S_1}$. Thus, we may assume that $\mathrm{rad}I_S = P_{S_1}/\mathrm{soc}P_{S_1}$. Moreover, $S = \mathrm{top}(\mathrm{rad}P_{S_1})$; see (1.1). By Lemma 1.2, we obtain an almost split sequence

$$0 \longrightarrow \mathrm{rad}P_{S_1} \xrightarrow{(q_3,p_3)^T} P_{S_1} \oplus S \xrightarrow{(p_2,q_2)} \mathrm{rad}I_S \longrightarrow 0.$$

On the other hand, since q_2 and p_1 are irreducible, $(\star)$ is an almost split sequence. This yields a mesh-complete diagram as stated in Statement (2). A seen above, $\mathrm{dp}(q_1q_2) = 2$ and we may assume that $\iota_S = q_1q_2$. This establishes Statement (2).

Suppose now that P_{S_1} is wedged. Then S_2 is simple. Thus, $\mathrm{rad}P_{S_1} = S_2 \oplus S_0$, where S_0 is simple. Since $p_2(\mathrm{rad}P_{S_1}) \neq 0$, we see that $p_2(S_0) \neq 0$, and hence, $q_1(p_2(S_0)) \neq 0$. Therefore, $S_0 \cong \mathrm{soc}I_S = S$. We may assume that $S_0 = S$ with an inclusion map $f : S \to P_{S_1}$. Since P_{S_1} is wedged, it follows from Lemma 3.2 that (†) is an almost split sequence. In particular, $\tau^- S_2 = \mathrm{rad}I_S$. Combining the mesh-complete diagram stated in Lemma 3.4 and the almost split sequence $(*)$, we obtain a mesh-complete diagram as stated in Statement (3) with $\mathrm{dp}(p_2f) = 2$. Since I_S is projective, $d_l(q_1) = \infty$, and hence, $\mathrm{dp}(q_1p_2f) = 3$. Since $q_1p_2f \neq 0$, we may assume that $\iota_S = q_1p_2f$. The proof of the lemma is completed.

Remark. The dual statements of Lemmas 3.8 and 3.9 hold true, which are left for the reader to formulate.

3.10. Corollary. *Let A be a wedged string algebra with radical cubed zero. If S is a simple module in* $\mathrm{mod}A$*, then* $\mathrm{dp}(\pi_S) \leq 3$ *and* $\mathrm{dp}(\iota_S) \leq 3$.

Proof. Let S be a simple module in $\mathrm{mod}A$. By Lemma 3.7, $\ell(I_S) \leq 3$. If $\ell(I_S) = 1$, then $\mathrm{dp}(\iota_S) = 0$. If I_S is uniserial of length 2 or 3, then it follows from Lemmas 3.8 and 3.9 that $\mathrm{dp}(\iota_S) \leq 3$. If I_S is co-wedged, by the dual of Lemma 3.4, $\mathrm{dp}(\iota_S) = 2$. Dually, $\mathrm{dp}(\pi_S) \leq 3$. The proof of the corollary is completed.

4. Module categories of radical nilpotency at most four

The objective of this section is to divide the class of artin algebras such that the radical of their module category has a vanishing fourth power into two subclasses: hereditary algebras of type $\mathbb{A}_4$ and tri-string algebras. We begin with the following fact that these algebras need to be string algebras.

4.1. Proposition. *Let A be an artin algebra. If* $\mathrm{rad}^4(\mathrm{mod}A) = 0$*, then A is a string algebra and Γ_A is planar.*

Proof. Assume that $\mathrm{rad}^4(\mathrm{mod}A) = 0$. Then, A is representation-finite; see [2, (V.7.6)]. We shall first show that Γ_A is planar. Suppose on the contrary that there exists in $\mathrm{mod}A$ an almost split sequence

$$0 \longrightarrow X \xrightarrow{(f_1,\ldots,f_r)^T} Y_1 \oplus \cdots \oplus Y_r \xrightarrow{(g_1,\ldots,g_r)} Z \longrightarrow 0,$$

where $r \geq 3$ and $Y_1, \ldots, Y_r$ are indecomposable. Applying Lemmas 1.7 and 1.8, we deduce easily that $d_l(g_i) \geq 3$ and $d_r(f_i) \geq 3$, for $i = 1, \ldots, r$.

We claim that g_1, g_2, g_3 are all monomorphisms. Otherwise, we may assume that g_1 is an epimorphism. By Lemma 2.3, $d_l(g_1) \leq 3$. Then, by Lemma 1.8, $(f_2, f_3)^T : X \to Y_2 \oplus Y_3$ is of left degree ≤ 2, and by Lemma 1.9, we obtain an irreducible map $v = (v_2, v_3) : \tau Y_2 \oplus \tau Y_3 \to X$ of left degree one, which is a sink epimorphism by Lemma 1.7. This implies that Y_1 is projective. In particular, f_1 is a monomorphism. In a dual manner, we obtain a source monomorphism $w = (w_2, w_3)^T : Z \to \tau^- Y_2 \oplus \tau^- Y_3$. In particular, Z is not injective. Observing that $\tau Y_2 \to X \to Y_3$ is a pre-sectional path in Γ_A, by Proposition 1.10, we may assume that f_3v_2 is of depth 2. Since $d_l(g_3) \geq 3$, the composite $g_3f_3v_2$ is of depth 3. By Lemma 2.1, Z is injective, a contradiction. This establishes our claim. Dually, f_1, f_2, f_3 are all epimorphisms.

If $r \geq 4$, then one of the Y_i, say Y_1, is projective-injective; see [5] or [16, Theorem 7]. In particular, g_1 is an epimorphism, a contradiction. Thus, $r = 3$.

Since the g_i are all monomorphisms, Y_1, Y_2, Y_3 are not injective. Hence, we have an irreducible map $h = (h_1, h_2, h_3)^T : Z \to \tau^- Y_1 \oplus \tau^- Y_2 \oplus \tau^- Y_3$. Consider a source map $(h, h_4)^T : Z \to \tau^- Y_1 \oplus \tau^- Y_2 \oplus \tau^- Y_3 \oplus M$. If M is non-zero, since (g_1, g_2, g_3) is a sink map, M is projective, and consequently, Z is not injective. This yields an almost split sequence starting with Z, whose middle term is a direct sum of at least four indecomposable modules, a contradiction. Thus, h is a source map. Since the f_i are all epimorphisms, we obtain in a dual fashion a sink map $p = (p_1, p_2, p_3) : \tau Y_1 \oplus \tau Y_2 \oplus \tau Y_3 \to X$. For each $1 \leq i \leq 3$, choose φ_i to be one of the two composites $g_j f_j$ with $j \neq i$. As argued above, we see that $\varphi_i p_i$ is of depth 3, and by Lemma 2.1, Z is injective and τY_i is projective, for $i = 1, 2, 3$. And dually, X is projective and $\tau^- Y_i$ is injective, for $i = 1, 2, 3$.

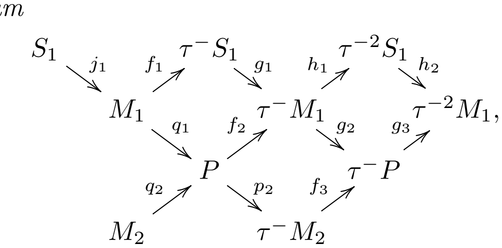
If τY_i is not simple for some $1 \leq i \leq 3$, then we obtain a radical monomorphism $q_i : S_i \to \tau Y_i$, where S_i is simple. Since p_i and $(f_1, f_2, f_3)^T$ are also monomorphisms, $f_j p_i q_i \neq 0$ for some $1 \leq j \leq 3$. Since g_j is a monomorphism, $g_j f_j p_i q_i \neq 0$, a contradiction. Thus, $\tau Y_1, \tau Y_2$ and τY_3 are all simple. Dually, $\tau^- Y_1, \tau^- Y_2$ and $\tau^- Y_3$ are all simple. Since h is a source epimorphism and p is a source monomorphism, we conclude that $\ell(X) = \ell(Z) = 4$.

Furthermore, given $1 \leq i \leq 3$, we see that $h\varphi_i p_i = 0$. Hence, $\varphi_i p_i$ factors through the simple socle S of Z. Since τY_i is simple, $\tau Y_i \cong S$. That is, $Y_i \cong \tau^- S$, for $i = 1, 2, 3$. In view of the above almost split sequence, we get $8 = 3 \cdot \ell(\tau^- S)$, absurd. This shows that Γ_A is indeed planar. Since A is representation-finite, the radical of any indecomposable projective module, as well as the socle-factor of any indecomposable injective module, is uniserial or a direct sum of two uniserial modules; see [1, (4.6)]. That is, A is a string algebra. The proof of the proposition is completed.

Remark. If A is a string algebra given by a quiver with relations, then Γ_A is planar; see [7]. We do not know, however, if this is still true for a general string artin algebra.

We shall study hereditary algebras of type $\mathbb{A}_4$ according to their Loewy length.

4.2. Proposition. *Let A be a connected artin algebra. Then A is hereditary of type $\mathbb{A}_4$ with Loewy length three if and only if* $\mathrm{rad}^4(\mathrm{mod}A) = 0$ *and there exists a projective module P in* $\mathrm{ind}A$ *or* $\mathrm{ind}A^{\mathrm{op}}$ *such that* $\mathrm{rad}P = M_1 \oplus M_2$, *where M_1, M_2 are uniserial and M_1 is not simple. In this case, there exists in* $\mathrm{ind}A$ *or* $\mathrm{ind}A^{\mathrm{op}}$ *a mesh-complete diagram*



where S_1, M_1, P, M_2 are projective, while $\tau^{-2}S$, $\tau^{-2}M_1$, $\tau^- P$, $\tau^- M_2$ are injective.

Proof. Let A be hereditary of type $\mathbb{A}_4$ such that $\mathrm{rad}^3 A = 0$ and $\mathrm{rad}^2 A \neq 0$. By Proposition 2.5, $\mathrm{rad}^4(\mathrm{mod} A) = 0$. Moreover, the projective modules in $\varGamma_A$ or those in $\varGamma_{A^{\rm op}}$ generate a section $\varDelta\colon P_0 \longrightarrow P_1 \longrightarrow P \longleftarrow P_2$. Thus, $\mathrm{rad} P = P_1 \oplus P_2$. Since A is hereditary, P_0 and P_2 are simple. Thus, P_1 is uniserial of length two.

Suppose now that $\mathrm{rad}^4(\mathrm{mod} A) = 0$ and P is a projective module in $\mathrm{ind} A$ such that $\mathrm{rad} P = M_1 \oplus M_2$, where M_1, M_2 are uniserial and M_1 is not simple. Then, A is neither a hereditary algebra of type $\vec{\mathbb{A}}_4$ nor a diamond algebra. By Theorem 2.6, $\mathrm{rad}^3 A = 0$, and hence, $l(M_2) = 2$. Note that P is not injective; see (1.2) and the inclusion maps $q_i : M_i \to P$ are irreducible, $i = 1, 2$. Put $S_1 = \mathrm{rad} M_1$, which is simple. The inclusion map $j_1 : S_1 \to M_1$ is radical. Since $\mathrm{rad}^4(S_1, P) = 0$ and $q_1 j_1 \neq 0$, $\mathrm{dp}(j_1) \leq 2$. If $\mathrm{dp}(j_1) = 2$ then, by Lemma 2.1, P is injective, a contradiction. Starting with the irreducible monomorphisms j_1, q_1, q_2 and applying Lemma 1.5 repeatedly, we obtain a fitting diagram in $\mathrm{ind} A$, consisting of all modules except $\tau^{-2} M_1$ and all irreducible maps except h_2, g_3 of the diagram stated in the proposition. We shall complete the construction of the desired diagram.

(1) *The modules S_1, M_2 are simple projective, M_1 is projective, while $\tau^{-2} S_1$, $\tau^- M_2$ and $\tau^- P$ are injective.*

Note that $M_2 \to P \to \tau^- M_1 \to \tau^{-2} S_1$ and $S_1 \to M_1 \to P \to \tau^- M_2$ are pre-sectional paths in $\varGamma_A$. By Lemma 2.4, S_1 and M_2 are projective, while $\tau^{-2} S_1$ and $\tau^- M_2$ are injective. If M_2 is not simple, then we may find a pre-sectional path $U \to M_2 \to P \to \tau^- M_1 \to \tau^{-2} S_1$ in $\varGamma_A$, a contradiction; see (2.4). Moreover, since S_1 is simple projective, M_1 is projective. Finally, considering the ladder of height 2 from M_1 to $\tau^- P$, we obtain a map $f : M_1 \to \tau^- P$ with $\mathrm{dp}(f) = 3$; see (1.11). By Lemma 2.1, $\tau^- P$ is injective.

(2) *The maps j_1 and g_1 are source monomorphisms.*

Suppose that $j_1 : S_1 \to M_1$ is not a source map. This yields an irreducible map $(g, j_1) : S_1 \to N \oplus M_1$, where N is indecomposable. Thus, we can construct a ladder of height 3 from S_1 to $\tau^- P$, and by Lemma 1.11, we obtain a map $h : S_1 \to \tau^- P$ of depth 4, a contradiction. Next, since q_1 is a monomorphism, so is g_1. If g_1 is not a source map, then we get an irreducible map $(u, h_1) : L \oplus \tau^- M_1 \to \tau^{-2} S_1$, where L is indecomposable. Observing that $u : L \to \tau^{-2} S_1$ is an irreducible monomorphism; see (1.5), we obtain pre-sectional path $M_2 \to P \to \tau^- M_1 \to \tau^{-2} S_1 \to \tau^- L$ in $\varGamma_A$, a contradiction; see (2.4).

(3) *There exists an injective module $\tau^{-2} M_1$ together with a sink epimorphism $(h_2, g_3) : \tau^{-2} S_1 \oplus \tau^- P \to \tau^{-2} M_1$.*

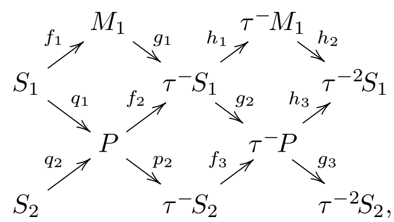
Since $\ell(M_1) = 2$, by Statement (2), $\tau^- S_1$ is simple and isomorphic to a submodule of $\tau^- M_1$. On the other hand, since $M_2 \to P \to \tau^- M_1$ is a pre-sectional path in $\varGamma_A$, we get a non-zero map $w : M_2 \to \tau^- M_1$; see (1.10). Thus, M_2 is isomorphic to a simple submodule of $\tau^- M_1$. Since $M_2 \not\cong \tau^- S_1$, we see that $\tau^- M_1$ is not injective. Thus, we may complete the construction of the fitting diagram stated in the proposition. By Proposition 4.1, $(h_2, g_3) : \tau^{-2} S_1 \oplus \tau^- P \to \tau^{-2} M_1$ is a sink map. Considering the ladder of height 2 from P to $\tau^{-2} M_1$, we obtain a map $v : P \to \tau^{-2} M_1$ of depth 3; see (1.11). By Lemma 2.1, $\tau^{-2} M_1$ is injective.

(4) *The maps $(g_1, f_2) : \tau^- S_1 \oplus P \to \tau^- M_1$ and $(g_2, f_3) : \tau^- M_1 \oplus \tau^- M_2 \to \tau^- P$ are sink epimorphisms, while q_2 is a source monomorphism.*

By Proposition 4.1, we have sink epimorphisms $(g_1, f_2) : \tau^- S_1 \oplus P \to \tau^- M_1$ and $(g_2, f_3) : \tau^- M_1 \oplus \tau^- M_2 \to \tau^- P$. If q_2 is not a source map, then we can find an irreducible map $(q_2, w)^T : M_2 \to P \oplus W$, where W is indecomposable. This yields a ladder of height 3 from M_2 to $\tau^{-2}M_1$, and by Lemma 1.11, there exists a map $\theta : M_2 \to \tau^{-1}M_1$ of depth 4, absurd.

By the above statements, the diagram constructed above is mesh-complete. Forgetting the irreducible maps yields a translation subquiver of Γ_A with all the properties stated in Proposition 1.13. Thus, A is hereditary of type $\mathbb{A}_4$. Since $\mathrm{rad}^2 P \neq 0$, we see that $\ell\ell(A) = 3$. The proof of the proposition is completed.

4.3. Proposition. *Let A be a connected artin algebra. Then A is hereditary of type $\mathbb{A}_4$ with Loewy length two if and only if* $\mathrm{rad}^4(\mathrm{mod}A) = 0$ *and there exists a projective module P in* $\mathrm{ind}A$ *or* $\mathrm{ind}A^{\mathrm{op}}$ *with* $\mathrm{rad}P \cong S_1 \oplus S_2$*, where S_1, S_2 are simple such that* $\mathrm{soc}(I_{S_1}/S_1)$ *not simple. In this case, there exists in* $\mathrm{ind}A$ *or* $\mathrm{ind}A^{\mathrm{op}}$ *a mesh-complete diagram*



where M_1, S_1, P, S_2 are projective, and τ^-M_1, $\tau^{-2}S_1$, τ^-P, $\tau^{-2}S_2$ are injective.

Proof. Let A be a hereditary algebra of type $\mathbb{A}_4$ with $\mathrm{rad}^2 A = 0$. By Proposition 2.5, $\mathrm{rad}^4(\mathrm{mod}A) = 0$. Moreover, the indecomposable projective modules in Γ_A or those in $\Gamma_{A^{\mathrm{op}}}$ generate a section $\Delta :\ P_0 \longrightarrow S_1 \longleftarrow P \longrightarrow S_2,$ where S_1, S_2 are simple such that $S_1 \oplus S_2 \cong \mathrm{rad}P$. Since the inclusion map $q_1 : S_1 \to P$ is not a source map, $\mathrm{soc}(I_{S_1}/S_1)$ is not simple; see (3.2).

Suppose now that $\mathrm{rad}^4(\mathrm{mod}A) = 0$ and there exists a projective P in $\mathrm{ind}A$ with $\mathrm{rad}P = S_1 \oplus S_2$, where S_1, S_2 are simple such that $\mathrm{soc}(I_{S_1}/S_1)$ is not simple. Note that the inclusion mapi $q_i : S_i \to P$ is an irreducible monomorphism, for $i = 1, 2$, and q_1 is not a source map; see (3.2). Since A is not a diamond algebra, by Proposition 4.1, there exists a source map $(f_1, q_1)^T : S_1 \to M_1 \oplus P$, where M_1 is indecomposable and $f_1 : S_1 \to M_1$ is an irreducible monomorphism. Starting with f_1, q_1, q_2 and applying Lemma 1.5 repeatedly, we obtain a fitting diagram in $\mathrm{ind}A$, consisting of all modules except $\tau^{-2}S_1$ and all irreducible maps except h_2, h_3 of the diagram stated in the proposition. We shall complete the construction of the desired mesh-complete diagram.

(1) *The modules S_1, S_2, M_1 are projective, and τ^-M_1, τ^-P, $\tau^{-2}S_2$ are injective.*

Note that $M_1 \to \tau^-S_1 \to \tau^-P \to \tau^{-2}S_2$ and $S_2 \to P \to \tau^-S_1 \to \tau^-M_1$ are pre-sectional paths in Γ_A. By Lemma 2.4, M_1 and S_2 are projective, while $\tau^{-2}S_2$ and $\tau^{-1}M_1$ are injective. Considering the ladder of height two from S_1 to τ^-P, we obtain a map $\theta : S_1 \to \tau^-P$ of depth 3; see (1.11). Hence, τ^-P is injective; see (2.1). If S_1 is not projective, then we can find a pre-sectional path $X \to S_1 \to P \to \tau^-S_2$ in Γ_A. By Lemma 2.4, τ^-S_2 is injective, a contradiction. Thus, S_1 is projective.

(2) *The maps $(g_1, f_2) : M_1 \oplus P \to \tau^-S_1$ and $(g_2, f_3) : \tau^-S_1 \oplus \tau^-S_2 \to \tau^-P$ are sink epimorphisms, while g_1, q_2, f_3 are source monomorphisms.*

By Proposition 4.1, (g_1, f_2) and (g_2, f_3) are sink epimorphisms. Since q_1 is a monomorphism, so is g_1. If g_1 is not a source map, then there exists an irreducible map $(g, h_1) : N \oplus \tau^-S_1 \to \tau^-M_1$, where N is indecomposable. Observing that $g : N \to \tau^-M_1$ is a monomorphism; see (1.5), we obtain a pre-sectional path

$S_2 \to P \to \tau^- S_1 \to \tau^- M_1 \to \tau^- N$ in Γ_A, a contradiction; see (2.4). Finally, since f_1 is a monomorphism, so is f_3; see (1.5). Suppose that q_2 or f_3 is not a source monomorphism. Since S_2 is simple, we see easily that there exists an irreducible map $(f_3, h)^T : \tau^- S_2 \to \tau^- P \oplus L$, where L is indecomposable. This yields a ladder of height 3 from S_1 to $\tau^{-2} S_2$, a contradiction; see (1.11).

(3) *There exists an injective module* $\tau^{-2} S_1$ *together with a sink epimorphism* $(h_2, h_3) : \tau^- M_1 \oplus \tau^- P \to \tau^{-2} S_1$.

By Lemma 1.11, $g_1 f_1 \neq 0$. Thus, S_1 is isomorphic to a simple submodule of $\tau^- S_1$. Since f_1 is a monomorphism, so is f_2. Thus, $f_2 q_2 \neq 0$, and hence, S_2 isomorphic to a simple submodule of $\tau^- S_1$. Clearly, $S_1 \not\cong S_2$. Having a non-simple socle, $\tau^- S_1$ is not injective. Thus, we may complete the construction of the fitting diagram stated in the proposition. By Proposition 4.1, (h_2, h_3) is a sink epimorphism. Considering the ladder of height 2 from P to $\tau^{-2} S_1$, we obtain a map $v : P \to \tau^{-2} M_1$ of depth 3; see (1.11). By Lemma 2.1, $\tau^{-2} S_1$ is injective.

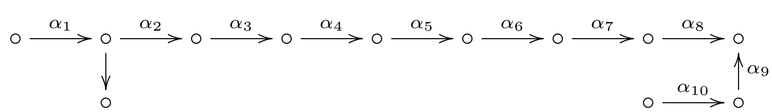
In view of the above statements, the diagram constructed above is mesh-complete. By Proposition 1.13, A is hereditary of type $\mathbb{A}_4$ with S_1, S_2, P, M_1 being essentially the only projective modules in ind$A$. Since S_1, S_2 are simple, we see that $\ell\ell(A) = 2$. The proof of the proposition is completed.

For our purpose, we need to put more conditions on wedged string algebras.

4.4. Definition. A wedged string algebra A is called a *tri-string algebra* provided that the following conditions are satisfied.

(1) The cube of the radical of A is zero.
(2) If S is a simple module in mod$A$, then $\ell(P_S) + \ell(I_S) \leq 5$.
(3) If S is a simple direct summand of the radical of a wedged projective module or the socle-factor of a co-wedged injective module in ind$A$, then $\ell(P_S) + \ell(I_S) \leq 4$.
(4) A wedged projective module and a co-wedged injective module in ind$A$ have no common composition factor.

Example. (1) A local Nakayama algebra of Loewy length three satisfies all but the second conditions stated in Definition 4.4.
(2) A tri-string algebra over a field is given by the following quiver with relations $\alpha_6 \alpha_5 \alpha_4 = 0$ and $\alpha_{i+1} \alpha_i = 0$, for $i \in \{1, 2, 3, 6, 7, 9\}$:



We shall show that $\text{rad}^4(\text{mod}A) = 0$ in case A is a tri-string algebra. It amounts to study the depth of θ_S for each simple module S in mod$A$.

4.5. Lemma. *Let A be a tri-string algebra with S a simple module in* mod$A$. *If* P_S *or* I_S *is uniserial of length* 3, *then* $\text{dp}(\theta_S) \leq 3$.

Proof. We consider only the case where I_S is uniserial of length 3. Then, by Definition 4.4(2), $\ell(P_S) \leq 2$. Since $\text{dp}(\iota_S) \leq 3$; see (3.10), we may assume that $\ell(P_S) = 2$. Put $S_0 = \text{rad} P_S$, which is simple. Then, S is a direct summand of $\text{soc}(I_{S_0}/S_0)$; see (1.1). Since $\ell(P_S) + \ell(I_S) = 5$, by Definition 4.4(3), I_{S_0} is not co-wedged. Hence, I_{S_0} is uniserial. By Lemma 3.2, the inclusion map $j : S_0 \to P_S$ is a source monomorphism, whose co-kernel $\pi_S : P_S \to S$ is a sink epimorphism.

Write $S_1 = \mathrm{soc}(I_S/S)$. Then, S is a direct summand of the top of $\mathrm{rad}P_{S_1}$; see (1.1). Since $\ell(P_S)+\ell(I_S) = 5$, by Definition 4.4(3), P_{S_1} is not wedged. Thus, P_{S_1} is uniserial of length 2 or 3; see (3.7). By Lemma 3.9(1) and (2), there exists sectional a path of irreducible maps $S \xrightarrow{q_2} M \xrightarrow{q_1} I_S$ in $\mathrm{ind}A$ such that $\iota_S = q_1q_2$. Since P_{S_1} and I_S are projective, viewing the diagrams in Lemma 3.9(1) and (2), we see that $d_l(q_2) = d_l(q_1) = \infty$; see (1.7) and (1.10). Therefore, $\mathrm{dp}(\theta_S) = \mathrm{dp}(q_1q_2\pi_S) = 3$. The proof of the lemma is completed.

4.6. Lemma. *Let A be a tri-string algebra with S a simple module in* $\mathrm{mod}A$. *If S is a direct summand of the radical of a wedged projective module or the socle-factor of a co-wedged injective module, then* $\mathrm{dp}(\theta_S) \le 3$.

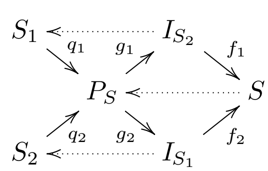
Proof. We consider only the case where P is a wedged projective module with $\mathrm{rad}P = S \oplus S_2$ and $\mathrm{top}P = S_1$. Then, I_S is uniserial of length 2 or 3; see (3.4). If $\ell(I_S) = 3$, then $\mathrm{dp}(\theta_S) \le 3$; see (4.5). Let $\ell(I_S) = 2$. Since $S_1 = \mathrm{soc}(I_S/S)$; see (1.1), by Lemma 3.8(3), we obtain a path of irreducible maps $S \xrightarrow{q_1} P \xrightarrow{p_2} I_{S_1}$ in $\mathrm{ind}A$, where q_1 is a source map, such that $\iota_S = p_2q_1$ and $\mathrm{dp}(\iota_S) = 2$.

On the other hand, by Definition 4.4(3), $\ell(P_S) \le 2$. We need only to consider the case where $\ell(P_S) = 2$. Write $S_0 = \mathrm{rad}P_S$, which is simple. Then, S is a direct summand of $\mathrm{soc}(I_{S_0}/S_0)$; see (1.1). By Definition 4.4(4), I_{S_0} is not co-wedged, that is, I_{S_0} is uniserial. By Proposition 3.2, the inclusion map $j : S_0 \to P_S$ is a source monomorphism, whose co-kernel $\pi_S : P_S \to S$ is a sink epimorphism. Since $q_1 : S \to P$ is a source map, P_S is injective, and hence, $d_r(\pi_S) = \infty$. Therefore, $\mathrm{dp}(\theta_S) = \mathrm{dp}(\iota_S\pi_S) = 3$. The proof of the lemma is completed.

4.7. Lemma. *Let A be a tri-string algebra with S a simple module in* $\mathrm{mod}A$. *If P_S is wedged or I_S is co-wedged, then* $\mathrm{dp}(\theta_S) \le 3$.

Proof. We consider only the case where P_S is wedged with $\mathrm{rad}P_S = S_1 \oplus S_2$, where S_1, S_2 are simple. Then, $\mathrm{dp}(\pi_S) = 2$; see (3.4). By Definition 4.4(2), $\ell(I_S) \le 2$. We may assume that $\ell(I_S) = 2$. Moreover, I_{S_1}, I_{S_2} are uniserial of length 2 or 3; see (3.4) and (3.7). Then, $S = \mathrm{soc}(I_{S_i}/S_i)$; see (1.1), for $i = 1, 2$.

Suppose first that I_{S_1} or I_{S_2} is of length 3, say $\ell(I_{S_1}) = 3$. Then, I_{S_1} is projective-injective with $S = \mathrm{soc}(I_{S_1}/S_1) = \mathrm{rad}I_{S_1}/S_1$. In view of the almost split sequence as stated in Lemma 1.2, we obtain an irreducible monomorphism $q_1 : S \to I_{S_1}/S_1$ with $d_l(q_1) = \infty$; see (1.10). Since $\ell(I_{S_1}/S_1) = 2 = \ell(I_S)$, we find an isomorphism $u : I_{S_1}/S_1 \to I_S$ such that $\iota_S = uq_1$. Thus, $\mathrm{dp}(\theta_S) = \mathrm{dp}(q_1\pi_S) = 3$. Suppose now that $\ell(I_{S_1}) = \ell(I_{S_2}) = 2$. By Lemma 3.4, we obtain a mesh-complete diagram



in $\mathrm{ind}\,A$. Since $\ell(I_S) = 2$, there exists an irreducible map $g : S \to N$, where N is indecomposable. Since $(f_1, f_2) : I_{S_2} \oplus I_{S_1} \to S$ is a sink map, N is projective. Hence, $d_l(g) = \infty$ and S is a direct summand of $\mathrm{rad}N$. Since $\ell(P_S) + \ell(I_S) = 5$, by Definition 4.4(3), N is uniserial. Then, $\mathrm{rad}N = S = \mathrm{soc}N$, and consequently, $\ell(N) = 2 = \ell(I_S)$. Thus, $\iota_S = wg$, for some isomorphism $w : N \to I_S$. Hence, $\mathrm{dp}(\theta_S) = \mathrm{dp}(g\pi_S) = 3$. The proof of the lemma is completed.

The following is our promised result.

4.8. PROPOSITION. *Let A be a tri-string artin algebra. Then* $\mathrm{rad}^4(\mathrm{mod}A) = 0$.

Proof. It suffices to show that $\mathrm{dp}(\theta_S) \le 3$, for every simple module S in $\mathrm{mod}A$; see [9, (2.7)]. By Lemmas 4.5 and 4.7, we may assume that $\ell(P_S) \le 2$ and $\ell(I_S) \le 2$. By Corollary 3.10, we may further assume that $\ell(P_S) = \ell(I_S) = 2$. Set $S_1 = \mathrm{soc}P_S$ and $S_0 = \mathrm{top}I_S$. Observing that $S_1 = \mathrm{rad}P_S$ and $S_0 = I_S/S$, we see that S is a direct summand of each of $\mathrm{soc}(I_{S_1}/S_1)$ and $\mathrm{top}(\mathrm{rad}P_{S_0})$; see (1.1). If I_{S_1} is co-wedged, then $\mathrm{soc}(I_{S_1}/S_1) = I_{S_1}/S_1$, and by Lemma 4.7, $\mathrm{dp}(\theta_S) \le 3$. Similarly, if P_{S_0} is wedged, then $\mathrm{dp}(\theta_S) \le 3$. Suppose now that I_{S_1} and P_{S_0} are uniserial. Since $S_1 = \mathrm{rad}P_S$, by Proposition 3.2, the inclusion map $j : S_1 \to P_S$ is a source monomorphism, whose co-kernel $\pi_S : P_S \to S$ is a sink epimorphism. Similarly, $\iota_S : S \to I_S$ is a source monomorphism. By Lemma 1.7(1), $d_l(\iota_S) \ge 2 > \mathrm{dp}(\pi_S)$. Thus, $\mathrm{dp}(\theta_S) = \mathrm{dp}(\iota_S\pi_S) = 2$. The proof of the proposition is completed.

We are ready to state and prove the main result of this section.

4.9. THEOREM. *Let A be a connected artin algebra. Then* $\mathrm{rad}^4(\mathrm{mod}A) = 0$ *if and only if A is a hereditary algebra of type $\mathbb{A}_4$ or a tri-string algebra.*

Proof. By Propositions 2.5 and 4.8, we need only to prove the necessity. Suppose that $\mathrm{rad}^4(\mathrm{mod}A) = 0$ and A is not a hereditary algebra of type $\mathbb{A}_4$. By Proposition 4.1, A is a string algebra. Let P be a non-uniserial projective module in $\mathrm{ind}A$. Then, $\mathrm{rad}P = S_1 \oplus S_2$, where S_1, S_2 are uniserial. By Propositions 4.2 and 4.3, S_i is simple and the inclusion map $q_i : S_i \to P$ is a source map, and hence, $\mathrm{soc}(I_{S_i}/S_i)$ is simple; see (3.2), for $i = 1, 2$. That is, P is wedged. Dually, every non-uniserial injective module is co-wedged. Thus, A is a wedged string algebra. Now, we shall verify one by one the conditions stated in Definition 4.4.

(1) Note that A is representation-finite and $\mathrm{rad}^4A = 0$. If $\mathrm{rad}^3A \ne 0$, then $\mathrm{rad}^3(\mathrm{mod}A) \ne 0$. That is, A is of Loewy length 4 and $\mathrm{rad}(\mathrm{mod}A)$ is of nilpotency 4. By Theorem 2.6, A is hereditary of type $\vec{\mathbb{A}}_4$, a contradiction. Thus, $\mathrm{rad}^3A = 0$.

(2) Let S be a simple module in $\mathrm{mod}A$. The projective cover $\pi_S : P_S \to S$ and the injective envelope $\iota_S : S \to I_S$ are such that $\mathrm{dp}(\iota_S\pi_S) \le 3$; see (2.1). By Lemma 3.7, $\ell(P_S) \le 3$ and $\ell(I_S) \le 3$. Suppose that $\ell(I_S) = 3$. If I_S is co-wedged then, by the dual of Lemma 3.4, $\mathrm{dp}(\iota_S) = 2$. If I_S is uniserial of length 3, then $\mathrm{dp}(\iota_S) \ge 2$; see (3.9). Dually, if $\ell(P_S) = 3$, then $\mathrm{dp}(\pi_S) \ge 2$. If $\ell(P_S) = \ell(I_S) = 3$, then $\mathrm{dp}(\iota_S\pi_S) \ge 4$, a contradiction. Therefore, $\ell(P_S) + \ell(I_S) \le 5$.

(3) Let S be a simple direct summand of the radical of a wedged projective module P in $\mathrm{ind}A$. Write $\mathrm{rad}P = S \oplus S_2$ and $S_1 = \mathrm{top}P$. Then, I_S is uniserial of length ≥ 2; see (3.4). In particular, $S_1 = \mathrm{soc}(I_S/S)$; see (1.1). By Lemmas 3.8(3) and 3.9(3), $\mathrm{dp}(\iota_S) = \ell(I_S)$. Suppose first that $\ell(I_S) = 3$. Since $\mathrm{dp}(\iota_S) = 3$ and $\mathrm{dp}(\iota_S\pi_S) \le 3$, we see that π_S is an isomorphism. Therefore, $\ell(P_S) + \ell(I_S) = 4$.

Suppose now that $\ell(I_S) = 2$. Then, $\mathrm{dp}(\iota_S) = 2$. If π_S is an isomorphism, then $\ell(P_S)+\ell(I_S) = 3$. Otherwise, $\mathrm{dp}(\iota_S\pi_S) \ge 3$. Thus, $\mathrm{dp}(\iota_S\pi_S) = 3$. As a consequence, $\mathrm{dp}(\pi_S) = 1$. That is, π_S is irreducible. On the other hand, since P_1 is wedged, the inclusion map $q : S \to P_1$ is a source map; see (3.2). Thus, P_S is projective-injective. In particular, P_S is uniserial and the canonical projection $p : P_S \to P_S/\mathrm{soc}P_S$ is a source map. Since π_S is irreducible, $P_S/\mathrm{soc}P_S \cong S$. Therefore, $\ell(P_S) = 2$, and consequently, $\ell(P_S) + \ell(I_S) = 4$. Dually, if S is a simple direct summand of the socle-factor of a co-wedged injective module in $\mathrm{ind}A$, then $\ell(P_S) + \ell(I_S) = 4$.

(4) Consider a wedged projective module P_0 and a co-wedged injective module I_0. Write $\mathrm{top}P_0 = S_0$ and $\mathrm{rad}P_0 \cong S_1 \oplus S_2$, and $\mathrm{soc}I_0 = T_0$ and $I_0/T_0 = T_1 \oplus T_2$, where S_i, T_j are simple, $0 \le i, j \le 2$. As mentioned above, I_{S_i} is uniserial of length ≥ 2 and $\mathrm{dp}(\iota_{S_i}) \ge 2$, $i = 1, 2$. Dually, P_{T_j} is uniserial and $\mathrm{dp}(\pi_{T_j}) \ge 2$, $j = 1, 2$.

Since $\ell(P_0) + \ell(I_0) = 6$, as shown above, $S_0 \not\cong T_0$. Since I_{S_i} is uniserial, $S_i \not\cong T_0$ for $i = 1, 2$. And since P_{T_j} is uniserial, $S_0 \cong T_j$ for $j = 1, 2$. If $S_i \cong T_j$ for some $1 \le i, j \le 2$, then $\mathrm{dp}(\pi_{S_i}) = \mathrm{dp}(\pi_{T_j}) \ge 2$. This yields that $\mathrm{dp}(\iota_{S_i}\pi_{S_i}) \ge 4$, a contradiction. The proof of the theorem is completed.

5. Main statements

The objective of this section is to provide, for each $2 \le n \le 4$, an explicit list of connected artin algebras whose module category is of radical nilpotency n and describe the indecomposable modules and the almost split sequences in their module category. We start with the easy case where $n = 2$.

5.1. Proposition. *Let A be a connected artin algebra. The radical of* $\mathrm{mod}A$ *is of nilpotency two if and only if A is hereditary of type $\vec{\mathbb{A}}_2$.*

Proof. We need only to prove the necessity; see (2.5). If $\mathrm{rad}(\mathrm{mod}A) \ne 0$ and $\mathrm{rad}^2(\mathrm{mod}A) = 0$, then $\mathrm{rad}A \ne 0$ and $\mathrm{rad}^2A = 0$. By Theorem 2.6, A is hereditary of type $\vec{\mathbb{A}}_2$. The proof of the proposition is completed.

Remark. In case A is given by a quiver with relations, the above result is stated in a master dissertation under the supervision of the first named author; see [18].

The following statement is the list of algebras whose module category is of radical nilpotency three.

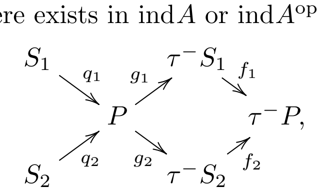
5.2. Theorem. *Let A be a connected artin algebra. The radical of* $\mathrm{mod}A$ *is of nilpotency three if and only if A is a hereditary algebra of type $\mathbb{A}_3$ or $\mathbb{B}_2$, or else, a non-hereditary Nakayama algebra with radical squared zero.*

Proof. If A is hereditary of type $\mathbb{A}_3$ or $\mathbb{B}_2$, then $\mathrm{rad}^2(\mathrm{mod}A) \ne 0$; see (5.1) and $\mathrm{rad}^3(\mathrm{mod}A) = 0$; see (2.5) and (3.6). If A is a non-hereditary Nakayama algebra with $\mathrm{rad}^2A = 0$, then $\ell\ell(A) = 2$. By Corollary 2.8(1), $\mathrm{rad}(\mathrm{mod}A)$ is of nilpotency m with $3 \le m \le 3$. This establishes the sufficiency.

Conversely, assume that $\mathrm{rad}(\mathrm{mod}A)$ is of nilpotency 3. By Theorem 2.5, A is not a hereditary algebra of type $\mathbb{A}_4$. By Theorem 4.9, A is a tri-string algebra. In particular, A is a wedged string algebra.

Suppose first that A is a Nakayama algebra. By Corollary 2.8(1), $\ell\ell(A) = 3$ if A is hereditary, and otherwise, $\ell\ell(A) = 2$. That is, A is hereditary of type $\vec{\mathbb{A}}_3$ or a non-hereditary Nakayama algebra with $\mathrm{rad}^2A = 0$.

Suppose now that A is not a Nakayama algebra. Then, there exists a wedged projective module P in $\mathrm{ind}A$ or $\mathrm{ind}A^{\mathrm{op}}$ with $\mathrm{rad}P = S_1 \oplus S_2$, where S_1, S_2 are simple. By Lemma 3.4, there exists in $\mathrm{ind}A$ or $\mathrm{ind}A^{\mathrm{op}}$ a mesh-complete diagram



where $\tau^- P \cong \mathrm{top}P$, such that $\mathrm{dp}(g_2q_1) = \mathrm{dp}(q_1g_2) = \mathrm{dp}(f_1g_1) = 2$. Since $\mathrm{rad}^3(\mathrm{mod}A) = 0$, by Lemma 2.1, S_1, S_2 are projective and $\tau^- P$ is injective. In view of Proposition 3.6, we conclude that A is hereditary of type $\mathbb{B}_2$ or $\hat{\mathbb{A}}_3$. The proof of the theorem is completed.

Remark. The hereditary Nakayama algebras with radical squared zero are hereditary algebras of type $\mathbb{A}_1$ or $\mathbb{A}_2$.

We are ready to obtain the list of connected artin algebras whose module category is of radical nilpotency four.

5.3. Theorem. *Let A be a connected artin algebra. The radical of* $\mathrm{mod}A$ *is of nilpotency four if and only if A is a hereditary algebra of type $\mathbb{A}_4$, a non-hereditary Nakayama algebra of Loewy length three, or a non-hereditary non-Nakayama tri-string algebra.*

Proof. If A is hereditary of type $\mathbb{A}_4$, then $\mathrm{rad}(\mathrm{mod}A)$ is of nilpotency 4; see (2.5). In view of Proposition 4.8 and Theorem 4.9, we may assume that A is a tri-string algebra. Since $\mathrm{rad}^3 A = 0$, we see that $\ell\ell(A) \le 3$. And since $\mathrm{rad}^4(\mathrm{mod}A) = 0$, $\mathrm{rad}(\mathrm{mod}A)$ is of nilpotency $m \le 4$.

Suppose first that A is a Nakayama algebra. By Corollary 2.8, $m = \ell\ell(A) < 4$ in case A is hereditary; and otherwise, $\ell\ell(A) + 1 \le m \le 2 \cdot \ell\ell(A) - 1$. Therefore, $m = 4$ if and only if A is non-hereditary of Loewy length three. The theorem holds in this case.

Suppose now that A is not a Nakayama algebra. Being a tri-string algebra, A is a wedged string algebra. If A is hereditary then, by Proposition 3.6, $\mathrm{rad}^3(\mathrm{mod}A) = 0$, and hence, $m < 4$. Otherwise, by Proposition 5.1 and Theorem 5.2, we see that $m \notin \{1, 2, 3\}$, that is, $m = 4$. Therefore, $m = 4$ if and only if A is not hereditary. The proof of the theorem is completed.

Remark. The hereditary Nakayama algebras of Loewy length three are hereditary algebras of type $\vec{\mathbb{A}}_3$ and the hereditary non-Nakayama tri-string algebras are hereditary algebras of type $\mathbb{B}_2$ or $\hat{\mathbb{A}}_3$.

Next, we shall describe the module categories whose radical is nilpotent of nilpotency up to four. For the hereditary case, this is done in Theorem 2.6 and Propositions 3.6, 4.2 and 4.3. In view of Theorem 4.9, it suffices to consider tri-string algebras. We start with describing their indecomposable modules.

5.4. Theorem. *Let A be a connected tri-string algebra.*

(1) *If M is a module in* $\mathrm{ind}A$*, then M is of length at most three; and if M is not projective or injective, then M is uniserial of length at most two.*
(2) *If M, N are non-isomorphic non-uniserial modules in* $\mathrm{ind}A$*, then they have no common composition factor.*
(3) *If A is not a local Nakayama algebra of Loewy length* 2*, then* $\mathrm{Ext}^1_A(S, S) = 0$ *for every simple module S in* $\mathrm{mod}A$.

Proof. Let M be a module in $\mathrm{ind}A$, which is neither projective nor injective. Then, there exist radical maps $f : P \to M$ and $g : M \to I$ in $\mathrm{ind}A$ such that $gf \ne 0$, where P is projective and I is injective. Since $\mathrm{rad}^4(\mathrm{mod}A) = 0$; see (4.8), $\mathrm{dp}(gf) \le 3$. As a consequence, f or g is irreducible. We may assume that $g : M \to I$ is irreducible. Since M is not projective, I is not simple. That is, $2 \le \ell(I) \le 3$; see (3.7). Note that $I = I_S$, where $S = \mathrm{soc}I$. Suppose first that I is co-wedged with $I/S = S_1 \oplus S_2$ where S_1, S_2 are simple. By the dual of Proposition 3.4, we may assume that $M \cong \tau S_1$. Since $\ell(I) = 3$, we obtain $\ell(M) = 2$. Being indecomposable, M is uniserial. Suppose now that I is uniserial. If $\ell(I) = 3$, then I is projective-injective; see (3.7), and hence, $M = \mathrm{rad} I$, which is uniserial of length two. Let $\ell(I) = 2$. Since M is not projective, by Proposition 3.8, $M = S$. Since the projective or injective modules in $\mathrm{ind} A$ are of length ≤ 3, Statement (1) holds.

To prove Statement (2), note that the non-uniserial modules in $\mathrm{ind} A$ are wedged projective or co-wedged injective modules. By Definition 4.4(4), we only need to consider two non-isomorphic wedged projective modules P_1 and P_2 in $\mathrm{ind} A$. Write $\mathrm{top} P_1 = S_0$ and $\mathrm{rad} P_1 = S_1 \oplus S_2$, where S_1, S_2 are simple; and $\mathrm{top} P_2 = T_0$ and $\mathrm{rad} P_2 = T_1 \oplus T_2$, where T_1, T_2 are simple. Then, $T_0 \ncong S_0$ by the assumption. Since I_{S_i}, I_{T_j} are uniserial of length ≥ 2; see (3.4), P_{S_i}, P_{T_j} are of length ≤ 2 by Definition 4.4(3). Hence, $T_0 \ncong S_i$ and $S_0 \ncong T_j$, for $1 \leq i, j \leq 2$. Suppose that $S_1 \cong T_1$. Then, S_1 is a direct summand of each of $\mathrm{rad} P_{S_0}$ and $\mathrm{rad} P_{T_0}$. Thus, S_0 and T_0 are direct summands of I_{S_1}/S_1. Since $S_0 \ncong T_0$, we see that I_{S_1} is not uniserial, a contradiction. This establishes Statement (2).

Suppose that $\mathrm{Ext}^1_A(S, S) \neq 0$, for some simple module S in $\mathrm{mod} A$. That is, S is a direct summand of $\mathrm{rad} P_S$. Observing that $\ell(I_S) \geq 2$, we deduce from Definition 4.4(3) that P_S is uniserial. Assume that $\ell(P_S) = 3$. Then, P_S is projective-injective; see (3.7), and $S = \mathrm{top}(\mathrm{rad} P_S) \cong \mathrm{soc}(P_S/S_1)$, where $S_1 = \mathrm{soc} P_S$. Moreover, $I_{S_1} \cong P_S$ with $S \cong \mathrm{soc}(I_{S_1}/S_1)$. By Lemma 1.1, $S_1 \cong \mathrm{top}(\mathrm{rad} P_S) = S$. Hence, $I_S \cong P_S$, and consequently, $\ell(P_S) + \ell(I_S) = 6$, a contradiction to Definition 4.4(2). Thus, $\ell(P_S) = 2$. Dually, $\ell(I_S) = 2$. Since $S = \mathrm{soc} P_S$, we see that $P_S \cong I_S$. That is, P_S is projective-injective. Being connected, A is a local Nakayama algebra of Loewy length two. The proof of the theorem is completed.

Remark. In view of Definition 4.4 and Theorem 5.4, we obtain some necessary and sufficient combinatorial conditions for a string algebra given by a quiver with relations to be a tri-string algebra. This is left for the reader to formulate explicitly.

Example. Let A be an algebra over a field given by the quiver

$$\begin{array}{ccccc} \circ & \xrightarrow{\ \beta\ } & \circ & \xrightarrow{\ \gamma\ } & \circ \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \delta} & & \\ \circ & & \circ & & \end{array}$$

with relations $\gamma\beta = 0$ and $\delta\beta = 0$. Then $\mathrm{rad}^4(\mathrm{mod} A) \neq 0$.

Finally, we shall describe the almost split sequences in the module category of a tri-string algebra. Recall that a non-injective module in $\mathrm{ind} A$ is either a wedged projective module or a uniserial module of length one or two; see (5.4).

5.5. Theorem. *Let A be a tri-string artin algebra. Then every almost split sequence in $\mathrm{mod} A$ is isomorphic to one of the following ones.*

(1) *If P is a wedged projective module in $\mathrm{ind} A$ with $\mathrm{rad} P = S_1 \oplus S_2$, where S_1, S_2 are simple, then there exists an almost split sequence*

$$0 \longrightarrow P \longrightarrow P/S_1 \oplus P/S_2 \longrightarrow \mathrm{top} P \longrightarrow 0.$$

(2) *If M is a non-injective uniserial module of length two with an injective envelope I_M, then there exists an almost split sequence*

$$0 \longrightarrow M \longrightarrow I_M \longrightarrow I_M/M \longrightarrow 0$$

in case I_M is co-wedged; and otherwise, an almost most split sequence

$$0 \longrightarrow M \longrightarrow I_M \oplus \mathrm{top}M \longrightarrow I_M/\mathrm{soc}M \longrightarrow 0.$$

(3) *If S is the socle of a co-wedged injective module I with $I/S = S_1 \oplus S_2$, where S_1, S_2 are simple, then there exists an almost split sequence*

$$0 \longrightarrow S \longrightarrow M_1 \oplus M_2 \longrightarrow I \longrightarrow 0,$$

where $\mathrm{soc}M_i = \mathrm{rad}M_i = S$ and $\mathrm{top}M_i = S_i$, for $i = 1, 2$.

(4) *If S is a simple direct summand of the radical of a wedged projective module P, then there exists an almost split sequence*

$$0 \longrightarrow S \longrightarrow P \longrightarrow P/S \longrightarrow 0.$$

(5) *If S is the socle of a non-simple uniserial injective module I and is not a direct summand of the radical of any wedged projective module, then there exists an almost split sequence*

$$0 \longrightarrow S \longrightarrow N \longrightarrow N/S \longrightarrow 0,$$

where $N = I$ in case $\ell(I) = 2$, and $N = \mathrm{rad}I$ in case $\ell(I) = 3$.

Proof. Let $\delta:\ 0 \longrightarrow M \longrightarrow N \longrightarrow L \longrightarrow 0$ be an almost split sequence in $\mathrm{mod}A$. By Theorem 5.4, $\ell(M) \leq 3$. Consider first the case where $\ell(M) = 3$. Not being injective, M is not uniserial; see (3.7), and consequently, $M = P$, a wedged projective module in $\mathrm{ind}A$ with $\mathrm{rad}P = S_1 \oplus S_2$, where S_1, S_2 are simple. By Lemma 3.4, δ is of the form as stated in Statement (1).

Consider now the case where $\ell(M) = 2$. Being indecomposable, M is uniserial. Write $S = \mathrm{soc}M$ and $S_1 = \mathrm{top}M = M/S$. Then, $\mathrm{soc}I_M = S$ and $I_M \cong I_S$. Since M is not injective, $\ell(I_M) = 3$. Suppose first that I_M is co-wedged. Then, S_1 is a direct summand of I_M/S, say $I_M/S = S_1 \oplus S_2$, where S_2 is simple. Observe that $I_M/M \cong S_2$. By the dual of Lemma 3.2, δ is of the first form as stated in Statement (2). Suppose now that I_M is uniserial. Then, I_M is projective-injective with $M = \mathrm{rad}I_M$ and $\mathrm{rad}I_M/S = \mathrm{top}M$. By Lemma 1.2, δ is of the second form as stated in Statement (2).

Consider finally the case where $M = S$, a non-injective simple module in $\mathrm{mod}A$ with $I_S = I$. In case I is co-wedged, by the dual of Lemma 3.4, δ is of the form as stated in Statement (3). Otherwise, I is uniserial of length 2 or 3. Write $S_1 = \mathrm{soc}(I/S)$ and $P = P_{S_1}$. If P is wedged, then S is a direct summand of $\mathrm{rad}P$, and by Propositions 3.8(3) and 3.9(3), δ is of the form as stated in Statement (4). In case P is uniserial, viewing the first two statements of each of Propositions 3.8 and 3.9, we conclude that δ is of the form as stated in Statement (5). The proof of the theorem is completed.

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Corresponding author: Shiping Liu, Département de mathématiques, Université de Sherbrooke, Sherbrooke, Québec, Canada
Email address: `shiping.liu@usherbrooke.ca`

Youqi Yin, Department of mathematics, University of Shaoxing, Shaoxing 312000, China
Email address: `yinyouqi@usx.edu.cn`