Stably Triangulated Algebras

Shiping Liu and Charles Paquette

Throughout this note, A stands for a connected artin algebra and $\mathrm{mod}\,A$ for the category of finitely generated right A-modules in which the maps are composed from the left to the right. Moreover, denote by $\overline{\mathrm{mod}}\,A$ the injectively stable category of $\mathrm{mod}\,A$ (that is, the quotient of $\mathrm{mod}\,A$ modulo the ideal of maps which factor through an injective module), and by $\underline{\mathrm{mod}}\,A$ the projectively stable category.

It is well known that $\mathrm{mod}\,A$ is triangulated if and only if A is semi-simple. On the other hand, it follows from Happel's result in [2] that $\overline{\mathrm{mod}}\,A$ is triangulated if A is stably equivalent to a self-injective algebra. Our objective is to establish the converse of this statement. More precisely, we shall show that if $\overline{\mathrm{mod}}\,A$ is pre-triangulated, then A is either self-injective or directed Nakayama of Loewy length two. In this way, we recover Reiten's characterization of algebras stably equivalent to a self-injective algebra.

Recall that an additive category $\mathcal{C}$ is *triangulated* if it is equipped with an automorphism T and a class $\mathcal{T}$ of sextuples (called *exact triangles*)

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T(X),$$

which satisfies the axioms TR1, TR2, TR3 and TR4 stated in [2, (1.1)]. We shall say that $\mathcal{C}$ is *pre-triangulated* if $\mathcal{T}$ satisfies the axioms TR1, TR2 and TR3. We refer to [1] for the Auslander-Reiten theory in $\mathrm{mod}\,A$ such as irreducible maps, almost split sequences, and Auslander-Reiten quiver.

Lemma. *Let* $0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$ *be an almost split sequence in* $\mathrm{mod}\,A$. *If* $\overline{\mathrm{mod}}\,A$ *is pre-triangulated, then the following statements hold true*:

(1) *If N injective, then so is M.*

(2) *If L is projective, then so is M.*

(3) *M is injective if and only if M is projective.*

Proof. Assume that $\overline{\mathrm{mod}}\,A$ is pre-triangulated. For a map $e : X \to Y$ in $\mathrm{mod}\,A$, we denote by $\bar{e} : X \to Y$ the corresponding map in $\overline{\mathrm{mod}}\,A$. Suppose that N is injective while M is not. Write first $M = M_1 \amalg M_2$ with M_1 indecomposable and non-injective, then $f = (f_1, f_2)$ and $g = (g_1, g_2)^T$ accordingly. It follows from the axiom TR1 that $\overline{f}_1$ embeds in an exact triangle

$$L \xrightarrow{\overline{f}_1} M_1 \xrightarrow{\overline{h}} E \longrightarrow T(L)$$

in $\overline{\mathrm{mod}}\,A$. Now the three axioms of a pre-triangulated category ensure that $\bar{f}_1$ is a pseudo-kernel of $\bar{h}$; see [2, (1.2)]. In particular, $f_1 h = uv$, where $u : L \to I$ and $v : I \to E$ are maps in $\mathrm{mod}\,A$ with I injective. Since L is not injective, u is not a section. Hence there exists $(w_1, w_2)^T : M_1 \amalg M_2 \to I$ such that $u = f(w_1, w_2)^T$. Therefore, $f_1 h = f_1 w_1 v + f_2 w_2 v$, that is, $(f_1, f_2)(w_1 v - h, w_2 v)^T = 0$. This induces a map $w : N \to E$ such that $(w_1 v - h, w_2 v)^T = (g_1, g_2)^T w$. Since N is injective, we get $\bar{h} = \bar{w}_1 \bar{v} - \bar{g}_1 \bar{w} = \bar{0}$. Being a pseudo-kernel of a zero morphism, $\bar{f}_1$ is a retraction in $\overline{\mathrm{mod}}\,A$. Since M_1 is indecomposable and not injective, f_1 is a retraction in $\mathrm{mod}\,A$. This is absurd since f_1 is irreducible. This proves (1). Being equivalent to $\overline{\mathrm{mod}}\,A$; see [1, (IV.1.9)], $\underline{\mathrm{mod}}\,A$ is also pre-triangulated. Thus, (2) holds dually.

Suppose next that M is injective but not projective. Let N_1 be an indecomposable non-projective direct summand of M. Then $\mathrm{mod}\,A$ admits an almost split sequence $0 \longrightarrow \tau N_1 \longrightarrow U \longrightarrow N_1 \longrightarrow 0$. Note that L is a direct summand of U. Since N_1 is injective, by (1), U is injective. Thus L is injective, which is absurd. Similarly, it follows from (2) that M is injective whenever it is projective. The proof of the lemma is completed.

Recall that a representation-finite artin algebra is *directed* if its Auslander-Reiten quiver contains no oriented cycle. We call A *stably triangulated* if $\overline{\mathrm{mod}}\,A$, or equivalently $\underline{\mathrm{mod}}\,A$, is triangulated.

THEOREM. *Let A be a connected artin algebra. The following are equivalent:*
(1) *A is stably triangulated.*
(2) *$\overline{\mathrm{mod}}\,A$ is pre-triangulated.*
(3) *A is either self-injective or directed Nakayama of Loewy length two.*
(4) *A is stably equivalent to a self-injective algebra.*

Proof. Assume that $\overline{\mathrm{mod}}\,A$ is pre-triangulated while A is not self-injective. Let I be an indecomposable injective module in $\mathrm{mod}\,A$ which is not projective. Then $\mathrm{mod}\,A$ admits an almost split sequence $0 \longrightarrow \tau I \longrightarrow P_1 \longrightarrow I \longrightarrow 0$. By the lemma, P_1 is projective-injective. Let $n > 0$ be an integer for which $\mathrm{mod}\,A$ admits almost split sequences $0 \longrightarrow \tau^i I \longrightarrow P_i \longrightarrow \tau^{i-1} I \longrightarrow 0$ with P_i projective-injective, $i = 1, \ldots, n$. Then the $\tau^i I$ with $0 \le i \le n$ are all simple; see [1, (V.3.3)], which are pairwise non-isomorphic since I is injective. Hence the P_i with $1 \le i \le n$ are pairwise non-isomorphic and of Loewy length two. If $\tau^n I$ is not projective, then $\mathrm{mod}A$ has an almost split sequence $0 \longrightarrow \tau^{n+1} I \longrightarrow P_{n+1} \longrightarrow \tau^n I \longrightarrow 0$. Suppose that P_{n+1} has a non-injective direct summand M. Then $\tau^- M$ is a non-projective direct summand of P_n, which is contrary to the assumption that P_n is projective. This shows that P_{n+1} is injective, and hence projective-injective by the lemma. Since $\mathrm{mod}A$ has only finitely many non-isomorphic simple modules, we may assume that $\tau^n I$ is projective. Since A is connected, we see that the P_i and the $\tau^j I$ with $1 \le i \le n$ and $0 \le j \le n$ are the non-isomorphic indecomposable modules in $\mathrm{mod}\,A$. In particular, A is directed. Since the simple modules $\tau^i I$ with $0 \le i \le n$ form a τ-orbit, A is a Nakayama algebra; see [1, (IV.2.10)]. Moreover, the P_i with

$1 \leq i \leq n$ are of Loewy length two while $\tau^n I$ is simple. Thus, A is of Loewy length two. This proves that (2) implies (3).

Suppose that A is directed Nakayama of Loewy length two. Let $S_1, \ldots, S_n$ be the non-isomorphic simple A-modules. Then $\overline{\mathrm{mod}}\, A \cong \mathrm{mod}\, k_1 \times \cdots \times \mathrm{mod}\, k_n$, where $k_i = \mathrm{End}_A(S_i)$, $i = 1, \ldots, n$. If B_i denotes the trivial extension of k_i by k_i, then $B = B_1 \Pi \cdots \Pi B_n$ is symmetric with $\overline{\mathrm{mod}}\, B \cong \mathrm{mod}\, k_1 \times \cdots \times \mathrm{mod}\, k_n$. This shows that (3) implies (4). Finally, it follows from [2, (2.6)] that (4) implies (1). The proof of the theorem is completed.

Remark. The equivalence of the statements (3) and (4) in the theorem is due to Reiten; see [3].

Acknowledgement. The first author is supported in part while the second author in full by the Natural Sciences and Engineering Research Council of Canada. Moreover, the first author holds a guest researchership at the Beijing Normal University financed through the 985-project.

References

[1] M. Auslander, I. Reiten and S.O. Smalø "Representation Theory of Artin Algebras", *Cambridge studies in advanced mathematics* **36** (Cambridge University Press, Cambridge, 1995).

[2] D. Happel, "Triangulated Categories in the Representation Theory of Finite Dimensional Algebras", *London Mathematical Society Lectures Note Series* **119** (Cambridge University Press, Cambridge, 1988).

[3] I. Reiten, "Stable equivalence of self-injective algebras", *J. Algebra* **40** (1976) 63-74.

Shiping Liu, Département de mathématiques, Université de Sherbrooke, Sherbrooke, Québec, Canada J1K 2R1, and School of Mathematics, Beijing Normal University, Beijing, China.

Email: shiping.liu@usherbrooke.ca

Charles Paquette, Département de mathématiques, Université de Sherbrooke, Sherbrooke, Québec, Canada J1K 2R1.

Email: charles.paquette@usherbrooke.ca