Artin Algebras with Loops
but no Outer Derivations

Ragnar-Olaf Buchweitz and Shiping Liu

1. The Result

If Λ is a finite dimensional algebra over an algebraically closed field k, then there exists a unique finite quiver Q_Λ, the *ordinary quiver* of Λ, such that Λ is Morita equivalent to a quotient of the path algebra kQ_Λ by an admissible ideal I; see, for example, [6] or [1].

The first Hochschild cohomology group of Λ over k is

$$\mathrm{HH}^1(\Lambda) = \mathrm{HH}^1(\Lambda / k) = \frac{\mathrm{Der}_k(\Lambda, \Lambda)}{\mathrm{Inn}(\Lambda)},$$

the k-vectorspace of *outer k-derivations*, that is, the quotient of all k-derivations of Λ modulo the inner ones.

It has been suspected for some time, and [4] seems to be the earliest reference, that vanishing of the first Hochschild cohomology precludes the existence of oriented cycles in the ordinary quiver. There are various supporting partial results, among them [8, (2.3), (3.2)], [2, (2.2)], [7, (1.3)].

An algebra Λ without oriented cycles in its ordinary quiver is of finite global dimension. In turn, finite global dimension implies that there are no loops in the ordinary quiver, see [10] or [9].

The following result shows that even the existence of loops in the ordinary quiver is no guarantee for the existence of non trivial outer derivations, thus refuting the above suspicion.

THEOREM 1.1. *Let Q be the quiver with three vertices $\mathbf{1}, \mathbf{2}, \mathbf{3}$ and three arrows $\alpha \colon \mathbf{2} \to \mathbf{1}$, $\beta \colon \mathbf{2} \to \mathbf{2}$, $\gamma \colon \mathbf{3} \to \mathbf{2}$. For any field k there exist finite dimensional k-algebras with ordinary quiver Q that admit only inner derivations. These algebras are necessarily of infinite global dimension and of infinite representation type.*

The proof will occupy the rest of this paper. We first describe the basic structure of the finite dimensional algebras Λ with ordinary quiver Q, then identify $\mathrm{HH}^1(\Lambda)$ as a space of first order differential operators in one variable, give in (4.3) a quick proof for fields whose prime field is large enough and present finally in (5.6) a family of examples that works over any field.

2. The Basic Structure

We begin by exhibiting the path algebra of the quiver Q from the Theorem as a matrix algebra and determine its admissible ideals. For the moment, k can be any field. Regarding the multiplication of arrows we follow the convention from [1]: With respect to the primitive idempotents e_i, corresponding to the vertices **i** for $i = 1, 2, 3$, the path algebra $kQ = k\langle e_1, e_2, e_3; \alpha, \beta, \gamma \rangle$ is defined by the relations

$$e_1\alpha = \alpha = \alpha e_2 \ , \quad e_2\beta = \beta = \beta e_2 \ , \quad e_3\gamma = \gamma = \gamma e_2 \ , \quad 0 = \gamma\alpha = \beta\alpha = \gamma\beta \ .$$

Throughout, $k[\beta]$ denotes the polynomial algebra in one variable β over k, and (β^n) denotes the ideal generated in $k[\beta]$ by β^n.

LEMMA 2.1. *The path algebra kQ can be realized as a triangular matrix algebra*

$$\varphi : kQ \xrightarrow{\cong} \begin{pmatrix} k & k[\beta] & k[\beta] \\ 0 & k[\beta] & k[\beta] \\ 0 & 0 & k \end{pmatrix}$$

where φ is uniquely determined through

$$\varphi(xe_1 + ye_2 + ze_3 + u\alpha + v\beta + w\gamma) = \begin{pmatrix} x & u & 0 \\ 0 & y + v\beta & w \\ 0 & 0 & z \end{pmatrix}$$

for $x, y, z, u, v, w \in k$. Under this isomorphism the powers of the path ideal (α, β, γ) are mapped to

$$\varphi\left((\alpha, \beta, \gamma)^n\right) = \begin{pmatrix} 0 & (\beta^{n-1}) & (\beta^{\max(0,n-2)}) \\ 0 & (\beta^n) & (\beta^{n-1}) \\ 0 & 0 & 0 \end{pmatrix} \quad for \quad n > 0 \, .$$

$\square$

An ideal I in kQ is *admissible* if $(\alpha, \beta, \gamma)^N \subseteq I \subseteq (\alpha, \beta, \gamma)^2$ for some $N \geq 2$. If k is algebraically closed, the quotients kQ/I of kQ by admissible ideals I are up to Morita equivalence precisely the finite dimensional k-algebras with that ordinary quiver. Hochschild (co-)homology is Morita invariant, see [11, (1.2.4), (1.5.6)], a fact established for outer derivations of matrix algebras already by P. Dirac [5] in 1925!

LEMMA 2.2. *The admissible ideals in kQ are precisely*

$$I(n; n', n''; V) = (\alpha\beta^{n'}, \beta^n, \beta^{n''}\gamma, \alpha V\gamma) \subset kQ$$

where $2 \leq n$; $1 \leq n', n'' \leq n$ and $V \subseteq k[\beta]$ is a k-subspace that contains (β^m) with $m = \min(n', n'')$. Such an ideal is mapped to

$$\varphi\left(I(n; n', n''; V)\right) = \begin{pmatrix} 0 & (\beta^{n'}) & V \\ 0 & (\beta^n) & (\beta^{n''}) \\ 0 & 0 & 0 \end{pmatrix} \, .$$

If $\Lambda = kQ/I(n; n'; n''; V)$ is the corresponding algebra, then its Cartan matrix *is*

$$(\dim_k e_i \, \Lambda \, e_j)_{1 \leq i,j \leq 3} = \begin{pmatrix} 1 & n' & m - d \\ 0 & n & n'' \\ 0 & 0 & 1 \end{pmatrix}$$

where $d = \dim_k V/(\beta^m)$.

$\square$

The vectorspace V is uniquely determined by its image $V/(\beta^m) \subseteq k[\beta]/(\beta^m)$, equivalently, by the corresponding point

$$\left(\frac{V}{(\beta^m)} \subseteq \frac{k[\beta]}{(\beta^m)} \right) \in \mathrm{Grass}_k(d, m)$$

in the Grassmanian of vector subspaces of dimension d of the m-dimensional k-vectorspace $k[\beta]/(\beta^m)$. This *continuous invariant* ranges hence over the k-rational points of an irreducible projective algebraic variety of dimension $d(m - d)$.

2.3. The determinant of the Cartan matrix being equal to $n \geq 2$, this matrix is not invertible over the integers, whence the global dimension of A is necessarily infinite. This fact is as well implied immediately by the more general affirmative solution to the *"no loops conjecture"* due to H. Lenzing [10] (see also K. Igusa [9]).

REMARK 2.4. The Hochschild homology of $\Lambda = kQ/I(n; n', n''; V)$ over k depends solely upon its Loewy length; as k-vectorspaces,

$$\mathrm{HH}_*(\Lambda) \cong \mathrm{HH}_*(k) \oplus \mathrm{HH}_* \left(k[\beta]/(\beta^n) \right) \oplus \mathrm{HH}_*(k).$$

This follows for example from applying Theorem 1.2.15 in [11] twice. In particular, $\mathrm{HH}_i(\Lambda) \cong \mathrm{HH}_i \left(k[\beta]/(\beta^n) \right)$ for $i > 0$ and $\mathrm{HH}_i \left(k[\beta]/(\beta^n) \right) \neq 0$ for any i as soon as $n \geq 2$; [11, Prop. 5.4.15] gives an explicit description of these groups.

REMARK 2.5. Note also that for any field extension k' of k and any finite dimensional k-algebra Λ one has a natural k'-linear isomorphism $\mathrm{HH}^*(\Lambda /k) \otimes_k k' \cong \mathrm{HH}^* \left((\Lambda \otimes_k k')/k' \right)$ whence vanishing can be tested over any field containing k.

3. The Derivations

Now we investigate derivations. First note that for a finite dimensional basic algebra Λ over k, modulo inner derivations every derivation vanishes on the primitive idempotents. A derivation that vanishes on these idempotents is *normalized*; it respects the two-sided Pierce decomposition of Λ. In particular, $\mathrm{HH}^1(\Lambda)$ is isomorphic to the vectorspace of normalized derivations modulo the inner derivations vanishing on the primitive idempotents, see [8, (3.1)]. Recall as well that $\mathrm{HH}^1(\Lambda)$ inherits the Lie k-algebra structure from the Lie algebra of all (normalized) derivations.

As the arrows together with the primitive idempotents generate the path algebra P of any quiver, a normalized derivation D of such algebra has a unique representation

$$D = \sum_{\alpha: \mathbf{i} \to \mathbf{j}} x_\alpha \frac{\partial}{\partial \alpha}$$

where $x_\alpha := D(\alpha)$ lies in the same two-sided Pierce component as the arrow α. If $J \subset P$ is an ideal, then D descends to a derivation of P/J if and only if $D(J) \subseteq J$.

To take care of the possibly finite characteristic of k, we introduce two secondary invariants of $\Lambda = kQ/I(n; n', n''; V)$:

$$\delta = \gcd(n, n', n'') \quad \text{and} \quad c = \dim_k(k/\delta k) = \begin{cases} 0 & \text{if } \delta \neq 0 \text{ in } k, \\ 1 & \text{if } \delta = 0 \text{ in } k. \end{cases}$$

LEMMA 3.1. *With notation as introduced above, one has:*

1. *The normalized k-derivations of the path algebra kQ are of the form*

$$D = \alpha a(\beta)\frac{\partial}{\partial\alpha} + b(\beta)\frac{\partial}{\partial\beta} + c(\beta)\gamma\frac{\partial}{\partial\gamma}$$

   with $a(\beta), b(\beta), c(\beta) \in k[\beta]$.
2. *The normalized derivation D of kQ descends to a derivation on* Λ *if and only if* $\delta b(0) = 0$ *and the first order differential operator*

$$\Theta_D = (a(\beta) + c(\beta)) + b(\beta)\frac{\partial}{\partial\beta}$$

   on $k[\beta]$ *maps V to itself.*
3. *The normalized derivation D induces an inner derivation on* Λ *if and only if*

$$a(\beta) + c(\beta) \equiv a(0) + c(0) \bmod \beta^{\min(n', n'')} \quad and \quad b(\beta) \equiv 0 \bmod \beta^n .$$

PROOF. In kQ, the twosided Pierce component of α is $e_1(kQ)e_2 = \alpha k[\beta]$, that of β is $e_2(kQ)e_2 = k[\beta]$ and that of γ is $e_2(kQ)e_3 = k[\beta]\gamma$, whence (1).

We now check what it means for D to map (each twosided Pierce component of) $I = I(n; n', n''; V)$ to itself: As $e_2Ie_2 = (\beta^n)$, we get

$$D(\beta^n) = nb(\beta)\beta^{n-1} \in (\beta^n) \quad \text{if and only if} \quad nb(0) = 0 .$$

Applying D to $\alpha\beta^{n'}$ yields

$$D(\alpha\beta^{n'}) = \alpha\left(a(\beta)\beta^{n'} + n'b(\beta)\beta^{n'-1}\right) ,$$

which is in $\alpha(\beta^{n'}) = e_1Ie_2$ if and only if $n'b(0) = 0$. Similarly applying D to $\beta^{n''}\gamma$ yields the condition $n''b(0) = 0$. Now $nb(0) = n'b(0) = n''b(0) = 0$ if and only if $\delta b(0) = 0$. Finally, for $\alpha v\gamma$ with $v \in V \subseteq k[\beta]$, one has

$$D(\alpha v\gamma) = \alpha\left(a(\beta)v + b(\beta)\frac{\partial v}{\partial\beta} + c(\beta)v\right)\gamma = \alpha\Theta_D(v)\gamma$$

and (2) follows.

If $a = a_{11}e_1 + a_{22}e_2 + a_{33}e_3 + \alpha a_{12} + a_{23}\gamma + \alpha a_{13}\gamma \in kQ$ with $a_{11}, a_{33} \in k$ and the remaining coefficients in $k[\beta]$, then the inner derivation $[a,\ ]$ vanishes on the primitive idempotents if and only if $\alpha a_{12} = a_{23}\gamma = \alpha a_{13}\gamma = 0$ in Λ. Calculating its value on the arrows yields

$$[a,\ ] = \alpha(a_{11} - a_{22})\frac{\partial}{\partial\alpha} + (a_{22} - a_{33})\gamma\frac{\partial}{\partial\gamma}$$

and (3) follows.   □

To summarize this detailed information succinctly, we interpret it in terms of differential operators of first order on $k[\beta]$. If p is a polynomial in β, its derivative with respect to β is denoted by p' as usual.

LEMMA 3.2. *Consider* $J_1 \subseteq J_2 \subseteq V \subseteq k[\beta]$ *where* $J_\nu = (\beta^{e_\nu}), \nu = 1, 2,$ *are proper ideals and V is some vector subspace in* $k[\beta]$.
If $e \in \mathbb{N}$ *divides* e_ν *for* $\nu = 1, 2,$ *then the k-vectorspace*

$$\mathcal{D}_{V,e} = \left\{\Theta = A(\beta) + B(\beta)\frac{\partial}{\partial\beta}\ \middle|\ A, B \in k[\beta], eB(0) = 0, \Theta(V) \subseteq V\right\}$$

is a Lie subalgebra of the Lie k-algebra of all first order differential operators on $k[\beta]$ that contains $J = (k \oplus J_2) \bigoplus J_1 \dfrac{\partial}{\partial \beta}$ as an ideal and $\mathcal{D}_{V,1}$ as a Lie subalgebra.

The quotient Lie k-algebra $\mathcal{D}(V, e; J_1, J_2) = \mathcal{D}_{V,e}/J$ is finite dimensional and contains an ideal isomorphic to $(J_2/J_1) \dfrac{\partial}{\partial \beta}$. Its dimension satisfies

$$0 \le e_1 - e_2 \le \dim_k \mathcal{D}(V, e; J_1, J_2) - \epsilon \le e_1 + e_2 - 2$$

where $\epsilon = \dim_k(k/ek)$.

PROOF. It suffices to recall that the Lie bracket of two first order differential operators on $k[\beta]$ is given by

$$\left[A_1(\beta) + B_1(\beta)\frac{\partial}{\partial \beta}, A_2(\beta) + B_2(\beta)\frac{\partial}{\partial \beta} \right] = \det \begin{pmatrix} B_1 & A_1' \\ B_2 & A_2' \end{pmatrix} + \det \begin{pmatrix} B_1 & B_1' \\ B_2 & B_2' \end{pmatrix} \frac{\partial}{\partial \beta}.$$

Now the ideal $J_\nu = (\beta^{e_\nu})$ is mapped to itself by the vectorfield $B(\beta)\dfrac{\partial}{\partial \beta}$ if and only if $e_\nu B(0) = 0$. The final assertion follows from the inclusions

$$(k \oplus J_2) \bigoplus J_1 \frac{\partial}{\partial \beta} \subseteq (k \oplus J_2) \bigoplus J_2 \frac{\partial}{\partial \beta} \subseteq \mathcal{D}_{V,1} \subseteq \mathcal{D}_{V,e}.$$

$\square$

For a finite dimensional algebra $\Lambda = kQ/I(n; n', n''; V)$ as before we have thus the following description of $\mathrm{HH}^1(\Lambda)$ where we set again $m = \min(n', n'')$, $\delta = \gcd(n, n', n'')$ and $c = \dim_k(k/\delta k)$.

COROLLARY 3.3. *Associating to a normalized derivation D on kQ the first order differential operator Θ_D on $k[\beta]$ as in (3.1.2) induces an isomorphism of Lie k-algebras*

$$\mathrm{HH}^1(\Lambda) \cong \mathcal{D}(V, \delta; \beta^n, \beta^m).$$

In particular, $\mathcal{D}'(\Lambda) = \mathcal{D}(V, 1; \beta^m, \beta^m)$ is a Lie algebra subquotient of $\mathrm{HH}^1(\Lambda)$ and

$$\dim_k \mathrm{HH}^1(\Lambda) = \dim \mathcal{D}'(\Lambda) + n - m + c \le (m-1) + (n-1) + c.$$

$\square$

EXAMPLE 3.4. *If V is an ideal in $k[\beta]$, so that $V = (\beta^{m-d})$, and if δ divides d, then the upper bound is achieved,*

$$\dim \mathrm{HH}^1(\Lambda) = (n-1) + (m-1) + c > 0.$$

Note that the hypothesis is trivially satisfied if $m = 1$, in which case $c = 0$ and $\dim \mathrm{HH}^1(\Lambda) = n - 1$.

3.5. To investigate the structure of $\mathcal{D}'(\Lambda) = \mathcal{D}(V, 1; \beta^m, \beta^m)$, choose polynomials $p_1, p_2, \ldots, p_d \in k[\beta]$ that generate the vectorspace V minimally modulo (β^m). As the classes of the polynomials p_i in $k[\beta]/(\beta^m)$ are k-linearly independent, $\mathcal{D}'(\Lambda)$ is isomorphic as vectorspace to the solutions of the system of d equations

(1) $$\begin{pmatrix} p_1 & p_1' \\ p_2 & p_2' \\ \vdots & \vdots \\ p_d & p_d' \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} \equiv \begin{pmatrix} a_{11} & \ldots & a_{1d} \\ a_{21} & \ldots & a_{2d} \\ \vdots & \ddots & \vdots \\ a_{d1} & \ldots & a_{dd} \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_d \end{pmatrix} \bmod \beta^m$$

with $a_{ij} \in k$ and $A, B \in k[\beta]$; $A(0) = B(0) = 0$; $\deg A, \deg B \leq m - 1$.

Comparing in each of these equations the coefficients of β^i for $i = 0, \ldots, m-1$ yields m linear equations over k; thus we obtain in total dm equations in $d^2 + 2(m-1)$ unknowns, whence the dimension of $\mathcal{D}'(\Lambda)$ satisfies

$$(2) \qquad \dim \mathcal{D}'(\Lambda) \geq d^2 + 2(m-1) - dm = (d-2)(d+2-m) + 2 \,.$$

Evaluating this lower bound yields immediately the following.

COROLLARY 3.6. *Whenever* $d \leq 2 \leq m$ *or* $d > 2$ *and* $d + \dfrac{2}{d-2} > m - 2$, *then* $\mathcal{D}'(\Lambda) \neq 0$ *and so* $\dim_k \operatorname{HH}^1(\Lambda) > n - m + c \geq 0$.

PROOF. If $d = 0$, then the right hand side of (2) evaluates to $2m - 2 \geq 2$, if $d = 1$ it evaluates to $m - 1 \geq 1$, if $d = 2$ it evaluates to 2. The second case simply rewrites $(d-2)(d+2-m) + 2 > 0$. $\qquad\square$

Thus to find examples with $\operatorname{HH}^1(\Lambda) = 0$, one needs $d \geq 3$ and $m = n \geq 7$. Note that in this case the algebra $\Lambda = kQ/I(n; n, n; V)$ is of infinite representation type; see for example the classification of the maximal algebras with 2 simple modules in [**3**]. So we get the following.

COROLLARY 3.7. *Let* Λ *be a finite dimensional* k-*algebra with ordinary quiver* Q *as in Theorem* (1.1). *If* Λ *is of finite representation type, then* $\operatorname{HH}^1(\Lambda)$ *does not vanish.*

4. Examples in Large Characteristic

If the field k has a large enough prime field, we can use "finite Fourier analysis" to find examples where $\mathcal{D}'(\Lambda) = 0$. For $y \in k$, let $e^{y\beta} \bmod \beta^m$ denote the image of

$$e^{y\beta} = \sum_{\nu} y^{\nu} \frac{\beta^{\nu}}{\nu!} \in k[[\beta]]$$

in $k[[\beta]]/(\beta^m) \cong k[\beta]/(\beta^m)$ for $m \in \mathbb{N}$. This class is well defined as soon as $(m-1)! \neq 0$ in k. Clearly, if defined, $e^{y\beta} \bmod \beta^m$ is a unit in $k[\beta]/(\beta^m)$ with inverse $e^{-y\beta} \bmod \beta^m$.

LEMMA 4.1. *If* $(m-1)! \neq 0$ *in* k *and* $Y \subseteq k$ *is a subset of cardinality* μ, *then the classes* $\{e^{y\beta} \bmod \beta^m\}_{y \in Y}$ *span a vectorspace of dimension* $d = \min(m, \mu)$ *in* $k[\beta]/(\beta^m)$.

PROOF. Vandermonde's determinant. $\qquad\square$

PROPOSITION 4.2. *Let* k *be a field and* d, m *integers with* $d \geq 3, m \geq 3d - 2$, *and* $(m-1)! \neq 0$ *in* k. *Assume* $Y \subseteq k$ *is a finite subset of cardinality* d *such that the set of differences*

$$Y - Y = \{y_1 - y_2 \mid y_1, y_2 \in Y\} \subseteq k$$

has maximal cardinality, equal to $1 + 2\binom{d}{2}$. *Let* V *be the* k-*vectorspace given by* $(\beta^m) \subseteq V \subseteq k[\beta]$ *and*

$$V/(\beta^m) = \sum_{y \in Y} k\left(e^{y\beta} \bmod \beta^m\right) \subseteq k[\beta]/(\beta^m) \,.$$

There is then no nonzero differential operator $\Theta = A(\beta) + B(\beta)\dfrac{\partial}{\partial\beta}$ with $A, B \in (\beta)$ and $\deg A, \deg B < m$ that transforms the k-vectorspace V into itself.

PROOF. The condition on (the characteristic of) k ensures that V is well defined. If $\Theta = A(\beta) + B(\beta)\dfrac{\partial}{\partial\beta}$ is a differential operator on $k[\beta]$ with $B(0) = 0$, then

$$(3) \qquad \Theta(e^{y\beta} \bmod \beta^m) = (A(\beta) + yB(\beta))\, e^{y\beta} \bmod \beta^m .$$

Now assume

$$(4) \qquad (A(\beta) + yB(\beta))\, e^{y\beta} \bmod \beta^m = \sum_{y'\in Y} a_{yy'} e^{y'\beta} \bmod \beta^m$$

for some matrix $(a_{yy'})_{y,y'\in Y}$ of elements from k. Multiplying by $e^{-y\beta} \bmod \beta^m$ and subtracting yields an equation

$$(5) \qquad (y_1 - y_2)\, B = \sum_{y'\in Y} a_{y_1 y'} e^{(y'-y_1)\beta} - \sum_{y'\in Y} a_{y_2 y'} e^{(y'-y_2)\beta} \bmod \beta^m$$

for each pair of elements $(y_1, y_2) \in Y \times Y$. If y_3 is a third element from Y, there result equations

$$(y_1 - y_2) \left(\sum_{y'\in Y} a_{y_1 y'} e^{(y'-y_1)\beta} - \sum_{y'\in Y} a_{y_3 y'} e^{(y'-y_3)\beta} \right) \bmod \beta^m$$

$$= (y_1 - y_3) \left(\sum_{y'\in Y} a_{y_1 y'} e^{(y'-y_1)\beta} - \sum_{y'\in Y} a_{y_2 y'} e^{(y'-y_2)\beta} \right) \bmod \beta^m$$

in $k[\beta]/(\beta^m)$. The (classes of) exponential functions involved are $1 = e^{0\beta}$ and $e^{y_i - y_j}$ for $y_i \in Y \setminus \{y_j\}$ with $j = 1, 2, 3$. By the assumption on the set of differences, if y_1, y_2, y_3 are pairwise distinct, the set

$$Z := \{0,\, y_i - y_j;\ y_i \in Y \setminus \{y_j\};\ \ j = 1, 2, 3\} \subseteq Y - Y \subseteq k$$

has cardinality equal to $1 + 3(d-1) = 3d - 2$, and as $m \geq 3d - 2$, the corresponding classes $\{e^{z\beta} \bmod \beta^m\}_{z\in Z}$ are linearly independent in $k[\beta]/(\beta^m)$ by Lemma (4.1). It follows in particular that $a_{y_i y_2} = 0$ for $y_i \neq y_2$. As the choice of y_2 was arbitrary and as there are at least 3 distinct elements in Y, it follows that $a_{yy'} = 0$ whenever $y \neq y'$. In turn, the system (4) evaluates now at $\beta = 0$ to

$$A(0) + yB(0) = \sum_{y'\in Y} a_{yy'} = a_{yy}, \quad y \in Y,$$

and so $A(0) = B(0) = 0$ implies that the diagonal elements a_{yy} vanish as well. Finally, equation (5) shows that $B = 0$ and any equation in (4) yields $A = 0$. $\square$

COROLLARY 4.3. *Let $I \subset kQ$ be an admissible ideal with discrete invariants (n, n', n'', d) and continuous invariant $(V/(\beta^m)) \in \mathrm{Grass}_k(d, m)$. If $m! \neq 0$ in k for $m = \min(n', n'')$, and if $m \geq 3d - 2 \geq 7$, then there exists a Zariski open and dense subset $\mathfrak{U} \subset \mathrm{Grass}_k(d, m)$ such that $\Lambda = kQ/I$ satisfies*

$$\dim_k \mathrm{HH}^1(\Lambda) = n - m$$

whenever $(V/(\beta^m)) \in \mathfrak{U}$.

PROOF. The conditions guarantee that d, m satisfy the hypotheses of Proposition (4.2) and that $\delta = \gcd(n, n', n'') \leq m$ is a unit in k, whence $c = 0$. As $\dim_k \mathrm{HH}^1(\Lambda)$ is upper semicontinuous on $\mathrm{Grass}_k(d, m)$ it remains to show that the minimal value is taken on generically, for example over the purely transcendental field extension $k' = k(y_1, \ldots, y_d)$ of k. There Proposition (4.2) applies. $\square$

EXAMPLE 4.4. The simplest case occurs for $d = 3, m = 7$. With $Y = \{0, 1, 3\}$, the set of differences $Y - Y = \{-3, -2, -1, 0, 1, 2, 3\}$ has maximal cardinality and so for any field k of characteristic greater than 7 or of characteristic zero, the k-algebra

$$\Lambda = kQ/(\beta^7, \alpha\gamma, \alpha(e^\beta \bmod \beta^7)\gamma, \alpha(e^{3\beta} \bmod \beta^7)\gamma)$$

of dimension 27 has a loop in its ordinary quiver but satisfies $\mathrm{HH}^1(\Lambda) = 0$.

5. Arbitrary Characteristic

To find examples in arbitrary characteristic, we analyze a bit further the system of equations (1). One may choose a k-basis $(p_i(\beta))_{i \in \mathbb{N}}$ of $V \subseteq k[\beta]$ such that the *orders* $\mathrm{ord}\, p_i$ of these polynomials satisfy

$$0 \leq \nu_1 = \mathrm{ord}\, p_1 < \nu_2 = \mathrm{ord}\, p_2 < \cdots < \nu_i = \mathrm{ord}\, p_i < \cdots .$$

As $(\beta^m) \subseteq V$ one has $\nu_i \leq m + i - 1$; the classes of $p_1, \ldots, p_d$ constitute a basis of $V/(\beta^m)$. In view of Example (3.4) we assume henceforth that $m \geq 2$.

LEMMA 5.1. *With the preceding notations and assumptions, the dimension of the Lie algebra $\mathcal{D}'(\Lambda)$ from (3.5) satisfies*

$$\dim \mathcal{D}'(\Lambda) \geq 2m - \max(1, \nu_1, m - \nu_2 + 1) - \max(1, m - \nu_1).$$

In particular, $\mathcal{D}'(\Lambda) = 0$ implies $\nu_1 = 0$ and $\nu_2 = 1$.

PROOF. Consider first the integers μ with $m \geq \mu > \max(1, \nu_1, m - \nu_2 + 1)$. As $\mu > \nu_1$, each $B_\mu(\beta) = -\beta^{\mu - \nu_1 - 1}p_1$ is a nonzero polynomial. As $m \geq \mu > 1$ the order satisfies

$$m > \mathrm{ord}\, B_\mu = (\mu - \nu_1 - 1) + \nu_1 = \mu - 1 > 0$$

whence $B_\mu(0) = 0$ and the classes $\{B_\mu(\beta) \bmod \beta^m\}_\mu$ are k-linearly independent in $k[\beta]/(\beta^m)$. For the differential operator $\Theta_\mu = \beta^{\mu - \nu_1 - 1}\left(p_1' - p_1 \frac{\partial}{\partial \beta}\right)$ one has $\Theta_\mu(p_1) = 0$ by construction and for $i \geq 1$ one finds

$$\mathrm{ord}(\Theta_\mu(p_i)) = \mathrm{ord}\left(\beta^{\mu - \nu_1 - 1}\left(p_1' p_i - p_1 p_i'\right)\right)$$
$$\geq (\mu - \nu_1 - 1) + (\nu_1 + \nu_i - 1) \geq \mu + \nu_2 - 2 \geq m$$

in view of $\nu_i \geq \nu_2$ and $\mu \geq m - \nu_2 + 2$. It follows that $\Theta_\mu(p_i) \in (\beta^m) \subseteq V$ for each i, whence the subset $\{\Theta_\mu\}_\mu \subseteq \mathcal{D}'(\Lambda)$ is k-linearly independent of cardinality $m - \max(1, \nu_1, m - \nu_2 + 1)$.

Now consider the integers μ' with $m > \mu' \geq \max(1, m - \nu_1)$. The polynomials $A(\beta) = \beta^{\mu'}$ satisfy then $A(0) = 0$ and $\mathrm{ord}\,(Ap_i) = \mu' + \nu_i \geq \mu' + \nu_1 \geq m$, whence the set of (scalar) differential operators $\{\Theta^{\mu'} = \beta^{\mu'}\}_{\mu'}$ defines a k-linearly independent subset of $\mathcal{D}'(\Lambda)$ of cardinality $m - \max(1, m - \nu_1)$. As the union $\{\Theta_\mu\}_\mu \cup \{\Theta^{\mu'}\}_{\mu'}$ is clearly still linearly independent over k, the lower bound on $\dim \mathcal{D}'(\Lambda)$ follows.

For the final assertion observe that $0 \leq \nu_1 \leq m$ whence $1 \leq \max(1, m - \nu_1) \leq m$, with equality on the right if and only if $\nu_1 = 0$. Similarly $1 \leq \nu_2 \leq m + 1$ and so $1 \leq \max(1, \nu_1, m - \nu_2 + 1) \leq m$ with equality on the right if and only if $\nu_1 = m$ or $\nu_2 = 1$. The claim follows. $\square$

5.2. Due to the preceding result we assume henceforth that $\nu_1 = 0, \nu_2 = 1$. We normalize further the polynomials p_i so that

$$p_i(\beta) = \beta^{\nu_i} + \sum_{j=1}^{m-1-\nu_i} p_{i,j}\beta^{\nu_i+j} \quad , \quad p_{i,j} \in k\,.$$

As $\nu_2 = 1$, we may moreover assume $p_{1,1} = 0$. Indeed, if the *order sequence* $(0 < 1 < \nu_3 < \ldots < \nu_d)$ is fixed, we can impose $p_{i,j} = 0$ whenever $j = \nu_{i'} - \nu_i$ for some $i' > i$, and in that case the undetermined $p_{i,j}$ serve as $d(m-d)$ affine coordinates on $\mathrm{Grass}_k(d,m)$ for the classes of vectorspaces with the given order sequence.

5.3. Just comparing orders and lowest order terms of the polynomials on both sides of the equations in (1) shows that $A(0) = B(0) = 0$ implies
- (a) $a_{ij} = 0$ for $1 \le j < i \le d$,
- (b) $a_{11} = 0$,
- (c) $B = a_{22}\beta + O(\beta^2)$, and
- (d) $a_{ii} = \nu_i a_{22}$ for $i = 3, \ldots, d$.

5.4. Now consider the polynomials

$$(6) \qquad \Delta_{ij} = \det \begin{pmatrix} p_i & p_i' \\ p_j & p_j' \end{pmatrix}$$
$$= \nu_{ji}\beta^{\nu_j+\nu_i-1} + [(\nu_{ji}+1)\,p_{j,1} + (\nu_{ji}-1)\,p_{i,1}]\,\beta^{\nu_j+\nu_i} + O(\beta^{\nu_j+\nu_i+1})$$

in $k[\beta]$ where $\nu_{ji} = \nu_j - \nu_i$. The assumption $\mathrm{ord}\,p_1 = 0, \mathrm{ord}\,p_2 = 1$ yields that Δ_{12} is a unit in $k[\beta]/(\beta^m)$. Moreover

$$(7) \qquad (\Delta_{2i}, -\Delta_{1i}, \Delta_{12}) \begin{pmatrix} p_1 & p_1' \\ p_2 & p_2' \\ p_i & p_i' \end{pmatrix} = (0,0)\,.$$

Multiplying the given system (1) from the left by the $d \times d$-matrix of polynomials

$$\begin{pmatrix} p_2' & -p_1' & 0 & \ldots & 0 & \ldots & 0 \\ -p_2 & p_1 & 0 & \ldots & 0 & \ldots & 0 \\ \Delta_{23} & -\Delta_{13} & \Delta_{12} & \ldots & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & & \vdots \\ \Delta_{2i} & -\Delta_{1i} & 0 & \ldots & \Delta_{12} & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots & \ddots & \vdots \\ \Delta_{2d} & -\Delta_{1d} & 0 & \ldots & 0 & \ldots & \Delta_{12} \end{pmatrix}$$

produces the equivalent system

$$\Delta_{12}A \equiv \sum_{j=1}^{d} (a_{1j}p_2' - a_{2j}p_1')\,p_j \bmod \beta^m$$

$$(8) \qquad \Delta_{12}B \equiv \sum_{j=1}^{d} (-a_{1j}p_2 + a_{2j}p_1)\,p_j \bmod \beta^m$$

$$0 \equiv \sum_{j=1}^{d} (a_{1j}\Delta_{2i} - a_{2j}\Delta_{1i} + a_{ij}\Delta_{12})\,p_j \bmod \beta^m \; ; \; i = 3, \ldots, d\,.$$

Using relation (7) to write $\Delta_{1i}p_2 = \Delta_{2i}p_1 + \Delta_{12}p_i$ and employing the conditions found in (5.2), the last $d-2$ equations in (8) become

$$(9) \quad 0 \equiv -a_{22}\left(\Delta_{2i}p_1 + (1-\nu_i)\Delta_{12}p_i\right) +$$
$$\sum_{j=2}^{d} a_{1j}\Delta_{2i}p_j - \sum_{j=3}^{d} a_{2j}\Delta_{1i}p_j + \sum_{j=i+1}^{d} a_{ij}\Delta_{12}p_j \bmod \beta^m \; ; \; i = 3,\ldots,d\,.$$

Consider in particular the equation for $i = d$,

$$0 \equiv -a_{22}\left(\Delta_{2d}p_1 + (1-\nu_d)\Delta_{12}p_d\right) + \sum_{j=2}^{d} a_{1j}\Delta_{2d}p_j - \sum_{j=3}^{d} a_{2j}\Delta_{1d}p_j \bmod \beta^m\,.$$

If the occurring $2(d-1)$ polynomials

$$q_d := \Delta_{2d}p_1 + (1-\nu_d)\Delta_{12}p_d$$
$$(10) \qquad q_{1dj} := \Delta_{1d}p_j \qquad \text{for} \quad j = 3,\ldots,d,$$
$$q_{2dj} := \Delta_{2d}p_j \qquad \text{for} \quad j = 2,\ldots,d,$$

are linearly independent modulo (β^m), then the coefficients a_{1j}, a_{2j} have to vanish for all j. Remembering that Δ_{12} is a unit, looking back at system (8) shows then that $A = B = 0$, and so $\mathcal{D}'(A) = 0$.

If $d = 3$, this condition is clearly as well necessary, whence we get the following result.

COROLLARY 5.5. *If $d = 3$ and V is generated modulo (β^n) by 3 polynomials of the form $p_1 = 1 + O(\beta)$, $p_2 = \beta + O(\beta^2)$ and $p_3 = \beta^{\nu_3} + O(\beta^{\nu_3+1})$, then $\Lambda = kQ/I(n;n,n;V)$ satisfies $\mathrm{HH}^1(\Lambda) = 0$ if and only if $n \neq 0$ in k and the classes of the four polynomials*

$$q_3 = \Delta_{23}p_1 + (1-\nu_3)\Delta_{12}p_3, \quad q_{133} = \Delta_{13}p_3, \quad q_{232} = \Delta_{23}p_2, \quad q_{233} = \Delta_{23}p_3$$

are k-linearly independent in $k[\beta]/(\beta^n)$. $\qquad\qquad\square$

To investigate the polynomials in (10), look at their low order terms. Using (6) and assuming, as we may according to (5.2), that $p_{1,1} = 0$ we obtain

$$q_d = (p_{d,1} - \nu_d p_{2,1})\,\beta^{\nu_d+1} + O(\beta^{\nu_d+2})$$
$$q_{1dj} = \nu_d \beta^{\nu_d+\nu_j-1} + O(\beta^{\nu_d+\nu_j}) \qquad \text{for } j = 3,\ldots,d\,,$$
$$q_{2dj} = (\nu_d - 1)\beta^{\nu_d+\nu_j} + [\nu_d p_{d,1} + (\nu_d-1)p_{j,1} + (\nu_d-2)p_{2,1}]\beta^{\nu_d+\nu_j+1}$$
$$+ O(\beta^{\nu_d+\nu_j+2}) \qquad \text{for } j = 2,\ldots,d\,.$$

Now assume that char $k = p > 0$ and that $\nu_d = \mathrm{ord}\,p_d \equiv 1 \bmod p$. In this case the preceding expressions simplify to

$$q_d = (p_{d,1} - p_{2,1})\,\beta^{\nu_d+1} + O(\beta^{\nu_d+2})$$
$$q_{1dj} = \beta^{\nu_d+\nu_j-1} + O(\beta^{\nu_d+\nu_j+1}) \qquad \text{for } j = 3,\ldots,d\,,$$
$$q_{2dj} = (p_{d,1} - p_{2,1})\,\beta^{\nu_d+\nu_j+1} + O(\beta^{\nu_d+\nu_j+2}) \qquad \text{for } j = 2,\ldots,d\,.$$

This shows the following result.

COROLLARY 5.6. *Assume* $\operatorname{char} k = p > 0$, $d \geq 3$ *and* $\nu_d = \operatorname{ord} p_d \equiv 1 \bmod p$. *If* $p_{d,1} - p_{2,1} \neq 0$, *and if furthermore the set*

$$\{\nu_d + 1, \nu_d + \nu_j - 1; j = 3, \dots, d; \nu_d + \nu_j + 1; j = 2, \dots, d\}$$

contains $2(d-1)$ *distinct integers smaller than* n, *then* $\Lambda = kQ/(n; n, n; V)$ *satisfies* $\mathrm{HH}^1(\Lambda) = 0$ *whenever* $n \neq 0$ *in* k.

EXAMPLE 5.7. For $d = 3$ the set in question is just $\nu_3 + \{1, \nu_3 - 1, 2, \nu_3 + 1\}$ and so the Corollary (5.6) applies to the algebra $\Lambda = kQ/I(6p+1, 6p+1, 6p+1, V)$ where still $\operatorname{char} k = p > 0$, and $V/(\beta^{6p+1})$ is generated by $p_1 = 1$, $p_2 = \beta + \beta^2$, $p_3 = \beta^{2p+1}$. Explicitly,

$$\Lambda = kQ/\left(\beta^{6p+1}, \alpha\gamma, \alpha(\beta + \beta^2)\gamma, \alpha\beta^{2p+1}\gamma\right).$$

To summarize, we now have the

PROOF OF THEOREM (1.1): Combine (2.3), (3.7), (4.4) and (5.7).   □

6. Concluding Remarks

6.1. An unsatisfactory feature of the example in (5.7) is perhaps that its Loewy length grows with the characteristic of the coefficient field. One might thus want to analyze directly the situation for $d = 3$ in any characteristic. For

$$p_1 = 1 + p_{1,3}\beta^3 + p_{1,4}\beta^4 + p_{1,5}\beta^5 + p_{1,6}\beta^6 + p_{1,7}\beta^7 + \dots$$
$$p_2 = \beta + p_{2,2}\beta^3 + p_{2,3}\beta^4 + p_{2,4}\beta^5 + p_{2,5}\beta^6 + p_{2,6}\beta^7 + \dots$$
$$p_3 = \beta^2 + p_{3,1}\beta^3 + p_{3,2}\beta^4 + p_{3,3}\beta^5 + p_{3,4}\beta^6 + p_{3,5}\beta^7 + \dots$$

the four polynomials $q_3, q_{133}, q_{232}, q_{233}$ from (5.5) are of order at least $\nu_3 + 1 = 3$ and a quick calculation in, say, MAPLE shows that the matrix of their coefficients with respect to $\beta^3, \beta^4, \beta^5, \beta^6$ has determinant

$$-3\left(p_{1,3} - 2p_{2,3} + p_{3,3} + p_{3,1}(3p_{2,2} + 2p_{3,1}^2 - 3p_{3,2})\right)^2$$

which yields the following computer aided result: There are examples with vanishing first Hochschild cohomology of type $kQ/I(7; 7, 7; V)$ and $d = 3$ if and only if the characteristic of k is different from 3 or 7. If one looks at the same coefficient matrix but with respect to $\beta^3, \beta^4, \beta^5, \beta^6, \beta^7$, then a similar computer calculation shows that the cofactor of β^5 is a primitive polynomial in the $p_{i,j}$, whence there are examples with vanishing first Hochschild cohomology of type $kQ/I(8; 8, 8; V)$ with $d = 3$ in any characteristic different from 2.

6.2. We finish with an exercise for the diligent reader: Let $Q_{(r)}$ be the quiver that has again vertices $\mathbf{1}, \mathbf{2}, \mathbf{3}$ and arrows $\alpha: \mathbf{2} \to \mathbf{1}$, $\gamma: \mathbf{3} \to \mathbf{2}$, but this time r loops $\beta_\rho: \mathbf{2} \to \mathbf{2}$ for $\rho = 1, \dots, r$.

Extending the method of (4.2) to several variables shows that in large enough or zero characteristic there are finite dimensional k-algebras Λ on $Q_{(r)}$ that satisfy $\mathrm{HH}^1(A) = 0$. Thus there is not even an upper bound on the number of loops in the ordinary quiver of an Artin algebra without outer derivations.

References

[1] M. Auslander, I. Reiten, and S. Smalø: *Representation theory of Artin algebras*, Cambridge Studies in Advanced Mathematics, **36** (University Press, Cambridge 1995).

[2] M. J Bardzell and E. N. Marcos, *Induced boundary maps for the cohomology of monomial and Auslander algebras*, Canadian Mathematical Society Conference Proceedings, **24** (1998), 47–54.

[3] K. Bongartz and P. Gabriel, *Covering Spaces in representation Theory*, Invent. Math., **65** (1982), 331–378.

[4] C. Cibils: *Review of D. Happel: Hochschild cohomology of finite-dimensional algebras. Séminaire d'algèbre P. Dubreil et M.-P. Malliavin, Proc., Paris/Fr. 1987/88, Springer Lect. Notes Math.* **1404** *(1989), 108–126*; Math. Reviews **91 b:16012** (1991).

[5] P. A. M. Dirac: *The Fundamental Equations of Quantum Mechanics.* Proc. Roy. Soc. A **109**, p. 642; reprinted in: *Sources of Quantum Mechanics*, ed. by B. L. van der Warden, Dover Publications, Inc., New York 1968

[6] P. Gabriel, *Auslander-Reiten sequences and representation-finite algebras*, Lecture Notes in Mathematics, **831**, (Springer-Verlag, Berlin, 1980), 1–71.

[7] S. Gastamina, J. A. de la Peña, M. I Platzeck, M. J. Redondo and S. Trepode, *Finite dimensional algebras with vanishing Hochschild cohomology*, preprint.

[8] D. Happel: *Hochschild cohomology of finite-dimensional algebras.* Séminaire d'algèbre P. Dubreil et M.-P. Malliavin, Proc., Paris/Fr. 1987/88, Lecture Notes in Mathematics, **1404** (Springer, Berlin, 1989), 108–126.

[9] K. Igusa: *Notes on the no loop conjecture.* J. Pure and Applied Algebra, **69** (1990), 161–176.

[10] H. Lenzing: *Nilpotente Elemente in Ringen von endlicher globaler Dimension*, Math. Z., **108** (1969), 313–324.

[11] J. L. Loday: *Cyclic Homology.* Grundlehren der math. Wissenschaften, **301**, Springer-Verlag, New York 1992.

Department of Mathematics, University of Toronto, Toronto, Ontario, Canada M5S 3G3

E-mail address: `ragnar@math.utoronto.ca`

Département de mathématiques et d'informatique, Université de Sherbrooke, Sherbrooke, Québec, Canada J1K 2R1

E-mail address: `shiping.liu@dmi.usherb.ca`