STANDARD COMPONENTS OF A KRULL-SCHMIDT CATEGORY

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Abstract. First, for a general Krull-Schmidt category, we provide criteria for an Auslander-Reiten component with sections to be standard. Specializing to the category of finitely presented representations of a strongly locally finite quiver and its bounded derived category, we obtain many new types of standard Auslander-Reiten components. Finally, specialized to the module category of a finite-dimensional algebra, our criteria become particularly nice and yield some new results.

Introduction

Standard Auslander-Reiten components of the module category of a finite dimensional algebra are extremely interesting, since the maps between modules in such a component can be described in a simple combinatorial way; see [4, 13]. This kind of components appears mainly for representation-finite algebras, hereditary algebras, tubular algebras and tilted algebras; see [13], and each of them has at most finitely many non-periodic Auslander-Reiten orbits; see [14]. In particular, the regular ones are stable tubes or of shape $\mathbb{Z}\Delta$ with Δ a finite acyclic quiver.

On the other hand, the Auslander-Reiten theory has been extended to Krull-Schmidt categories; see [2, 11]. It is natural to expect that new types of standard Auslander-Reiten components will appear in this context. Indeed, in the most general setup, we shall find various criteria for such an Auslander-Reiten component having sections to be standard. In particular, an Auslander-Reiten component which is a wing or of shape $\mathbb{N}\mathbb{A}_\infty^+$, $\mathbb{N}^-\mathbb{A}_\infty^-$ or $\mathbb{Z}\mathbb{A}_\infty$ is standard if and only if its quasi-simple objects are pairwise orthogonal bricks. Specializing to $\operatorname{rep}^+(Q)$, the category of finitely presented representations of a connected strongly locally finite quiver Q, we prove that the preprojective component and the preinjective components are standard; and every component is standard in case Q is of finite or infinite Dynkin type. Applying this to the bounded derived category $D^b(\operatorname{rep}^+(Q))$ of $\operatorname{rep}^+(Q)$, we show that the connecting component is standard; and every component is standard in the Dynkin case. All these particularly establish the existence of standard Auslander-Reiten components which are wings or of shapes $\mathbb{N}\mathbb{A}_\infty^+$, $\mathbb{N}^-\mathbb{A}_\infty^-$ and $\mathbb{Z}\Delta$ with Δ any strongly locally finite quiver without infinite paths. Furthermore, specialized to the module category $\operatorname{mod} A$ of a finite dimensional algebra A, our criteria become surprisingly nice and easy to verify; see (3.1). As a consequence, an Auslander-Reiten component with sections of $\operatorname{mod} A$ is standard if and only if it is generalized standard, if and only if it is the connecting component of a tilted factor algebra of A. Finally, we remark that some of our results will be applied in the future to study cluster categories of infinite Dynkin types.

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1. Standard components having sections

Throughout this paper, k stands for an arbitrary field. A k-category is a category in which the morphism sets are k-vector spaces and the composition of morphisms is k-bilinear. A k-category is called *Hom-finite* if its morphism spaces are all finite dimensional over k, and *Krull-Schmidt* if every non-zero object is a finite direct sum of objects with a local endomorphism algebra.

For the rest of this section, let $\mathcal{C}$ stand for a Hom-finite Krull-Schmidt additive k-category. The *radical* morphisms in $\mathcal{C}$ are those in the Jacobson radical $\mathrm{rad}(\mathcal{C})$. One calls $\mathrm{rad}^{\infty}(\mathcal{C}) = \cap_{n\ge 1}\mathrm{rad}^{n}(\mathcal{C})$ the *infinite radical* of $\mathcal{C}$, where $\mathrm{rad}^{n}(\mathcal{C})$ is the n-th power of $\mathrm{rad}(\mathcal{C})$. Two objects X, Y in $\mathcal{C}$ are said to be *orthogonal* if $\mathrm{Hom}_{\mathcal{C}}(X,Y)=0$ and $\mathrm{Hom}_{\mathcal{C}}(Y,X)=0$. If $X\in\mathcal{C}$ is indecomposable, then the division algebra $k_{_X} = \mathrm{End}(X)/\mathrm{rad}(X,X)$ is called the *automorphism field* of X, and we shall call X a *brick* provided that $\mathrm{End}_{\mathcal{C}}(X)$ is trivial, that is, $\mathrm{End}_{\mathcal{C}}(X)\cong k$. Let $f: X\to Y$ be a morphism in $\mathcal{C}$. One says that f is *irreducible* if it is neither a section nor a retraction, and any factorization $f = gh$ implies that h is a section or g is a retraction. Moreover, f is called *left almost split* if it is not a section and every non-section morphism $g : X \to M$ in $\mathcal{C}$ factors through f; *left minimal* if every endomorphism h of Y such that $f = hf$ is an automorphism. In a dual manner, one defines f to be *right almost split* and *right minimal.* Further, f is called a *source morphism for* X if it is left minimal and left almost split, and a *sink morphism for* Y if it is right minimal and right almost split. A sequence of morphisms

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

in $\mathcal{C}$ with $Y\neq 0$ is called *almost split* provided that f is a source morphism and a pseudo-kernel of g, while g is a sink morphism and a pseudo-cokernel of f; see [11, (1.3)]. In case $\mathcal{C}$ is abelian or triangulated, the definition of an almost split sequence given here coincides somehow with the classical one; see [11, (1.5), (6.1)].

We shall make a frequent use of the following easy result.

1.1. Lemma. *Let $\mathcal{C}$ have an almost split sequence as follows*:

$$X \xrightarrow{\binom{f_1}{f_2}} Y_1 \amalg Y_2 \xrightarrow{(g_1,g_2)} Z.$$

(1) *There exists a k-linear isomorphism $k_{_X}\cong k_{_Z}$.*
(2) *If $u : M \to Y_1$ is a morphism in $\mathcal{C}$ such that $g_1u = 0$, then there exists some $w : M \to X$ such that $u = f_1w$ and $f_2w = 0$.*
(3) *If $v : Y_1 \to N$ is a morphism in $\mathcal{C}$ such that $vf_1 = 0$, then there exists some $w : Z \to N$ such that $v = wg_1$ and $wg_2 = 0$.*

Proof. Statement (1) is implicitly stated and proved in the proof of [11, (2.1)]. Let $u : M \to Y_1$ be such that $g_1u = 0$. Then $(g_1, g_2)\binom{u}{0} = 0$, and hence there exists some $w : M \to X$ such that $\binom{u}{0} = \binom{f_1}{f_2}w$. This proves Statement (2). Dually, we can show Statement (3). The proof of the lemma is completed.

The *Auslander-Reiten quiver* $\Gamma_{_\mathcal{C}}$ of $\mathcal{C}$ is first defined to be a valued translation quiver as follows. The vertex set is a complete set of the representatives of the isomorphism classes of the indecomposable objects in $\mathcal{C}$. For vertices X and Y, we write $d'_{_{XY}}$ and $d_{_{XY}}$ for the dimensions of

$$\operatorname{irr}(X,Y) = \operatorname{rad}(X,Y)/\operatorname{rad}^2(X,Y)$$

over k_X and k_Y respectively, and draw a unique valued arrow $X \to Y$ with valuation (d_{XY}, d'_{XY}) if and only if $d_{XY} > 0$. The translation τ is defined so that $\tau Z = X$ if and only if $\mathcal{C}$ has an almost split sequence $X \longrightarrow Y \longrightarrow Z$. A valuation (d_{XY}, d'_{XY}) is called *symmetric* if $d_{XY} = d'_{XY}$, and *trivial* if $d_{XY} = d'_{XY} = 1$. Next, $\Gamma_{\mathcal{C}}$ is modified in such a way that each symmetrically valued arrow $X \to Y$ is replaced by d_{XY} unvalued arrows from X to Y. That is, $\Gamma_{\mathcal{C}}$ becomes a partially valued translation quiver in which all valuations are non-symmetric; see [11, (2.1)].

Let Σ be a convex subquiver of $\Gamma_{\mathcal{C}}$ in which every object has a trivial automorphism field. In particular, $d_{XY} = d'_{XY}$ for all $X, Y \in \Sigma$. By our construction, Σ is a non-valued translation quiver with possible multiple arrows. Thus, one can define the *path category* $k\Sigma$ and the *mesh category* $k(\Sigma)$ of Σ over k; see, for example, [13, (2.1)]. In the sequel, for $u \in k\Sigma$, we shall write $\overline{u}$ for its image in $k(\Sigma)$.

1.2. Definition. Let Σ be a convex subquiver of $\Gamma_{\mathcal{C}}$, and let $\mathcal{C}(\Sigma)$ be the full subcategory of $\mathcal{C}$ generated by the objects in Σ. We shall say that Σ is *standard* provided that every object in Σ has a trivial automorphism field and there exists a k-equivalence $F : k(\Sigma) \xrightarrow{\sim} \mathcal{C}(\Sigma)$, which acts identically on the objects.

1.3. Lemma. *Let Σ be a convex subquiver of $\Gamma_{\mathcal{C}}$, and let $F : k(\Sigma) \xrightarrow{\sim} \mathcal{C}(\Sigma)$ be a k-equivalence acting identically on the objects. If $X, Y \in \Sigma$, then the classes $F(\overline{\alpha}) + \operatorname{rad}^2(X,Y)$ form a k-basis of $\operatorname{irr}(X,Y)$, where α ranges over the set of arrows from X to Y.*

Proof. Let $X, Y \in \Sigma$. For $1 \le i \le 2$, consider the k-subspace $\mathcal{I}^{(i)}(X,Y)$ of $k(\Sigma)(X,Y)$ generated by the $\overline{p}$, where p ranges over the set of paths of length $\ge i$ from X to Y. Write $\Sigma_1(X,Y)$ for the set of arrows from X to Y. Since the mesh relations are sums of paths of length two, the classes $\overline{\alpha} + \mathcal{I}^{(2)}(X,Y)$, with $\alpha \in \Sigma_1(X,Y)$, are k-linearly independent, and hence, they form a k-basis for $\mathcal{I}^{(1)}(X,Y)/\mathcal{I}^{(2)}(X,Y)$. Thus, $\mathcal{I}^{(1)}(X,Y)/\mathcal{I}^{(2)}(X,Y)$ and $\operatorname{irr}(X,Y)$ are of the same k-dimension. Since F induces a k-isomorphism $F : k(\Sigma)(X,Y) \to \operatorname{Hom}_{\mathcal{C}}(X,Y)$, it is easy to see that F induces a k-epimorphism $F : \mathcal{I}^{(1)}(X,Y) \to \operatorname{rad}(X,Y)$. In particular, F maps $\mathcal{I}^{(2)}(X,Y)$ into $\operatorname{rad}^2(X,Y)$. This yields a k-epimorphism

$$\overline{F} : \mathcal{I}^{(1)}(X,Y)/\mathcal{I}^{(2)}(X,Y) \to \operatorname{irr}(X,Y) : u + \mathcal{I}^{(2)}(X,Y) \mapsto F(u) + \operatorname{rad}^2(X,Y),$$

which is necessarily an isomorphism. The proof of the lemma is completed.

Given a quiver Σ with no oriented cycle, one constructs a stable translation quiver $\mathbb{Z}\Sigma$; see, for example, [13, (2.1)]. We denote by $\mathbb{N}\Sigma$ the full translation subquiver of $\mathbb{Z}\Sigma$ generated by the vertices (n,x) with $n \ge 0$ and $x \in \Sigma$, and by $\mathbb{N}^{-}\Sigma$ the one generated by the vertices (n,x) with $n \le 0$ and $x \in \Sigma$. Now, let Γ be a connected component of $\Gamma_{\mathcal{C}}$. A connected full subquiver Δ of Γ is called a *section* if it is convex in Γ, contains no oriented cycle, and meets every τ-orbit in Γ exactly once. In this case, every object in Γ is uniquely written as $\tau^n X$ with $n \in \mathbb{Z}$ and $X \in \Delta$, and there exists a translation-quiver embedding $\Gamma \to \mathbb{Z}\Delta : \tau^n X \mapsto (-n, X)$; see [10, (2.3)]. We denote by Δ^- the full subquiver of Γ generated by the vertices $\tau^n X$ with $n > 0$ and $X \in \Delta$, and by Δ^+ the one generated by the vertices $\tau^n X$ with $n < 0$ and $X \in \Delta$. One says that Δ is a *right-most section* if $\Delta^+ = \emptyset$; and *left-most section* if $\Delta^- = \emptyset$.

In order to state and prove the following main result of this section, we need some terminology and notation. Firstly, an infinite path in a quiver is called *left infinite* if it has no starting point; and *right infinite* if it has no ending point. Secondly, given two (possibly empty) subquivers Σ, Ω of $\Gamma_{\mathcal{C}}$, we shall write $\mathrm{Hom}_{\mathcal{C}}(\Sigma, \Omega) = 0$ in case $\mathrm{Hom}_{\mathcal{C}}(X, Y) = 0$ for all possible objects $X \in \Sigma$ and $Y \in \Omega$.

1.4. Theorem. *Let $\mathcal{C}$ be a Hom-finite Krull-Schmidt additive k-category, and let Γ be a connected component of $\Gamma_{\mathcal{C}}$ having a section Δ. If Δ^+ has no left infinite path and Δ^- has no right infinite path, then Γ is standard if and only if Δ is standard such that* $\mathrm{Hom}_{\mathcal{C}}(\Delta^+, \Delta \cup \Delta^-) = 0$ *and* $\mathrm{Hom}_{\mathcal{C}}(\Delta, \Delta^-) = 0$.

Proof. Suppose that Δ^+ has no left infinite path and Δ^- has no right infinite path. Assume first that Γ is standard. In particular, Δ is standard. Since Γ embeds in $\mathbb{Z}\Delta$, we see that Γ has no path from X to Y in case $X \in \Delta^+$ and $Y \in \Delta \cup \Delta^-$, or $X \in \Delta$ and $Y \in \Delta^-$. This shows the necessity.

Assume conversely that Δ is standard such that $\mathrm{Hom}_{\mathcal{C}}(\Delta^+, \Delta \cup \Delta^-) = 0$ and $\mathrm{Hom}_{\mathcal{C}}(\Delta, \Delta^-) = 0$. In particular, every object in Δ has a trivial endomorphism algebra. Being of the form $\tau^n X$ with $n \in \mathbb{Z}$ and $X \in \Delta$, by Lemma 1.1(1), every object in Γ has a trivial automorphism field. Since every object in Δ^+ has a sink morphism and Δ is a section, Δ^+ is locally finite. By König's Lemma, Δ^+ has only finitely many paths ending in any pre-fixed object. Thus, for each object $M \in \Delta \cup \Delta^+$, we may define an integer $n_M \geq 0$ in such a way that $n_M = 0$ if $M \in \Delta$; and otherwise, $n_M - 1$ is the maximal length of the paths of Δ^+ which end in M. The following statement is evident.

(1) *Let $p : X \rightsquigarrow Y$ be a non-trivial path in Γ. If $X \in \Delta \cup \Delta^+$, then $Y \in \Delta \cup \Delta^+$ with $n_X \leq n_Y$, and the equality occurs if and only if $X, Y \in \Delta$.*

For each $n \geq 0$, denote by Γ^n the full subquiver of Γ generated by the vertices $X \in \Delta \cup \Delta^+$ with $n_X \leq n$, which is clearly convex in Γ. Moreover, denote by Γ^+ the union of the Γ^n with $n \geq 0$, that is, the full subquiver of Γ generated by $\Delta^+ \cup \Delta$. The following statement is an immediate consequence of Statement (1).

(2) *If $p : X \rightsquigarrow Y$ is a non-trivial path in Γ^{n+1} with $n \geq 0$, then $X \in \Gamma^n$, and consequently, $p \notin \Gamma^n$ if and only if $Y \notin \Gamma^n$.*

Now, let $F^0 : k(\Delta) \xrightarrow{\sim} \mathcal{C}(\Delta)$ be a k-linear equivalence, acting identically on the objects. Since Δ contains no mesh of Γ, we have $k(\Delta) = k\Delta$. Assume that $n \geq 0$ and F^0 extends to a full k-linear functor $F^n : k\Gamma^n \to \mathcal{C}(\Gamma^n)$, acting identically on the objects and having a kernel generated by the mesh relations. In order to extend F^n to $k\Gamma^{n+1}$, we shall need the following statement.

(3) *If $f : X \to Y$ is a non-zero radical morphism in $\mathcal{C}(\Gamma^{n+1})$, then Γ^{n+1} has a non-trivial path from X to Y.* To show this, we claim that there exists $M \in \Delta$, which is a predecessor of Y, such that $\mathrm{Hom}_{\mathcal{C}}(X, M) \neq 0$. Indeed, suppose that this claim was false. Then, $Y \in \Delta^+$. Since Δ is a section in Γ, every immediate predecessor of an object Δ^+ lies in $\Delta^+ \cup \Delta$. Since every object in Δ^+ admits a sink morphism in $\mathcal{C}$, by factorizing the radical morphism f, we get a left infinite path

$$\cdots \longrightarrow Y_i \longrightarrow Y_{i-1} \longrightarrow \cdots \longrightarrow Y_1 \longrightarrow Y$$

in Δ^+ such that $\mathrm{Hom}_{\mathcal{C}}(X, Y_i) \neq 0$ for all $i > 0$, contrary to the hypothesis on Δ^+. Thus, Δ does contain an object M as claimed. Since $\mathrm{Hom}_{\mathcal{C}}(\Delta^+, \Delta) = 0$, we have $X \in \Delta$. Since $k\Delta \cong \mathcal{C}(\Delta)$, there exists a path in Δ from X to M. This yields a non-trivial path in Γ^{n+1} from X to Y. Statement (3) is established.

Fix $Z \in \Gamma^{n+1} \backslash \Gamma^n$. Observe that $Z \in \Delta^+$ and $\tau Z \in \Delta^+ \cup \Delta$. Thus, $k\Gamma^{n+1}$ has a mesh relation $\delta_Z = \sum_{i=1}^r \beta_i \alpha_i$, where $\alpha_i : \tau Z \to Y_i$, $i = 1, \ldots, r$, are the arrows starting in τZ and $\beta_i : Y_i \to Z$, $i = 1, \ldots, r$, are the arrows ending in Z. By Statement (2), $\tau Z, Y_1, \ldots, Y_r \in \Gamma^n$. Since τZ admits a source morphism in $\mathcal{C}$, it follows from Lemma 1.3 that $f = (F^n(\alpha_1), \ldots, F^n(\alpha_r))^T : \tau Z \to Y_1 \oplus \cdots \oplus Y_r$ is a source morphism, which embeds in an almost split sequence

$$(*) \qquad \tau Z \xrightarrow{f} Y_1 \oplus \cdots \oplus Y_r \xrightarrow{(g_1, \ldots, g_r)} Z$$

in $\mathcal{C}$; see [11, (1.4)]. Set $F^{n+1}(Z) = Z$, $F^{n+1}(\varepsilon_Z) = \mathbf{1}_Z$, where ε_Z is the *trivial* path at Z, and $F^{n+1}(\beta_i) = g_i$, for $i = 1, \ldots, s$. In view of Statement (2), we have defined F^{n+1} on the vertices, the trivial paths, and the arrows in Γ^{n+1}. In an evident manner, we may extend F^n to a k-functor $F^{n+1} : k\Gamma^{n+1} \to \mathcal{C}(\Gamma^{n+1})$, acting identically on the objects.

Let $u : Y \to Z$ be a non-zero radical morphism in $\mathcal{C}(\Gamma^{n+1})$. By Statement (3), Γ^{n+1} has a non-trivial path from Y to Z, and hence $Y \in \Gamma^n$ by Statement (2). If $Z \in \Gamma^n$ then, by the induction hypothesis, $u = F^n(\rho)$ for some morphism $\rho : Y \to Z$ in $k\Gamma^n$. Otherwise, Z is the ending term of an almost split sequence $(*)$ as stated above. Then $u = \sum_{i=1}^r g_i u_i$, with morphisms $u_i : Y \to Y_i$ in $\mathcal{C}$. Since $Y_i \in \Gamma^n$, there exists $\rho_i : Y \to Y_i$ in $k\Gamma^n$ such that $u_i = F^n(\rho_i)$, for $i = 1, \ldots, r$. This yields $u = F^{n+1}(\sum_{i=1}^r \beta_i \rho_i)$. That is, F^{n+1} is full.

Next we shall show, for $\theta \in k\Gamma^{n+1}$, that $F^{n+1}(\theta) = 0$ if and only if θ lies in the mesh ideal of $k\Gamma^{n+1}$. In view of the induction hypothesis, we may assume that θ is non-zero of the form $\theta : Y \to Z$ with $Z \in \Gamma^{n+1} \backslash \Gamma^n$. In particular, Γ^{n+1} has a non-trivial path from Y to Z. By Statement (2), $Y \in \Gamma^n$. Suppose first that θ lies in the mesh ideal of $k\Gamma^{n+1}$. For simplicity, we may assume that $\theta = \zeta\,\delta\,\sigma$, where $\sigma, \delta, \zeta \in k\Gamma^{n+1}$ with δ a mesh relation. If ζ has as a non-zero summand a multiple of a non-trivial path, then $\delta \in k\Gamma^n$ by Statement (2). Hence, $F^{n+1}(\theta) = 0$ by the induction hypothesis. Otherwise, δ is the mesh relation δ_Z as stated above, and $\theta = (\sum_{i=1}^r \beta_i \alpha_i)\eta$, where $\eta : Y \to \tau Z$ is a morphism in $k\Gamma^n$. Since $(*)$ is an almost split sequence, we obtain $F^{n+1}(\theta) = 0$.

Suppose conversely that $F^{n+1}(\theta) = 0$. Consider the mesh relation δ_Z and the almost split sequence $(*)$ as stated above. Then $\theta = \sum_{i=1}^r \beta_i \theta_i$, where $\theta_i : Y \to Y_i$ is in $k\Gamma^n$. Since $\sum_{i=1}^r F^{n+1}(\beta_i) F^n(\theta_i) = F^{n+1}(\theta) = 0$, there exists $v : Y \to \tau Z$ in $\mathcal{C}$ such that $F^n(\theta_i) = F^n(\alpha_i)\, v$, for $i = 1, \ldots, r$. Since F^n is full, $v = F^n(\eta)$ with $\eta : Y \to \tau Z$ in $k\Gamma^n$. Hence $F^n(\theta_i) = F^n(\alpha_i \eta)$, and by the induction hypothesis, $\theta_i - \alpha_i \eta$ lies in the mesh ideal of $k\Gamma^n$, $i = 1, \ldots, r$. As a consequence,

$$\theta = \textstyle\sum_{i=1}^r \beta_i(\theta_i - \alpha_i \eta) + (\sum_{i=1}^r \beta_i \alpha_i)\eta$$

lies in the mesh ideal of $k\Gamma^{n+1}$. This shows that F^{n+1} is full and its kernel is generated by the mesh relations. By induction, F^0 extends to a full k-functor $F^+ : k\Gamma^+ \to \mathcal{C}(\Gamma^+)$, acting identically on the objects and having a kernel generated by the mesh relations.

Finally, for each object $N \in \Gamma$, we may define $m_N \geq 0$ so that $m_N = 0$ if $N \in \Gamma^+$; and otherwise, $m_N - 1$ is the maximal length of the paths of Δ^- which start in N. For $m \geq 0$, denote by $\Gamma^{(m)}$ the full subquiver of Γ generated by the objects Y with $m_Y \leq m$. Then Γ is the union of the $\Gamma^{(m)}$ with $m \geq 0$. In a dual manner, we may apply the induction on m to show that F^+ extends to a full

k-functor $F : k\Gamma \to \mathcal{C}(\Gamma)$, which acts identically on the objects and has a kernel generated by the mesh relations. The proof of the theorem is completed.

1.5. Lemma. *Let Γ be a connected component of $\Gamma_{\mathcal{C}}$, containing a section Δ.*
(1) *If Δ has no left infinite path, then Δ^+ has no left infinite path.*
(2) *If Δ has no right infinite path, then Δ^- has no right infinite path.*

Proof. It suffices to prove Statement (1). Suppose that Δ^+ has a left infinite path

$$\cdots \longrightarrow \tau^{-n_i} X_i \longrightarrow \cdots \longrightarrow \tau^{-n_1} X_1 \longrightarrow \tau^{-n_0} X_0,$$

where $X_i \in \Delta$ and $n_i > 0$. Since Γ embeds in $\mathbb{Z}\Delta$; see [10, (2.3)], we see that $n_i \le n_{i-1}$ for all $i > 0$. As a consequence, there exists $r \ge 0$ such that $n_i = n_r$ for $i \ge r$. Thus $\cdots \longrightarrow X_i \longrightarrow \cdots \longrightarrow X_r$ is a left infinite path in Δ. The proof of the lemma is completed.

We shall say that a sink morphism in $\mathcal{C}$ is *proper* if it either is a monomorphism or fits in an almost slit sequence; and dually, a source morphism is *proper* if it either is an epimorphism or fits in an almost slit sequence. Observe that sink or source morphisms in an abelian category are all propre. The following result is a generalization of Lemma 3 stated in [13, (2.3)].

1.6. Theorem. *Let $\mathcal{C}$ be a Hom-finite Krull-Schmidt additive k-category. Let Γ be a connected component of $\Gamma_{\mathcal{C}}$, and let Δ be a section of Γ in which every object has a trivial automorphism field and admits a proper sink morphism as well as a proper source morphism. If Δ has no infinite path, then Γ is standard if and only if* $\mathrm{Hom}_{\mathcal{C}}(\Delta^+, \Delta^-) = 0$.

Proof. Suppose that Δ has no infinite path. By Lemma 1.5, Δ^+ has no left infinite path and Δ^- has no right infinite path. We shall need the following statement.

Sub-Lemma: *Let $M \in \Gamma$ with* $\mathrm{Hom}_{\mathcal{C}}(M, \Delta^-) = 0$, *and let $N \in \Delta$. If $\mathcal{C}$ has a non-zero radical morphism $f : M \to N$, then Γ has a non-trivial path $M \rightsquigarrow N$.*

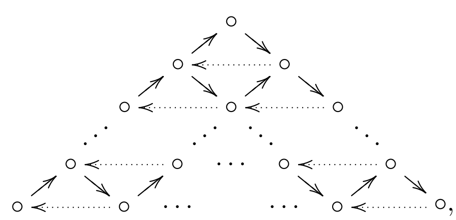
Indeed, suppose that Γ has no non-trivial path from M to N. By assumption, N admits a sink morphism $g = (g_1, \cdots, g_r) : N_1 \oplus \cdots \oplus N_r \to N$, where $N_i \in \Gamma$. If $f : M \to N$ is non-zero and radical, then $f = \sum_{i=1}^r g_i f_i$, with $f_i : M \to N_i$ in $\mathcal{C}$. We may assume that f_1 is non-zero. Since Δ is a section, $N_1 \in \Delta \cup \tau\Delta$; see [10, (2.2)]. Since $\mathrm{Hom}_{\mathcal{C}}(M, \Delta^-) = 0$, we have $N_1 \in \Delta$. Since Γ has no path from M to N_1, we see that f_1 is radical. Repeating this process, we see that Δ contains an infinite path ending in N, a contradiction. This proves the sub-lemma.

Now, assume that $\mathrm{Hom}_{\mathcal{C}}(\Delta^+, \Delta^-) = 0$. We deduce from the above sub-lemma that $\mathrm{Hom}_{\mathcal{C}}(\Delta^+, \Delta) = 0$. Using the dual statement, we obtain $\mathrm{Hom}_{\mathcal{C}}(\Delta, \Delta^-) = 0$. It remains to construct a k-linear equivalence $F : k\Delta \to \mathcal{C}(\Delta)$. Since the objects in Δ have a trivial automorphism field, so do the objects in Γ. Set $F(X) = X$ and $F(\varepsilon_X) = \mathbb{1}_X$ for $X \in \Delta$. Let $X, Y \in \Delta$ with $d = d_{XY} > 0$. If $\alpha_i : X \to Y$, $i = 1, \ldots, d$, are the arrows from X to Y, then we choose irreducible morphisms $f_{\alpha_i} : X \to Y$ such that $f_{\alpha_1} + \mathrm{rad}^2(X, Y), \ldots, f_{\alpha_r} + \mathrm{rad}^2(X, Y)$ form a k-basis of $\mathrm{irr}(X, Y)$, and set $F(\alpha_i) = f_{\alpha_i}$, $i = 1, \ldots, d$. In an evident manner, we obtain a k-linear functor $F : k\Delta \to \mathcal{C}(\Delta)$.

We claim that F induces a k-isomorphism $F_{XY} : \mathrm{Hom}_{k\Delta}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, Y)$, for any $X, Y \in \Delta$. Since every object in Δ admits a sink morphism and a source morphism, Δ is locally finite. Having no infinite path, by König's Lemma, Δ has at most finitely many paths from X to Y. Define an integer n_{XY} in such a way that $n_{XY} = -1$ if Δ has no path from X to Y; and otherwise, n_{XY} is the maximal length of the paths from X to Y. If $n_{XY} = -1$, then the claim follows easily from the above statement. If $n_{XY} = 0$, then $\mathrm{Hom}_{k\Delta}(X, Y) = k\varepsilon_Y$. On the other hand, $\mathrm{Hom}_{\mathcal{C}}(X, Y) = k\mathbf{1}_Y$ by the above sub-lemma, and the claim follows.

Suppose that $n_{XY} > 0$. Let $\beta_i : Z_i \to Y$, $i = 1, \ldots, s$, be the arrows in Δ ending in Y. Then $n_{XZ_i} < n_{XY}$, and $(f_{\beta_1}, \cdots, f_{\beta_s}) : Z_1 \oplus \cdots \oplus Z_s \to Y$ is irreducible; see [2, (3.4)]. Since Y admits a proper sink morphism, there exists a morphism $u : U \to Y$ such that $v = (f_{\beta_1}, \cdots, f_{\beta_s}, u) : Z_1 \oplus \cdots \oplus Z_s \oplus U \to Y$ is a proper sink morphism. Let $h : X \to Y$ be a morphism in $\mathcal{C}$. Being radical, h factors through v. Since Δ is a section, every indecomposable summand of U lies in $\tau\Delta$; and since $\mathrm{Hom}_{\mathcal{C}}(X, \Delta^-) = 0$, we have $h = f_{\beta_1}h_1 + \cdots + f_{\beta_s}h_s$, with morphisms $h_i : X \to Z_i$ in $\mathcal{C}$. For each $1 \le i \le s$, by the induction hypothesis, h_i is a sum of composites of the chosen irreducible morphisms. Therefore, h is a sum of composites of the chosen irreducible morphisms. Hence, F_{XY} is surjective. Next, let $\rho : X \to Y$ be in $k\Delta$ such that $F(\rho) = 0$. Then $\rho = \beta_1\rho_1 + \cdots + \beta_s\rho_s$, where the $\rho_i : X \to Z_i$ are in $k\Delta$. Set $w = (F(\rho_1), \cdots, F(\rho_s))^T : X \to Z_1 \oplus \cdots \oplus Z_s$. Then $(f_{\beta_1}, \cdots, f_{\beta_s})w = F(\rho) = 0$. If $\mathcal{C}$ has an almost slit sequence ending in Y then, by Lemma 1.1(2), w factors through τY; and since $\mathrm{Hom}_{\mathcal{C}}(X, \Delta^-) = 0$, we have $w = 0$. Otherwise, v is a monomorphism, and hence, $w = 0$. That is, in any case, $F(\rho_i) = 0$, and by the inductive hypothesis, $\rho_i = 0$, $i = 1, \ldots, s$. As a consequence, $\rho = 0$. Thus, F_{XY} is injective. This implies that F is an equivalence. By Theorem 1.4, Γ is standard. This establishes the sufficiency, and the necessity is evident. The proof of the theorem is completed.

Let Σ be a convex subquiver of $\Gamma_{\mathcal{C}}$. We shall say that Σ is *schurian* if, for any objects X, Y in Σ, the k-space $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is of dimension at most one; and it vanishes whenever Y is a not successor of X in Σ. Moreover, we call Σ a *wing of rank n* if it is trivially valued of the following shape:



where the dotted arrows indicate the action of τ, the objects are pairwise distinct and the number of τ-orbits is n; see [13, (3.3)]. In this case, the object on the top is called the *wing vertex* and the objects at the bottom are said to be *quasi-simple*.

1.7. Lemma. *Let $\mathcal{W}$ be a wing of $\Gamma_{\mathcal{C}}$. If the quasi-simple objects in $\mathcal{W}$ are pairwise orthogonal bricks, then $\mathcal{W}$ is schurian.*

Proof. Assume that the quasi-simple objects in $\mathcal{W}$ are pairwise orthogonal bricks. Let n be the rank of $\mathcal{W}$. If $n = 1$, then the lemma holds trivially. Suppose that $n > 1$ and the lemma holds for wings of rank $n - 1$. Write the objects in $\mathcal{W}$ as X_{ij} with $1 \le j \le n$ and $j \le i \le n$ so that X_{11} is the wing vertex and the X_{nj} with $1 \le j \le n$ are the quasi-simple objects. Observe that X_{21} is the wing vertex of a schurian wing $\mathcal{W}_1$, while X_{22} is the wing vertex of a schurian wing $\mathcal{W}_2$. It is evident that we may choose irreducible morphisms $f_{ij} : X_{ij} \to X_{i+1,j}$ for $j \le i < n$

and $1 \le j < n$; and irreducible morphisms $g_{pq} : X_{pq} \to X_{p-1,q-1}$ for $q \le p \le n$ and $2 \le q \le n$ such that

$$\mathcal{E}(X_{nj}) : \quad X_{n,j+1} \xrightarrow{g_{n,j+1}} X_{n-1,j} \xrightarrow{f_{n-1,j}} X_{nj}$$

is an almost split sequence, for $j = 1, \ldots, n-1$; and

$$\mathcal{E}(X_{ij}) : \qquad X_{i,j+1} \xrightarrow{(g_{i,j+1},\, f_{i,j+1})} X_{i-1,j} \oplus X_{i+1,j+1} \xrightarrow{\begin{pmatrix} f_{i-1,j} \\ g_{i+1,j+1} \end{pmatrix}} X_{ij}$$

is an almost split sequence for $1 \le j < n$ and $j < i < n$. Next, we shall divide the proof into several sub-lemmas.

(1) $\mathrm{Hom}_{\mathcal{C}}(X_{n1}, X_{ii}) = 0$ *and* $\mathrm{Hom}_{\mathcal{C}}(X_{i1}, X_{nn}) = 0$, *for* $1 \le i \le n$. Suppose that $\mathcal{C}$ has a non-zero morphism $f : X_{n1} \to X_{rr}$ for some $1 \le r \le n$. Assume that r is maximal. Since X_{n1}, X_{nn} are orthogonal, we have $r < n$. Since $\mathcal{W}_1$ is schurian, $f_{rr}f = 0$. Applying Lemma 1.1(2) to the almost split sequence $\mathcal{E}(X_{r+1,r})$, we see that f factors through $g_{r+1,r+1} : X_{r+1,r+1} \to X_{rr}$, which contradicts the maximality of r. The first part of the statement is established. In a dual manner, we may prove the second part.

(2) $\mathrm{Hom}_{\mathcal{C}}(X_{i1}, \mathcal{W}_2) = 0$ *and* $\mathrm{Hom}_{\mathcal{C}}(\mathcal{W}_1, X_{ii}) = 0$, *for* $1 \le i \le n$. Suppose that $f : X_{s1} \to X$ is a non-zero morphism with $1 \le s \le n$ and $X \in \mathcal{W}_2$, which is necessarily radical. If $X \ne X_{jj}$ for any $2 \le j \le n$, then X admits a sink morphism whose domain is a direct sum of one or two objects in $\mathcal{W}_2$. Factorizing f through this sink morphism, we obtain a non-zero morphism $g : X_{s1} \to X_{ii}$ with $2 \le i \le n$. Assume that s is maximal for this property. By Statement (1), $s < n$. Since $\mathcal{W}_2$ is schurian, $gg_{s+1,2} = 0$. Applying Lemma 1.1(3) to $\mathcal{E}(X_{s+1,1})$, we see that g factors through $f_{s1} : X_{s1} \to X_{s+1,1}$, which contradicts the maximality of s. The first part of the statement is established. In a dual fashion, we may establish the second part.

(3) $\mathrm{Hom}_{\mathcal{C}}(X_{nn}, \mathcal{W}_1) = 0$ *and* $\mathrm{Hom}_{\mathcal{C}}(\mathcal{W}_2, X_{n1}) = 0$. Suppose that $\mathcal{C}$ has a non-zero morphism $f : X_{nn} \to X_{pq}$ with $2 \le p \le n$ and $1 \le q < p$. We may assume that p is maximal for this property. Since the quasi-simple objects are orthogonal, $p < n$. By the maximality of p, we have $f_{pq}f = 0$. Applying Lemma 1.1(2) to $\mathcal{E}(X_{p+1,q})$, we see that f factors through $g_{p+1,q+1}$, contrary to the maximality of p. The first part of the statement is established, and the second part follows dually.

(4) *If* $f : X_{ii} \to X_{11}$ *with* $1 \le i < n$ *is such that* $fg_{i+1,i+1} \cdots g_{nn} = 0$, *then* $f = 0$. *Dually, if* $g : X_{11} \to X_{i1}$ *is a morphism with* $1 \le i < n$ *such that* $f_{n-1,1} \cdots f_{i1}g = 0$, *then* $g = 0$. Suppose that $fg_{i+1,i+1} \cdots g_{nn} = 0$ but $f \ne 0$. Let r with $i+1 \le r \le n$ be minimal such that $fg_{i+1,i+1} \cdots g_{rr} = 0$. Write $fg_{i+1,i+1} \cdots g_{rr} = gg_{rr}$, where $g : X_{r-1,r-1} \to X_{11}$ is a non-zero morphism. Applying Lemma 1.1(3) to $\mathcal{E}(X_{r,r-1})$, we see that g factors through $f_{r-1,r-1}$, which contradicts Statement (2). This establishes the first par of the statement.

(5) $\mathrm{Hom}_{\mathcal{C}}(X_{ii}, X_{11})$ *and* $\mathrm{Hom}_{\mathcal{C}}(X_{11}, X_{i1})$ *are one-dimensional, for* $1 \le i \le n$. It suffices to prove the first part of the statement, since the second part follows dually. Let $f : X_{nn} \to X_{11}$ be a morphism. By Statement (3), $f_{11}f = 0$. Applying Lemma 1.1(2) to $\mathcal{E}(X_{21})$, we obtain some $f_1 : X_{nn} \to X_{22}$ such that $f = g_{22}f_1$. Since $f_{22}f_1 = 0$ by Statement (3), we may repeat this process to obtain a morphism $f_{n-1} : X_{nn} \to X_{nn}$ such that $f = g_{22} \cdots g_{nn}f_{n-1}$. Since $\mathcal{W}_1$ is schurian, $f_{n-1} = \lambda \mathbf{1}$ for some $\lambda \in k$, and hence, $f = \lambda g_{22} \cdots g_{nn}$. Since $g_{22} \cdots g_{nn} \ne 0$, we see that $\{g_{22} \cdots g_{nn}\}$ is a k-basis for $\mathrm{Hom}_{\mathcal{C}}(X_{nn}, X_{11})$. Write $g_{11} = \mathbf{1}_{X_{11}}$. If $g : X_{ii} \to X_{11}$ is a morphism with $1 \le i < n$, then $gg_{i+1,i+1} \cdots g_{nn} = \mu g_{22} \cdots g_{nn} = \mu g_{11} \cdots g_{nn}$,

for some $\mu \in k$. This yields that $(g - \mu g_{11} \cdots g_{ii}) g_{i+1,i+1} \cdots g_{nn} = 0$. By the first part of Statement (4), $g = \mu g_{11} \cdots g_{ii}$. Being non-zero, $g_{11} \cdots g_{ii}$ forms a k-basis for $\mathrm{Hom}_{\mathcal{C}}(X_{ii}, X_{11})$.

Now, suppose that $\mathrm{Hom}_{\mathcal{C}}(X, Y) \neq 0$ for some $X, Y \in \mathcal{W}$. We claim that Y is a successor of X and $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is one-dimensional. If $X \in \mathcal{W}_1$, then $Y \in \mathcal{W}_1$ by Statement (2). Since $\mathcal{W}_1$ is schurian, our claim follows. Otherwise, $X = X_{ss}$ for some $1 \leq s \leq n$. If $s = n$ then, by Statement (3), $Y = X_{ii}$ for some $1 \leq i \leq n$. Combining Statement (5) and the fact that $\mathcal{W}_2$ is schurian, we see that $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is one-dimensional. If $s = 1$, then $Y = X_{j1}$ for some $1 \leq j \leq n$, and hence, $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is one-dimensional by Statement (5). Finally, suppose that $1 < s < n$. If $Y \in \mathcal{W}_2$, since $\mathcal{W}_2$ is schurian, our claim follows. Otherwise, by Statement (3), $Y = X_{t1}$ for some $1 \leq t < n$. If $t = 1$, then $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is one-dimensional by Statement (5). It remains to consider the case where $1 < t < n$. Let $f : X_{ss} \to X_{t1}$ be a non-zero morphism with $1 < s, t < n$. Factorizing f along the $\mathcal{E}(X_{j1})$ with $2 \leq j \leq t$, we get $g : X_{ss} \to X_{t+1,2}$ and $h : X_{ss} \to X_{11}$ such that $f = g_{t+1,2}\, g + f_{t-1,1} \cdots f_{11} h$. By Statement (5), $h = \lambda g_{22} \cdots g_{ss}$ with $\lambda \in k$. This yields $f = g_{t+1,2} u$, where $u : X_{ss} \to X_{t+1,2}$ is a non-zero morphism. Since $\mathcal{W}_2$ is schurian, $X_{t+1,2}$ is a successor of X_{ss} and $\mathrm{Hom}_{\mathcal{C}}(X_{ss}, X_{t+1,2})$ has a k-basis $\{v\}$. Therefore, $f = \mu g_{t+1,2} v$ with $\mu \in k$. This shows that $\{g_{t+1,2} v\}$ is a k-basis for $\mathrm{Hom}_{\mathcal{C}}(X_{ss}, X_{t1})$. This establishes our claim. The proof of the lemma is completed.

Let $\mathbb{A}_\infty^+$ and $\mathbb{A}_\infty^-$ denote the linearly oriented quivers of type $\mathbb{A}_\infty$ having a unique source and having a unique sink, respectively. If Γ is a connected component of $\Gamma_{\mathcal{C}}$ of shape $\mathbb{Z}\mathbb{A}_\infty$, $\mathbb{N}\mathbb{A}_\infty^+$ or $\mathbb{N}^-\mathbb{A}_\infty^-$, then the objects in Γ having at most one immediate predecessor and at most one immediate successor are called *quasi-simple.*

1.8. Theorem. *Let $\mathcal{C}$ be a Hom-finite Krull-Schmidt additive k-category with Γ a connected component of $\Gamma_{\mathcal{C}}$. If Γ is a wing or of shape $\mathbb{Z}\mathbb{A}_\infty$, $\mathbb{N}\mathbb{A}_\infty^+$ or $\mathbb{N}^-\mathbb{A}_\infty^-$, then it is standard if and only if its quasi-simple objects are pairwise orthogonal bricks.*

Proof. We shall need only to prove the sufficiency. Let Γ be a wing or of shape $\mathbb{Z}\mathbb{A}_\infty$, $\mathbb{N}\mathbb{A}_\infty^+$ or $\mathbb{N}^-\mathbb{A}_\infty^-$ with the quasi-simple objects being pairwise orthogonal bricks. Then any two objects in Γ lie in a wing whose quasi-simples are pairwise orthogonal bricks. By Lemma 1.7, Γ is schurian. Choose a section Δ of Γ so that Δ is the right-most section if Γ is a wing or of shape $\mathbb{N}^-\mathbb{A}_\infty^-$; and Δ is the left-most section if Γ is of shape $\mathbb{N}\mathbb{A}_\infty^+$; and Δ is any section with an alternating orientation if Γ is of shape $\mathbb{Z}\mathbb{A}_\infty$. Then Δ^- has no right infinite path and Δ^+ has no left infinite path such that $\mathrm{Hom}_{\mathcal{C}}(\Delta^+, \Delta \cup \Delta^-) = 0$ and $\mathrm{Hom}_{\mathcal{C}}(\Delta, \Delta^-) = 0$.

For each arrow $\alpha : X \to Y$ in Δ, we choose an irreducible morphism $f_\alpha : X \to Y$ in $\mathcal{C}$. Since every path in Δ is sectional, the composite of any chain of the chosen irreducible morphisms is non-zero; see [11, (2.7)]. Therefore, for any $M, N \in \Delta$, $\mathrm{Hom}_{\mathcal{C}}(M, N)$ is one-dimensional if and only if N is a successor of M in Δ, and in this case, the composite of the chain of the chosen irreducible morphisms corresponding to the path from M to N forms a k-basis for $\mathrm{Hom}_{\mathcal{C}}(X, Y)$. It is now easy to see that $k\Delta \cong \mathcal{C}(\Delta)$. By Theorem 1.4, Γ is standard. This establishes the sufficiency, and the necessity is trivial. The proof of the theorem is completed.

2. Specialization to representation categories of quivers

Throughout this section, we fix a connected quiver $Q = (Q_0, Q_1)$, where Q_0 is the set of vertices and Q_1 is the set of arrows, which is assumed to be strongly locally finite, that is, Q is locally finite such that the number of paths between any two given vertices is finite. A *k-representation* M of Q consists of a family of k-spaces $M(x)$ with $x \in Q_0$, and a family of k-maps $M(\alpha) : M(x) \to M(y)$ with $\alpha : x \to y \in Q_1$. For such a representation M, one defines its *support* $\operatorname{supp} M$ to be the full subquiver of Q generated by the vertices x for which $M(x) \neq 0$, and one calls M *locally finite dimensional* if $\dim_k M(x)$ is finite for all $x \in Q_0$, and *finite dimensional* if $\Sigma_{x\in Q_0} \dim_k M(x)$ is finite. The locally finite dimensional k-representations of Q form a hereditary abelian k-category $\operatorname{rep}(Q)$. The subcategory of $\operatorname{rep}(Q)$ of finite dimensional representations is written as $\operatorname{rep}^b(Q)$. For each $x \in Q_0$, one constructs an indecomposable projective representation P_x and an indecomposable injective representation I_x; see [3, Section 1]. Since Q is strongly locally finite, P_x and I_x lie in $\operatorname{rep}(Q)$. One says that $M \in \operatorname{rep}(Q)$ is *finitely presented* if M has a minimal projective presentation $P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$, where P_1, P_0 are finite direct sums of some P_x with $x \in Q_0$; and *finitely co-presented* if M has a minimal injective co-presentation $0 \longrightarrow M \longrightarrow I_0 \longrightarrow I_1$, where I_0, I_1 are finite direct sums of some I_x with $x \in Q_0$. Let $\operatorname{rep}^+(Q)$ and $\operatorname{rep}^-(Q)$ be the full subcategories of $\operatorname{rep}(Q)$ of finitely presented representations and of finitely co-presented representations, respectively. Then $\operatorname{rep}^b(Q)$ is the intersection of $\operatorname{rep}^+(Q)$ and $\operatorname{rep}^-(Q)$. In particular, $I_x \in \operatorname{rep}^+(Q)$ if and only if $I_x \in \operatorname{rep}^b(Q)$. One denotes by Q^+ the full subquiver of Q generated by the vertices x for which $I_x \in \operatorname{rep}^b(Q)$.

It is known that $\operatorname{rep}^+(Q)$ and $\operatorname{rep}^-(Q)$ are hereditary, abelian and Hom-finite; see [3, (1.15)]. In particular, they are Krull-Schmidt. The shapes of the their Auslander-Reiten components have been well described. Indeed, the Auslander-Reiten quiver $\Gamma_{\operatorname{rep}^+(Q)}$ of $\operatorname{rep}^+(Q)$ has a unique *preprojective component*, which has a left-most section generated by the P_x with $x \in Q_0$; see [3, (4.6)] and [13, (2.4)]. The connected components of $\Gamma_{\operatorname{rep}^+(Q)}$ containing some of the I_x with $x \in Q^+$ are called *preinjective*, which correspond bijectively to the connected components of the quiver Q^+. Note that every preinjective component has a right-most section generated by its injective representations I_x; see [13, (2.4)] and [3, (4.7)]. The other connected components of $\Gamma_{\operatorname{rep}^+(Q)}$ are called *regular*, which are wings, stable tubes or of shapes $\mathbb{Z}\mathbb{A}_\infty, \mathbb{N}\mathbb{A}_\infty^+$ and $\mathbb{N}^-\mathbb{A}_\infty^-$; see [3, (4.14)], [12] and [13].

The following easy fact is well known in the finite case.

2.1. Lemma. *Let X and Y be representations lying in $\Gamma_{\operatorname{rep}^+(Q)}$. If τX and τY are defined in $\Gamma_{\operatorname{rep}^+(Q)}$, then* $\operatorname{Hom}_{\operatorname{rep}^+(Q)}(X, Y) \cong \operatorname{Hom}_{\operatorname{rep}^+(Q)}(\tau X, \tau Y)$.

Proof. Assume that τX and τY are defined in $\Gamma_{\operatorname{rep}^+(Q)}$. In view of the proof stated in [3, (2.8)], we have $\operatorname{Hom}(\tau X, \tau Y) \cong D\operatorname{Ext}^1(Y, \tau X)$. Dually, since τX is not injective and finite-dimensional; see [3, (3.6)], $\operatorname{Hom}(X, Y) \cong D\operatorname{Ext}^1(Y, \tau X)$. The proof of the lemma is completed.

Recall that Q is of *infinite Dynkin type* if its underlying graph is $\mathbb{A}_\infty$, $\mathbb{A}_\infty^\infty$ or $\mathbb{D}_\infty$. In this case, a reduced walk is a *string* if it contains at most finitely many, but at least one, sinks or sources. To each string w, one associates a *string representation* M_w defined as follows: for $x \in Q_0$, one sets $M_w(x) = k$ if x appears in w, and otherwise, $M_w(x) = 0$; and for $\alpha \in Q_1$, one sets $M_w(\alpha) = \mathbf{1}$ if α arrears in w, and otherwise, $M_w(\alpha) = 0$; see [3, Section 5]. It is easy to see that every string representation has a trivial endomorphism algebra.

2.2. Theorem. *Let Q be a connected quiver which is strongly locally finite.*

(1) *The preprojective component and the preinjective components of $\Gamma_{\mathrm{rep}^+(Q)}$ are standard.*

(2) *If Q is of finite or infinite Dynkin type, then every connected component of $\Gamma_{\mathrm{rep}^+(Q)}$ is standard.*

(3) *If Q is infinite but not of infinite Dynkin type, then $\Gamma_{\mathrm{rep}^+(Q)}$ has infinitely many non-standard regular components.*

Proof. (1) The preprojective component $\mathcal{P}_Q$ of $\Gamma_{\mathrm{rep}^+(Q)}$ has a left-most section Δ which is generated by the P_x with $x \in Q_0$ and isomorphic to Q^{op}; see [3, (4.6)] and [13, (2.4)]. Hence, $\Delta^- = \emptyset$. Moreover, Δ^+ has no left infinite path; see [3, (4.8)] and (1.5). If $f : X \to Y$ is a non-zero morphism with $X \in \mathcal{P}_Q$ and $Y \in \Delta$, then X is a predecessor of Y in $\mathcal{P}_Q$; see [3, (4.9)], and hence, $X \in \Delta$. Therefore, $\mathrm{Hom}_{\mathrm{rep}^+(Q)}(\Delta^+, \Delta) = 0$. Let $\mathcal{P}(Q)$ be the full subcategory of $\mathrm{rep}^+(Q)$ generated by the P_x with $x \in Q_0$. For each arrow $\alpha : y \to x$ in Q, denote by $P_\alpha : P_x \to P_y$ the morphism given by the right multiplication by α. It is easy to see that

$$F : kQ^{\mathrm{op}} \to \mathcal{P}(Q) : x \mapsto P_x;\ \alpha^{\mathrm{o}} \mapsto P_\alpha$$

is a faithful k-functor, which is also full by Proposition 1.3 stated in [3]. Thus, Δ is standard. By Theorem 1.4, $\mathcal{P}_Q$ is standard. Dually, the preinjective component $\mathcal{I}_Q$ of $\Gamma_{\mathrm{rep}^-(Q)}$ is standard. By the dual of Lemma 4.5(1) stated in [3], $\mathcal{I}_Q$ has a left-most section Θ generated by its infinite-dimensional representations. Now the preinjective components of $\Gamma_{\mathrm{rep}^+(Q)}$ are the connected components of the complement of Θ in $\mathcal{I}_Q$. Hence, each preinjective component of $\Gamma_{\mathrm{rep}^+(Q)}$ is a convex translation subquiver of $\mathcal{I}_Q$, and in particular, it is standard.

(2) Suppose that Q is of infinite Dynkin type. Let Γ be a regular component of $\Gamma_{\mathrm{rep}^+(Q)}$. Then Γ is a wing or of shape $\mathbb{Z}A_\infty$, $\mathbb{N}^-\mathbb{A}_\infty^-$ or $\mathbb{N}\mathbb{A}_\infty^+$; see [3, (4.14)]. Moreover, Q is of type $\mathbb{A}_\infty^\infty$ or $\mathbb{D}_\infty$; see [3, (5.16)]. Assume first that Q is of type $\mathbb{A}_\infty^\infty$. By Proposition 5.9 stated in [3], the representations in $\Gamma_{\mathrm{rep}^+(Q)}$ are all string representations, and hence, they are all bricks. Moreover, the quasi-simple representations in Γ have pairwise disjoint supports; see [3, (5.15)]. In particular, they are pairwise orthogonal. By Theorem 1.8, Γ is standard.

Assume next that Q is of type $\mathbb{D}_\infty$. Then Γ is of shape $\mathbb{Z}\mathbb{A}_\infty$, $\mathbb{N}\mathbb{A}_\infty^+$ or $\mathbb{N}^-\mathbb{A}_\infty^-$; see [3, (5.22)]. In particular, τ or τ^- is defined everywhere in Γ. We shall consider only the first case, since the second case can treated in a dual manner. Let $a \in Q_0$ be one of the two vertices of weight one, which lies in the support of at most two quasi-simple representations; see [3, (5.20)]. Thus, there exists a quasi-simple representation $S \in \Gamma$ such that $(\tau^n S)(a) = 0$, for all $n \geq 0$.

Let M, N be quasi-simple representations in Γ. There exists $m \geq 0$ such that $\tau^m M = \tau^r S$ and $\tau^m N = \tau^s S$ with $r, s \geq 0$. We may assume that $r \geq s$. By Lemma 2.1, $\mathrm{Hom}(M, N) \cong \mathrm{Hom}(\tau^m M, \tau^m N) = \mathrm{Hom}(\tau^r S, \tau^s S)$. Since $(\tau^r S)(a) = 0$, we see that $\tau^r S$ is a string representation; see [3, (5.19)]. Thus, $\tau^r S$ is a brick. Taking $N = M$, we see that M is a brick. Suppose that $M \neq N$. Then $r > s$. Setting $t = r - s$, there exists a sectional path $S_t \longrightarrow S_{t-1} \longrightarrow \cdots \longrightarrow S_1 \longrightarrow \tau^s S$ in Γ. For $x \in Q_0$, we have $\dim S_t(x) = \sum_{i=s}^r \dim \tau^i S(x)$. Since $(\tau^i S)(a) = 0$ for $i \geq 0$, $\dim S_t(a) = 0$. Hence, S_t is a string representation; see [3, (2.19)]. If the supports of $\tau^r S$ and $\tau^s S$ have a common vertex b, then $\dim S_t(b) \geq \dim \tau^r S(b) + \dim \tau^s S(b) \geq 2$, contrary to S_t being a string representation. Thus, $\tau^r S$ and $\tau^s S$ have disjoint supports. In particular, they are orthogonal, and so are M and N. By Theorem 1.8, Γ is standard. In view of Statement (1), we have established Statement (2).

(3) Suppose that Q is infinite but not of infinite Dynkin type. Then Q has a finite subquiver Σ of Euclidean type. Then we can find a homogeneous tube $\mathcal{T}$ in $\Gamma_{\mathrm{rep}^b(\Sigma)}$; see, for example, [3, (6.3)]. Let M_i with $i \geq 1$ be the representations in $\mathcal{T}$ which are not quasi-simple. Regarded as representations of Q, the M_i are distributed in infinitely many regular components of $\Gamma_{\mathrm{rep}^+(Q)}$; see [3, (6.1), (6.2)]. These regular components are not standard, since the M_i have non-trivial endomorphism algebras. The proof of the theorem is completed.

Remark. (1) In view of Theorem 5.17 stated in [3], we see that wings and the translation quivers $\mathbb{Z}\mathbb{A}_\infty$, $\mathbb{N}\mathbb{A}_\infty^+$ and $\mathbb{N}^-\mathbb{A}_\infty^-$ all occur as standard Auslander-Reiten components of Krull-Schmidt categories.

(2) Let Q be finite of Euclidean type. If k is not algebraically closed, then some indecomposable k-representations of Q have a non-trivial automorphism field; see the proof in [3, (6.3)]. As a consequence, every connected component of $\Gamma_{\mathrm{rep}^b(Q)}$ is standard if and only if k is algebraically closed.

We conclude this section with an application to the bounded derived category $D^b(\mathrm{rep}^+(Q))$ of $\mathrm{rep}^+(Q)$. Since $\mathrm{rep}^+(Q)$ is hereditary, the vertices of $\Gamma_{D^b(\mathrm{rep}^+(Q))}$ can be chosen to be the shifts of those in $\Gamma_{\mathrm{rep}^+(Q)}$. If Q is not of finite Dynkin type, then the connected components of $\Gamma_{D^b(\mathrm{rep}^+(Q))}$ are the shifts of the regular components of $\Gamma_{\mathrm{rep}^+(Q)}$ and the shifts of the *connecting component*, which is obtained by gluing the preprojective component together with the shift by -1 of the preinjective components of $\Gamma_{\mathrm{rep}^+(Q)}$; see [5, (4.4)] and [3, (7.10)]. In case Q is of finite Dynkin type, $\Gamma_{D^b(\mathrm{rep}^+(Q))}$ is connected of shape $\mathbb{Z}Q^{\mathrm{op}}$, which is obtained by gluing, for each integer i, the shift by i of $\Gamma_{\mathrm{rep}^b(Q)}$ together with its shift by $i+1$; see [5, (4.5)]. In this case, we also call $\Gamma_{D^b(\mathrm{rep}^+(Q))}$ the *connecting component*.

2.3. Theorem. *If Q is a connected strongly locally finite quiver, then the connecting component of $\Gamma_{D^b(\mathrm{rep}^+(Q))}$ is standard; and every connected component is standard in case Q is of finite or infinite Dynkin type.*

Proof. Assume that Q is a connected strongly locally finite quiver and $\mathcal{C}_Q$ is the connecting component of $\Gamma_{D^b(\mathrm{rep}^+(Q))}$. Let Δ be the full subquiver of $\mathcal{C}_Q$ generated by the representations $P_x \in \Gamma_{\mathrm{rep}^+(Q)}$ with $x \in Q_0$, which is isomorphic to Q^{op}. It follows from Lemma 7.8 stated in [3] that Δ is a section of $\mathcal{C}_Q$. Since $\mathrm{rep}^+(Q)$ fully embeds in $D^b(\mathrm{rep}^+(Q))$, by Theorem 2.2, Δ is standard. Let $M, N \in \mathrm{rep}^+(Q)$. Since $\mathrm{rep}^+(Q)$ is hereditary, $\mathrm{Hom}_{D^b(\mathrm{rep}^+(Q))}(M[m], N[n]) = 0$ for $m > n$; see [7, (3.1)]. Combining this fact with the standardness of the preprojective component of $\Gamma_{\mathrm{rep}^+(Q)}$, we deduce easily that $\mathrm{Hom}_{D^b(\mathrm{rep}^+(Q))}(\Delta^+, \Delta) = 0$ and $\mathrm{Hom}_{D^b(\mathrm{rep}^+(Q))}(\Delta \cup \Delta^+, \Delta^-) = 0$.

If Q is not of finite Dynkin type, then Δ^+ coincides with the full subquiver of the preprojective component of $\Gamma_{\mathrm{rep}^+(Q)}$ generated by the non-projective representations, while Δ^- coincides with the shift by -1 of the preinjective components of $\Gamma_{\mathrm{rep}^+(Q)}$. Thus, Δ^+ contains no left infinite path and Δ^- contains no right infinite path by Lemma 4.8 stated in [3]. This is also the case if Q is of finite Dynkin type; see (1.5). Thus $\mathcal{C}_Q$ is standard by Theorem 1.4. This establishes the first part of the theorem. Combining this with Theorem 2.2(2), we obtain the second part. The proof of the theorem is completed.

Remark. Let Q have no infinite path. If Q is not of finite Dynkin type, then $\Gamma_{\mathrm{rep}^+(Q)}$ has a unique preinjective component of shape $\mathbb{N}Q^{\mathrm{op}}$ and its proprojective component is of shape $\mathbb{N}^-Q^{\mathrm{op}}$; see [3, (4.7)]. Thus, in any case, the connecting component of $\Gamma_{D^b(\mathrm{rep}^+(Q))}$ is standard of shape $\mathbb{Z}Q^{\mathrm{op}}$.

3. Specialization to module categories of algebras

Throughout this section, assume that k is algebraically closed. Let A stand for a finite-dimensional k-algebra and $\mathrm{mod}A$ for the category of finite-dimensional left A-modules. In this classical situation, we have the following easy criteria for an Auslander-Reiten component with sections to be standard.

3.1. Theorem. *Let A be a finite-dimensional algebra over an algebraically closed field, and let Γ be a connected component of $\Gamma_{\mathrm{mod}A}$. If Δ is a section of Γ, then Γ is standard if and only if* $\mathrm{Hom}_A(\Delta, \tau\Delta) = 0$ *if and only if* $\mathrm{Hom}_A(\tau^-\Delta, \Delta) = 0$.

Proof. Let Δ be a section of Γ. Note that every module in Δ admits a proper sink map and a proper source map. Moreover, since the base field is algebraically closed, every module in Δ has a trivial automorphism field.

Suppose that $\mathrm{Hom}_A(\Delta, \tau\Delta) = 0$. Then Δ is finite; see [14, (2.1)]. By Lemma 1.5, Δ^+ has no left infinite path and Δ^- has no right infinite path. Assume that $\mathrm{Hom}_A(X, Y) \neq 0$ for some $X \in \Delta^+$ and $Y \in \Delta^-$. Since every module in Δ^+ admits a sink epimorphism, we obtain an arrow $X_1 \to X$ in Γ such that $\mathrm{Hom}_A(X_1, Y) \neq 0$. Observe that $X_1 \in \Delta \cup \Delta^+$. If $X_1 \in \Delta^+$, then Γ has an arrow $X_2 \to X_1$ such that $\mathrm{Hom}_A(X_2, Y) \neq 0$. Since Δ^+ has no left infinite path, there exists a module M in Δ such that $\mathrm{Hom}_A(M, Y) \neq 0$. Similarly, since Δ^- has no right infinite path and every module in Δ^- has a source monomorphism, there exists a module N in $\tau\Delta$ such that $\mathrm{Hom}_A(M, N) \neq 0$, a contradiction. This shows that $\mathrm{Hom}_A(\Delta^+, \Delta^-) = 0$. By Theorem 1.6, Γ is standard. If $\mathrm{Hom}_A(\tau^-\Delta, \Delta) = 0$, one shows in a dual manner that Γ is standard. Conversely, it is evident that $\mathrm{Hom}_A(\Delta, \tau\Delta) = 0$ and $\mathrm{Hom}_A(\tau^-\Delta, \Delta) = 0$ if Γ is standard. The proof of the theorem is completed.

Let Γ be a connected component of $\Gamma_{\mathrm{mod}A}$. Recall that Γ is *generalized standard* if $\mathrm{rad}^\infty(\mathrm{mod}A)$ vanishes in Γ; see [14]. It is known that Γ is generalized standard if it is standard; see [9], and the converse holds true in case Γ has no projective module or no injective module; see [16]. Observing that the conditions on Δ stated in Theorem 3.1 are trivially verified in case Γ is generalized standard, we obtain the following consequence.

3.2. Corollary. *Let Γ be a connected component of $\Gamma_{\mathrm{mod}A}$. If Γ has a section, then it is standard if and only if it is generalized standard.*

The algebra A is called *tilted* if $A = \mathrm{End}_H(T)$, where H is a finite-dimensional hereditary algebra and T is a tilting H-module. In this case, $\mathrm{mod}A$ contains slices; see [6], and a connected component of $\Gamma_{\mathrm{mod}A}$ containing the indecomposable modules of a slice is called a *connecting component*. It is shown that a connecting component of a tilted algebra is standard; see [1, (5.7)].

3.3. Corollary. *If Γ is a connected component of $\Gamma_{\mathrm{mod}A}$, then Γ is standard with sections if and only if it is a connecting component of a tilted factor algebra of A.*

Proof. Let Γ be a connected component of $\Gamma_{\mathrm{mod}\,A}$. Suppose first that Γ is standard with a section Δ. In particular, we have $\mathrm{Hom}_A(\Delta, \tau\Delta) = 0$. If I is the intersection of the annihilators of the modules in Γ, then $B = A/I$ is a tilted algebra with Γ a connecting component of $\Gamma_{\mathrm{mod}\,B}$; see [8, (2.2)], and also [15].

Suppose next that there exists a tilted algebra $B = A/I$ with Γ being a connecting component of $\Gamma_{\mathrm{mod}\,B}$. Then Γ has a section Δ generated by the non-isomorphic indecomposable modules of a slice of $\mathrm{mod}B$. By the defining property of a slice, $\mathrm{Hom}_B(\Delta, \tau\Delta) = 0$. Thus, by Theorem 3.1, Γ is a standard component of $\Gamma_{\mathrm{mod}\,B}$. Since $\mathrm{mod}B$ fully embeds in $\mathrm{mod}A$, we see that Γ is a standard component of $\Gamma_{\mathrm{mod}\,A}$. The proof of the corollary is completed.

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