KOSZUL DUALITY FOR NON-GRADED DERIVED CATEGORIES

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ABSTRACT. We are concerned with relating derived categories of all modules of two dual Koszul algebras. First, we give a complete account of Koszul complexes, Koszul algebras and Koszul duals in terms of locally finite quivers with relations. Then, we generalize the well-known Acyclic Assembly Lemma and formalize an old method of extending a functor from an additive category into a complex category to its complex category. Applying these techniques to a Koszul algebra defined by a gradable quiver, we extend Beilinson, Ginzburg and Soergel's Koszul duality to dualities between a 2-real-parameter family of pairs of categories derived from subcategories of the complex categories of all modules of the Koszul algebra and its Koszul dual. In case the Koszul algebra is locally bounded on one side and its Koszul dual is locally bounded on the other side, our Koszul duality restricts to an equivalence of the bounded derived categories of finitely supported modules, and an equivalence of the bounded derived categories of finite dimensional modules.

INTRODUCTION

The history of Koszul theory traces back to Cartan and Eilenberg's computing the cohomology groups of a Lie algebra using the Koszul resolution; see [8, Chapter 8, Section 7]. Later, various Koszul resolutions were used to compute the homology and the cohomology of Hopf algebras, restricted Lie algebras and Steenrod algebra; see [7, 24]. In dealing with graded algebras arising from algebraic topology, Priddy formalized the Koszul theory of Koszul complexes and Koszul algebras and discovered a duality among homology algebras of certain Koszul algebras; see [27]. This beautiful theory has applications in many branches of mathematics such as algebraic topology; see [14, 28], algebraic geometry; see [4, 5], quantum group; see [19], commutative algebra; see [9], the representation theory of Lie algebras; see [5, 30] and that of associative algebras; see [11, 12, 20, 21].

Beilinson, Ginzburg and Soergel described in [5] the Koszul duality in terms of graded derived categories of two dual Koszul algebras; see also [4, 10, 15, 23]. More precisely, they established an equivalence between the category derived from a subcategory of the bounded-above complex category of the graded module category of a left finite Koszul algebra and the category derived from a subcategory of the bounded-below complex category of the graded module category of its Koszul dual.

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Their duality restricts to an equivalence of the bounded derived categories of finitely generated graded modules if the Koszul algebra is of finite length and its Koszul dual is left noetherian. Under the setting of positively graded categories, Mazorchuk, Ovsienko and Stroppel generalized in [25] the Koszul duality to Koszul categories; see, for a similar consideration, [22].

The classic Koszul duality deals with derived categories of graded module categories. However, it is also important to study derived categories of non-graded modules of Koszul algebras, for instance, those arising from mixed geometry; see [5, (1.4.2)]. This is even more interesting from the representation theoretic viewpoint; see [1, 2]. Indeed, given an algebra with radical squared zero defined by a gradable quiver, which turns out to be a Koszul algebra, Bautista and Liu established an equivalence between its bounded derived category of finite dimensional modules and the bounded derived category of finitely co-presented modules of the path algebra of the opposite quiver, that is the Koszul dual; see [2, (3.9)].

The objective of this paper is to present a self-contained complete account of the Koszul theory of Koszul complexes, Koszul algebras, Koszul duals and Koszul duality from a combinatorial viewpoint. In particular, our *Koszul algebra* is an algebra defined by a locally finite quiver with homogeneous relations such that every principal simple module has a linear projective resolution; see (2.14) and compare [20]. In case the quiver is gradable, we shall extend the classic Koszul duality, by establishing equivalences between a 2-real-parameter family of pairs of categories derived from subcategories of the complex categories of all modules of a Koszul algebra and its Koszul dual. In contrast to the highly sophisticated technique of spectral sequences used in [5], our technique is elementary with detailed arguments. Let us outline the content of the paper section by section.

In Section 1, we shall lay down the foundation of the paper. Besides collecting and proving some preliminary results, we shall introduce some new classes of algebras defined by a locally finite quiver with relations, which include the locally bounded categories; see [6, (2.1)] and path algebras of strongly locally finite quivers; see [3, Page 100]. Their representation theory is worth future investigation.

In Section 2, we shall prepare for constructing linear projective resolutions and linear injective co-resolutions. We shall start with projective covers in the most general case; see (2.3, 2.4). Then, we shall obtain a class of *principal* injective modules in the locally finite dimensional case; see (2.5), and study injective envelopes in the strongly locally finite dimensional case; see (2.9, 2.10). Finally, we shall show that a graded algebra is quadratic if and only if every principal simple module admits a linear projective 2-presentation; see (2.13) and compare [5, (2.3.3)].

In Section 3, we shall present a description of Koszul complexes, Koszul algebras and Koszul duals in terms of locally finite quivers with quadratic relations. Given a quadratic algebra Λ, we shall define a *local Koszul complex* for each principal simple Λ-module S; see (3.3) and compare [5, (2.6)], which is a projective resolution of S if and only if S has a linear projective resolution; see (3.4). Next, we shall define the *quadratic dual* $\Lambda^!$ of Λ by the opposite quiver with dual quadratic relations; see (3.7) and compare [5, (2.8.1)], and show that $\Lambda^!$ is Koszul if and only if Λ is

Koszul; see (3.10) and compare [5, (2.9.1)], [25, Theorem 30]. In case Λ is locally finite dimensional, we show that Λ is Koszul if and only if its opposite algebra is Koszul, or equivalently, every simple Λ-module admits a particular linear injective co-resolution; see (3.13) and compare [5, (2.2.1)].

In Section 4, we shall provide tools for constructing Koszul duality. Let $\mathcal{A}$ be an additive category with countable direct sums. First, we relate by taking total complex the double complex category $DC(\mathcal{A})$ to the complex category $C(\mathcal{A})$; see (4.1), and obtain a generalization of the Acyclic Assembly Lemma; see [31, (2.7.1)], which ensures the acyclicity of the total complex of a substantially larger family of double complexes; see (4.3). Next, we introduce a homotopy theory in $DC(\mathcal{A})$, which is compatible with taking total complex; see (4.4, 4.5). Finally, we formalize an old method for extending a functor from an additive category $\mathcal{B}$ into $C(\mathcal{A})$ to the complex category $C(\mathcal{B})$. Such extended an functor descends to the homotopy category $K(\mathcal{B})$; see (4.8), but only to categories derived from some possible subcategories of $C(\mathcal{B})$; see (4.9).

In Section 5, we shall describe our Koszul duality. Let Λ be a quadratic algebra Λ defined by a locally finite gradable quiver. We first construct two *Koszul functors* : each sends one of the module categories $\mathrm{Mod}\,\Lambda$ and $\mathrm{Mod}\,\Lambda^!$ into the complex category of the other; see (5.1). As explained above, they are extended to two *complex Koszul functors*: each sends one of the complex categories $C(\mathrm{Mod}\,\Lambda)$ and $C(\mathrm{Mod}\,\Lambda^!)$ into the other one. Our generalized Acyclic Assembly Lemma ensures that they descend to a 2-real-parameter family of pairs of *derived Koszul functors*: each pair interchanges a pair of categories derived from subcategories of $C(\mathrm{Mod}\,\Lambda)$ and $C(\mathrm{Mod}\,\Lambda^!)$; see (5.3), all but the classical pair considered in [5, 25] contain doubly infinite complexes.

In case Λ is Koszul, the Koszul functors send an indecomposable injective $\Lambda^!$-module to the minimal projective resolution of a simple Λ-module and an indecomposable projective Λ-module to the minimal injective co-resolution of a simple $\Lambda^!$-module, respectively; see (5.4). Moreover, the composites of one Koszul functor and the extension of the other one send respectively a bounded-above $\Lambda^!$-module to its minimal injective co-resolution and a Λ-module to its minimal projective resolution; see (5.5, 5.6). Using this fact, we show that each pair of derived Koszul functors is a pair of mutually quasi-inverse triangle equivalences; see (5.7). If Λ is locally bounded on one side and $\Lambda^!$ is locally bounded on the other side, then our Koszul duality restricts to an equivalence of the bounded derived categories of finitely supported modules, and an equivalence of the bounded derived categories of finite dimensional modules; see (5.8) and compare [5, (2.12.6)]. This case occurs, for instance, when the quiver has no right infinite path or no left infinite path.

1. Preliminaries

The objective of this section is to recall some background and collect and prove some preliminary results. The terminology and notation introduced in this section will be used throughout the paper.

I. Linear algebra. Throughout, k denotes a commutative field. All tensor products will be over k unless the otherwise is explicitly stated. The k-vector space freely spanned by a set $\mathcal{S}$ will be written as $k\mathcal{S}$. Let Mod$k$ stand for the category of all k-spaces and mod$k$ for the category of finite dimensional k-spaces. We shall make a frequent use of the exact functor $D = \mathrm{Hom}_k(-, k) : \mathrm{Mod}k \to \mathrm{Mod}k$. The following result is well-known.

1.1. Lemma. *Given $U, V \in \mathrm{mod}k$ and $M, N \in \mathrm{Mod}k$, we obtain an isomorphism*

$$\rho : \mathrm{Hom}_k(U, V) \otimes \mathrm{Hom}_k(M, N) \to \mathrm{Hom}_k(U \otimes M, V \otimes N) : f \otimes g \mapsto \rho(f \otimes g),$$

natural in all variables, where $\rho(f \otimes g)(u \otimes m) = f(u) \otimes g(m)$ for $u \in U$ and $m \in M$.

Remark. In the sequel, we shall identify the map $\varphi(f \otimes g)$ with $f \otimes g$.

As a consequence of Lemma 1.1, we obtain the following well-known result.

1.2. Corollary. *Given $U \in \mathrm{mod}k$ and $M, N \in \mathrm{Mod}k$, we obtain*

(1) *a natural isomorphism $\sigma : DU \otimes N \to \mathrm{Hom}_k(U, N) : f \otimes n \mapsto \sigma(f \otimes n)$, where $\sigma(f \otimes n)(u) = f(u)n$, for $u \in U$ and $n \in N$;*

(2) *a natural isomorphism $\varphi : DU \otimes DM \to D(M \otimes U) : f \otimes g \mapsto \varphi(f \otimes g)$, where $\varphi(f \otimes g)(m \otimes u) = g(m)f(u)$, for $u \in U$ and $m \in M$.*

We shall need the following statement later.

1.3. Lemma. *Given morphisms $f : U \to M$ and $g : N \to V$ in mod$k$, we obtain a commutative diagram with vertical isomorphisms as follows*:

$$\begin{array}{ccc} U \otimes DV & \xrightarrow{f \otimes Dg} & M \otimes DN \\ \downarrow{\scriptstyle \theta_{U,V}} & & \downarrow{\scriptstyle \theta_{M,N}} \\ D(V \otimes DU) & \xrightarrow{D(g \otimes Df)} & D(N \otimes DM). \end{array}$$

Proof. Composing the isomorphism $U \otimes DV \to D^2U \otimes DV$, induced from the canonical isomorphism $U \to D^2U$, with the isomorphism $D^2U \otimes DV \to D(V \otimes DU)$; see (1.2), we obtain an isomorphism $\theta_{U,V}$ such that $\theta_{U,V}(u \otimes \zeta)(v \otimes \xi) = \zeta(v)\xi(u)$, for all $u \in U$, $v \in V$, $\zeta \in DV$ and $\xi \in DU$. Similarly, we obtain an isomorphism $\theta_{M,N}$, making the diagram stated in the lemma commute. Indeed, given $u \in U$, $\zeta \in DV$, $n \in N$ and $\xi \in DM$, we obtain

$$\theta_{M,N}((f \otimes Dg)(u \otimes \zeta))(n \otimes \xi) = \theta_{M,N}(f(u) \otimes \zeta g)(n \otimes \xi) = \zeta(g(n))\, \xi(f(u))$$

and

$$\begin{aligned} D(g \otimes Df)(\theta_{U,V}(u \otimes \zeta))(n \otimes \xi) &= \theta_{U,V}(u \otimes \zeta)((g \otimes Df)(n \otimes \xi)) \\ &= \theta_{U,V}(u \otimes \zeta)(g(n) \otimes \xi f) \\ &= \zeta(g(n))\, \xi(f(u)). \end{aligned}$$

The proof of the lemma is completed.

Let $U \in \mathrm{Mod}k$. Given a subspace V of U, we shall denote by $V^\perp$ the subspace of DU of linear forms vanishing on V, called the *perpendicular* of V in DU.

1.4. LEMMA. *Let U be a finite dimensional k-space.*

(1) *If V, W are subspaces of U, then $(V+W)^\perp = V^\perp \cap W^\perp$, and on the other hand, $(V \cap W)^\perp = V^\perp + W^\perp$.*
(2) *If $\{u_1, \ldots, u_n\}$ and $\{v_1, \ldots, v_n\}$ are bases of U with dual bases $\{u_1^\star, \ldots, u_n^\star\}$ and $\{v_1^\star, \ldots, v_n^\star\}$ respectively, then $\sum_{i=1}^n u_i \otimes u_i^\star = \sum_{i=1}^n v_i \otimes v_i^\star$ in $U \otimes DU$.*

Proof. Statement (1) is evident. By Corollary 1.2, we obtain an isomorphism $\theta : U \otimes \mathrm{Hom}_k(U,k) \to \mathrm{End}_k(U) : u \otimes f \to \theta(u \otimes f)$. Given a basis $\{u_1, \ldots, u_n\}$ of U, it is easy to see that $\theta\left(\sum_{i=1}^n u_i \otimes u_i^\star\right) = \mathbf{1}_U$. The proof of the lemma is completed.

II. QUIVERS. Let $Q = (Q_0, Q_1)$ be a locally finite quiver, where Q_0 is a set of vertices and Q_1 is a set of arrows. Given an arrow $\alpha : x \to y$, we write $x = s(\alpha)$ and $y = e(\alpha)$. Given $x \in Q_0$, one has a *trivial path* ε_x of length 0 with $s(\varepsilon_x) = e(\varepsilon_x) = x$. A *path* of length $n > 0$ is a sequence $\rho = \alpha_n \cdots \alpha_1$, with $\alpha_i \in Q_1$, such that $s(\alpha_{i+1}) = e(\alpha_i)$, for $i = 1, \ldots, n-1$; and in this case, we write $s(\rho) = s(\alpha_1)$ and $t(\rho) = t(\alpha_n)$, and call α_n the *terminal arrow* of ρ. An infinite path in Q is called *right infinite* if it has no ending point and *left infinite* if it has no starting point.

The *opposite* quiver of Q is a quiver Q^{o} defined in such a way that $(Q^{\mathrm{o}})_0 = Q_0$ and $(Q^{\mathrm{o}})_1 = \{\alpha^{\mathrm{o}} : y \to x \mid \alpha : x \to y \in Q_1\}$. A non-trivial path $\rho = \alpha_n \cdots \alpha_1$ in $Q(x,y)$, where $\alpha_i \in Q_1$, corresponds to a non-trivial path $\rho^{\mathrm{o}} = \alpha_1^{\mathrm{o}} \cdots \alpha_n^{\mathrm{o}}$ in $Q^{\mathrm{o}}(y,x)$. However, the trivial path in Q at a vertex x will be identified with the trivial path in Q^{o} at x.

Fix an integer $n \geq 0$ and some vertices x, y of Q. We shall denote by Q_n the set of paths of length n and by $Q(x,y)$ the set of paths from x to y. Moreover, we shall write $Q_n(x,y)$, $Q_{\leq n}(x,y)$, and $Q_{\geq n}(x,y)$ for the subsets of $Q(x,y)$ of paths of length n, of length $\leq n$, and of length $\geq n$, respectively. Further, we put $Q_n(x,-) = \cup_{z \in Q_0} Q_n(x,z)$ and $Q_n(-,x) = \cup_{z \in Q_0} Q_n(z,x)$. Finally, we define $Q_{\leq n}(x,-) = \cup_{z \in Q_0} Q_{\leq n}(x,z)$ and $Q_{\leq n}(-,x) = \cup_{z \in Q_0} Q_{\leq n}(z,x)$, and similarly, $Q_{\geq n}(x,-) = \cup_{z \in Q_0} Q_{\geq n}(x,z)$ and $Q_{\geq n}(-,x) = \cup_{z \in Q_0} Q_{\geq n}(z,x)$. For convenience, we shall put $Q_s(x,y) = \emptyset$ for an integer $s < 0$.

We say that Q is *strongly locally finite* if $Q(x,y)$ is finite for all $x, y \in Q_0$; see [3], and *gradable* if $Q_0 = \cup_{n \in \mathbb{Z}} Q^n$, a disjoint union called a *grading*, such that every arrow is of the form $x \to y$, where $x \in Q^n$, $y \in Q^{n+1}$ and $n \in \mathbb{Z}$; see [1, (7.1)].

III. PATH ALGEBRAS. An algebra in this paper does not necessarily have an identity, and an ideal in an algebra is always a two-sided ideal. Let $Q = (Q_0, Q_1)$ be a locally finite quiver. We denote by kQ the path algebra of Q over k, whose opposite algebra is kQ^{o}. Given $\gamma = \sum_{i=1}^s \lambda_i \rho_i \in kQ$, where $\lambda_i \in k$ and ρ_i are paths, we shall write $\gamma^{\mathrm{o}} = \sum_{i=1}^s \lambda_i \rho_i^{\mathrm{o}} \in kQ^{\mathrm{o}}$. This yields an algebra anti-isomorphism $kQ \to kQ^{\mathrm{o}} : \gamma \mapsto \gamma^{\mathrm{o}}$.

Let R be an ideal in kQ. We shall say that R is *weakly admissible* if $R \subseteq (kQ^+)^2$, where kQ^+ is the ideal in kQ generated by the arrows. A weakly admissible ideal R is called *locally admissible* if, for any $x, y \in Q_0$, there exists $n_{xy} > 0$ such that $kQ_n(x,y) \subseteq R$ for all $n \geq n_{xy}$; *right* (respectively, *left*) *admissible* if, for any $x \in Q_0$, there exists $n_x > 0$ such that $kQ_n(x,-) \subseteq R$ (respectively $kQ_n(-,x) \subseteq R$) for all $n \geq n_x$; and *admissible* if it is right and left admissible; compare [6, (2.1)].

Let R be a weakly admissible ideal in kQ. In this case, the pair (Q, R) a called *bound quiver*. For $n \geq 0$, we shall put $R_n = R \cap kQ_n$; and for $x, y \in Q_0$, we write $R(x, y) = R \cap kQ(x, y)$ and $R_n(x, y) = R \cap kQ_n(x, y)$. An element $\rho \in R(x, y)$ is called a *relation* in R from x to y. Such a relation ρ is called *quadratic* if $\rho \in kQ_2(x, y)$; *homogeneous* if $\rho \in kQ_n(x, y)$ for some $n \geq 2$; *monomial* if $\rho \in Q(x, y)$; and *primitive* if $\rho = \sum_{i=1}^{s} \lambda_i \rho_i$, where $\lambda_i \in k$ and $\rho_i \in Q(x, y)$ are such that $\sum_{i \in \Sigma} \lambda_i \rho_i \notin R$ for any $\Sigma \subset \{1, \ldots, s\}$. We shall say that R is *quadratic* (respectively, *homogeneous*, *monomial*) if it is generated by a set of quadratic (respectively, homogeneous, monomial) relations. A *minimal generating set* Ω of R is a set of primitive relations in R such that R is generated by Ω but not by any proper subset of Ω; and in this case, we put $\Omega(x, y) = \Omega \cap kQ(x, y)$ and $\Omega(x, -) = \cup_{z \in Q_0} \Omega(x, z)$.

1.5. Lemma. *Let Q be a locally finite quiver with R a homogenous ideal in kQ. If Ω is a minimal generating set of R, the the classes of ρ modulo $(kQ^+)R + R(kQ^+)$, with $\rho \in \Omega$, are k-linearly independent.*

Proof. Let Ω be a minimal generating set of R. Assume that $\lambda_1 \rho_1 + \cdots + \lambda_r \rho_r$ lies in $(kQ^+)R + R(kQ^+)$, where $\lambda_i \in k$ are non-zero and $\rho_i \in \Omega(x, y)$ are pairwise distinct, for some $x, y \in Q_0$. Then, $\rho_1 = \sum_{i=1}^{s} \gamma_i \rho_1 \delta_i + \sum_{j=1}^{t} \xi_j \sigma_j \zeta_j$, where $\sigma_j \in \Omega \backslash \{\rho_1\}$, and $\gamma_i, \delta_i, \xi_j, \zeta_j \in kQ$ are homogeneous such that γ_i or δ_i is of positive degree for every $1 \leq i \leq s$. Since ρ_1 and the σ_j are homogeneous, $\rho_1 = \sum_{j \in \Theta} \xi_j \sigma_j \zeta_j$, where Θ is the set of indices j for which $\xi_j \sigma_j \zeta_j$ is of the same degree as ρ_1, a contradiction to the minimality of Ω. The proof of the lemma is completed.

IV. Algebras and modules. In this subsection we fix $\Lambda = kQ/R$, where Q is a locally finite quiver and R is a weakly admissible ideal in kQ. Write $\bar{\gamma} = \gamma + R \in \Lambda$, for $\gamma \in kQ$. Then, $\{e_x = \bar{\varepsilon}_x \mid x \in Q_0\}$ is a *complete* set of pairwise orthogonal idempotents, that is $\Lambda = \oplus_{x \in Q_0} \Lambda e_x = \oplus_{x \in Q_0} e_x \Lambda$. The opposite algebra of Λ is $\Lambda^{\rm o} = kQ^{\rm o}/R^{\rm o}$, where $R^{\rm o} = \{\rho^{\rm o} \mid \rho \in R\}$. We shall write $\bar{\gamma}^{\rm o} = \gamma^{\rm o} + R^{\rm o}$ for $\gamma \in kQ$, but $e_x = \varepsilon_x + R^{\rm o}$ for $x \in Q_0$. This yields an anti-isomorphism $\Lambda \to \Lambda^{\rm o} : \bar{\gamma} \to \bar{\gamma}^{\rm o}$.

We shall say that Λ is *locally finite dimensional* if $e_y \Lambda e_x$ is finite dimensional for all $x, y \in Q_0$; compare [6, (2.1)]; *strongly locally finite dimensional* if R is locally admissible; *right* (respectively, *left*) *locally bounded* if R is right (respectively, left) admissible; and *locally bounded* if R is admissible; compare [6, (2.1)]. Clearly, a left or right locally bounded algebra is strongly locally finite dimensional.

We shall write J for the ideal in Λ generated by $\bar{\alpha}$ with $\alpha \in Q_1$, and say that J is *locally nilpotent* if, for each pair $(x, y) \in Q_0 \times Q_0$, there exists an integer $n_{xy} > 0$ such that $e_y J^{n_{xy}} e_x = 0$. We shall need the following easy result.

1.6. Lemma. *Let $\Lambda = kQ/R$, where Q is locally finite and R is weakly admissible.*

(1) *As a k-vector space, $\Lambda = \Lambda_0 \oplus \Lambda_1 \oplus J^2$, where Λ_0 has a k-basis $\{e_x \mid x \in Q_0\}$ and Λ_1 has a k-basis $\{\bar{\alpha} \mid \alpha \in Q_1\}$.*
(2) *The ideal R is locally admissible if and only if J is locally nilpotent*; *and in this case, J contains only nilpotent elements.*

Proof. We shall prove only the second part of Statement (2). Given $u \in J$, write $u = \sum_{i=1}^{s} u_i$ with $u_i \in e_{y_i} J e_{x_i}$, for some $x_i, y_i \in Q_0$. If J is locally nilpotent, then

$e_{y_j} J^n e_{x_i} = 0$ for some $n > 0$ and all $1 \le i, j \le s$, and consequently, $u^n = 0$. The proof of the lemma is completed.

Example. (1) If Q is a strongly locally finite quiver, then kQ is strongly locally finite dimensional.

(2) Let $\Lambda = kQ/R$, where Q is a single loop α and R is generated by $\alpha^2 - \alpha^3$. Then Λ is locally finite dimensional, but not strongly locally finite dimensional.

A left Λ-module M is called *unitary* if $M = \sum_{x \in Q_0} e_x M$. Such a unitary module M is called *finitely supported* if $e_x M = 0$ for all but finitely many $x \in Q_0$ and *locally finite dimensional* if $e_x M$ is finite dimensional for all $x \in Q_0$. We shall denote by $\mathrm{Mod}\Lambda$ the category of all left unitary Λ-modules, and by $\mathrm{Mod}^b\Lambda$, $\mathrm{mod}\Lambda$ and $\mathrm{mod}^b\Lambda$ its full subcategories of finitely supported modules, of locally finite dimensional modules and of finite dimensional modules, respectively.

Let $M \in \mathrm{Mod}\Lambda$. We shall write $\mathrm{rad}M$ for the Jacobson radical, and $\mathrm{soc}M$ for the socle, of M. We shall call $S_J(M) = \{u \in M \mid Ju = 0\}$ the *J-socle*, JM the *J-radical*, and $T_J(M) = M/JM$ the *J-top*, of M. Recall that a submodule of M is *essential* if it intersects non-trivially every non-zero submodule of M. Associated with $a \in Q_0$, we have a *principal* projective module $P_a = \Lambda e_a$ and a *principal* simple module $S_a = P_a/JP_a$ in $\mathrm{Mod}\Lambda$.

1.7. Lemma. *Let $\Lambda = kQ/R$ be a strongly locally finite dimensional algebra.*

(1) *If $a \in Q_0$, then JP_a is the largest proper submodule of P_a.*
(2) *The non-isomorphic simple modules in $\mathrm{Mod}\Lambda$ are S_a with $a \in Q_0$; and consequently, $S_J(M) = \mathrm{soc}M$, for all $M \in \mathrm{Mod}\Lambda$.*
(3) *If $M \in \mathrm{Mod}\Lambda$ has a finitely supported essential socle, then every quotient module of M has an essential socle.*

Proof. (1) If N is a submodule of P_a not contained in JP_a, then $e_a - u \in N$ for some $u \in JP_a$. Since $u = ue_a$, we see that $(e_a + e_a u + \cdots + e_a u^{n-1})(e_a - u) = e_a \in N$.

(2) Let S be a simple module in $\mathrm{Mod}\Lambda$. Being unitary, S is generated by an element u in $e_a S$, for some $a \in Q_0$. By Statement (1), we have an epimorphism $f : P_a \to S$ with $J = \mathrm{Ker}(f)$, and hence, $S \cong S_a$.

(3) Let $\mathrm{soc}M$ be essential in M and supported by $a_1, \ldots, a_r \in Q_0$. Consider a submodule N of M such that M/N has a non-zero element $w + N \in M/N$, where $w \in e_{b_1} M + \cdots + e_{b_s} M$. Since J is locally nilpotent, $e_{a_j} J^t e_{b_i} = 0$ for some $t > 0$, and for all $i = 1, \ldots, s$; $j = 1, \ldots, r$. Suppose that $v(w + N) \ne 0$, for some $v \in J^t$. Since $\mathrm{soc}M$ is essential in M, there exists some $u \in \Lambda$ such that $0 \ne (uv)w \in \mathrm{soc}M$. In particular, $e_{a_j}(uv)e_{b_i} \ne 0$ for some $1 \le i \le s$ and $1 \le j \le r$, a contradiction. Thus, there exists some maximal $0 \le n < t$ such that $J^n(w + N) \ne 0$. Then, $0 \ne J^n(w + N) \subseteq \mathrm{soc}(M/N)$. The proof of the proposition is completed.

Remark. In case Λ is strongly locally finite dimensional, by Lemma 1.7(1), P_a is indecomposable for every $a \in Q_0$.

Example. Let Λ be the locally finite dimensional algebra defined by a loop α with a relation $\alpha^2 - \alpha^3$. Then, the principal projective module Λ is decomposable. If α acts identically on k, then k is a non-principal simple module with a zero J-socle.

A *representation* M of the bound quiver (Q, R) consists of a family of k-spaces $M(x)$ with $x \in Q_0$ and a family of k-linear maps $M(\alpha) : M(x) \to M(y)$ with $\alpha : x \to y \in Q_1$, such that $M(\rho) = 0$ for all $\rho \in R(x, y)$ with $x, y \in Q_0$. Here, $M(\gamma) = \sum_i \lambda_i M(\alpha_{i,m_i}) \circ \cdots \circ M(\alpha_{i,1})$ for any $\gamma = \sum_i \lambda_i \alpha_{i,m_i} \cdots \alpha_{i,1} \in kQ(x, y)$ with $\lambda_i \in k$ and $\alpha_{ij} \in Q_1$. In particular, we may write $M(\bar{\gamma}) = M(\gamma)$, for $\gamma \in kQ$. A morphism $f : M \to N$ of representations consists of a family of k-linear maps f_x with $x \in Q_0$ such that $f_y \circ M(\alpha) = N(\alpha) \circ f_x$, for every $\alpha : x \to y$ in Q. We shall denote by $\text{Rep}(Q, R)$ the category of all representations of (Q, R).

It is well known that a module $M \in \text{Mod}\Lambda$ can be regarded as a representation $M \in \text{Rep}(Q, R)$ such that $M(x) = e_x M$ for $x \in Q_0$, and $M(\alpha) : M(x) \to M(y)$ is the left multiplication by $\bar{\alpha}$ for $\alpha \in Q_1(x, y)$. A morphism $f : M \to N$ in $\text{Mod}\Lambda$ can be regarded as a morphism $(f_x)_{x \in Q_0} : M \to N$ in $\text{Rep}(Q, R)$, where $f_x : M(x) \to N(x)$ is obtained by restricting f. Taking this point of view, we shall define an exact functor $D : \text{Mod}\Lambda \to \text{Mod}\Lambda^{\text{o}}$ as follows. Given a module M, we define a module DM by $(DM)(x) = \text{Hom}_k(M(x), k)$ for $x \in Q_0$, and $(DM)(\alpha^{\text{o}}) = \text{Hom}_k(M(\alpha), k)$ for $\alpha \in Q_1$. Given a morphism $f : M \to N$, we define a morphism $Df : DN \to DM$ by $(Df)_x = \text{Hom}(f_x, k)$, for every $x \in Q_0$.

1.8. Lemma. *Let $\Lambda = kQ/R$, where Q is locally finite and R is weakly admissible.*

(1) *The functor $D : \text{mod}\Lambda \to \text{mod}\Lambda^{\text{o}}$ is an equivalence.*

(2) *If $M \in \text{Mod}\Lambda$ and $V \in \text{mod}k$, then $D(M \otimes V) \cong DM \otimes DV$.*

Proof. Statement (1) is evident, and Statement (2) follows from Corollary 1.2. The proof of the lemma is completed.

V. Graded algebras. Let $\Lambda = kQ/R$, where Q is a locally finite quiver and R is a homogeneous ideal in kQ. Then, Λ is positively graded with a *J-grading* $\Lambda = \oplus_{n \geq 0} \Lambda_n$, where $\Lambda_n = \{\bar{\gamma} \mid \gamma \in kQ_n\}$. Observe that Λ^{o} is also positively graded as $\Lambda^{\text{o}} = \oplus_{n \geq 0} \Lambda^{\text{o}}_n$, where $\Lambda^{\text{o}}_n = \{\bar{\gamma}^{\text{o}} \mid \gamma \in kQ_n\}$. One says that Λ is *quadratic* if R is a quadratic ideal.

1.9. Proposition. *Let $\Lambda = kQ/R$, where Q is a locally finite quiver and R is a homogeneous ideal in kQ. Then Λ is locally finite dimensional if and only if Λ is strongly locally finite dimensional.*

Proof. Assume that Λ is locally finite dimensional but R is not locally admissible. Then $Q(x, y)$, for some $x, y \in Q_0$, has arbitrarily long paths not lying in R. Since $e_y \Lambda e_x$ is finite dimensional, $\lambda_1 \delta_1 + \cdots + \lambda_n \delta_n \in R(x, y)$, where $\lambda_i \in k$ are non-zero and $\delta_i \in Q(x, y) \backslash R$ are of pairwise different lengths. Since R is homogeneous, $\lambda_1 \delta_1 + \cdots + \lambda_n \delta_n = \rho_1 + \cdots + \rho_s$, where $\rho_1, \ldots, \rho_s \in R(x, y)$ are homogeneous of pairwise different degrees. Then, each δ_i is a summand of a unique ρ_j, say ρ_i. Thus, $\sum_{i=1}^n (\rho_i - \lambda_i \delta_i) + (\sum_{j>n} \rho_j) = 0$, and $\lambda_i \delta_i = \rho_i$ for $i = 1, \ldots, n$, a contradiction. The proof of the proposition is completed.

A module $M \in \text{Mod}\Lambda$ is said to be *graded* if $M = \oplus_{i \in \mathbb{Z}} M_i$, where the M_i are k-spaces such that $\Lambda_i M_j \subseteq M_{i+j}$ for all $i, j \in \mathbb{Z}$. Such a graded module M is said to be *generated in degree* n if $M = \Lambda M_n$. A morphism $f : M \to N$ between graded modules is called *homogeneous of degree* n if $f(M_i) \subseteq N_{i+n}$ for all $i \in \mathbb{Z}$; and in this case, we shall write $f_{i,x} : M_i(x) \to N_{i+n}(x)$, where $i \in \mathbb{Z}$ and $x \in Q_0$, for the map obtained by restricting f. A *graded morphism* is a homogeneous morphism of degree 0. Observe that the shifts of a graded module are isomorphic to each other by homogeneous isomorphisms. The following statement is evident.

1.10. LEMMA. *Let $\Lambda = kQ/R$, where Q is locally finite and R is homogeneous. A sequence of homogeneous morphisms of degree n between graded modules*

$$L \xrightarrow{f} M \xrightarrow{g} N$$

is exact if and only if the sequence

$$L_{i-n}(x) \xrightarrow{f_{i-n,x}} M_i(x) \xrightarrow{g_{i+n,x}} N_{i+n,x}(x)$$

is an exact sequence, for all $i \in \mathbb{Z}$ and $x \in Q_0$.

VI. DERIVED CATEGORIES. Throughout the paper, we shall compose morphisms in any category from the right to the left. All functors between additive categories are additive. Let $\mathcal{A}$ be a full additive subcategory of an abelian category $\mathfrak{A}$. We shall denote by $C(\mathcal{A})$ and $C^b(\mathcal{A})$ the *complex category* and the *bounded complex category* of $\mathcal{A}$ respectively, whose shift functor is written as [1]. By identifying an object M with the stalk complex $M[0]$, we shall regard $\mathcal{A}$ as a full subcategory of $C(\mathcal{A})$. Moreover, $K(\mathcal{A})$ and $K^b(\mathcal{A})$ will stand for the *homotopy category* and the *bounded homotopy category* of $\mathcal{A}$, respectively. Let $(X^\bullet, d_X^\bullet)$ be a complex in $C()A$. The *twist* $\mathfrak{t}(X^\bullet)$ of $X^\bullet$ is the complex $(M^\bullet, d_M^\bullet)$ defined by $M^n = X^n$ and $d_M^\bullet = -d_X^n$; see [2, Section 4]. Clearly, $\mathfrak{t}(M^\bullet) \cong X^\bullet$. One calls $X^\bullet$ *acyclic* if all its cohomological objects $\mathrm{H}^n(X^\bullet)$ with $n \in \mathbb{Z}$, which are objects in $\mathfrak{A}$, vanish. Given a morphism $f^\bullet : X^\bullet \to Y^\bullet$ in $C(\mathcal{A})$, its *mapping cone* $C_{f^\bullet}$ is defined by $C^n_{f^\bullet} = X^{n+1} \oplus Y^n$ and

$$d^n_{C_{f^\bullet}} = \begin{pmatrix} -d_X^{n+1} & 0 \\ f^{n+1} & d_Y^n \end{pmatrix}.$$

A full additive subcategory $\mathscr{A}$ of $C(\mathcal{A})$ is called *derivable* if it is closed under the shifts and taking cones. In this case, the quotient category $\mathcal{K}(\mathscr{A})$ of $\mathscr{A}$ modulo null-homotopic morphisms is a triangulated subcategory of the triangulated category $K(\mathfrak{A})$, and the localization $\mathcal{D}(\mathscr{A})$ of $\mathcal{K}(\mathscr{A})$ at quasi-isomorphisms is a triangulated category; see [26, Chapter 2, Sections 1.6 and 1.7], which we call the *category derived* from $\mathscr{A}$. In particular, we shall write $D(\mathcal{A})$ and $D^b(\mathcal{A})$ for the categories derived from $C(\mathcal{A})$ and $C^b(\mathcal{A})$ and call them the *derived category* and the *bounded derived category* of $\mathcal{A}$, respectively.

2. PROJECTIVE COVERS AND INJECTIVE ENVELOPES

The objective of this section is to obtain some preparatory results for constructing linear projective resolution and linear injective co-resolution, most of them are generalizations of classical results for modules over a locally bounded category; see [6, 13], or for representations of a strongly locally finite quiver; see [3].

Let $\Lambda = kQ/R$, where Q is a locally finite quiver and R is a weakly admissible ideal in kQ. We shall denote by $\mathrm{Proj}\Lambda$ the full additive subcategory of $\mathrm{Mod}\Lambda$ generated by the modules isomorphic to $P_a \otimes V$ with $a \in Q_0$ and $V \in \mathrm{Mod}k$, and by $\mathrm{proj}\,\Lambda$ the one generated by the modules isomorphic to P_a with $a \in Q_0$.

We start with describing morphisms involving modules in $\mathrm{Proj}\,\Lambda$. It is necessary to fix some notation, which will be used for the rest of the paper. Let $\gamma \in kQ(x,y)$ and $\bar\gamma = \gamma + R \in \Lambda$, where $x, y \in Q_0$. The left multiplication by $\bar\gamma$ yields a k-linear map $P_a(\bar\gamma) : P_a(x) \to P_a(y)$ for every $a \in Q_0$, while the right multiplication by $\bar\gamma$ yields a Λ-linear morphism $P[\bar\gamma] : P_y \to P_x$, which restricts to a k-linear map $P[\bar\gamma]_a : P_y(a) \to P_x(a)$ for every $a \in Q_0$.

2.1. PROPOSITION. *Let $\Lambda = kQ/R$, where Q is locally finite and R is weakly admissible. Let $M \in \mathrm{Mod}\Lambda$ and $V \in \mathrm{Mod}k$. Given $a, b \in Q_0$, we obtain*

(1) *a k-linear isomorphism $\mathcal{P}_{x,y} : e_b\Lambda e_a \to \mathrm{Hom}_\Lambda(P_b, P_a) : u \mapsto P[u]$;*
(2) *a k-linear isomorphism $\mathcal{M}_a : \mathrm{Hom}_\Lambda(P_a, M) \to e_aM : f \mapsto f(e_a)$;*
(3) *a k-linear isomorphism $\psi_M : \mathrm{Hom}_\Lambda(P_a \otimes V, M) \to \mathrm{Hom}_k(V, e_aM)$;*
(4) *a k-linear map $\mathcal{M}_{a,b} : e_b\Lambda e_a \to \mathrm{Hom}_k(e_aM, e_bM) : u \mapsto M(u)$, where $M(u)$ denotes the left multiplication by u.*

Proof. Statements (1), (2) and (4) are evident. Observing that P_a is a Λ-k-bimodule, we deduce Statement (3) from the adjoint isomorphism and Statement (2). The proof of the proposition is completed.

In the locally finite dimensional case, the morphisms in $\mathrm{Proj}\,\Lambda$ are completely described in the following statement; compare [1, (7.6)].

2.2. LEMMA. *Let $\Lambda = kQ/R$ be a locally finite dimensional algebra. Given $a, b \in Q_0$ and $V, W \in \mathrm{Mod}k$, every Λ-linear morphism $f : P_a \otimes V \to P_b \otimes W$ is uniquely written as $f = \sum P[u] \otimes f_u$, where u runs over a basis of $e_a\Lambda e_b$ and $f_u \in \mathrm{Hom}_k(V, W)$.*

Proof. Let $f : P_a \otimes V \to P_b \otimes W$ be Λ-linear. Then, $f(e_a \otimes V) \subset e_a\Lambda e_b \otimes W$. Let $\{u_1, \ldots, u_n\}$ be a finite basis of $e_a\Lambda e_b$. If $v \in V$, then $f(e_a \otimes v) = \sum_{i=1}^n u_i \otimes w_i$, for some unique $w_1, \ldots, w_n \in W$. This yields k-linear maps $f_i : V \to W : v \mapsto w_i$, for $i = 1, \ldots, n$. We see easily that $f = \sum_{i=1}^n P[u_i] \otimes f_i$, and this expression is unique. The proof of the lemma is completed.

Let $M \in \mathrm{Mod}\Lambda$. An epimorphism $d : P \to M$ with $P \in \mathrm{proj}\,\Lambda$ will be called a *J-minimal projective cover* over $\mathrm{proj}\,\Lambda$ if $\mathrm{Ker}(d) \subseteq JP$. For instance, the canonical projection $d_a : P_a \to S_a$ is a J-minimal projective cover of S_a, for every $a \in Q_0$. A generating set $\{u_1, \ldots, u_s\}$ of M is called a *J-top basis* if $\{u_1 + JM, \ldots, u_s + JM\}$ is a k-basis of $T_J(M)$. The following statement is well-known in the finite dimensional case; see [17, (1.1)], and its proof is left to the reader.

2.3. LEMMA. *Let $\Lambda = kQ/R$, where Q is locally finite and R is weakly admissible. A module $M \in \mathrm{Mod}\Lambda$ has a J-top basis $\{u_1, \ldots, u_s\}$ with $u_i \in e_{a_i}M$ if and only if it has a J-minimal projective cover $d : P_{a_1} \oplus \cdots \oplus P_{a_s} \to M$ with $d(e_{a_i}) = u_i$, where $a_1, \ldots, a_s \in Q_0$.*

Let M be a module in $\mathrm{Mod}\Lambda$. Given an integer $n \geq 1$, a *projective n-presentation* over $\mathrm{proj}\,\Lambda$ of M is an exact sequence

$$P^{-n} \xrightarrow{d^{-n}} P^{-n+1} \longrightarrow \cdots \longrightarrow P^{-1} \xrightarrow{d^{-1}} P^0 \xrightarrow{d^0} M \longrightarrow 0$$

with $P^{-i} \in \operatorname{proj}\Lambda$, for $i = 0, \ldots, n$. Such a projective n-presentation is called *J-minimal* if $\operatorname{Ker}(d^{-i}) \subseteq JP^{-i}$, for $i = 0, \ldots, n$. The following statement is well known in case Q is finite; compare [11, (2.5)].

2.4. Corollary. *Let $\Lambda = kQ/R$, where Q is locally finite and R is weakly admissible. If $a \in Q_0$ with $Q_1(a, -) = \{\alpha_i : a \to b_i \mid i = 1, \ldots, r\}$, then S_a admits a J-minimal projective presentation*

$$P_{b_1} \oplus \cdots \oplus P_{b_r} \xrightarrow{(P[\bar{\alpha}_1], \cdots, P[\bar{\alpha}_r])} P_a \xrightarrow{d_a} S_a \longrightarrow 0.$$

Proof. Let $a \in Q_0$ with $Q_1(a, -) = \{\alpha_i : a \to b_i \mid i = 1, \ldots, r\}$. It is evident that $\operatorname{Ker}(d_a) = JP_a$ with a J-top basis $\{\bar{\alpha}_1, \cdots, \bar{\alpha}_r\}$. Let $j : JP_a \to P_a$ be the inclusion map. By Lemma 2.3, we obtain a J-minial projective cover $d : P_{b_1} \oplus \cdots \oplus P_{b_r} \to JP_a$ such that $(P[\bar{\alpha}_1], \cdots, P[\bar{\alpha}_r]) = j \circ d$. The proof of the corollary is completed.

Next, we shall study injective envelopes. Let us fix some notation. Given $a \in Q_0$, we shall write $P_a^{\rm o} = \Lambda^{\rm o} e_a \in \operatorname{Mod}\Lambda^{\rm o}$ and $I_a = DP_a^{\rm o} \in \operatorname{Mod}\Lambda$. As a representation, $I_a(x) = \operatorname{Hom}_k(e_x \Lambda^{\rm o} e_a, k)$ for all $x \in Q_0$; and $I_a(\alpha)$, with $\alpha \in Q_1$, sends $f \in I_a(x)$ to $I_a(\alpha)(f) \in I_a(y)$ so that $I_a(\alpha)(f)(v^{\rm o}) = f(\bar{\alpha}^{\rm o} v^{\rm o})$, for all $v \in e_a \Lambda e_y$.

2.5. Proposition. *Let $\Lambda = kQ/R$ be a locally finite dimensional algebra. Let $M \in \operatorname{Mod}\Lambda$ and $V \in \operatorname{Mod}k$. Given $a \in Q_0$, we obtain a k-linear isomorphism*

$$\phi_{_M} : \operatorname{Hom}_\Lambda(M, I_a \otimes V) \to \operatorname{Hom}_k(e_a M, V).$$

Proof. Fix $a \in Q_0$. We have a k-linear map $\psi_a : \operatorname{Hom}_k(e_a \Lambda^{\rm o} e_a, V) \to V : g \mapsto g(e_a)$. For $x \in Q_0$, we deduce from Corollary 1.2(1) a k-linear isomorphism

$$\sigma_x : I_a(x) \otimes V = \operatorname{Hom}_k(e_x \Lambda^{\rm o} e_a, k) \otimes V \to \operatorname{Hom}_k(e_x \Lambda^{\rm o} e_a, V)$$

such that $\sigma_x(h \otimes v)(u^{\rm o}) = h(u^{\rm o})v$, for $h \in I_a(x)$, $v \in V$ and $u \in e_a \Lambda e_x$. Recall that a Λ-linear morphism $f : M \to I_a \otimes V$ consists of a family of k-linear maps $f_x : e_x M \to I_a(x) \otimes V$ with $x \in Q_0$. In particular, we obtain a k-linear map

$$\phi_{_M} : \operatorname{Hom}_\Lambda(M, I_a \otimes V) \to \operatorname{Hom}_k(e_a M, V) : f \to \psi_a \circ \sigma_a \circ f_a.$$

Suppose that $\phi_{_M}(f) = 0$. We claim that $f = 0$, that is, $f_x = 0$, for all $x \in Q_0$. Indeed, for any $m \in e_x M$, write $f_x(m) = \sum_{i=1}^s h_i \otimes v_i$, where $h_i \in \operatorname{Hom}_k(e_x \Lambda^{\rm o} e_a, k)$ and $v_i \in V$ such that $v_1, \ldots, v_s$ are k-linearly independent. Given any $u \in e_a \Lambda e_x$, we obtain $um \in e_a M$ such that $f_a(um) = u f_x(m) = \sum_{i=1}^s (uh_i) \otimes v_i$. Thus,

$$0 = \phi_{_M}(f)(um) = \textstyle\sum_{i=1}^s \sigma_a(uh_i \otimes v_i)(e_a) = \sum_{i=1}^s (uh_i)(e_a) v_i = \sum_{i=1}^s h_i(u^{\rm o}) v_i.$$

Since the v_i are k-linearly independent, $h_i(u^{\rm o}) = 0$, for $i = 1, \ldots, s$. Hence, $h_i = 0$, for $i = 1, \ldots, s$. Thus, $f_x(m) = 0$, and hence, $f_x = 0$. This establishes our claim.

Next, consider a k-linear map $g_a : e_a M \to V$. Given $x \in Q_0$ and $m \in e_x M$, we have a k-linear map $g_x(m) : e_x \Lambda^{\rm o} e_a \to V : u^{\rm o} \to g_a(um)$, and then, a k-linear map $f_x : e_x M \to I_a(x) \otimes V : m \mapsto \sigma_x^{-1}(g_x(m))$. Let $w \in e_y \Lambda e_x$ with $y \in Q_0$. For any $u \in e_a \Lambda e_y$, we have $\sigma_y(f_y(wm))(u^{\rm o}) = g_y(wm)(u^{\rm o}) = g_a((uw)m)$. On the other hand, writing $g_x(m) = \sum_{i=1}^s \sigma_x(h_i \otimes v_i)$ with $h_i \in I_a(x)$ and $v_i \in V$, we obtain $w f_x(m) = w \sigma_x^{-1}(g_x(m)) = \sum_{i=1}^s (wh_i) \otimes v_i$. Then,

$$\sigma_y(w f_x(m))(u^{\rm o}) = \textstyle\sum_{i=1}^s h_i(w^{\rm o} u^{\rm o}) v_i = \sum_{i=1}^s \sigma_x(h_i \otimes v_i)((uw)^{\rm o}) = g_a((uw)m),$$

which is $\sigma_y(f_y(wm))(u^{\rm o})$. Since σ_y is bijective, $wf_x(m) = f_y(wm))$. This shows that the f_x with $x \in Q_0$ form a Λ-linear morphism $f : M \to I_a \otimes V$ such that $\phi_{_M}(f) = g_a$. The proof of the proposition is completed.

Remark. In case Λ is locally finite dimensional, by Proposition 2.5, $I_a \otimes V$ is injective in ModΛ, for $a \in Q_0$ and $V \in {\rm Mod}k$; compare [3, (1.3)]. We shall call I_a the *principal* injective module associated with a. In case Λ is strongly finite dimensional, by Lemmas 1.7 and 1.8, I_a indecomposable.

In general, I_a is probably not injective. By abuse of notation, however, we shall denote by InjΛ the full additive subcategory of ModΛ generated by the modules isomorphic to $I_a \otimes V$, where $a \in Q_0$ and $V \in {\rm Mod}k$, and by injΛ the one generated by the modules isomorphic to I_a with $a \in Q_0$. To describe the morphisms in InjΛ, we shall fix some notation. Given $u \in e_b\Lambda e_a$ with $a, b \in Q_0$, the right multiplication by $u^{\rm o}$yields a $\Lambda^{\rm o}$-linear morphism $P[u^{\rm o}] : P^{\rm o}_a \to P^{\rm o}_b$. Applying $D : {\rm Mod}\,\Lambda^{\rm o} \to {\rm Mod}\,\Lambda$, we obtain a Λ-linear morphism $I[u] = DP[u^{\rm o}] : I_b \to I_a$ such that $I[u](f)(v^{\rm o}) = f(v^{\rm o}u^{\rm o})$, for all $f \in I_a(x)$ with $x \in Q_0$ and $v \in e_a\Lambda e_x$.

2.6. Lemma. *Let $\Lambda = kQ/R$ be a locally finite dimensional algebra. Given $a, b \in Q_0$ and $V, W \in {\rm Mod}k$, every Λ-linear morphism $f : I_a \otimes V \to I_b \otimes W$ is uniquely written as $f = \sum I[u] \otimes f_u$, where u runs over a k-basis of $e_a\Lambda e_b$ and $f_u \in {\rm Hom}_k(V, W)$.*

Proof. Fix $a, b \in Q_0$. Since $e_a\Lambda e_b$ is finite dimensional, we have a k-isomorphism

$$\theta_{a,b} : e_a\Lambda e_b \to {\rm Hom}_k({\rm Hom}_k(e_b\Lambda^{\rm o}e_a, k), k) : u \mapsto \theta_{a,b}(u)$$

such that $\theta_{a,b}(u)(f) = f(u^{\rm o})$, for $f \in {\rm Hom}_k(e_b\Lambda^{\rm o}e_a, k) = I_a(b)$ Let $V, W \in {\rm Mod}k$. Consider the following k-linear isomorphisms

$$\begin{array}{ccc} e_a\Lambda e_b \otimes {\rm Hom}_k(V, W) & \xrightarrow{\theta_{a,b}\otimes 1} & {\rm Hom}_k(I_a(b), k) \otimes {\rm Hom}_k(V, W) \\ & & \big\downarrow \rho \\ {\rm Hom}_\Lambda(I_a \otimes V, I_b \otimes W) & \xrightarrow{\phi} & {\rm Hom}_k(I_a(b) \otimes V, W), \end{array}$$

where ρ and ϕ are as defined in Lemma 1.1 and Proposition 2.5, respectively. For $u \in e_a\Lambda e_b$ and $h \in {\rm Hom}_k(V, W)$, we claim that $\phi(I[u] \otimes h) = (\rho \circ (\theta_{a,b} \otimes \mathbf{1}))(u \otimes h)$. Indeed, $\phi(I[u] \otimes h)$ is the composite of the maps in the sequence

$$I_a(b) \otimes V \xrightarrow{I[u]\otimes h} I_b(b) \otimes W \xrightarrow{\sigma_b} {\rm Hom}_k(e_b\Lambda^{\rm o}e_b, W) \xrightarrow{\psi_b} W,$$

where σ_b and ψ_b are as defined in the proof of Proposition 2.5. Given $g \in I_a(b)$ and $v \in V$, we obtain $(\varphi(\theta_{a,b}(u) \otimes h))\,(g \otimes v) = \theta_{a,b}(u)(g)h(v) = g(u^{\rm o})h(v)$ and

$$\phi(I[u] \otimes h)(g \otimes v) = \sigma_b(I[u](g) \otimes h(v))(e_b) = I[u](g)(e_b)\,h(v) = g(u^{\rm o})\,h(v).$$

This establishes our claim. As a consequence, we obtain a k-linear isomorphism

$$\phi^{-1} \circ \rho \circ (\theta_{a,b} \otimes 1) : e_a\Lambda e_b \otimes {\rm Hom}_k(V, W) \to {\rm Hom}_\Lambda(I_a \otimes V, I_b \otimes W) : u \otimes h \to I[u] \otimes h.$$

The proof of the lemma is completed.

We shall calculate explicitly the J-socle for I_a and $I_a/S_J(I_a)$.

2.7. Lemma. *Let $\Lambda = kQ/R$, where Q is a locally finite quiver and R is a weakly admissible ideal. If $a \in Q_0$, then*

(1) *$S_J(I_a)$ has a k-basis $\{e_a^\star\}$, where $e_a^\star \in I_a(a)$ with $e_a^\star(e_a) = 1$ and $e_a^\star(e_a J^{\rm o} e_a) = 0$;*
(2) *$S_J(I_a/S_J(I_a))$ has a k-basis $\{\alpha^\star + S_J(I_a) \mid \alpha : x \to a \in Q_1(-, a)\}$, where $\alpha^\star \in I_a(x)$ such that $\alpha^\star(\bar\alpha^{\rm o}) = 1$ and $\alpha^\star(\bar\gamma^{\rm o}) = 0$ for all $\gamma \in Q(x, a)\backslash\{\alpha\}$.*

Proof. Fix $a \in Q_0$. Clearly, $e_a^\star \in S_J(I_a)$. If $f \in I_a(x)$ for some $x \in Q_0$, which is neither zero nor a multiple of $e_a^\star$, then $f(u^{\rm o}) \neq 0$ for some $u \in e_a J e_x$, that is, $(u \cdot f)(e_a) \neq 0$. Hence, $f \notin S_J(I_a)$. Thus, $S_J(I_a) = k e_a^\star$.

Fix some vertex $x \in Q_0$. Consider first $\alpha \in Q_1(x, a)$. The existence of $\alpha^\star$ follows from Proposition 1.6(1). Observe that $\bar\alpha \cdot \alpha^\star = e_a^\star$. Let $\beta \in Q_1(x, y)$ with $\beta \neq \alpha$. For $\delta \in Q(y, a)$, since $\delta\beta \neq \alpha$, we obtain $(\bar\beta \cdot \alpha^\star)(\bar\delta^{\rm o}) = \alpha^\star(\bar\beta^{\rm o}\bar\delta^{\rm o}) = 0$. Therefore, $\alpha^\star + S_J(I_a) \in S_J(I_a/S_J(I_a))$. Now, assume that $Q_1(x, a) = \{\alpha_1, \ldots, \alpha_r\}$. If $\sum_{i=1}^r \lambda_i \alpha_i^\star \in S_J(I_a)$ for some $\lambda_i \in k$, then

$$\lambda_j = \textstyle\sum_{i=1}^r \lambda_i \cdot \alpha_i^\star(\bar\alpha_j^{\rm o}) = \sum_{i=1}^r \lambda_i \cdot (\bar\alpha_j \alpha_i^\star)(e_a) = \left(\bar\alpha_j \cdot \left(\sum_{i=1}^r \lambda_i \alpha_i^\star\right)\right)(e_a) = 0,$$

for $j = 1, \ldots, r$. As a consequence, the classes $\alpha^\star + S_J(I_a)$ with $\alpha \in Q_1(-, a)$ are k-linearly independent in $S_J(I_a/S_J(I_a))$.

Finally, consider $g + S_J(I_a) \in S_J(I_a/S_J(I_a))$, where $g \in I_a(x)$ for some $x \in Q_0$. Let $\rho \in Q_{\geq 2}(x, a)$. Write $\rho = \delta\alpha$, where $\alpha : x \to y$ is an arrow and $\delta : y \rightsquigarrow a$ is non-trivial. Since $\bar\alpha g \in S_J(I_a)$ and δ is non-trivial, $g(\bar\rho^{\rm o}) = (\bar\alpha g)(\bar\delta^{\rm o}) = 0$. Hence, $g(e_x (J^{\rm o})^2 e_a) = 0$. By Lemma 1.6(1), $g = \sum_{\gamma \in Q_{\leq 1}(-, a)} \lambda_\gamma \gamma^\star$, where $\lambda_\gamma \in k$. Thus, $g + S_J(I_a) = \sum_{\alpha \in Q_1(-, a)} \lambda_\alpha (\alpha^\star + S_J(I_a))$. The proof of the lemma is completed.

The following statement is well-known in the finite dimensional case.

2.8. Corollary. *Let $\Lambda = kQ/R$ be strongly locally finite dimensional. If $a \in Q_0$, then $S_J(I_a)$ and $S_J(I_a/S_J(I_a))$ are essential socles of I_a and $I_a/S_J(I_a)$.*

Proof. By Lemma 1.7(2), the J-socle of a module is its socle. Let $h \in I_a(x)\backslash S_J(I_a)$, for some $x \in Q_0$. Then, $h(e_x J^{\rm o} e_a) \neq 0$. Since $J^{\rm o}$ is locally nilpotent, there exists a maximal positive integer s such that $h(e_x (J^{\rm o})^s e_a) \neq 0$. Then, $h(\bar\zeta^{\rm o}) = \lambda \neq 0$ for some $\zeta \in Q_s(x, a)$. Note that $\bar\zeta h \in I_a(a)$ with $(\bar\zeta h)(e_a) = h(\bar\zeta^{\rm o}) = \lambda$. By the maximality of s, we see that $(\bar\zeta h)(e_a J^{\rm o} e_a) = 0$. Hence, $\bar\zeta h = \lambda e_a^\star \in S_J(I_a)$. Thus, $S_J(I_a)$ is essential in I_a.

Write $\zeta = \beta\xi$, where $\beta \in Q_1(b, a)$ and $\xi \in Q_{s-1}(x, b)$ with $b \in Q_0$. Then, $\bar\xi h \in I_a(b)$ with $(\bar\xi h)(\bar\beta^{\rm o}) = h(\bar\zeta^{\rm o}) \neq 0$. Therefore, $\bar\xi(h + S_J(I_a)) = \bar\xi h + S_J(I_a) \neq 0$. By the maximality of s, we see that $(\bar\xi h)(e_b (J^{\rm o})^2 e_a) = 0$. By Lemma 1.6(1), $\bar\xi h + S_J(I_a) = \sum_{\alpha \in Q_1(-, a)} \lambda_\alpha \cdot (\alpha^\star + S_J(I_a)) \in S_J(I_a/S_J(I_a))$, where $\lambda_\alpha \in k$. That is, $S_J(I_a/S_J(I_a))$ is essential in $I_a/S_J(I_a)$. The proof of the corollary is completed.

Example. Let Λ be a locally finite dimensional algebra given by a loop α with a relation $\alpha^2 - \alpha^3$. Then $S_J(D(\Lambda)) = 0$, which is not essential in $D(\Lambda)$.

Let $M \in \operatorname{Mod}\Lambda$. A subset $\{u_1, \ldots, u_s\}$ of M is called an *essential socle basis* if M has an essential socle, of which $\{u_1, \ldots, u_s\}$ is a k-basis. The following result is well-known in case Λ is finite dimensional, and its proof is left to the reader.

2.9. Lemma. *Let $\Lambda = kQ/R$ be a strongly locally finite dimensional algebra. A module $M \in \operatorname{Mod}\Lambda$ has an essential socle basis $\{u_1, \ldots, u_s\}$ with $u_i \in e_{a_i} M$ if and only if M has an injective envelope $j : M \to I_{a_1} \oplus \cdots \oplus I_{a_s}$ with $j(u_i) = e_{a_i}^\star$, where $a_1, \ldots, a_s \in Q_0$.*

The following statement is well-known in case Q is finite.

2.10. Corollary. *Let $\Lambda = kQ/R$ be a strongly locally finite dimensional algebra. If $a \in Q_0$ with $Q_1(-,a) = \{\beta_i : b_i \to a \mid i = 1, \ldots, s\}$, then*

$$0 \longrightarrow S_a \xrightarrow{j_a} I_a \xrightarrow{\left(I[\bar\beta_1],\ldots,I[\bar\beta_s]\right)^t} I_{b_1} \oplus \cdots \oplus I_{b_s},$$

is a minimal injective co-presentation of S_a, where j_a sends $e_a + Je_a$ to $e_a^\star$.

Proof. Let $a \in Q_0$ with $Q_1(-,a) = \{\beta_i : b_i \to a \mid i = 1, \ldots, s\}$. By Corollary 2.8 and Lemma 2.9, j_a is an injective envelope of S_a with $\mathrm{Im}(j_a) = S_J(I_a)$. By Lemma 2.7 and Corollary 2.8, $\{\beta_1^\star + S_J(I_a), \ldots, \beta_s^\star + S_J(I_a)\}$ is an essential socle basis for I_a. By Lemma 2.9, we obtain an injective envelope $j : I_a/S(I_a) \to I_{b_1} \oplus \cdots \oplus I_{b_s}$, sending $\beta_i^\star + S_J(I_a)$ to $(0, \ldots, e_{b_i}^\star, \ldots, 0)$, for $i = 1, \ldots, s$. Since $I[\bar\beta_i](\beta_i^\star) = e_{b_i}^\star$, we see that $\left(I[\bar\beta_1], \ldots, I[\bar\beta_s]\right)^t$ is the composite of the canonical projection $I_a \to I_a/S_J(I_a)$ and the injective envelope j. The proof of the corollary is completed.

For the rest of this section, assume that $\Lambda = kQ/R$, where R is homogeneous. Given $a \in Q_0$, in view of the J-grading $\Lambda = \oplus_{n\geq 0}\Lambda_n$, we see that P_a and S_a are graded and generated in degree one. In case Λ is locally finite dimensional, then I_a is negatively graded as $I_a = \oplus_{n\geq 0}(I_a)_{-n}$, where $(I_a)_{-n} = \mathrm{Hom}_k(\Lambda_n^{\mathrm{o}} e_a, k)$, for all $n \geq 0$. However, I_a is not graded in general. For instance, if Λ is the path algebra of a single loop α, then $D\Lambda = \mathrm{Hom}_k(\oplus_{n\geq 0} k\alpha^n, k) \not\cong \oplus_{n\geq 0}\mathrm{Hom}_k(\Lambda_n, k)$. The following statement is a variation of a classical result on graded projective covers; see, for example, [11, (2.4)], and its proof is left to the reader.

2.11. Lemma. *Let $\Lambda = kQ/R$, where Q is locally finite and R is homogeneous. Let M be a finitely generated graded module in* $\mathrm{Mod}\Lambda$*. If $f : P \to M$ and $f' : P' \to M$ are homogeneous J-minimal projective covers, then $f' = f \circ g$, where $g : JP' \to P$ is a graded isomorphism.*

The following result describes a J-minimal projective 2-presentation of a principal simple module in the graded case; compare [13, (2.4)].

2.12. Lemma. *Let $\Lambda = kQ/R$, where Q is locally finite and R is homogeneous with a minimal generating set Ω. Let $a \in Q_0$ with $Q_1(a,-) = \{\alpha_i : a \to b_i \mid i = 1, \ldots, r\}$ and $\Omega(a,-) = \{\rho_1, \ldots, \rho_s\}$. If $\rho_j = \sum_{j=1}^r \gamma_{ij}\alpha_i$ with $\gamma_{ij} \in kQ(b_i, c_j)$, then S_a has a J-minimal projective 2-presentation*

$$P_{c_1} \oplus \cdots \oplus P_{c_s} \xrightarrow{(P[\bar\gamma_{ij}])_{r\times s}} P_{b_1} \oplus \cdots \oplus P_{b_r} \xrightarrow{(P[\bar\alpha_1],\cdots,P[\bar\alpha_r])} P_a \xrightarrow{d_a} S_a \longrightarrow 0.$$

Proof. Let $\rho_j = \sum_{j=1}^r \gamma_{ij}\alpha_i$, where $\gamma_{ij} \in kQ(b_i, c_j)$. Write $d_1 = (P[\bar\alpha_1], \cdots, P[\bar\alpha_r])$ and $d_2 = (P[\bar\gamma_{ij}])_{r\times s}$. By Corollary 2.4, it suffices to show that d_2 co-restricts to a J-minimal projective cover of $\mathrm{Ker}(d_1)$. Since $u_j = (\bar\gamma_{1j}, \ldots, \bar\gamma_{rj}) \in \mathrm{Ker}(d_1)$, by Lemma 2.3, it amounts to show that $\{u_1, \ldots, u_s\}$ is a J-top basis of $\mathrm{Ker}(d_1)$.

Let $v = (\bar\delta_1, \ldots, \bar\delta_r) \in \mathrm{Ker}(d_1)$, where $\delta_i \in kQ(b_i, -)$. We may assume that $\delta_i \in kQ(b_i, c)$, for some $c \in Q_0$. Since $d_1(v) = 0$, we obtain $\sum_{i=1}^r \delta_i\alpha_i \in R(a,c)$, and hence, $\sum_{i=1}^r \delta_i\alpha_i = \sum_{j=1}^s \omega_j\rho_j + \sum_{i=1}^r \eta_i\alpha_i = \sum_{i=1}^r (\sum_{j=1}^s \omega_j\gamma_{ij} + \eta_i)\alpha_i$, where $\omega_j \in kQ(c_j, c)$ and $\eta_i \in R(b_i, c)$. This yields $\delta_i = \sum_{j=1}^s \omega_j\gamma_{ij} + \eta_i$, and hence,

$\bar{\delta}_i = \sum_{j=1}^{s} \bar{\omega}_j \bar{\gamma}_{ij}$, for $i = 1, \ldots, r$. As a consequence, $v = \sum_{j=1}^{s} \bar{\omega}_j u_j$. This shows that $\mathrm{Ker}(d_1) = \sum_{i=1}^{n} \Lambda u_i$.

Assume next that $\sum_{j=1}^{s} \lambda_j u_j \in J\mathrm{Ker}(d_1) = \sum_{i=1}^{n} J u_i$, where $\lambda_j \in k$. Write $\sum_{j=1}^{s} \lambda_j u_j = \sum_{j=1}^{s} \bar{\nu}_j u_j$, with $\nu_j \in kQ^+$. Then, $\sum_{j=1}^{s} \lambda_j \gamma_{ij} = \sum_{j=1}^{s} (\nu_j \gamma_{ij} + \eta_{ij})$, where $\eta_{ij} \in R(b_i, c_j)$, for $i = 1, \ldots, r$. Calculating $\sum_{i=1}^{r} \sum_{j=1}^{s} \lambda_j \gamma_{ij} \alpha_i$, we obtain $\sum_{j=1}^{s} \lambda_j \rho_j = \sum_{j=1}^{s} \nu_j (\rho_j + \zeta_j)$, where $\zeta_j \in R(a, c_j)$. By Lemma 1.5, $\lambda_j = 0$, for $i = 1, \ldots, s$. The proof of the lemma is completed.

A projective n-presentation over $\mathrm{proj}\,\Lambda$ of a module is called *linear* if the morphisms between the projective modules are homogenous of degree one. The following statement extends a well-known result, saying that a classical Koszul algebra is quadratic; see [5, (2.3.3)].

2.13. Theorem. *Let $\Lambda = kQ/R$, where Q is locally finite and R is homogeneous. Then, Λ is quadratic if and only if every principal simple Λ-module admits a J-minimal linear projective 2-presentation over* $\mathrm{proj}\,\Lambda$.

Proof. Let Ω be a minimal generating set of R. Fix $a \in Q_0$. Since $\Omega(a, -)$ contains only finitely many quadratic relations, the necessity follows immediately from Lemma 2.12. Assume that S_a admits a linear projective 2-presentation over $\mathrm{proj}\Lambda$. Letting $Q_1(a, -) = \{\alpha_i : a \to b_i \mid i = 1, \ldots, r\}$, we deduce from Lemmas 2.6, 2.11 and 2.12 a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
P_2 & \xrightarrow{\quad d_2 \quad} & P_1 & \xrightarrow{\quad d_1 \quad} & P_a & \xrightarrow{d_0} & S_a & \longrightarrow & 0 \\
\| & & \downarrow f_1 & & \downarrow f_0 & & \| & & \\
P_{c_1} \oplus \cdots \oplus P_{c_s} & \xrightarrow{(P[\bar{\gamma}_{ij}])_{r \times s}} & P_{b_1} \oplus \cdots \oplus P_{b_r} & \xrightarrow{(P[\bar{\alpha}_1], \cdots, P[\bar{\alpha}_r])} & P_a & \xrightarrow{d_a} & S_a & \longrightarrow & 0,
\end{array}$$

where the upper row is a linear projective 2-presentation, f_0, f_1 are graded isomorphisms, and $\gamma_{ij} \in kQ(b_i, c_j)$. Since $f_1 \circ d_2$ is homogeneous of degree one, $\gamma_{ij} \in kQ_1(b_i, c_j)$ and $\eta_j = \sum_{i=1}^{r} \gamma_{ij} \alpha_i \in R_2(a, c_j)$, for $j = 1, \ldots, s$. By Lemma 2.3, $\{u_j = (\bar{\gamma}_{1j}, \ldots, \bar{\gamma}_{rj}) \mid j = 1, \ldots, s\}$ is a J-top basis of $\mathrm{Ker}(P[\bar{\alpha}_1], \cdots, P[\bar{\alpha}_r])$.

Let $\rho \in \Omega(a, c)$ be a relation of degree $n > 2$. Write $\rho = \sum_{i=1}^{r} \gamma_i \alpha_i$, for some $\gamma_i \in kQ_{n-1}(b_i, c)$. Since $(\bar{\gamma}_1, \ldots, \bar{\gamma}_r) \in \mathrm{Ker}(P[\bar{\alpha}_1], \cdots, P[\bar{\alpha}_r])$, we see that $(\bar{\gamma}_1, \ldots, \bar{\gamma}_r) = \sum_{j=1}^{s} \bar{\delta}_j u_j$, for some $\delta_j \in kQ_{n-2}(c_j, c)$. Then, $\gamma_i = \sigma_i + \sum_{j=1}^{s} \delta_j \gamma_{ij}$, where $\sigma_i \in R(b_i, c)$, for $i = 1, \ldots, r$. This yields $\rho = \sum_{i=1}^{r} \sigma_i \alpha_i + \sum_{j=1}^{s} \delta_j \eta_j$. Since $n > 2$, we see that $\rho \in R(kQ^+) + (kQ^+)R$, a contradiction to Lemma 1.5. The proof of the theorem is completed.

A complex $P^{\bullet}$ over $\mathrm{proj}\Lambda$ is called a *projective resolution* over $\mathrm{proj}\Lambda$ of a module M if $P^i = 0$ for all $i > 0$, and $\mathrm{H}^0(P^{\bullet}) \cong M$ and $\mathrm{H}^i(P^{\bullet}) = 0$ for $i < 0$. The following definition is a variation of the classical one; see, for example, [5].

2.14. Definition. Let $\Lambda = kQ/R$, where Q is a locally finite quiver and R is a homogeneous ideal in kQ.

(1) A complex over $\mathrm{Mod}\Lambda$ is called *linear* if the differentials are homogeneous morphisms of degree one between indecomposable modules.

(2) The algebra Λ is called *Koszul* if S_a admits a linear projective resolution over $\mathrm{proj}\Lambda$, for every $a \in Q_0$.

REMARK. By Theorem 2.13, a Koszul algebra is quadratic; compare [5, (2.3.3)].

EXAMPLE. Given any locally finite quiver Q, it is evident that $\Lambda = kQ/(kQ^+)^2$ is a Koszul algebra.

3. Koszul complexes and Koszul duals

The objective of this section is to present a combinatorial account of Koszul complexes, Koszul algebras and Koszul duals. Although our main results will be similar to those stated in [5], we shall take an elementary approach with a local viewpoint and provide detailed arguments.

Let $\Lambda = kQ/R$, where Q is a locally finite quiver and R is a quadratic ideal in kQ. In order to define the local Koszul complexes, we need to introduce some notation. Given $\alpha \in Q_1$, we obtain a *derivation* $\partial_\alpha : kQ \to kQ$, that is a k-linear map, sending a path ρ to δ if $\rho = \alpha\delta$; and to 0 if α is not a terminal arrow of ρ. In particular, ∂_α vanishes on kQ_0 and sends kQ_p to kQ_{p-1} for all $p > 0$. Fix some $a, x \in Q_0$ and $n \geq 0$. Recall that $R_n = R \cap kQ_n$ and $R_n(a,x) = R(a,x) \cap kQ_n(a,x)$. We shall define a subspace $R^{(n)}(a,x)$ of $kQ_n(a,x)$ by $R^{(n)}(a,x) = kQ_n(a,x)$, for $n = 0, 1$, and $R^{(n)}(a,x) = \cap_{0 \leq j \leq n-j} kQ_{n-j-2}(-,x) \cdot R_2 \cdot kQ_j(a,-)$, for $n \geq 2$. Observe that $R^{(2)}(a,x) = R_2(a,x)$. We shall write $R^{(n)}(a,-) = \oplus_{x \in Q_0} R^{(n)}(a,x)$. As shown below, these subspaces are stable under the derivations.

3.1. LEMMA. *Let Q be a locally finite quiver with R a quadratic ideal in kQ. Consider an element $\gamma \in R^{(n)}(a,x)$ for some $n > 0$ and $a, x \in Q_0$.*

(1) *If $\alpha \in Q_1(y,x)$, then $\partial_\alpha(\gamma) \in R^{(n-1)}(a,y)$.*
(2) *If $\gamma = \alpha_1\gamma_1 + \cdots + \alpha_m\gamma_m$, where $\gamma_i \in kQ_{n-1}(a,y_i)$ and $\alpha_i \in Q_1(y_i,x)$, then $\gamma_i \in R^{(n-1)}(a,y_i)$, for $i = 1, \ldots, m$.*

Proof. Let $\alpha \in Q_1(y,x)$. Clearly, $\partial_\alpha(\beta\delta) = \partial_\alpha(\beta)\,\delta$, for $\beta \in Q_1$ and $\delta \in kQ$. Since $\partial_\alpha(kQ_n(a,x)) \subseteq kQ_{n-1}(a,x)$, we may assume that $n \geq 3$. Given $0 \leq j \leq n-3$, write $\gamma = \sum_{i=1}^r \alpha_i\zeta_i\rho_i\delta_i$, where $\alpha_i \in Q_1(y_i,x)$; $\zeta_i \in kQ_{n-3-j}(-,y_i)$; $\rho_i \in R_2$; $\delta_i \in kQ_j(a,-)$. Thus, $\partial_\alpha(\gamma) = \sum_{i=1}^r \partial_\alpha(\alpha_i)\zeta_i\rho_i\delta_i$. Suppose that $\partial_\alpha(\gamma) \neq 0$. We may assume that $\alpha_i = \alpha$ if and only if $1 \leq i \leq s$, for some $1 \leq s \leq r$. Then,

$$\partial_\alpha(\gamma) = \textstyle\sum_{i=1}^s \zeta_i\rho_i\delta_i \in kQ_{n-3-j}(-,y) \cdot R_2 \cdot kQ_j(a,-).$$

Thus, $\partial_\alpha(\gamma) \in R^{(n-1)}(a,y)$. This establishes Statement (1), from which Statement (2) follows immediately. The proof of the lemma is completed.

As a consequence, we obtain the following statement.

3.2. LEMMA. *Let $\Lambda = kQ/R$, where Q is locally finite and R is quadratic. Given $a, x, y \in Q_0$ and $n > 0$, we obtain a Λ-linear morphism*

$$\partial_a^{-n}(y,x) = \textstyle\sum_{\alpha \in Q_1(y,x)} P[\bar{\alpha}] \otimes \partial_\alpha : P_x \otimes R^{(n)}(a,x) \to P_y \otimes R^{(n-1)}(a,y).$$

Moreover, if $\rho = \sum_{i=1}^m \zeta_i\delta_i \in R^{(n)}(a,x)$, where $\delta_i \in kQ_{n-1}(a,y_i)$ and $\zeta_i \in kQ_1(y_i,x)$, then $\partial_a^{-n}(y,x)(u \otimes \rho) = \sum_{i=1}^m u\bar{\zeta_i} \otimes \delta_i$, for all $u \in P_x$.

Proof. Fix $a, x, y \in Q_0$ and $n > 0$. By Lemma 3.1(1), we do have a Λ-linear morphism $\partial_a^{-n}(y,x)$ as defined in the lemma. Consider $\rho = \sum_{i=1}^m \zeta_i\delta_i \in R^{(n)}(a,x)$, where $\delta_i \in kQ_{n-1}(a, y_i)$ and $\zeta_i \in kQ_1(y_i, x)$. Write $\zeta_i = \sum_{j=1}^{s} \lambda_{ij}\alpha_j$, where $\lambda_{ij} \in k$ and $\alpha_1, \ldots, \alpha_s$ are the arrows in $Q_1(y, x)$. For any $u \in P_x$, we obtain

$$\begin{array}{rcl} \partial_a^{-n}(y,x)(u \otimes \rho) & = & \sum_{l=1}^{s}(P[\bar{\alpha}_l] \otimes \partial_{\alpha_l})(u \otimes \rho) \\ & = & \sum_{l,j=1}^{s}\sum_{i=1}^{m} u\lambda_{ij}\bar{\alpha}_l \otimes \partial_{\alpha_l}(\alpha_j)\delta_i \\ & = & \sum_{i=1}^{m} u(\sum_{j=1}^{s}\lambda_{ij}\bar{\alpha}_j) \otimes \delta_i. \end{array}$$

The proof of the lemma is completed.

Fix $a \in Q_0$. Since $R^{(n)}(a, x)$ is finite dimensional and vanishes for almost all $x \in Q_0$, by Lemma 3.2, we obtain a sequence $K_a^{\bullet}$ over projΛ as follows:

$$\cdots \longrightarrow K_a^{-n} \xrightarrow{\partial_a^{-n}} K_a^{-n+1} \longrightarrow \cdots \longrightarrow K_a^{-1} \xrightarrow{\partial_a^{-1}} K_a^{0} \longrightarrow 0 \longrightarrow \cdots,$$

where $K_a^{-n} = \oplus_{x\in Q_0} P_x \otimes R^{(n)}(a, x)$ for every $n \geq 0$, and

$$\partial_a^{-n} = (\partial_a^{-n}(y,x))_{(y,x)\in Q_0\times Q_0} : \oplus_{x\in Q_0} P_x \otimes R^{(n)}(a,x) \to \oplus_{y\in Q_0} P_y \otimes R^{(n-1)}(a,y),$$

which is homogeneous of degree one, for every $n > 0$. Observing that $K_a^0 = P_a \otimes k\varepsilon_a$, we obtain an augmented Λ-linear morphism $\partial_a^0 : K_a^0 \to S_a : e_a \otimes \varepsilon_a \mapsto e_a + JP_a$.

3.3. Lemma. *Let $\Lambda = kQ/R$ with Q locally finite and R quadratic. If $a \in Q_0$, then*

(1) $\mathrm{Ker}(\partial_a^{-n}) \subseteq JK_a^{-n}$ *for $n \geq 0$;*

(2) $K_a^{\bullet}$ *is a linear complex over* projΛ;

(3) S_a *has as a linear projective* 2*-presentation the sequence*

$$K_a^{-2} \xrightarrow{\partial_a^{-2}} K_a^{-1} \xrightarrow{\partial_a^{-1}} K_a^{0} \xrightarrow{\partial_a^{0}} S_a \longrightarrow 0.$$

Proof. Fix $a \in Q_0$. We have $\mathrm{Ker}(\partial_a^0) = JK_a^0$. Let $w \in \mathrm{Ker}(\partial_a^{-n})$, for some $n > 0$. Then, $e_x w \in \mathrm{Ker}(\partial_a^{-n})$ for every $x \in Q_0$. Since $e_x w \in P_x \otimes R^{(n)}(a, x)$, by definition, $\partial_a^{-n}(e_x w) = \sum_{y\in Q_0} \partial_a^{-n}(y, x)(e_x w) = 0$, where $\partial_a^{-n}(y, x)(e_x w) \in P_y \otimes R^{(n-1)}(a, y)$. Thus, $\partial_a^{-n}(y, x)(e_x w) = 0$, for every $y \in Q_0$.

Write $e_x w = \sum_{i=0}^{s} w_i$, where $w_i \in J^i P_x \otimes R^{(n)}(a, x)$. Since $\partial_a^{-n}(y, x)$ is homogeneous of degree one, $\partial_a^{-n}(y, x)(w_0) = 0$. Now, $w_0 = e_x \otimes \gamma$, where $\gamma \in R^{(n)}(a, x)$. Write $\gamma = \sum_{z\in Q_0}(\sum_{\beta_z\in Q_1(z,x)} \beta_z\xi_{\beta_z})$, where $\xi_{\beta_y} \in kQ_{n-1}(a, y)$. By definition,

$$\partial_a^{-n}(y,x)(e_x\otimes\gamma) = \sum_{\alpha\in Q_1(y,x);\, z\in Q_0;\, \beta_z\in Q_1(z,x)} \bar{\alpha}\otimes\partial_\alpha(\beta_z\xi_{\beta_z}) = \sum_{\beta_z\in Q_1(z,x)} \bar{\beta}_y\otimes\xi_{\beta_y}.$$

Since the $\bar{\beta}_y$ are k-linearly independent, $\xi_{\beta_y} = 0$, for all $y \in Q_0$. This implies that $w_0 = 0$. That is, $e_x w \in JP_x \otimes R^{(n)}(a, x)$ for all $x \in Q_0$. As a consequence, $w \in JK_a^{-n}$. This establishes Statement (1).

Next, we shall show that $\partial_a^{1-n} \circ \partial_a^{-n} = 0$, for $n > 1$. Indeed, let $v \in P_x$ and $\gamma \in R^{(n)}(a, x)$, where $x \in Q_0$. By the definition of $R^{(n)}(a, x)$, we may assume that $\gamma = \rho\delta$, for some $\rho \in R_2(z, x)$ and $\delta \in kQ_{n-2}(a, z)$ with $z \in Q_0$. Write $\rho = \sum_{i=1}^{s} \lambda_i\beta_i\alpha_i$, where $\lambda_i \in k$, $\alpha_i \in Q_1(z, y_i)$ and $\beta_i \in Q_1(y_i, x)$ with $y_i \in Q_0$. By Lemma 3.2, we obtain

$$(\partial_a^{1-n} \circ \partial_a^{-n})(v \otimes \gamma) = \sum_{i=1}^{s}(\partial_a^{1-n} \circ \partial_a^{-n})\,(v \otimes \lambda_i\beta_i\alpha_i\delta) = v\left(\sum_{i=1}^{s}\lambda_i\bar{\beta}_i\bar{\alpha}_i\right) \otimes \delta = 0.$$

This establishes Statement (2).

Finally, assume that $\alpha_i : a \to b_i$, $i = 1, \ldots, r$ are the arrows in $Q_1(a, -)$. Then, $K_a^{-1} = \otimes_{i=1}^{r} P_{b_i} \otimes k\alpha_i$. Let Ω be a minimal generating set for R with $\rho_j : a \rightsquigarrow c_j$, $j = 1, \ldots, s$, the relations in $\Omega(a, -)$. Then, $K_a^{-2} = \oplus_{j=1}^{s} P_{c_j} \otimes k\rho_j$. Writing

$\rho_j = \sum_{j=1}^{r} \gamma_{ij}\alpha_i$ for some $\gamma_{ij} \in kQ_1(b_i, c_j)$, in view of Lemma 3.2, we obtain a commutative diagram

$$\begin{array}{ccccccccc}
P_{c_1} \oplus \cdots \oplus P_{c_s} & \xrightarrow{(P[\bar\gamma_{ij}])_{r\times s}} & P_{b_1} \oplus \cdots \oplus P_{b_r} & \xrightarrow{(P[\bar\alpha_1],\cdots,P[\bar\alpha_r])} & P_a & \xrightarrow{d_a} & S_a & \longrightarrow & 0 \\
\downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_a} & & \| & & \\
K_a^{-2} & \xrightarrow{\partial_a^{-2}} & K_a^{-1} & \xrightarrow{\partial_a^{-1}} & K_a^{0} & \xrightarrow{\partial_a^{0}} & S_a & \longrightarrow & 0
\end{array}$$

with f_a, f_1 and f_2 graded isomorphisms such that $f_a(e_a) = e_a \otimes \varepsilon_a$; $f_1(e_{b_i}) = e_{b_i} \otimes \alpha_i$ and $f_2(e_{c_j}) = e_{c_j} \otimes \rho_j$, for $i = 1, \ldots, r$; $j = 1, \ldots, s$. By Lemma 2.12, the lower row is a linear projective 2-presentation of S_a. The proof of the lemma is completed.

In the sequel, the linear complex $K_a^{\bullet}$ will be called the *local Koszul complex* of Λ at a. The following statement is a local version under the combinatorial setting of a well-known result in [5, (2.6.1)].

3.4. Theorem. *Let $\Lambda = kQ/R$, where Q is locally finite and R is quadratic. If $a \in Q_0$, then S_a has a linear projective resolution over* projΛ *if and only if $K_a^{\bullet}$ is a projective resolution of S_a.*

Proof. By Lemma 3.3, it suffices to show the necessity. Suppose that S_a has a linear projective resolution over projΛ. By Lemmas 2.11 and 3.3, there exists a commutative diagram

$$\begin{array}{ccccccccccccc}
\cdots \longrightarrow & P^{-p-1} & \xrightarrow{d^{-p-1}} & P^{-p} & \xrightarrow{d^{-p}} & P^{1-p} & \longrightarrow \cdots \longrightarrow & P^{-1} & \xrightarrow{d^{-1}} & P^{0} & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle f^{-p}} & & \downarrow{\scriptstyle f^{1-p}} & & \downarrow{\scriptstyle f^{-1}} & & \downarrow{\scriptstyle f^{0}} & & \\
\cdots \longrightarrow & K_a^{-p-1} & \xrightarrow{\partial_a^{-p-1}} & K_a^{-p} & \xrightarrow{\partial_a^{-p}} & K_a^{1-p} & \longrightarrow \cdots \longrightarrow & K_a^{-1} & \xrightarrow{\partial_a^{-1}} & K_a^{0} & \longrightarrow & 0,
\end{array}$$

where $p \geq 2$, the upper row is a linear projective resolution of S_a, and $f^{-p}, \cdots, f^0$ are graded isomorphisms. In particular, ∂_a^{-i} co-restricts to a J-minimal projective cover of $\mathrm{Ker}(\partial_a^{1-i})$, for $i = 1, \ldots, p$.

We claim that ∂_a^{-p-1} co-restricts to J-minimal projective cover of $\mathrm{Ker}(\partial_a^{-p})$. By Lemma 3.3(1), it suffices to show that it is surjective. We may assume that K_a^{-p} is non-zero. Then, $K_a^{-p} = \oplus_{j=1}^{n} P_{y_j} \otimes k\rho_j$, where $\rho_j \in R^{(p)}(a, y_j)$, $j = 1, \ldots, n$, form a basis of $R^{(p)}(a, -)$; while $K_a^{1-p} = \oplus_{i=1}^{m} P_{x_i} \otimes k\zeta_i$, where $\zeta_i \in R^{(p-1)}(a, x_i)$, $i = 1$, $\ldots$, m, form a basis of $R^{(p-1)}(a, -)$. Observe that $f^{-p} \circ d^{-p-1}$ is a J-minimal projective cover of $\mathrm{Ker}(\partial_a^{-p})$. By Lemma 2.3, $\mathrm{Ker}(\partial_a^{-p})$ admits a normalized J-top basis T^p. Since $f^{-p} \circ d^{-p-1}$ is homogeneous of degree one, T^p consists of homogeneous elements of degree one.

Consider $u = (u_1, \ldots, u_n) \in T^p \cap e_z K_a^{-p}$, where $z \in Q_0$ and $u_j \in P_{y_j} \otimes k\rho_j$. Then $u_j = \bar\gamma_j \otimes \rho_j$, where $\gamma_j \in kQ_1(y_j, z); j = 1, \ldots, n$. Since $\rho_j \in R^{(p)}(a, y_j)$, by Lemma 3.1(2), we may write $\rho_j = \sum_{i=1}^{m} \delta_{ij}\, \zeta_i$, where $\delta_{ij} \in kQ_1(x_i, y_j)$. Since $\partial_a^{-p}(u) = 0$, by Lemma 3.2, we obtain

$$\textstyle\sum_{i=1}^{m} (\sum_{j=1}^{n} \bar\gamma_j \bar\delta_{ij}) \otimes \zeta_i = \sum_{j=1}^{n} \sum_{i=1}^{m} \partial_a^{-p}(\bar\gamma_j \otimes \delta_{ij}\zeta_i) = 0.$$

Since the ζ_i are k-linearly independent, we deduce that $\sum_{j=1}^{n} \bar\gamma_j \bar\delta_{ij} = 0$. That is, $\eta_i = \sum_{j=1}^{n} \gamma_j \delta_{ij} \in R(x_i, z)$. Since R is quadratic, $\eta_i \in R_2(x_i, z)$, for $i = 1, \ldots, m$.

Setting $\omega = \sum_{j=1}^{n} \gamma_j$, we see that $\omega \in R^{(p+1)}(a, z)$. Indeed, $\omega = \sum_{i=1}^{m} \eta_i \zeta_i$ with $\eta_i \in R_2(x_i, z)$ and $\zeta_i \in kQ_{p-1}(a, x_i)$; and for $0 \leq s < p-1$, since $\rho_j \in R^{(p)}(a, y_j)$, we may write $\omega = \sum \mu_l \xi_l \delta_l$, where $\mu_l \in kQ_{p-2-s}(-, y_j)$, $\xi_l \in R_2$ and $\delta_l \in kQ_s(a, -)$. In particular, $e_z \otimes \omega \in K_a^{-p-1}$.

Let f_i be the composite of $\partial^{-p-1}(y_i, z) : P_z \otimes R^{(p+1)}(a, z) \to P_{y_i} \otimes R^{(p)}(a, y_i)$ and the canonical projection $p_i : P_{y_i} \otimes R^{(p)}(a, y_i) \to P_{y_i} \otimes k\rho_i$, for $i = 1, \ldots, n$. Since $\gamma_j \in kQ_1(y_j, z)$, we deduce from Lemma 3.2 that

$$f_i(e_z \otimes \omega) = p_i(\textstyle\sum_{j=1}^n \partial_a^{-p-1}(y_i, z)(e_z \otimes \gamma_j \rho_j)) = p_i(\sum_{y_j = y_i} \bar{\gamma}_j \otimes \rho_j) = \bar{\gamma}_i \otimes \rho_i = u_i,$$

and hence, $\partial_a^{-p-1}(e_z \otimes \omega) = (f_1(e_z \otimes \omega), \ldots, f_n(e_z \otimes \omega)) = (u_1, \ldots, u_n) = u$. This is establishes our claim. By Lemma 2.11, we obtain a graded isomorphism $f^{-p-1} : P^{-p-1} \to K_a^{-p-1}$ such that $f^{-p} \circ d^{-p-1} = \partial_a^{-p-1} \circ f^{-p-1}$. By induction, $K_a^{\bullet}$ is a projective resolution of S_a. The proof of the theorem is completed.

The classical quadratic dual of a quadratic algebra is defined by the tensor algebra of the dual space of the generating space under the left finiteness condition; see [5, (2.8.1)]. We shall define the quadratic dual of Λ by the opposite quiver $Q^{\rm o}$. For this, we need some preparation. Given $n \geq 0$, the finite basis Q_n of kQ_n has a dual basis $\{\xi^* \mid \xi \in Q_n\}$ in $D(kQ_n)$. Given $\gamma = \sum \lambda_i \xi_i$, where $\lambda_i \in k$ and $\xi_i \in Q_n$, we shall write $\gamma^* = \sum \lambda_i \xi_i^* \in D(kQ_n)$. This yields a k-linear isomorphism

$$\psi_n : kQ_n^{\rm o} \to D(kQ_n) : \gamma^{\rm o} \to \gamma^*.$$

Given $\xi \in kQ_n(x, y)$ with $x, y \in Q_0$, by abuse of notation, we shall identify ξ^* with its restriction to $kQ_n(x, y)$. In this way, $\{\xi^* \mid \xi \in Q_n(x, y)\}$ is the dual basis of $Q_n(x, y)$ in $D(kQ_n(x, y))$. We collect some basic properties as follows.

3.5. Lemma. *Let Q be a locally finite quiver with $\zeta \in kQ_s(x, y)$ and $\gamma \in kQ_t(y, z)$, for some $x, y, z \in Q_0$ and $s, t \geq 0$.*

(1) *If $\delta \in kQ_s$ and $\xi \in kQ_t$, then $(\gamma\zeta)^*(\xi\delta) = \gamma^*(\xi)\zeta^*(\delta)$.*

(2) *If $\gamma \in Q_1(y, z)$, then $(\gamma\zeta)^*(\eta) = \zeta^*(\partial_\gamma(\eta))$ for all $\eta \in kQ_{s+1}$.*

Proof. We may assume that $\zeta \in Q_s(x, y)$ and $\gamma \in Q_t(y, z)$. To prove Statement (1), we may assume $\delta \in Q_s$ and $\xi \in Q_t$. If $(\gamma\zeta)^*(\xi\delta) = 1$, then $\xi\delta = \gamma\zeta$. Since ξ and γ are of the same length, $\xi = \gamma$ and $\delta = \zeta$. Thus, $\gamma^*(\xi)\zeta^*(\delta) = 1$. If $(\gamma\zeta)^*(\xi\delta) = 0$, then $\xi\delta \neq \gamma\zeta$. In particular, $\xi \neq \gamma$ or $\delta \neq \zeta$, and hence, $\gamma^*(\xi)\zeta^*(\delta) = 0$.

Next, assume that $\gamma \in Q_1(y, z)$. To prove Statement (2), we may assume that $\eta \in Q_{s+1}$. Write $\eta = \alpha\delta$, for some $\alpha \in Q_1$ and $\delta \in Q_s$. By Statement (1), we see that $(\gamma\zeta)^*(\eta) = \gamma^*(\alpha)\zeta^*(\delta)$. If $\alpha \neq \gamma$, then $(\gamma\zeta)^*(\eta) = 0 = \zeta^*(\partial_\gamma(\eta))$. Otherwise, $\delta = \partial_\gamma(\eta)$, and hence, $(\gamma\zeta)^*(\eta) = \zeta^*(\partial_\gamma(\eta))$. The proof of the lemma is completed.

Let R be a quadratic ideal in kQ. For $x, y \in Q_0$, let $R_2^!(y, x)$ be the subspace of $kQ_2^{\rm o}(y, x)$ of elements $\rho^{\rm o}$, where $\rho \in kQ_2(x, y)$ such that ρ^* vanishes on $R_2(x, y)$. The ideal in $kQ^{\rm o}$ generated by the $R_2^!(y, x)$ with $x, y \in Q_0$ is denoted by $R^!$ and called the *quadratic dual* of R. The following statement describes explicitly $R^!$.

3.6. Lemma. *Let Q be a locally finite quiver and R be a quadratic ideal in kQ. If $\sigma \in kQ_n(x, y)$ with $x, y \in Q_0$ and $n \geq 0$, then $\sigma^{\rm o} \in R_n^!(y, x)$ if and only if $\sigma^* \in R^{(n)}(x, y)^\perp$, the perpendicular of $R^{(n)}(x, y)$ in $D(kQ_n(x, y))$.*

Proof. Let $\sigma \in kQ_n(x, y)$, with $x, y \in Q_0$ and $n \geq 0$. If $n = 0, 1$, then $R_n^!(y, x) = 0$, and since $R^{(n)}(x, y) = kQ_n(x, y)$, we have $R^{(n)}(x, y)^\perp = 0$. In case $n = 2$, since $R_2(x, y) = R^{(2)}(x, y)$, the lemma is the definition of $R_2^!(y, x)$. Let $n \geq 3$. Consider the k-isomorphism $\psi_n(x,y) : kQ_n^{\rm o}(y,x) \to D(kQ_n(x,y)) : \rho^{\rm o} \to \rho^*$. By definition, $R_n^!(y,x) = \sum_{j=0}^{n-2} R_{n,j}^!(y,x)$, where

$$R_{n,j}^!(y,x) = \textstyle\sum_{a,b\in Q_0} kQ_j^{\rm o}(a,x)\cdot R_2^!(b,a)\cdot kQ_{n-j-2}^{\rm o}(y,b),$$

and $R^{(n)}(x,y) = \cap_{j=0}^{n-2} R^{(n,j)}(x,y)$, where

$$R^{(n,j)}(x,y) = \textstyle\sum_{a,b\in Q_0} kQ_{n-j-2}(b,y)\cdot R_2(a,b)\cdot kQ_j(x,a).$$

First, assume that $\sigma^{\rm o} \in R_n^!(y,x)$. To show that $\sigma^* \in R^{(n)}(x,y)^\perp$, we may assume that $\sigma^{\rm o} \in kQ_j^{\rm o}(a,x)\cdot R_2^!(b,a)\cdot kQ_{n-j-2}^{\rm o}(y,b)$, for some $a,b \in Q_0$ and $0 \le j \le n-2$. Furthermore, we may assume that $\sigma^{\rm o} = (\delta\eta\gamma)^{\rm o}$, where $\gamma \in kQ_j(x,a)$, $\eta \in kQ_2(a,b)$ with $\eta^{\rm o} \in R_2^!(b,a)$, and $\delta \in kQ_{n-2-j}(b,y)$. Given any $w \in R^{(n)}(x,y)$, we may write

$$w = \textstyle\sum_{i=1}^t \delta_i\eta_i\gamma_i,$$

where $\gamma_i \in kQ_j(x,a_i)$, $\eta_i \in R_2(a_i,b_i)$, $\delta_i \in k_{n-j-2}Q(b_i,y)$, and $a_i, b_i \in Q_0$. Since $\eta^* \in R_2(a,b)^\perp$, we see that $\eta^*(\eta_i) = 0$ for all $1 \le i \le t$. By Lemma 3.5(1), $\sigma^*(w) = (\delta\eta\gamma)^*(w) = \sum_{i=1}^t \delta^*(\delta_i)\,\eta^*(\eta_i)\,\gamma^*(\gamma_i) = 0$. Therefore, $\sigma^* \in R^{(n)}(x,y)^\perp$.

Next, assume that $\sigma^* \in R^{(n)}(x,y)^\perp$. By Lemma 1.4(1), $\sigma^* \in \sum_{j=0}^{n-2} R^{(n,j)}(x,y)^\perp$. Since $D(kQ_n(x,y)) = \{\rho^* \mid \rho \in kQ_n(x,y)\}$, we may assume that $\sigma^* \in R^{(n,p)}(x,y)^\perp$, for some $0 \le p \le n-2$. Write $\sigma = \sum_{i=1}^m \sigma_i$, with $\sigma_i \in kQ_{n-p-2}(b_i,y)\cdot kQ_2(a_i,b_i)\cdot kQ_p(x,a_i)$, where $a_i, b_i \in Q_0$ such that $(a_i,b_i) \ne (a_j,b_j)$ for $i \ne j$. By Lemma 3.5(1), we see that σ_i^* vanishes on

$$kQ_{n-p-2}(b_j,y)\cdot kQ_2(a_j,b_j)\cdot kQ_p(x,a_j),$$

for any $i \ne j$. Therefore, $\sigma_i^* \in R^{(n,p)}(x,y)^\perp$, for $i = 1,\ldots,m$. Thus, we may assume that $\sigma = \delta\zeta\gamma$, where $\delta \in kQ_{n-p-2}(b,y)$, $\zeta \in kQ_2(a,b)$, $\gamma \in kQ_p(x,a)$, for some $a,b \in Q_0$, such that σ^* is non-zero. Then, δ^* and γ^* are non-zero, and hence, $\delta_i^*(\nu) = \gamma_i^*(\mu) = 1$, for some $\nu \in kQ_{n-p-2}(b,y)$ and $\mu \in kQ_p(x,a)$.

Choose a basis $\{\rho_1,\ldots,\rho_r;\rho_{r+1},\ldots,\rho_s\}$ of $kQ_2(a,b)$, where $\{\rho_1,\ldots,\rho_r\}$ is a basis of $R_2(a,b)$. Then, $kQ_2(a,b)$ has a basis $\{\eta_1,\ldots,\eta_r;\eta_{r+1},\ldots,\eta_s\}$ such that $\{\eta_1^*,\ldots,\eta_r^*;\eta_{r+1}^*,\ldots,\eta_s^*\}$ is the dual basis of $\{\rho_1,\ldots,\rho_r;\rho_{r+1},\ldots,\rho_s\}$. Observe that $\{\eta_{r+1}^{\rm o},\ldots,\eta_s^{\rm o}\}$ is a basis of $R_2^!(b,a)$. Write $\zeta = \sum_{i=1}^s \lambda_i\eta_i$, where $\lambda_i \in k$. Then, $\sigma^* = \sum_{i=1}^s \lambda_i(\delta\eta_i\gamma)^*$. By Lemma 1.4, $\sigma^* \in (kQ_{n-p-2}(b,y)\cdot R_2(a,b)\cdot kQ_p(x,a))^\perp$. Given any $1 \le i \le r$, applying Lemma 3.5(1), we obtain

$$0 = \sigma^*(\nu\rho_i\mu) = \textstyle\sum_{j=1}^s \lambda_j(\delta\eta_j\gamma)^*(\nu\rho_i\mu) = \sum_{j=1}^s \lambda_j\delta^*(\nu)\eta_j^*(\rho_i)\gamma^*(\mu) = \lambda_i.$$

Thus, $\sigma^* = \sum_{i=r+1}^s \lambda_i(\delta\eta_i\gamma)^*$, and consequently, $\sigma = \sum_{i=r+1}^s \lambda_i(\gamma\eta_i\delta)$. This implies that $\sigma^{\rm o} = \sum_{i=r+1}^s \lambda_i\gamma^{\rm o}\eta_i^{\rm o}\delta^{\rm o} \in R_n^!(y,x)$. The proof of the lemma is completed.

We are ready to define the quadratic dual of a quadratic algebra; compare [20, page 69] and [5, (2.8.1)]

3.7. Definition. Let $\Lambda = kQ/R$, where Q is a locally finite quiver and R is a quadratic ideal in kQ. The *quadratic dual* of Λ is defined to be $\Lambda^! = kQ^{\rm o}/R^!$, where $Q^{\rm o}$ is the opposite quiver of Q and $R^!$ is the quadratic dual of R.

3.8. Proposition. *Let $\Lambda = kQ/R$, where Q is locally finite and R is quadratic. Then $\Lambda^!$ and $\Lambda^{\rm o}$ are quadratic algebras with $(\Lambda^!)^! = \Lambda$ and $(\Lambda^{\rm o})^! = (\Lambda^!)^{\rm o}$.*

Proof. By definition, $\Lambda^{\rm o}$ and $\Lambda^!$ are quadratic algebras such that $(\Lambda^!)^! = kQ/(R^!)^!$ and $(\Lambda^{\rm o})^! = kQ/(R^{\rm o})^!$. Fix $x, y \in Q_0$, and consider the k-linear isomorphism

$$\psi_2^{\rm o}(y,x) : kQ_2(x,y) \to D(kQ_2^{\rm o}(y,x)) : \gamma \to (\gamma^{\rm o})^*.$$

Given $\gamma, \rho \in kQ_2(x,y)$, it is easy to see that $(\gamma^{\rm o})^*(\rho^{\rm o}) = \rho^*(\gamma) = \gamma^*(\rho)$. By definition, $\gamma \in (R^!)^!_2(x,y)$ if and only if $(\gamma^{\rm o})^*(\rho^{\rm o}) = 0$, for all $\rho^{\rm o} \in R^!_2(y,x)$. That is, $\rho^*(\gamma) = 0$, for all $\rho^* \in R_2(x,y)^\perp$. Since $R_2(x,y)$ is finite dimensional, the latter condition is equivalent to $\gamma \in R_2(x,y)$. Thus, $(R^!)^! = R$, and hence, $(\Lambda^!)^! = \Lambda$.

Next, $\gamma \in (R^{\rm o})^!_2(x,y)$ if and only if $(\gamma^{\rm o})^*(\rho^{\rm o}) = 0$, for all $\rho^{\rm o} \in R^{\rm o}_2(y,x)$. That is, $\gamma^*(\rho) = 0$, for all $\rho \in R_2(x,y)$. This is equivalent to $\gamma^{\rm o} \in R^!_2(y,x)$, that is, $\gamma \in (R^!)^{\rm o}_2(x,y)$. Hence, $(R^{\rm o})^! = (R^!)^{\rm o}$, and thus, $(\Lambda^{\rm o})^! = k(Q^{\rm o})^{\rm o}/(R^!)^{\rm o} = (\Lambda^!)^{\rm o}$. The proof of the proposition is completed.

Remark. It is known that a left finite quadratic algebra is the right quadratic dual of its left quadratic dual; see [5, (2.8.1)].

We shall give an alternative description of the local Koszul complexes of Λ in terms of $\Lambda^!$. We need some notation for $\Lambda^!$. Write $e_x = \varepsilon_x + R^!$ and $P^!_x = \Lambda^! e_x$, for $x \in Q_0$; and $\gamma^! = \gamma^{\rm o} + R^!$, for $\gamma \in kQ$. Then, $\Lambda^!$ is graded as $\Lambda^! = \oplus_{n \ge 0} \Lambda^!_n$, where $\Lambda^!_n = \{\gamma^! \mid \gamma \in kQ_n\}$. Fix $a \in Q_0$. Given $\alpha \in Q_1(y,x)$, the right multiplication by $\bar\alpha$ yields a Λ-linear map $P[\bar\alpha] : P_x \to P_y$; and the right multiplication by $\alpha^!$ yields a k-linear map $P[\alpha^!]_a : e_a \Lambda^!_{n-1} e_y \to e_a \Lambda^!_n e_x$. We define a sequence $L^{\bullet}_a$ as follows:

$$\cdots \longrightarrow L_a^{-n} \xrightarrow{d_a^{-n}} L_a^{1-n} \longrightarrow \cdots \longrightarrow L_a^{-1} \xrightarrow{d_a^{-1}} L_a^0 \longrightarrow 0 \longrightarrow \cdots$$

with $L_a^{-n} = \oplus_{x \in Q_0} P_x \otimes D(e_a \Lambda^!_n e_x)$ for $n \ge 0$; and $d_a^{-n} = (d_a^{-n}(y,x))_{(y,x) \in Q_0 \times Q_0}$ for $n > 0$, where

$$d_a^{-n}(y,x) = \textstyle\sum_{\alpha \in Q_1(y,x)} P[\bar\alpha] \otimes DP[\alpha^!]_a : P_x \otimes D(e_a \Lambda^!_n e_x) \to P_y \otimes D(e_a \Lambda^!_{n-1} e_y).$$

3.9. Lemma. *Let $\Lambda = kQ/R$, where Q is a locally finite quiver and R is a quadratic ideal. If $a \in Q_0$, then $L^{\bullet}_a$ is isomorphic to the local Koszul complex of Λ at a.*

Proof. Fix $a, x \in Q_0$ and $n \ge 0$. Recall that $D(kQ_n(a,x)) = \{\gamma^* \mid \gamma \in kQ_n(a,x)\}$ and $e_a \Lambda^!_n e_x = \{\gamma^! = \gamma^{\rm o} + R^! \mid \gamma \in kQ_n(a,x)\}$. By Lemma 3.6, $\gamma^* \in R^{(n)}(a,x)^\perp$ if and only if $\gamma^{\rm o} \in R^!_n(x,a)$. Thus, we obtain a k-bilinear form

$$<-,->: R^{(n)}(a,x) \times e_a \Lambda^!_n e_x \to k : (\delta, \gamma^!) \mapsto \gamma^*(\delta),$$

which is non-degenerate on the right. If $\delta \in R^{(n)}(a,x)$ is non-zero, then $\gamma^*(\delta) \ne 0$, that is, $<\delta, \gamma^!> \ne 0$, for some $\gamma \in kQ_n(a,x)$. Hence, $<-,->$ is non-degenerate. This yields a k-linear isomorphism

$$\phi_n(a,x) : R^{(n)}(a,x) \to D(e_a \Lambda^!_n e_x) : \delta \to <\delta, ->.$$

We claim, for $x, y \in Q_0$ and $n > 0$, that

$$\begin{array}{ccc}
R^{(n)}(a,x) & \xrightarrow{\ \sum_{\alpha \in Q_1(y,x)} \partial_\alpha\ } & R^{(n-1)}(a,y) \\
{\scriptstyle \phi_n(a,x)}\downarrow & & \downarrow{\scriptstyle \phi_{n-1}(a,y)} \\
D(e_a \Lambda^!_n e_x) & \xrightarrow{\ \sum_{\alpha \in Q_1(y,x)} DP[\alpha^!]_a\ } & D(e_a \Lambda^!_{n-1} e_y)
\end{array}$$

commutes. Given $\delta \in R^{(n)}(a,x)$ and $\zeta \in kQ_{n-1}(a,y)$, by Lemma 3.5(2), we obtain

$$\begin{aligned}\textstyle\sum_{\alpha\in Q_1(y,x)} DP[\alpha^!]_a(\phi_n(a,x)(\delta))(\zeta^!) &= \textstyle\sum_{\alpha\in Q_1(y,x)} \phi_n(a,x)(\delta)(\zeta^!\alpha^!)\\ &= \textstyle\sum_{\alpha\in Q_1(y,x)} (\alpha\zeta)^*(\delta)\\ &= \textstyle\sum_{\alpha\in Q_1(y,x)} \zeta^*(\partial_\alpha(\delta))\\ &= [\phi_{n-1}(a,y)\textstyle\sum_{\alpha\in Q_1(x,y)}\partial_\alpha(\delta)](\zeta^!).\end{aligned}$$

Thus, we obtain a commutative diagram with vertical isomorphisms

$$\begin{array}{ccc}\oplus_{x\in Q_0} P_x\otimes R^{(n)}(a,x) & \xrightarrow{\partial_a^{-n}} & \oplus_{y\in Q_0} P_y\otimes R^{(n-1)}(a,y)\\ {\scriptstyle \oplus(1\otimes\phi_n(a,x))}\downarrow & & \downarrow{\scriptstyle \oplus(1\otimes\phi_{n-1}(a,y))}\\ \oplus_{x\in Q_0} P_x\otimes D(e_a\Lambda_n^! e_x) & \xrightarrow{d_a^{-n}} & \oplus_{y\in Q_0} P_y\otimes D(e_a\Lambda_{n-1}^! e_y),\end{array}$$

for every $n>0$. The proof of the lemma is completed.

The following result is a generalization of Proposition 2.9.1 in [5], where Λ is assumed to be left finite; see also [25, Theorem 30].

3.10. THEOREM. *Let $\Lambda = kQ/R$, where Q is a locally finite quiver and R is a quadratic ideal. Then Λ is Koszul if and only if $\Lambda^!$ is Koszul.*

Proof. By Proposition 3.8, $(\Lambda^!)^! = \Lambda$. Thus, it suffices to prove the necessity. Suppose that Λ is Koszul. Fix $a\in Q_0$. By Lemma 3.9, the local Koszul complex of $\Lambda^!$ at a is isomorphic to the sequence

$$L^{\bullet}: \quad \cdots \longrightarrow L^{-n} \xrightarrow{d^{-n}} L^{1-n} \longrightarrow \cdots \longrightarrow L^{-1} \xrightarrow{d^{-1}} L^0 \longrightarrow 0 \longrightarrow \cdots$$

with $L^{-n} = \oplus_{x\in Q_0} P_x^!\otimes D(e_a\Lambda_n e_x)$ and $d^{-n}=(d^{-n}(y,x))_{(y,x)\in Q_0\times Q_0}$, where

$$d^{-n}(y,x) = \textstyle\sum_{\alpha\in Q_1(x,y)} P[\alpha^!]\otimes DP[\bar\alpha]_a : P_x^!\otimes D(e_a\Lambda_n e_x)\to P_y^!\otimes D(e_a\Lambda_{n-1}e_y).$$

We claim, for $n>0$, that $L^{\cdot}$ is exact at the degree $-n$. Since d^{-n-1} and d^{-n} are homogeneous of degree one, by Lemma 1.10, it amounts to establish, for all $b\in Q_0$ and $s\in\mathbb{Z}$, the exactness of the sequence

$$(*) \quad \oplus_{x\in Q_0} e_b\Lambda_{s-1}^! e_x\otimes D(e_a\Lambda_{n+1}e_x) \xrightarrow{d_{s-1,b}^{-n-1}} \oplus_{y\in Q_0} e_b\Lambda_s^! e_y\otimes D(e_a\Lambda_n e_y)$$
$$\xrightarrow{d_{s,b}^{-n}} \oplus_{z\in Q_0} e_b\Lambda_{s+1}^! e_z\otimes D(e_a\Lambda_{n-1}e_z)$$

with $d_{s,b}^{-n} = (d_b^{-n}(z,y))_{(z,y)\in Q_0\otimes Q_0}$, where $d_{s,b}^{-n}(z,y) = \sum_{\alpha\in Q_1(y,z)} P[\alpha^!]_b\otimes DP[\bar\alpha]_a$.

If $s<0$, then $e_b\Lambda_s^! e_y = 0$, and hence, $(*)$ is exact. In case $s=0$, it becomes

$$0 \longrightarrow e_b\Lambda_0^! e_b\otimes D(e_a\Lambda_n e_b) \xrightarrow{d_{s,b}^{-n}} \oplus_{z\in Q_0} e_b\Lambda_1^! e_z\otimes D(e_a\Lambda_{n-1}e_z)$$

with $d_{s,b}^{-n} = (d_{s,b}^{-n}(z,b))_{z\in Q_0}$, where $d_{s,b}^{-n}(z,b) = \sum_{\alpha\in Q_1(b,z)} P[\alpha^!]_b\otimes DP[\bar\alpha]_a$.

Let $f\in D(e_a\Lambda_n e_b)$ be a non-zero function. In particular, $f(u\bar\beta)\neq 0$, for some $\beta\in Q_1(b,z)$, $u\in e_a\Lambda_{n-1}e_z$ and $z\in Q_0$. That is, $(DP[\bar\beta]_a)(f)(u)\neq 0$, and hence, $(DP[\bar\beta]_a)(f)\neq 0$. Now, $d_{s,b}^{-n}(z,b)(e_b\otimes f) = \sum_{\alpha\in Q_1(b,z)} \alpha^!\otimes (DP[\bar\alpha]_a)(f)$, which is non-zero since the $\alpha^!$ with $\alpha\in Q_1(b,z)$ are k-linearly independent. Thus, the sequence $(*)$ is exact.

It remains to consider the case $s>0$. Since Λ is Koszul, by Theorem 3.4, the complex $L_b^{\bullet}$ as stated in Lemma 3.9 is exact at degree $-s$. By Lemma 1.10, we obtain an exact sequence

$$(**)\qquad \oplus_{z\in Q_0}\, e_a\Lambda_{n-1}e_z\otimes D(e_b\Lambda^!_{s+1}e_z)\xrightarrow{d^{-s-1}_{b,n-1,a}}\oplus_{y\in Q_0}e_a\Lambda_n e_y\otimes D(e_b\Lambda^!_s e_y)$$
$$\xrightarrow{d^{-s}_{b,n,a}}\oplus_{x\in Q_0}e_a\Lambda_{n+1}e_x\otimes D(e_b\Lambda^!_{s-1}e_x),$$

where $d^{-s-1}_{b,n-1,a}=(\sum_{\alpha\in Q_1(y,z)}P[\bar\alpha]_a\otimes DP[\alpha^!]_b)_{(y,z)\in Q_0\times Q_0}$. Applying the duality D to the exact sequence $(**)$, we obtain an exact sequence which, by Lemma 1.3, is isomorphic to $(*)$. The proof of the theorem is completed.

Remark. In case Λ is Koszul, one calls $\Lambda^!$ the *Koszul dual* of Λ.

We conclude this section by studying when the opposite algebra of a Koszul algebra is Koszul. By Proposition 3.8, $(\Lambda^{\rm o})^! = (\Lambda^!)^{\rm o} = kQ/(R^!)^{\rm o}$. We fix some notation for $(\Lambda^!)^{\rm o}$. Write $\hat\gamma=\gamma+(R^!)^{\rm o}$ for $\gamma\in kQ$; but $e_x=\varepsilon_x+(R^!)^{\rm o}$ for $x\in Q_0$. Then $(\Lambda^!)^{\rm o}=\oplus_{n\ge 0}(\Lambda^!)^{\rm o}_n$, where $(\Lambda^!)^{\rm o}_n=\{\hat\gamma\mid\gamma\in kQ_n\}$. Fix $a\in Q_0$. Given $\alpha\in Q_1(y,x)$, taking the dual of the right multiplication by $\bar\alpha^{\rm o}$ yields a Λ-linear map $I[\bar\alpha]=DP[\bar\alpha^{\rm o}]:I_x\to I_y$, and the left multiplication by $\alpha^!$ yields a k-linear map $P^!_a(\alpha^!):e_x\Lambda^!_n e_a\to e_y\Lambda^!_{n+1}e_a$. We define a sequence $T^{\bullet}_a$ over $\operatorname{inj}\Lambda$ as follows:

$$\cdots\longrightarrow 0\longrightarrow T^0_a\xrightarrow{d^0_a}T^1_a\longrightarrow\cdots\longrightarrow T^n_a\xrightarrow{d^n_a}T^{n+1}_a\longrightarrow\cdots$$

with $T^n_a=\oplus_{x\in Q_0}I_x\otimes e_x\Lambda^!_n e_a$ and $d^n_a=(d^n_a(y,x))_{(y,x)\in Q_0\times Q_0}$ for $n\ge 0$, where

$$d^n_a(y,x)=\textstyle\sum_{\alpha\in Q_1(y,x)}I[\bar\alpha]\otimes P^!_a(\alpha^!):I_x\otimes e_x\Lambda^!_n e_a\to I_y\otimes e_y\Lambda^!_{n+1}e_a.$$

3.11. Lemma. *Let $\Lambda=kQ/R$, where Q is locally finite and R is quadratic. If $a\in Q_0$, then $T^{\bullet}_a$ is isomorphic to the dual of the local Koszul complex of $\Lambda^{\rm o}$ at a.*

Proof. Fix $a\in Q_0$. By Proposition 3.8 and Lemma 3.9, the local Koszul complex of $\Lambda^{\rm o}$ at a is isomorphic to the complex $L^{\bullet}$ as follows:

$$\cdots\longrightarrow L^{-n}\xrightarrow{d^{-n}}L^{1-n}\longrightarrow\cdots\longrightarrow L^{-1}\xrightarrow{d^{-1}}L^0\longrightarrow 0,$$

with $L^{-n}=\oplus_{y\in Q_0}P^{\rm o}_y\otimes D(e_a(\Lambda^!)^{\rm o}_n e_y)$ and $d^{-n}=(d^{-n}(x,y))_{(x,y)\in Q_0\times Q_0}$, where

$$d^{-n}(x,y)=\textstyle\sum_{\alpha\in Q_1(y,x)}P[\bar\alpha^{\rm o}]\otimes DP[\hat\alpha]_a:P^{\rm o}_y\otimes D(e_a(\Lambda^!)^{\rm o}_{n+1}e_y)\to P^{\rm o}_x\otimes D(e_a(\Lambda^!)^{\rm o}_n e_x).$$

Since $e_a(\Lambda^!)^{\rm o}_n e_x$ is finite dimensional, we may compose the canonical k-isomorphism $D^2(e_a(\Lambda^!)^{\rm o}_n e_x)\to e_a(\Lambda^!)^{\rm o}_n e_x$ with the k-isomorphism $e_a(\Lambda^!)^{\rm o}_n e_x\to e_x\Lambda^!_n e_a$, sending $\hat\gamma$ to $\gamma^!$. This yields a k-isomorphism $\theta_n(a,x):D^2(e_a(\Lambda^!)^{\rm o}_n e_x)\to e_x\Lambda^!_n e_a$ such that

$$\begin{array}{ccc}
I_x\otimes D^2(e_a(\Lambda^!)^{\rm o}_n e_x) & \xrightarrow{I[\bar\alpha]\otimes DP[\hat\alpha]_a} & I_y\otimes D^2(e_a(\Lambda^!)^{\rm o}_{n+1}e_y)\\
{\scriptstyle 1\otimes\theta_n(a,x)}\downarrow & & \downarrow{\scriptstyle 1\otimes\theta_{n+1}(a,y)}\\
I_x\otimes e_x\Lambda^!_n e_a & \xrightarrow{I[\bar\alpha]\otimes P^!_a(\alpha^!)} & I_y\otimes e_y\Lambda^!_{n+1}e_a
\end{array}$$

commutes, for every $\alpha\in Q_1(y,x)$. Since the L^{-n} are finite direct sums, by Lemma 1.8(1), we see that $D(L^{\bullet})\cong T^{\bullet}_a$. The proof of the lemma is completed.

As another preparation, we need to consider the Yoneda Ext-groups in $\operatorname{Mod}\Lambda$ which are defined in a canonical way; see, for example, [18, Section III.5].

3.12. Lemma. *Let $\Lambda=kQ/R$ be a Koszul algebra. Then $\operatorname{Ext}^n_{\Lambda}(S_b,S_a)=e_b\Lambda^!_n e_a$, for all $a,b\in Q_0$ and $n\ge 0$.*

Proof. Let $a, b \in Q_0$. By Theorem 3.4 and Lemma 3.9, $L_b^{\bullet}$ is a J-minimal projective resolution of S_b. Thus, $\mathrm{Ext}^n_{\Lambda}(S_b, S_a) \cong \mathrm{Hom}_{\Lambda}(L_b^{-n}, S_a)$ for $n \geq 0$; see [18, (III.6.4)]. Since $e_b \Lambda^!_n e_a$ is finite dimensional, we deduce from Proposition 2.1(3) that

$$\mathrm{Ext}^n_{\Lambda}(S_b, S_a) \cong \mathrm{Hom}_{\Lambda}(P_a \otimes D(e_b \Lambda^!_n e_a), S_a) \cong \mathrm{Hom}_k(D(e_b \Lambda^!_n e_a), k) \cong e_b \Lambda^!_n e_a.$$

The proof of the lemma is completed.

In case Λ is locally finite dimensional, we obtain the following generalization of Proposition 2.2.1 stated in [5].

3.13. THEOREM. *Let $\Lambda = kQ/R$ be a locally finite dimensional qudratic algebra. The following statements are equivalent.*

(1) *The algebra Λ is Koszul.*
(2) *The opposite algebra Λ^{o} is Koszul.*
(3) *The complex $T_a^{\bullet}$ is an injective co-resolution of S_a, for every $a \in Q_0$.*

Proof. By Proposition 1.6(3), Λ is strongly locally finite dimensional, and by Proposition 2.5, $T_a^{\bullet}$ is a complex of injective modules. First, assume that $T_a^{\bullet}$ is an injective co-resolution of S_a for every $a \in Q_0$. Since Λ^{o} is locally finite dimensional, by Lemmas 1.8 and 3.11, every local Koszul complex of Λ^{o} is exact at all non-zero degrees. By Theorem 3.4, Λ^{o} is Koszul. Thus, Statement (3) implies Statement (2).

It suffices to show that Statement (1) implies Statement (3). Assume that Λ is Koszul. Fix $a \in Q_0$. Recall that $(T_a^{\bullet}, d^{\bullet})$ is defined by $T_a^i = \oplus_{x \in Q_0} I_x \otimes e_x \Lambda^!_i e_a$ and $d^i = (d^i(y,x))_{(y,x) \in Q_0 \times Q_0}$ for $i \geq 0$, where

$$d^i(y,x) = \textstyle\sum_{\alpha \in Q_1(y,x)} I[\bar{\alpha}] \otimes P^!_a(\alpha^!) : I_x \otimes e_x \Lambda^!_i e_a \to I_y \otimes e_y \Lambda^!_{i+1} e_a.$$

In particular, $T_a^0 = I_a \otimes k e_a$ and $T_a^1 = \oplus_{j=1}^s I_{b_j} \otimes k \beta_j^!$, where $\beta_j : b_j \to a$, $j = 1, \ldots, s$, are the arrows in $Q_1(-, a)$. Consider the Λ-linear morphism $d^{-1} : S_a \to T^0$, sending $e_a + Je_a$ to $e_a^{\star} \otimes e_a$. By Corollary 2.10, we have an exact sequence

$$0 \longrightarrow S_a \xrightarrow{d^{-1}} T_a^0 \xrightarrow{d^0} T_a^1 \longrightarrow \cdots \longrightarrow T_a^{n-1} \xrightarrow{d^{n-1}} T_a^n \xrightarrow{p^n} C^{n+1} \longrightarrow 0,$$

for some $n \geq 1$, such that $d^i = j^{i+1} p^i$, where $p^i : T_a^i \to C^{i+1}$ is the cokernel of d^{i-1}, and $j^{i+1} : C^{i+1} \to T_a^{i+1}$ is an injective envelope, for $i = 0, 1, \ldots, n-1$. Let $y \in Q_0$. It is well-known; see the proof of [18, (III.6.4)], and also [18, (III.8.2)], that

$$\mathrm{Ext}^{n+1}_{\Lambda}(S_y, S_a) \cong \mathrm{Hom}_{\Lambda}(S_y, C^{n+1}) / \mathrm{Im}(\mathrm{Hom}_{\Lambda}(S_y, p^n)).$$

On the other hand, applying $\mathrm{Hom}_{\Lambda}(S_y, -)$ to the short exact sequence

$$0 \longrightarrow C^n \xrightarrow{j^n} T_a^n \xrightarrow{p^n} C^{n+1} \longrightarrow 0,$$

we obtain an exact sequence

$$\mathrm{Hom}_{\Lambda}(S_y, C^n) \xrightarrow{j^n_*} \mathrm{Hom}_{\Lambda}(S_y, T_a^n) \xrightarrow{p^n_*} \mathrm{Hom}_{\Lambda}(S_y, C^{n+1}) \longrightarrow \mathrm{Ext}^{n+1}_{\Lambda}(S_y, S_a) \longrightarrow 0.$$

Since S_y is simple, j^n_* is surjective. Thus, $\mathrm{Hom}_{\Lambda}(S_y, C^{n+1}) \cong \mathrm{Ext}^{n+1}_{\Lambda}(S_y, S_a)$. By Lemma 3.12, we obtain $\dim_k \mathrm{Hom}_{\Lambda}(S_y, C^{n+1}) = \dim_k e_y \Lambda^!_{n+1} e_a$, and consequently, $S_J(C^{n+1}) \cong \oplus_{y \in Q_0} S_y \otimes e_y \Lambda^!_{n+1} e_a$. Since $e_y \Lambda^!_{n+1} e_a$ is finite dimensional, so is $S_J(C^{n+1})$. By Corollary 2.8 and Lemma 1.7(3), $S_J(C^{n+1})$ is essential in C^{n+1}. Thus, we obtain an injective envelope $j^{n+1} : C^{n+1} \to \oplus_{y \in Q_0} I_y \otimes e_y \Lambda^!_{n+1} e_a = T_a^{n+1}$; see 2.9. We claim that the sequence

$$0 \longrightarrow S_a \xrightarrow{d^{-1}} T_a^0 \xrightarrow{d^0} \cdots \xrightarrow{d^{n-2}} T_a^{n-1} \xrightarrow{d^{n-1}} T_a^n \xrightarrow{d^n} T_a^{n+1}$$

is exact with $S_J(T_a^{n+1}) \subseteq \operatorname{Im}(d^n)$. It suffices to show that $\operatorname{Ker}(d^n) = \operatorname{Im}(d^{n-1})$. Indeed, set $g = j^{n+1}p_n : T_a^n \to T_a^{n+1}$. Since $d^n d^{n-1} = 0$ and j^{n+1} is an injective envelope, $d^n = hg$ for some Λ-linear morphism $h : T_a^{n+1} \to T_a^{n+1}$. Write

$$g = (g(z,x))_{(z,x)\in Q_0\times Q_0} : \oplus_{x\in Q_0} I_x \otimes e_x\Lambda_n^! e_a \to \oplus_{z\in Q_0} I_z \otimes e_z\Lambda_{n+1}^! e_a,$$

where $g(z,x) : I_x \otimes e_x\Lambda_n^! e_a \to I_z \otimes e_z\Lambda_{n+1}^! e_a$ is Λ-linear, and

$$h = (h(y,z))_{(y,z)\in Q_0\times Q_0} : \oplus_{z\in Q_0} I_z \otimes e_z\Lambda_{n+1}^! e_a \to \oplus_{y\in Q_0} I_y \otimes e_y\Lambda_{n+1}^! e_a,$$

where $h(y,z) : I_z \otimes e_z\Lambda_{n+1}^! e_a \to I_y \otimes e_y\Lambda_{n+1}^! e_a$ is Λ-linear.

Given $x, y, z \in Q_0$, choose a basis $\{\bar\alpha \mid \alpha \in Q_1(z,x)\} \cup \mathcal{U}_{z,x}$ of e_xJe_z, where $\mathcal{U}_{z,x}$ consists of homogeneous elements of degrees > 1, and a basis $\mathcal{V}_{y,z}$ of homogeneous elements of e_zJe_y. By Lemma 2.6, $h(y,y) = \mathbf{1}_{I_y} \otimes h_{e_y} + \sum_{v\in\mathcal{V}_{y,y}} I[v] \otimes h_v$, where h_{e_y}, h_v are k-linear maps, and $h(y,z) = \sum_{v\in\mathcal{V}_{y,z}} I[v] \otimes h_v$ in case $z \neq y$. Since g vanishes on $S_J(T_a)$, we obtain

$$g(z,x) = \textstyle\sum_{\alpha\in Q_1(z,x)} I[\bar\alpha] \otimes g_\alpha + \sum_{u\in\mathcal{U}_{z,x}} I[u] \otimes g_u,$$

where g_α, g_u are k-linear maps. In view of Lemma 2.6, we can write $d^n(y,x)$ as $d^n(y,x) = \sum_{\alpha\in Q_1(y,x)} I[\bar\alpha] \otimes (h_{e_y} \circ g_\alpha)$. By the uniqueness, $(h_{e_y} \circ g_\alpha) = P_a^!(\alpha^!)$, for every $\alpha \in Q_1(y,x)$. Thus, we may assume that $h(y,y) = \mathbf{1}_{I_y} \otimes h_{e_y}$, and $h(y,z) = 0$ for $z \neq y$. Fix some $y \in Q_0$. Let $w \in e_y\Lambda_{n+1}^! e_a$, say $w = \xi^!$ for some $\xi \in Q_{n+1}(y,a)$. Writing $\xi = \zeta\alpha$, where $\alpha \in Q_1(y,x)$ and $\zeta \in Q_n(x,a)$ for some $x \in Q_0$, we see that

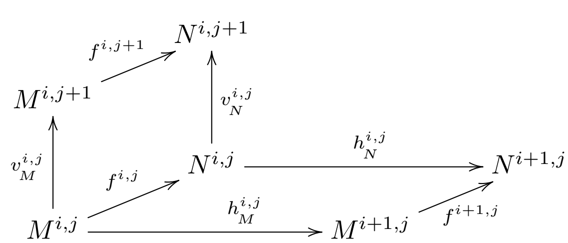
$$w = \alpha^!\zeta^! = P_a^!(\alpha^!)(\zeta^!) = h_{e_y}(g_\alpha(\zeta^!)).$$

Thus, h_{e_y} is surjective. Since $e_y\Lambda_{n+1}^! e_a$ is finite dimensional, h_{e_y} is bijective. Thus, h is a Λ-linear isomorphism. Then, $\operatorname{Ker}(d^n) = \operatorname{Ker}(g) = \operatorname{Ker}(p^n) = \operatorname{Im}(d^{n-1})$. Our claim is established. By induction, $T_a^\bullet$ is a minimal injective co-resolution of S_a. The proof of the theorem is completed.

4. Double complexes and extension of functors

The objective of this section is to provide tools for us to construct the Koszul duality. An additive category is called *concrete* if the objects are abelian groups and the morphisms are abelian group morphisms. Throughout this section, $\mathcal{A}, \mathcal{B}, \mathcal{C}$ stand for concrete additive categories, which are assumed to be full additive subcategories of concrete abelian categories.

Let $(M^{\bullet\bullet}, v_M^{\bullet\bullet}, h_M^{\bullet\bullet})$ be a double complex over $\mathcal{A}$, where $v_M^{\bullet\bullet}$ is the vertical differential and $h_M^{\bullet\bullet}$ is the horizontal one. We shall call $(M^{i,\bullet}, v_M^{i,\bullet})$ the *i-th column*, and $(M^{\bullet,j}, h_M^{\bullet,j})$ the *j-th row*, of $M^{\bullet\bullet}$. A *double complex morphism* $f^{\bullet\bullet} : M^{\bullet\bullet} \to N^{\bullet\bullet}$ consists of morphisms $f^{i,j} : M^{i,j} \to N^{i,j}$ in $\mathcal{A}$ making the diagram



commute, for $i, j \in \mathbb{Z}$, that is, $f^{i,\bullet} : M^{i,\bullet} \to N^{i,\bullet}$ and $f^{\bullet,j} : M^{\bullet,j} \to N^{\bullet,j}$ are complex morphisms, for $i, j \in \mathbb{Z}$. Thus, the double complexes over $\mathcal{A}$ form an additive category, written as $DC(\mathcal{A})$. Assume that $\mathcal{A}$ has countable direct sums. Given $M^{\bullet\bullet} \in DC(\mathcal{A})$, its *total complex* $\mathbb{T}(M^{\bullet\bullet})$ is defined by $\mathbb{T}(M^{\bullet\bullet})^n = \oplus_{i\in\mathbb{Z}} M^{i,n-i}$ and

$$d^n_{\mathbb{T}(M^{\bullet\bullet})} = (d^n_{\mathbb{T}(M^{\bullet\bullet})}(j,i))_{(j,i)\in\mathbb{Z}\times\mathbb{Z}} : \oplus_{i\in\mathbb{Z}} M^{i,n-i} \to \oplus_{j\in\mathbb{Z}} M^{j,n+1-j},$$

where

$$d^n_{\mathbb{T}(M^{\bullet\bullet})}(j,i) = \begin{cases} v_M^{i,n-i}, & j = i; \\ h_M^{i,n-i}, & j = i+1; \\ 0, & j \neq i, i+1. \end{cases}$$

Given a morphism $f^{\bullet\bullet} : M^{\bullet\bullet} \to N^{\bullet\bullet}$ in $DC(\mathcal{A})$, we put

$$\mathbb{T}(f^{\bullet\bullet})^n = (\mathbb{T}\,(f^{\bullet\bullet})^n(j,i))_{(j,i)\in\mathbb{Z}\times\mathbb{Z}} : \oplus_{i\in\mathbb{Z}} M^{i,n-i} \to \oplus_{j\in\mathbb{Z}} N^{j,n-j},$$

where

$$\mathbb{T}(f^{\bullet\bullet})^n(j,i) = \begin{cases} f^{i,n-i}, & j = i \\ 0, & j \neq i, \end{cases}$$

One verifies easily that $\mathbb{T}(f^{\bullet\bullet})^{n+1} \circ d^n_{\mathbb{T}(M^{\bullet\bullet})} = d^n_{\mathbb{T}(N^{\bullet\bullet})} \circ \mathbb{T}(f^{\bullet\bullet})^n$. This yields a morphism $\mathbb{T}(f^{\bullet\bullet}) = (\mathbb{T}(f^{\bullet\bullet})^n)_{n\in\mathbb{Z}} : \mathbb{T}(M^{\bullet\bullet}) \to \mathbb{T}(N^{\bullet\bullet})$ in $C(\mathcal{A})$, called the *total morphism* of $f^{\bullet\bullet}$.

4.1. LEMMA. *Let $\mathcal{A}$ be a concrete additive category with countable direct sums. The above construction yields a functor $\mathbb{T} : DC(\mathcal{A}) \to C(\mathcal{A})$.*

It is important to know when the total complex of a double complex is acyclic. We need some terminology. Let $M^{\bullet\bullet} \in DC(\mathcal{A})$. Given $n \in \mathbb{Z}$, the *n-diagonal* of $M^{\bullet\bullet}$ consists of the objects $M^{i,n-i}$, $i \in \mathbb{Z}$. We shall say that $M^{\bullet\bullet}$ is *n-diagonally bounded* (respectively, *bounded-above*, *bounded-below*) if $M^{i,n-i} = 0$ for all but finitely many (respectively, positive, negative) integers i. Moreover, $M^{\bullet\bullet}$ is called *diagonally bounded* (respectively, *bounded-above*, *bounded-below*) if it is n-diagonally bounded (respectively, bounded-above, bounded-below) for every $n \in \mathbb{Z}$. Finally, we say that $M^{\bullet\bullet}$ is *bounded* if there exists some $n > 0$ such that $M^{i,j} \neq 0$ only if $-n \le i, j \le n$.

4.2. LEMMA. *Let $\mathcal{A}$ be a concrete additive category with countable direct sums. Given $M^{\bullet\bullet} \in DC(\mathcal{A})$ and $n \in \mathbb{Z}$, we obtain $\mathrm{H}^n(\mathbb{T}(M^{\bullet\bullet})) = 0$ in case*

(1) *$M^{\bullet\bullet}$ is n-diagonally bounded-below with $\mathrm{H}^{n-j}(M^{\bullet,j}) = 0$ for all $j \in \mathbb{Z}$; or*

(2) *$M^{\bullet\bullet}$ is n-diagonally bounded-above with $\mathrm{H}^{n-i}(M^{i,\bullet}) = 0$ for all $i \in \mathbb{Z}$.*

Proof. Let $(M^{\bullet\bullet}, v^{\bullet\bullet}, h^{\bullet\bullet}) \in DC(\mathcal{A})$. We shall only consider the case where Statement (1) holds for some n. Then, there exists some $t < 0$ such that $M^{i,n-i} = 0$ for all $i < t$. Write $(X^\bullet, d^\bullet)$ for $\mathbb{T}(M^{\bullet\bullet})$. Consider $c = (c_{i,n-i})_{i\in\mathbb{Z}} \in \mathrm{Ker}(d^n)$, where $c_{i,n-i} \in M^{i,n-i}$. Then, $v^{i,n-i}(c_{i,n-i}) + h^{i-1,n-i+1}(c_{i-1,n-i+1}) = 0$, for $i \in \mathbb{Z}$. Since c has at most finitely many non-zero components, we may assume that $c_{i,n-i} = 0$ for all $i > 0$. Then, $h^{0,n}(c_{0,n}) = -v^{1,n-1}(c_{1,n-1}) = 0$. Since $\mathrm{H}^0(M^{\bullet,n}) = 0$, there exists some $x_{-1,n} \in M^{-1,n}$ such that $c_{0,n} = h^{-1,n}(x_{-1,n})$. This yields

$$h^{-1,n+1}(c_{-1,n+1} - v^{-1,n}(x_{-1,n})) = h^{-1,n+1}(c_{-1,n+1}) + v^{0,n}(c_{0,n}) = 0.$$

Since $\mathrm{H}^{-1}(M^{\bullet,n+1}) = 0$, we see that $c_{-1,n+1} - v^{-1,n}(x_{-1,n}) = h^{-2,n+1}(x_{-2,n+1})$, with $x_{-2,n+1} \in M^{-2,n+1}$. Continuing this process, we obtain $x_{i,n-1-i} \in M^{i,n-1-i}$ such that $c_{i,n-i} = v^{i,n-1-i}(x_{i,n-1-i}) + h^{i-1,n-i}(x_{i-1,n-i})$, for $i = -1, -2, \ldots, t$.

Since $M^{t-1,n-t+1}=0$, we see that $v^{t-1,n-t}(x_{t-1,n-t})=0=c_{t-1,n-1+1}$. Setting $x=(x_{i,n-1-i})_{i\in\mathbb{Z}}$, where $x_{i,n-1-i}=0$ for $i\geq 0$ or $i<t-1$, we obtain $c=d^{n-1}(x)$. The proof of the lemma is completed.

As an immediate consequence of Lemma 4.2, we obtain the promised generalization of the Acyclic Assembly Lemma stated, for example, in [31, (2.7.1)].

4.3. Proposition. *Let $\mathcal{A}$ be a concrete additive category with countable direct sums. If $M^{\bullet\bullet}\in DC(\mathcal{A})$, then $\mathbb{T}(M^{\bullet\bullet})$ is acyclic in case $M^{\bullet\bullet}$ is diagonally bounded-below with acyclic rows or diagonally bounded-above with acyclic columns.*

Now, we shall introduce a homotopy theory in $DC(\mathcal{A})$. Given a double complex $(M^{\bullet\bullet},v_{_M}^{\bullet\bullet},h_{_M}^{\bullet\bullet})$, we define its *horizontal shift* $M^{\bullet\bullet}[1]$ to be the double complex $(X^{\bullet\bullet},v_{_X}^{\bullet\bullet},h_{_X}^{\bullet\bullet})$ such that $X^{i,j}=M^{i+1,j}$, $v_{_X}^{i,j}=-v_{_M}^{i+1,j}$ and $h_{_X}^{i,j}=-h_{_M}^{i+1,j}$. We shall say that a morphism $f^{\bullet\bullet}:M^{\bullet\bullet}\to N^{\bullet\bullet}$ is *horizontally null-homotopic* if there exist $u^{i,j}:M^{i,j}\to N^{i-1,j}$, with $i,j\in\mathbb{Z}$, such that $u^{i+1,j}h_{_M}^{i,j}+h_{_N}^{i-1,j}u^{i,j}=f^{i,j}$ and $v_{_N}^{i-1,j}u^{i,j}+u^{i,j+1}v_{_M}^{i,j}=0$.

4.4. Lemma. *Let $\mathcal{A}$ be a concrete additive category with countable direct sums.*

(1) *If $M^{\bullet\bullet}\in DC(\mathcal{A})$, then $\mathbb{T}(M^{\bullet\bullet}[1])=\mathbb{T}(M^{\bullet\bullet})[1]$.*

(2) *If $f^{\bullet\bullet}:M^{\bullet\bullet}\to N^{\bullet\bullet}$ is horizontally null-homotopic, then $\mathbb{T}(f^{\bullet\bullet})$ is null-homotopic.*

Proof. We shall prove only Statement (2). Let $f^{\bullet\bullet}:M^{\bullet\bullet}\to N^{\bullet\bullet}$ be a horizontally null-homotopic morphism $DC(\mathcal{A})$. Let $u^{i,j}:M^{i,j}\to N^{i-1,j}$; $i,j\in\mathbb{Z}$ be morphisms such that $f^{i,j}=u^{i+1,j}\circ h_{_M}^{i,j}+h_{_N}^{i-1,j}\circ u^{i,j}$ and $v_{_N}^{i-1,j}u^{i,j}+u^{i,j+1}v_{_M}^{i,j}=0$. Define a morphism $h^n=(h^n(j,i))_{(j,i)\in\mathbb{Z}\times\mathbb{Z}}:\oplus_{i\in\mathbb{Z}}M^{i,n-i}\to\oplus_{j\in\mathbb{Z}}N^{j,n-j}$, where

$$h^n(j,i)=\begin{cases} u^{i,n-i}, & \text{if } j=n-i;\\ 0, & \text{if } j=n-i.\end{cases}$$

Given any $n,i,j\in\mathbb{Z}$, we obtain

$$\begin{aligned}\textstyle\sum_{p\in\mathbb{Z}} h^{n+1}(j,p)\circ d^n_{\mathbb{T}(M^{\bullet\bullet})}(p,i) &= h^{n+1}(j,j+1)\circ d^n_{\mathbb{T}(M^{\bullet\bullet})}(j+1,i)\\ &= \begin{cases} u^{i+1,n-i}\circ h_{_M}^{i,n-i}, & j=i;\\ u^{i,n+1-i}\circ v_{_M}^{i,n-i}, & j=i-1;\\ 0, & j\neq i,i-1,\end{cases}\end{aligned}$$

and

$$\begin{aligned}\textstyle\sum_{q\in\mathbb{Z}} d^{n-1}_{\mathbb{T}(N^{\bullet\bullet})}(j,q)\circ h^n(q,i) &= d^{n-1}_{\mathbb{T}(N^{\bullet\bullet})}(j,i-1)\circ h^n(i-1,i)\\ &= \begin{cases} h_{_N}^{i-1,n-i}\circ u^{i,n-i}, & j=i;\\ v_{_N}^{i-1,n-i}\circ u^{i,n-i}, & j=i-1;\\ 0, & j\neq i,i-1.\end{cases}\end{aligned}$$

This yields $\mathbb{T}(f^{\bullet\bullet})^n=h^{n+1}\circ d^n_{\mathbb{T}(M^{\bullet\bullet})}+d^{n-1}_{\mathbb{T}(N^{\bullet\bullet})}\circ h^n$. That is, $\mathbb{T}(f^{\bullet\bullet})$ is null-homotopic. The proof of the lemma is completed.

Let $f^{\bullet\bullet}:M^{\bullet\bullet}\to N^{\bullet\bullet}$ be a morphism in $DC(\mathcal{A})$. We define its *horizontal cone* $H_{f^{\bullet\bullet}}$ to be the double complex $(H^{\bullet\bullet},v^{\bullet\bullet},h^{\bullet\bullet})$ such that $H^{i,j}=M^{i+1,j}\oplus N^{i,j}$ and

$$v^{i,j}=\begin{pmatrix} -v_{_M}^{i+1,j} & 0\\ 0 & v_{_N}^{i,j}\end{pmatrix},\ h^{i,j}=\begin{pmatrix} -h_{_M}^{i+1,j} & 0\\ f^{\,i+1,j} & h_{_N}^{i,j}\end{pmatrix}.$$

This double complex is visualized as follows:

$$
\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \\
\cdots \longrightarrow & & M^{i+1,j+1} \oplus N^{i,j+1} & \xrightarrow{\begin{pmatrix} -h_{_M}^{i+1,j+1} & 0 \\ f^{\,i+1,j+1} & h_{_N}^{i,j+1} \end{pmatrix}} & M^{i+2,j+1} \oplus N^{i+1,j+1} & & \longrightarrow \cdots \\
 & \begin{pmatrix} -v_{_M}^{i+1,j} & 0 \\ 0 & v_{_N}^{i,j} \end{pmatrix} & \Big\uparrow & & \Big\uparrow & \begin{pmatrix} -v_{_M}^{i+2,j} & 0 \\ 0 & v_{_N}^{i+1,j} \end{pmatrix} & \\
\cdots \longrightarrow & & M^{i+1,j} \oplus N^{i,j} & \xrightarrow{\begin{pmatrix} -h_{_M}^{i+1,j} & 0 \\ f^{\,i+1,j} & h_{_N}^{i,j} \end{pmatrix}} & M^{i+2,j} \oplus N^{i+1,j} & & \longrightarrow \cdots \\
 & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & &
\end{array}
$$

whose j-th row is the mapping cone of $f^{\bullet,j} : M^{\bullet,j} \to N^{\bullet,j}$, for every $j \in \mathbb{Z}$. In a similar fashion, we may define the *vertical cone* $V_{f^{\bullet\bullet}}$ of $f^{\bullet\bullet}$ so that its i-th column is the mapping cone of $f^{i,\bullet} : M^{i,\bullet} \to N^{i,\bullet}$, for every $i \in \mathbb{Z}$.

4.5. Lemma. *Let $\mathcal{A}$ be a concrete additive category with countable direct sums. If $f^{\bullet\bullet} : M^{\bullet\bullet} \to N^{\bullet\bullet}$ is a morphism in $DC(\mathcal{A})$, then*

$$\mathbb{T}(H_{f^{\bullet\bullet}}) = C_{\mathbb{T}(f^{\bullet\bullet})} = \mathbb{T}(V_{f^{\bullet\bullet}}).$$

Proof. Let $f^{\bullet\bullet} : M^{\bullet\bullet} \to N^{\bullet\bullet}$ be a morphism in $DC(\mathcal{A})$. Given any $n \in \mathbb{Z}$, we obtain

$$\mathbb{T}(H_{f^{\bullet\bullet}})^n = \oplus_{i\in\mathbb{Z}}(M^{i+1,n-i} \oplus N^{i,n-i}) \ \text{ and } \ d^n_{\mathbb{T}(H_{f^{\bullet\bullet}})} = (d^n_{\mathbb{T}(H_{f^{\bullet\bullet}})}(j,i))_{(j,i)\in\mathbb{Z}\times\mathbb{Z}},$$

where $d^n_{\mathbb{T}(H_{f^{\bullet\bullet}})}(j,i) : M^{i+1,n-i} \oplus N^{i,n-i} \to M^{j+1,n+1-j} \oplus N^{j,n+1-j}$ is defined by

$$d^n_{\mathbb{T}(H_{f^{\bullet\bullet}})}(j,i) \;=\; \begin{cases} \begin{pmatrix} -v_{_M}^{i+1,n-i} & 0 \\ 0 & v_{_N}^{i,n-i} \end{pmatrix}, & j = i; \\ \begin{pmatrix} -h_{_M}^{i+1,n-i} & 0 \\ f^{\,i+1,n-i} & h_{_N}^{i,n-i} \end{pmatrix}, & j = i+1; \\ 0, & j \neq i, i+1. \end{cases}$$

On the other hand, $\mathbb{T}(f^{\bullet\bullet}) : \mathbb{T}(M^{\bullet\bullet}) \to \mathbb{T}(N^{\bullet\bullet})$ is a morphism in $C(\mathcal{A})$, whose mapping cone $C_{\mathbb{T}(f^{\bullet\bullet})}$ is defined by

$$C^n_{\mathbb{T}(f^{\bullet\bullet})} = \mathbb{T}(M^{\bullet\bullet})^{n+1} \oplus \mathbb{T}(N^{\bullet\bullet})^n = \oplus_{i\in\mathbb{Z}}(M^{i+1,n-i} \oplus N^{i,n-i}) = \mathbb{T}(H_{f^{\bullet\bullet}})^n,$$

and

$$d^n_{C_{\mathbb{T}(f^{\bullet\bullet})}} = \begin{pmatrix} -d^{n+1}_{\mathbb{T}(M^{\bullet\bullet})} & 0 \\ \mathbb{T}(f^{\bullet\bullet})^{n+1} & d^n_{\mathbb{T}(N^{\bullet\bullet})} \end{pmatrix} = (d^n_{C_{\mathbb{T}(f^{\bullet\bullet})}}(j,i))_{(j,i)\in\mathbb{Z}\times\mathbb{Z}},$$

where $d^n_{C_{\mathbb{T}(f^{\bullet\bullet})}}(j,i) : M^{i+1,n-i} \oplus N^{i,n-i} \to M^{j+1,n+1-j} \oplus N^{j,n+1-j}$ is defined to be

$$\begin{pmatrix} -d^{n+1}_{\mathbb{T}(M^{\bullet\bullet})}(j,i) & 0 \\ \mathbb{T}(f^{\bullet\bullet})^{n+1}(j,i) & d^n_{\mathbb{T}(N^{\bullet\bullet})}(j,i) \end{pmatrix} = \begin{cases} \begin{pmatrix} -v_{_M}^{i+1,n-i} & 0 \\ 0 & v_{_N}^{i,n-i} \end{pmatrix}, & j = i; \\ \begin{pmatrix} -h_{_M}^{i+1,n-i} & 0 \\ f^{\,i+1,n-i} & h_{_N}^{i,n-i} \end{pmatrix}, & j = i+1; \\ 0, & j \neq i, i+1. \end{cases}$$

Thus, $d^n_{C_{\mathbb{T}(f^{\bullet\bullet})}}(j,i) = d^n_{\mathbb{T}(H_{f^{\bullet\bullet}})}(j,i)$, for $i,j \in \mathbb{Z}$. This establishes the first part of the lemma, and the second part follows similarly. The proof of the lemma is completed.

As an application, we obtain a condition for the total morphism of double complex morphism is a quasi-isomorphism.

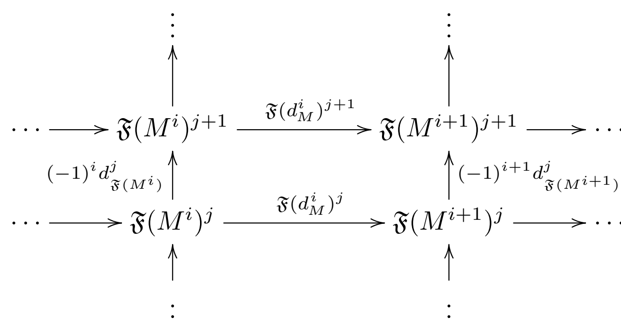
4.6. Lemma. *Let $\mathcal{A}$ be a concrete additive category with countable direct sums. Consider a morphism $f^{\bullet\bullet}: M^{\bullet\bullet} \to N^{\bullet\bullet}$ in $DC(\mathcal{A})$ such that $f^{i,\bullet}: M^{i,\bullet} \to N^{i,\bullet}$ is a quasi-isomorphism, for every $i \in \mathbb{Z}$. If $M^{\bullet\bullet}$ and $N^{\bullet\bullet}$ are diagonally bounded-above, then $\mathbb{T}(f^{\bullet\bullet})$ is a quasi-isomorphism.*

Proof. Assume that $M^{\bullet\bullet}$ and $N^{\bullet\bullet}$ are diagonally bounded-above. Then, the vertical cone $V_{f^{\bullet\bullet}}$ of $f^{\bullet\bullet}$ is also diagonally bounded-above. Given $i \in \mathbb{Z}$, since $f^{i,\bullet}: M^{i,\bullet} \to N^{i,\bullet}$ is a quasi-isomorphism, its cone, that is the i-th column of $V_{f^{\bullet\bullet}}$, is acyclic. By Proposition 4.3, $\mathbb{T}(V_{f^{\bullet\bullet}})$, that is $C_{\mathbb{T}(f^{\bullet\bullet})}$; see (4.5), is acyclic. Thus, $\mathbb{T}(f^{\bullet\bullet})$ is a quasi-isomorphism. The proof of the lemma is completed.

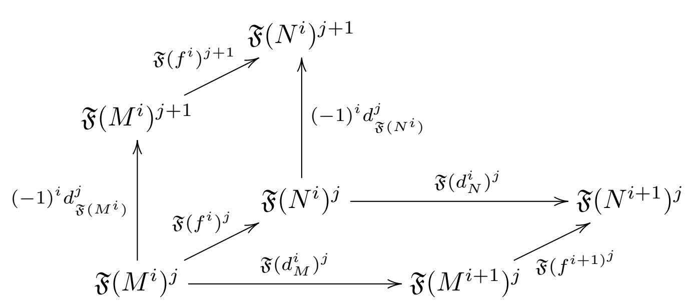
Let $\mathcal{B}$ have countable direct sums. Consider a functor

$$\mathfrak{F}: \mathcal{A} \to C(\mathcal{B}) : M \to \mathfrak{F}(M)^{\bullet}; f \mapsto \mathfrak{F}(f)^{\bullet}.$$

We shall extend it to to $C(\mathcal{A})$. First, we construct a functor $\mathfrak{F}^{DC}: C(\mathcal{A}) \to DC(\mathcal{B})$. Given a complex $M^{\bullet} \in C(\mathcal{A})$, applying $\mathfrak{F}$ to its components yields a double complex $\mathfrak{F}(M^{\bullet})^{\bullet}$ over $\mathcal{B}$ as follows:



whose i-th column is $\mathfrak{t}^i(\mathfrak{F}(M^i)^{\bullet})$, the i-th twist of $\mathfrak{F}(M^i)^{\bullet}$. Given a morphism $f^{\bullet}: M^{\bullet} \to N^{\bullet}$ in $C(\mathcal{A})$, we obtain a commutative diagram



for all $i,j \in \mathbb{Z}$. Thus, $\mathfrak{F}(f^{\bullet})^{\bullet} = (\mathfrak{F}(f^i)^j)_{i,j\in\mathbb{Z}} : \mathfrak{F}(M^{\bullet})^{\bullet} \to \mathfrak{F}(N^{\bullet})^{\bullet}$ is a morphism.

4.7. Proposition. *Let $\mathcal{A}, \mathcal{B}$ be concrete additive categories with $\mathcal{B}$ having countable direct sums. Then every functor $\mathfrak{F}: \mathcal{A} \to C(\mathcal{B})$ induces a functor*

$$\mathfrak{F}^{DC}: C(\mathcal{A}) \to DC(\mathcal{B}) : M^\bullet \mapsto \mathfrak{F}(M^\bullet)^\bullet; f^\bullet \mapsto \mathfrak{F}(f^\bullet)^\bullet.$$

(1) *If $M^\bullet$ is an object in $C(\mathcal{A})$, then $\mathfrak{F}^{DC}(M^\bullet[1]) = \mathfrak{F}^{DC}(M^\bullet)[1]$.*
(2) *If $f^\bullet$ is a morphism in $C(\mathcal{A})$, then $\mathfrak{F}^{DC}(C_{f^\bullet}) = H_{\mathfrak{F}^{DC}(f^\bullet)}$. Moreover, $\mathfrak{F}^{DC}(f^\bullet)$ is horizontally null-homotopic whenever $f^\bullet$ is null-homotopic.*

Proof. Statement (1) is evident. Let $f^\bullet : M^\bullet \to N^\bullet$ be a morphism in $C(\mathcal{A})$. Write its mapping cone as $(C^\bullet, d^\bullet_{_C})$. Then, $\mathfrak{F}(C^n)^\bullet = \mathfrak{F}(M^{n+1})^\bullet \oplus \mathfrak{F}(N^n)^\bullet$, and

$$d^\bullet_{\mathfrak{F}(C^n)} = \begin{pmatrix} d^\bullet_{\mathfrak{F}(M^{n+1})} & 0 \\ 0 & d^\bullet_{\mathfrak{F}(N^n)} \end{pmatrix}$$

and

$$\mathfrak{F}(d^n_{_C})^\bullet = \begin{pmatrix} -\mathfrak{F}(d^{n+1}_{_M})^\bullet & 0 \\ \mathfrak{F}(f^{n+1})^\bullet & \mathfrak{F}(d^n_{_N})^\bullet \end{pmatrix}.$$

Let $(H^{\bullet\bullet}, v^{\bullet\bullet}_{_H}, h^{\bullet\bullet}_{_H})$ be the horizontal cone of $\mathfrak{F}^{DC}(f^\bullet) : \mathfrak{F}^{DC}(M^\bullet) \to \mathfrak{F}^{DC}(N^\bullet)$. Then,

$$H^{i,j} = \mathfrak{F}(M^{i+1})^j \oplus \mathfrak{F}(N^i)^j = \mathfrak{F}(C^i)^j = \mathfrak{F}^{DC}(C^\bullet)^{i,j}$$

with horizontal differentials

$$h^{i,j}_{_H} = \begin{pmatrix} -\mathfrak{F}(d_{M^{i+1}})^j & 0 \\ \mathfrak{F}(f^{i+1})^j & \mathfrak{F}(d_{N^i})^j \end{pmatrix} = \mathfrak{F}(d^i_C)^j = h^{i,j}_{_{\mathfrak{F}^{DC}(C^\bullet)}}$$

and vertical differentials

$$v^{i,j}_{_H} = \begin{pmatrix} (-1)^i d^j_{\mathfrak{F}(M^{i+1})} & 0 \\ 0 & (-1)^i d^j_{\mathfrak{F}(N^i)} \end{pmatrix} = (-1)^i d^j_{\mathfrak{F}(C^i)} = v^{i,j}_{_{\mathfrak{F}^{DC}(C^\bullet)}}.$$

This shows that $C^\bullet = H^\bullet$, and the first part of Statement (2) is established. Suppose now that $f^\bullet$ is null-homotopic. Let $u^i : M^i \to N^{i-1}$ be morphisms such that $f^i = u^{i+1} \circ d^i_{_M} + d^{i-1}_{_N} \circ u^i$, for all $i \in \mathbb{Z}$. In particular, for any $j \in \mathbb{Z}$, we obtain

$$\mathfrak{F}(f^i)^j = \mathfrak{F}(u^{i+1})^j \circ \mathfrak{F}(d^i_{_M})^j + \mathfrak{F}(d^{i-1}_{_N})^j \circ \mathfrak{F}(u^i)^j.$$

Since $\mathfrak{F}(u^i)^\bullet : \mathfrak{F}(M^i) \to \mathfrak{F}(N^{i-1})^\bullet$ is a complex morphism, we obtain

$$(-1)^i \mathfrak{F}(u^i)^{j+1} \circ d^j_{\mathfrak{F}(M^i)} + (-1)^i d^j_{\mathfrak{F}(N^{i-1})} \circ \mathfrak{F}(u^i)^j = 0,$$

for all $j \in \mathbb{Z}$. Considering $\mathfrak{F}(u^i)^j : \mathfrak{F}(M^i)^j \to \mathfrak{F}(N^{i-1})^j$ with $i, j \in \mathbb{Z}$, we see that $\mathfrak{F}^{DC}(f^\bullet)$ is horizontally null-homotopic. The proof of the proposition is completed.

The following statement, which is a general version of Lemma 3.7 stated in [2], follows immediately from Lemma 4.5, Lemma 4.7 and Propositions 4.4.

4.8. Proposition. *Let $\mathcal{A}, \mathcal{B}$ be concrete additive categories with $\mathcal{B}$ having countable direct sums. Then, every functor $\mathfrak{F} : \mathcal{A} \to C(\mathcal{B})$ extends to a functor*

$$\mathfrak{F}^C = \mathbb{T} \circ \mathfrak{F}^{DC} : C(\mathcal{A}) \to C(\mathcal{B}).$$

(1) *If M is an object in $\mathcal{A}$, then $\mathfrak{F}^C(M) = \mathfrak{F}(M)^\bullet$.*
(2) *If $M^\bullet$ is a complex in $C(\mathcal{A})$, then $\mathfrak{F}^C(M^\bullet[1]) = \mathfrak{F}^C(M^\bullet)[1]$.*
(3) *If $f^\bullet$ is a morphism in $C(\mathcal{A})$, then $\mathfrak{F}^C(C_{f^\bullet}) = C_{\mathfrak{F}^C(f^\bullet)}$ and $\mathfrak{F}^C(f^\bullet)$ is null-homotopic whenever $f^\bullet$ is null-homotopic.*

REMARK. The method of extending a functor stated in Proposition 4.8 has benn already used by many authors under some special circumstances; see [2, 5, 16, 29],

The following result is essential for our construction of the Koszul duality.

4.9. THEOREM. *Let $\mathcal{A}, \mathcal{B}$ be concrete additive categories with $\mathcal{B}$ having countable direct sums. Let $\mathfrak{F} : \mathcal{A} \to C(\mathcal{B})$ be a functor such that $\mathfrak{F}^C$ sends a derivable subcategory $\mathscr{A}$ of $C(\mathcal{A})$ into a derivable subcategory $\mathscr{B}$ of $C(\mathcal{B})$.*

(1) *If $\mathfrak{F}$ is exact such that $\mathfrak{F}^{DC}$ sends complexes in $\mathscr{A}$ to diagonally bounded-below double complexes, then $\mathfrak{F}^C$ sends acyclic complexes in $\mathscr{A}$ to acyclic ones.*
(2) *If $\mathfrak{F}^C$ sends acyclic complexes in $\mathscr{A}$ to acyclic ones, then it induces a diagram*

$$\begin{array}{ccccc} \mathscr{A} & \longrightarrow & \mathcal{K}(\mathscr{A}) & \longrightarrow & \mathcal{D}(\mathscr{A}) \\ \mathfrak{F}^C \downarrow & & \downarrow \mathfrak{F}^K & & \downarrow \mathfrak{F}^D \\ \mathscr{B} & \longrightarrow & \mathcal{K}(\mathscr{B}) & \longrightarrow & \mathcal{D}(\mathscr{B}), \end{array}$$

which is commutative with $\mathfrak{F}^K$ and $\mathfrak{F}^D$ being triangle-exact.

Proof. (1) Let $\mathfrak{F}$ be exact such that, for every complex $M^\bullet \in \mathscr{A}$, the double complex $\mathfrak{F}(M^\bullet)^\bullet$ is diagonally bounded-below. If $M^\bullet$ is acyclic, then $\mathfrak{F}(M^\bullet)^\bullet$ has acyclic rows, and by Proposition 4.3, its total complex, that is $\mathfrak{F}^C(M^\bullet)$, is acyclic.

(2) By Proposition 4.8, there exists a triangle-exact functor $\mathfrak{F}^K : \mathcal{K}(\mathscr{A}) \to \mathcal{K}(\mathscr{B})$ making the left square commute. If $\mathfrak{F}^C$ sends acyclic complexes in $\mathscr{A}$ to acyclic ones in $\mathscr{B}$, then $\mathfrak{F}^K$ sends quasi-isomorphisms in $\mathcal{K}(\mathscr{A})$ to quasi-isomorphisms in $\mathcal{K}(\mathscr{B})$. Thus, there exists a triangle-exact functor $\mathfrak{F}^D : \mathcal{D}(\mathscr{A}) \to \mathcal{D}(\mathscr{B})$ making the right square commute. The proof of the theorem is completed.

We shall also need the following fact that the extension of functors is compatible with the composition of functors.

4.10. LEMMA. *Let $\mathfrak{F} : \mathcal{A} \to C(\mathcal{B})$ and $\mathfrak{G} : \mathcal{B} \to C(\mathcal{C})$ be functors, where $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are concrete additive categories. If $\mathcal{B}, \mathcal{C}$ have countable direct sums, then*

$$(\mathfrak{G}^C \circ \mathfrak{F})^C = \mathfrak{G}^C \circ \mathfrak{F}^C.$$

Proof. Assume that $\mathcal{B}, \mathcal{C}$ have countable direct sums. Fix $M^\bullet \in C(\mathcal{A})$. Given $n \in \mathbb{Z}$, by definition, we obtain $(\mathfrak{G}^C \circ \mathfrak{F})^C(M^\bullet)^n = \oplus_{i\in\mathbb{Z}} \mathfrak{G}^C(\mathfrak{F}(M^i)^\bullet)^{n-i}$ and $d^n_{(\mathfrak{G}^C\circ\mathfrak{F})^C(M^\bullet)} = (d^n_{(\mathfrak{G}^C\circ\mathfrak{F})^C(M^\bullet)}(j,i))_{(j,i)\in\mathbb{Z}\times\mathbb{Z}}$, where

$$d^n_{(\mathfrak{G}^C\circ\mathfrak{F})^C(M^\bullet)}(j,i) : \mathfrak{G}^C(\mathfrak{F}(M^i)^\bullet)^{n-i} \to \mathfrak{G}^C(\mathfrak{F}(M^j)^\bullet)^{n+1-j}$$

is given by

$$d^n_{(\mathfrak{G}^C\circ\mathfrak{F})^C(M^\bullet)}(j,i) = \begin{cases} (-1)^i d^{n-i}_{\mathfrak{G}^C(\mathfrak{F}(M^i)^\bullet)}, & j = i; \\ \mathfrak{G}^C(\mathfrak{F}(d^i_M)^\bullet)^{n-i}, & j = i+1 \\ 0, & j \neq i, i+1. \end{cases}$$

Furthermore, by definition, we obtain a diagram

$$\begin{array}{ccc} \mathfrak{G}^C(\mathfrak{F}(M^i)^\bullet)^{n-i} & \xrightarrow{\mathfrak{G}^C(\mathfrak{F}(d^i_M)^\bullet)^{n-i}} & \mathfrak{G}^C(\mathfrak{F}(M^{i+1})^\bullet)^{n-i} \\ \| & & \| \\ \oplus_{p\in\mathbb{Z}} \mathfrak{G}(\mathfrak{F}(M^i)^p)^{n-i-p} & \xrightarrow{\left(\mathfrak{G}^C(\mathfrak{F}(d^i_M)^\bullet)^{n-i}(q,p)\right)_{(q,p)\in\mathbb{Z}\times\mathbb{Z}}} & \oplus_{q\in\mathbb{Z}} \mathfrak{G}(\mathfrak{F}(M^{i+1})^q)^{n-i-q}, \end{array}$$

where

$$\mathfrak{G}^C(\mathfrak{F}(d_M^i)^\bullet)^{n-i}(q,p)=\begin{cases}\mathfrak{G}(\mathfrak{F}(d_M^i)^p)^{n-i-p}, & q=p;\\ 0, & q\neq p,\end{cases}$$

and a diagram

$$\begin{array}{ccc}\mathfrak{G}^C(\mathfrak{F}(M^i)^\bullet)^{n-i} & \xrightarrow{d^{n-i}_{\mathfrak{G}^C(\mathfrak{F}(M^i)^\bullet)}} & \mathfrak{G}^C(\mathfrak{F}(M^i)^\bullet)^{n+1-i}\\ \| & & \|\\ \oplus_{p\in\mathbb{Z}}\mathfrak{G}(\mathfrak{F}(M^i)^p)^{n-i-p} & \xrightarrow{\left(d^{n-i}_{\mathfrak{G}^C(\mathfrak{F}(M^i)^\bullet)}(q,p)\right)_{(q,p)\in\mathbb{Z}\times\mathbb{Z}}} & \oplus_{q\in\mathbb{Z}}\mathfrak{G}(\mathfrak{F}(M^i)^q)^{n+1-i-q},\end{array}$$

where

$$d^{n-i}_{\mathfrak{G}^C(\mathfrak{F}(M^i)^\bullet)}(q,p)=\begin{cases}(-1)^p d^{n-i-p}_{\mathfrak{G}(\mathfrak{F}(M^i)^p)}, & q=p;\\ \mathfrak{G}(d^p_{\mathfrak{F}(M^i)})^{n-i-p}, & q=p+1;\\ 0, & q\neq p,p+1.\end{cases}$$

Therefore, $(\mathfrak{G}^C\circ\mathfrak{F})^C(M^\bullet)$ is the complex described by the diagram

$$\begin{array}{ccc}(\mathfrak{G}^C\circ\mathfrak{F})^C(M^\bullet)^n & \xrightarrow{d^n_{(\mathfrak{G}^C\circ\mathfrak{F})^C(M^\bullet)}} & (\mathfrak{G}^C\circ\mathfrak{F})^C(M^\bullet)^{n+1}\\ \| & & \|\\ \oplus_{(i,p)\in\mathbb{Z}^2}\mathfrak{G}(\mathfrak{F}(M^i)^p)^{n-i-p} & \xrightarrow{(d^n_{(\mathfrak{G}^C\circ\mathfrak{F})^C(M^\bullet)}(j,q;i,p))_{(j,q;i,p)\in\mathbb{Z}^4}} & \oplus_{(j,q)\in\mathbb{Z}^2}\mathfrak{G}(\mathfrak{F}(M^j)^q)^{n+1-j-q}\end{array}$$

where

$$d^n_{(\mathfrak{G}^C\circ\mathfrak{F})^C(M^\bullet)}(j,q;i,p)=\begin{cases}(-1)^{i+p}d^{n-i-p}_{\mathfrak{G}(\mathfrak{F}(M^i)^p)}, & j=i;q=p;\\ (-1)^i\mathfrak{G}(d^p_{\mathfrak{F}(M^i)})^{n-i-p}, & j=i;q=p+1;\\ \mathfrak{G}(\mathfrak{F}(d_M^i)^p)^{n-i-p} & j=i+1,q=p;\\ 0, & \text{otherwise}.\end{cases}$$

Next, given any integer n, we obtain $\mathfrak{G}^C(\mathfrak{F}^C(M^\bullet))^n=\oplus_{s\in\mathbb{Z}}\mathfrak{G}(\mathfrak{F}^C(M^\bullet)^s)^{n-s}$ and $d^n_{\mathfrak{G}^C(\mathfrak{F}^C(M^\bullet))}=(d^n_{\mathfrak{G}^C(\mathfrak{F}^C(M^\bullet))}(t,s))_{(t,s)\in\mathbb{Z}\times\mathbb{Z}}$, where

$$d^n_{\mathfrak{G}^C(\mathfrak{F}^C(M^\bullet))}(t,s):\mathfrak{G}(\mathfrak{F}^C(M^\bullet)^s)^{n-s}\to\mathfrak{G}(\mathfrak{F}^C(M^\bullet)^t)^{n+1-t}$$

is given by

$$d^n_{\mathfrak{G}^C(\mathfrak{F}^C(M^\bullet))}(t,s)=\begin{cases}(-1)^s d^{n-s}_{\mathfrak{G}(\mathfrak{F}^C(M^\bullet)^s)}, & t=s;\\ \mathfrak{G}(d^s_{\mathfrak{F}^C(M^\bullet)})^{n-s}, & t=s+1;\\ 0, & t\neq s,s+1.\end{cases}$$

Furthermore, by definition, we obtain diagrams

$$\begin{array}{ccc}\mathfrak{G}(\mathfrak{F}^C(M^\bullet)^s)^{n-s} & \xrightarrow{d^{n-s}_{\mathfrak{G}(\mathfrak{F}^C(M^\bullet)^s)}} & \mathfrak{G}(\mathfrak{F}^C(M^\bullet)^s)^{n-s+1}\\ \| & & \|\\ \oplus_{i\in\mathbb{Z}}\mathfrak{G}(\mathfrak{F}(M^i)^{s-i})^{n-s} & \xrightarrow{(d^{n-s}_{\mathfrak{G}(\mathfrak{F}^C(M^\bullet)^s)}(j,i))_{(j,i)\in\mathbb{Z}\times\mathbb{Z}}} & \oplus_{j\in\mathbb{Z}}\mathfrak{G}(\mathfrak{F}(M^j)^{s-j})^{n-s+1},\end{array}$$

where

$$d^{n-s}_{\mathfrak{G}(\mathfrak{F}^C(M^\bullet)^s)}(j,i)=\begin{cases}d^{n-s}_{\mathfrak{G}(\mathfrak{F}(M^i)^{s-i})}, & j=i;\\ 0, & j\neq i,\end{cases}$$

and

$$\begin{array}{ccc}\mathfrak{G}\left(\mathfrak{F}^C(M^\bullet)^s\right)^{n-s} & \xrightarrow{\mathfrak{G}(d^s_{\mathfrak{F}^C(M^\bullet)})^{n-s}} & \mathfrak{G}\left(\mathfrak{F}^C(M^\bullet)^{s+1}\right)^{n-s} \\ \| & & \| \\ \oplus_{i\in\mathbb{Z}}\mathfrak{G}(\mathfrak{F}(M^i)^{s-i})^{n-s} & \xrightarrow{(\mathfrak{G}(d^s_{\mathfrak{F}^C(M^\bullet)})^{n-s}(j,i))_{(j,i)\in\mathbb{Z}\times\mathbb{Z}}} & \oplus_{j\in\mathbb{Z}}\mathfrak{G}(\mathfrak{F}(M^j)^{s+1-j})^{n-s},\end{array}$$

where

$$\mathfrak{G}(d^s_{\mathfrak{F}^C(M^\bullet)})^{n-s}(j,i) = \begin{cases} (-1)^i\,\mathfrak{G}(d^{s-i}_{\mathfrak{F}(M^i)})^{n-s}, & j=i; \\ \mathfrak{G}(\mathfrak{F}(d^i_M)^{s-i})^{n-s}, & j=i+1; \\ 0, & j\neq i,i+1. \end{cases}$$

Thus, $\mathfrak{G}^C(\mathfrak{F}^C(M^\bullet))$ is the complex described by the following diagram

$$\begin{array}{ccc}\mathfrak{G}^C(\mathfrak{F}^C(M^\bullet))^n & \xrightarrow{d^n_{\mathfrak{G}^C(\mathfrak{F}^C(M^\bullet))}} & \mathfrak{G}^C(\mathfrak{F}^C(M^\bullet))^{n+1} \\ \| & & \| \\ \oplus_{(i,s)\in\mathbb{Z}^2}\,\mathfrak{G}(\mathfrak{F}(M^i)^{s-i})^{n-s} & \xrightarrow{(d^n_{\mathfrak{G}^C(\mathfrak{F}^C(M^\bullet))}(j,t;i,s))_{(j,t;i,s)\in\mathbb{Z}^4}} & \oplus_{(j,t)\in\mathbb{Z}^2}\,\mathfrak{G}(\mathfrak{F}(M^j)^{t-j})^{n+1-t}\end{array}$$

where

$$d^n_{\mathfrak{G}^C(\mathfrak{F}^C(M^\bullet))}(j,t;i,s) = \begin{cases} (-1)^s d^{n-s}_{\mathfrak{G}(\mathfrak{F}(M^i)^{s-i})}, & t=s, j=i; \\ (-1)^i\mathfrak{G}(d^{s-i}_{\mathfrak{F}(M^i)})^{n-s}, & t=s+1, j=i; \\ \mathfrak{G}(\mathfrak{F}(d^i_M)^{s-i})^{n-s}, & t=s+1, j=i+1; \\ 0, & \text{otherwise.} \end{cases}$$

Setting $p=s-i$ and $q=t-j$, we see that $\mathfrak{G}^C(\mathfrak{F}^C(M^\bullet))$ is also described by

$$\begin{array}{ccc}\mathfrak{G}^C(\mathfrak{F}^C(M^\bullet))^n & \xrightarrow{d^n_{\mathfrak{G}^C(\mathfrak{F}^C(M^\bullet))}} & \mathfrak{G}^C(\mathfrak{F}^C(M^\bullet))^{n+1} \\ \| & & \| \\ \oplus_{(i,p)\in\mathbb{Z}^2}\mathfrak{G}(\mathfrak{F}(M^i)^p)^{n-i-p} & \xrightarrow{(d^n(j,q;i,p))_{(j,q;i,p)\in\mathbb{Z}^4}} & \oplus_{(j,q)\in\mathbb{Z}^2}\mathfrak{G}(\mathfrak{F}(M^j)^q)^{n+1-j-q}\end{array}$$

where

$$\begin{aligned} d^n(j,q;i,p) &= d^n_{\mathfrak{G}^C(\mathfrak{F}^C(M^\bullet))}(j,q+j;i,p+i) \\ &= \begin{cases} (-1)^{p+i} d^{n-i-p}_{\mathfrak{G}(\mathfrak{F}(M^i)^p)}, & q=p, j=i; \\ (-1)^i\mathfrak{G}(d^{s-i}_{\mathfrak{F}(M^i)})^{n-i-p}, & q=p+1, j=i; \\ \mathfrak{G}(\mathfrak{F}(d^i_M)^p)^{n-i-p}, & q=p, j=i+1; \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Thus, we see that $(\mathfrak{G}^C\circ\mathfrak{F})^C(M^\bullet) = (\mathfrak{G}^C\circ\mathfrak{F}^C)(M^\bullet)$. Similarly, we can verify that $(\mathfrak{G}^C\circ\mathfrak{F})^C(f^\bullet) = (\mathfrak{G}^C\circ\mathfrak{F}^C)(f^\bullet)$, for every morphism $f^\bullet: M^\bullet\to N^\bullet$ in $C(\mathcal{A})$. The proof of the proposition is completed.

To conclude this section, we shall study how to extend functorial morphisms.

4.11. LEMMA. *Let $\mathfrak{F},\mathfrak{G}:\mathcal{A}\to C(\mathcal{B})$ be functors, where $\mathcal{A},\mathcal{B}$ are concrete additive categories with $\mathcal{B}$ having countable direct sums. Then, every functorial morphism $\eta:\mathfrak{F}\to\mathfrak{G}$ induces functorial morphisms $\eta^{DC}:\mathfrak{F}^{DC}\to\mathfrak{G}^{DC}$ and $\eta^C:\mathfrak{F}^C\to\mathfrak{G}^C$.*

Proof. Let $\eta=(\eta^\bullet_M)_{M\in\mathcal{A}}:\mathfrak{F}\to\mathfrak{G}$ be a functorial morphism. Fix $M^\bullet\in C(\mathcal{A})$. Given $i,j\in\mathbb{Z}$, since $\eta^\bullet_M$ is natural in M, we obtain a commutative diagram

$$\begin{array}{ccccc}
 & & \mathfrak{G}(M^i)^{j+1} & & \\
 & \nearrow^{\eta^{j+1}_{M^i}} & \uparrow{\scriptstyle (-1)^i d^j_{\mathfrak{G}(M^i)}} & & \\
\mathfrak{F}(M^i)^{j+1} & & & & \\
\uparrow{\scriptstyle (-1)^i d^j_{\mathfrak{F}(M^i)}} & & \mathfrak{G}(M^i)^j & \xrightarrow{\mathfrak{G}(d^i_M)^j} & \mathfrak{G}(M^{i+1})^j. \\
 & \nearrow^{\eta^j_{M^i}} & & \nearrow^{\eta^j_{M^{i+1}}} & \\
\mathfrak{F}(M^i)^j & \xrightarrow{\mathfrak{F}(d^i_M)^j} & \mathfrak{F}(M^{i+1})^j & &
\end{array}$$

This yields a morphism $\eta^{\bullet}_{M^{\bullet}} = (\eta^j_{M^i})_{i,j\in\mathbb{Z}} : \mathfrak{F}^{DC}(M^{\bullet}) \to \mathfrak{G}^{DC}(M^{\bullet})$ in $DC(\mathcal{B})$ and a morphism $\eta_{M^{\bullet}} = \mathbb{T}(\eta^{\bullet}_{M^{\bullet}}) : \mathfrak{F}^{C}(M^{\bullet}) \to \mathfrak{G}^{C}(M^{\bullet})$ in $C(\mathcal{B})$. Let $f^{\bullet} : M^{\bullet} \to N^{\bullet}$ be a morphism in $C(\mathcal{A})$. Given $i, j \in \mathbb{Z}$, we obtain a commutative diagram

$$\begin{array}{ccc}
\mathfrak{F}(M^i)^j & \xrightarrow{\eta^j_{M^i}} & \mathfrak{G}(M^i)^j \\
\downarrow{\scriptstyle \mathfrak{F}(f^i)^j} & & \downarrow{\scriptstyle \mathfrak{G}(f^i)^j} \\
\mathfrak{F}(N^i)^j & \xrightarrow{\eta^j_{N^i}} & \mathfrak{G}(N^i)^j.
\end{array}$$

Hence, $\mathfrak{G}^{DC}(f^{\bullet}) \circ \eta^{\bullet}_{M^{\bullet}} = \eta^{\bullet}_{N^{\bullet}} \circ \mathfrak{F}^{DC}(f^{\bullet})$. Applying $\mathbb{T}$ to this equation, we obtain $\mathfrak{G}^{C}(f^{\bullet}) \circ \eta^{\bullet}_{M^{\bullet}} = \eta^{\bullet}_{N^{\bullet}} \circ \mathfrak{F}^{C}(f^{\bullet})$. Thus, $\eta^{\bullet}_{M^{\bullet}}$ and $\eta_{M^{\bullet}}$ are natural in $M^{\bullet}$. Therefore, $\eta^{DC} = (\eta^{\bullet}_{M^{\bullet}})_{M^{\bullet}\in C(\mathcal{A})}$ and $\eta^{C} = (\eta_{M^{\bullet}})_{M^{\bullet}\in C(\mathcal{A})}$ are desired functorial morphisms. The proof of the lemma is complete.

5. Koszul duality

The objective of this section is to describe the Koszul duality for a Koszul algebra defined by gradable quiver, which relates derived categories of modules over a Koszul algebra and those of modules over its Koszul dual.

Throughout this section, $\Lambda = kQ/R$, where Q is a locally finite gradable quiver and R is a quadratic ideal in kQ. We fix a grading $Q_0 = \cup_{n\in\mathbb{Z}} Q^n$, which will be used later without an explicit mention. Recall that $Q(x,y) = Q_{n-m}(x,y)$, for $x \in Q^n$ and $y \in Q^m$ with $m, n \in \mathbb{Z}$; see [1, (7.2)]. Here $Q_s(x,y) = \emptyset$ for $s < 0$. In particular, Λ is strongly locally finite dimensional. We shall regard modules in $\text{Mod}\Lambda$ as representations in $\text{Rep}(Q,R)$. Thus, every module M in $\text{Mod}\Lambda$ is graded as $M = \oplus_{n\in\mathbb{Z}} M_n$, where $M_n = \oplus_{x\in Q^n} M(x)$. Note that this grading for P_a with $a \in Q^n$ is the grading-shift by n of its J-grading. We say that M is *bounded-above* if $M_n = 0$ for $n \gg 0$, and *bounded-below* if $M_n = 0$ for $n \ll 0$. These notions are independent of the grading for Q_0; see [1, (7.1)]. The full subcategories of $\text{Mod}\Lambda$ of bounded-above modules and of bounded-below modules are written as $\text{Mod}^{-}\Lambda$ and $\text{Mod}^{+}\Lambda$, respectively.

Let $(M^{\bullet}, d^{\bullet})$ be a complex over $\text{Mod}\Lambda$. Given $x \in Q_0$, we obtain a complex $M^{\bullet}(x)$ over $\text{Mod}k$, whose n-th component is $M^n(x)$ and whose n-th differential is $d^n_x : M^n(x) \to M^{n+1}(x)$. Clearly, $M^{\bullet}$ is acyclic if and only if $M^{\bullet}(x)$ is acyclic, for every $x \in Q_0$. Let $f^{\bullet} : M^{\bullet} \to N^{\bullet}$ be a morphism in $C(\text{Mod}\Lambda)$. Given $x \in Q_0$, we obtain a morphism $f^{\bullet}(x) : M^{\bullet}(x) \to N^{\bullet}(x)$ in $C(\text{Mod}k)$, which is defined by $f^n(x) = f^n_x : M^n(x) \to N^n(x)$. Clearly, $f^{\bullet}$ is a quasi-isomorphism if and only if $f^{\bullet}(x)$ is a quasi-isomorphism, for every $x \in Q_0$. A similar consideration will be given to objects $M^{\bullet\bullet}$ and morphisms $f^{\bullet\bullet}$ in $DC(\mathrm{Mod}\Lambda)$ in such a way that $\mathbb{T}(M^{\bullet\bullet})(x) = \mathbb{T}(M^{\bullet\bullet})(x)$ and $\mathbb{T}(f^{\bullet\bullet})(x) = \mathbb{T}(f^{\bullet\bullet})(x)$, for every $x \in Q_0$.

Observe that Q^{o} admits a grading $(Q^{\mathrm{o}})_0 = \cup_{n\in\mathbb{Z}} (Q^{\mathrm{o}})^n$ with $(Q^{\mathrm{o}})^n = Q^{-n}$. Thus, the quadratic dual $\Lambda^!$ is defined by the gradable quiver Q^{o}. Given $a \in Q_0$, we denote by $S^!_a$, $P^!_a$ and $I^!_a$ the simple module, the indecomposable projective module and the indecomposable injective module in $\mathrm{Mod}\Lambda^!$ associated with a, respectively. Now, we define two *Koszul functors* $F : \mathrm{Mod}\Lambda^! \to C(\mathrm{Mod}\Lambda)$ and $G : \mathrm{Mod}\Lambda \to C(\mathrm{Mod}\Lambda^!)$. Indeed, given a module M in $\mathrm{Mod}\Lambda^!$, as shown below, we shall obtain a complex $F(M)^{\bullet}$ in $C(\mathrm{Mod}\Lambda)$ if, for $n \in \mathbb{Z}$, we put

$$F(M)^n = \oplus_{x\in(Q^{\mathrm{o}})^n} P_x \otimes M(x) = \oplus_{x\in Q^{-n}} P_x \otimes M(x)$$

and $d^n_{F(M)} = (d^n_{F(M)}(y,x))_{(y,x)\in Q^{-n-1}\times Q^{-n}} : F(M)^n \to F(M)^{n+1}$, where

$$d^n_{F(M)}(y,x) = \textstyle\sum_{\alpha\in Q_1(y,x)} P[\bar{\alpha}] \otimes M(\alpha^{\mathrm{o}}) : P_x \otimes M(x) \to P_y \otimes M(y).$$

And given a morphism $f : M \to N$ in $\mathrm{Mod}\Lambda^!$, we shall obtain a complex morphism $F(f)^{\bullet} : F(M)^{\bullet} \to F(N)^{\bullet}$ if, for any $n \in \mathbb{Z}$, we set

$$F(f)^n = \oplus_{x\in Q^{-n}} 1 \otimes f(x) \, : \oplus_{x\in Q^{-n}} P_x \otimes M(x) \to \oplus_{x\in Q^{-n}} P_x \otimes N(x).$$

On the other hand, given a module N in $\mathrm{Mod}\,\Lambda$, we shall obtain a complex $G(N)^{\bullet}$ in $C(\mathrm{Mod}\,\Lambda^!)$ provided that, for any integer n, we put

$$G(N)^n = \oplus_{x\in Q^n} I^!_x \otimes N(x)$$

and $d^n_{G(N)} = (d^n_{G(N)}(y,x))_{(y,x)\in Q^{n+1}\times Q^n}$, where

$$d^n_{G(N)}(y,x) = \textstyle\sum_{\alpha:x\to y} I[\alpha^!] \otimes N(\alpha) : I^!_x \otimes M(x) \to I^!_y \otimes N(y).$$

And given a morphism $g : M \to N$ in $\mathrm{Mod}\,\Lambda$, we shall obtain a complex morphism $G(g)^{\bullet} : G(M)^{\bullet} \to G(N)^{\bullet}$ if, for any $n \in \mathbb{Z}$, we put

$$G(g)^n = \oplus_{x\in Q^n} 1 \otimes g(x) \, : \oplus_{x\in Q^n} I^!_x \otimes M(x) \to \oplus_{x\in Q^n} I^!_x \otimes N(x).$$

5.1. Proposition. *Let $\Lambda = kQ/R$, where Q is a locally finite gradable quiver and R is a quadratic ideal. The above construction yields two exact functors*

(1) $F : \mathrm{Mod}\,\Lambda^! \to C(\mathrm{Mod}\Lambda) : M \to F(M)^{\bullet}; f \mapsto F(f)^{\bullet}$;

(2) $G : \mathrm{Mod}\,\Lambda \to C(\mathrm{Mod}\,\Lambda^!) : N \to G(N)^{\bullet}; g \mapsto G(g)^{\bullet}$.

Proof. We shall only prove Statement (1). Consider a module $M \in \mathrm{Mod}\,\Lambda^!$. We shall show that $F(M)^{\bullet}$ is a complex. Indeed, fix an integer n. Given $z \in Q^{-n-2}$ and $x \in Q^{-n}$, we write $Q(z,x) = \{\alpha_1\beta_1, \ldots, \alpha_s\beta_s\}$, where $\alpha_i, \beta_i \in Q_1$. Recall that $\Lambda^! = \{\gamma^! \mid \gamma \in kQ\}$, where $\gamma^! = \gamma^{\mathrm{o}} + R^!$. By definition, we obtain

$$(d^{n+1}_{F(M)} \circ d^n_{F(M)})(z,x) = \textstyle\sum_{i=1}^s P[\bar{\alpha}_i\bar{\beta}_i] \otimes M(\beta^!_i\alpha^!_i) : P_x \otimes M(x) \to P_z \otimes M(z).$$

As seen in the proof of Lemma 3.6, we may find bases $\{\rho_1, \ldots, \rho_r, \rho_{r+1}, \ldots, \rho_s\}$ and $\{\eta_1, \ldots, \eta_r, \eta_{r+1}, \ldots, \eta_s\}$ of $kQ(z,x)$ such that $\{\rho_1, \ldots, \rho_r\}$ is a basis of $R_2(z,x)$ and $\{\eta^{\mathrm{o}}_{r+1}, \ldots, \eta^{\mathrm{o}}_s\}$ is a basis of $R^!_2(x,z)$, while $\{\eta^*_1, \ldots, \eta^*_r, \eta^*_{r+1}, \ldots, \eta^*_s\}$ is the dual basis of $\{\rho_1, \ldots, \rho_r, \rho_{r+1}, \ldots, \rho_s\}$. In particular, $\bar{\rho}_i = 0$ for $1 \le i \le r$ and $\eta^! = 0$ for $r < i \le s$. By Lemma 1.4(2), we obtain

$$\textstyle\sum_{i=1}^s (\alpha_i\beta_i) \otimes (\alpha_i\beta_i)^* = \sum_{i=1}^s \rho_i \otimes \eta^*_i \in kQ(z,x) \otimes D(kQ(z,x)).$$

In view of the canonical projections $kQ(z,x) \to e_x \Lambda e_z$ and $kQ^{\rm o}(x,z) \to e_z \Lambda^! e_x$ and the isomorphism $D(kQ_2(z,x)) \to kQ_2^{\rm o}(x,z)$, we see from the above equation that

$$\sum_{i=1}^s \bar{\alpha}_i \bar{\beta}_i \otimes \beta_i^! \alpha_i^! = \sum_{i=1}^s \bar{\rho}_i \otimes \eta_i^!.$$

Applying to this equation the k-linear map

$$e_x \Lambda e_z \otimes e_z \Lambda^! e_x \to \mathrm{Hom}_\Lambda(P_x, P_z) \otimes \mathrm{Hom}_k(M(x), M(z))$$

obtained from Proposition 2.1, we conclude that

$$\sum_{i=1}^s P[\bar{\alpha}_i \bar{\beta}_i] \otimes M(\beta_i^! \alpha_i^!) = \sum_{i=1}^s P[\bar{\rho}_i] \otimes M(\eta_i^!) = 0.$$

That is, $d^{n+1}_{F(M)} \circ d^n_{F(M)} = 0$. Now, it is easy to see that F is a functor, which is exact because the tensor product is over k. The proof of the proposition is completed.

Remark. In case Q is finite, our Koszul functor F coincides with the one for $\Lambda^!$ defined in [5, page 489]. Indeed, $\Lambda = (\Lambda^!)^!$. Let $M = \oplus_{n\in\mathbb{Z}} M_n$ be a module in $\mathrm{Mod}\Lambda$, where $M_n = \oplus_{x\in Q^n} M(x)$. Since $e_y M_n = 0$ for all $y \notin Q^n$, we see that $\oplus_{x\in Q^n} P_x \otimes M(x) = \Lambda \otimes_{\Lambda/J\Lambda} M_n$,.

As has been seen in Section 4, the Koszul functors are extended to functors $F^C : C(\mathrm{Mod}\Lambda^!) \to C(\mathrm{Mod}\Lambda)$ and $G^C : C(\mathrm{Mod}\Lambda) \to C(\mathrm{Mod}\Lambda^!)$, call the *complex Koszul functors*, which descend to the homotopy categories; see (4.8). Since F^C does not send all acyclic complexes to acyclic ones, it does not descend to the full derived category of $\mathrm{Mod}\Lambda^!$. This forces us to consider subcategories of complex categories. For this purpose, we shall view a complex $M^\bullet$ over $\mathrm{Mod}\Lambda$ as a bigraded k-space $M^i_j = \oplus_{x\in Q^j} M^i(x)$, $i,j\in\mathbb{Z}$.

5.2. Definition. Let $\Lambda = kQ/R$, where Q is a locally finite gradable quiver and R is a quadratic ideal in kQ. Given $p,q\in\mathbb{R}$ with $p\geq 1$ and $q\geq 0$, we denote by

(1) $C^{\downarrow}_{p,q}(\mathrm{Mod}\Lambda)$ the full abelian subcategory of $C(\mathrm{Mod}\Lambda)$ of complexes $M^\bullet$ with $M^i_j = 0$ for $i+pj \gg 0$ or $i-qj \ll 0$; in other words, $M^\bullet$ concentrates in a lower triangle formed by two lines of slopes $-\frac{1}{p}$ and $\frac{1}{q}$ respectively;

(2) $C^{\uparrow}_{p,q}(\mathrm{Mod}\Lambda)$ the full abelian subcategory of $C(\mathrm{Mod}\Lambda)$ of complexes $M^\bullet$ with $M^i_j = 0$ for $i+pj \ll 0$ or $i-qj \gg 0$; in other words, $M^\bullet$ concentrates in a upper triangle formed by two lines of slopes $-\frac{1}{p}$ and $\frac{1}{q}$ respectively.

Remark. (1) Taking $p=1$ and $q=0$, we recover the categories $C^{\downarrow}(\Lambda)$ and $C^{\uparrow}(\Lambda)$ considered in [5, (2.12)], and also, [25, (2.4)].

(2) The $C^{\downarrow}_{p,q}(\mathrm{Mod}\Lambda)$ are pairwise distinct derivable subcategories of $C(\mathrm{Mod}^{-}\Lambda)$, while the $C^{\uparrow}_{p,q}(\mathrm{Mod}\Lambda)$ are pairwise distinct derivable subcategories of $C(\mathrm{Mod}^{+}\Lambda)$.

In the sequel, we shall write $K^{\downarrow}_{p,q}(\mathrm{Mod}\Lambda)$ and $K^{\uparrow}_{p,q}(\mathrm{Mod}\Lambda)$ for the quotients of $C^{\downarrow}_{p,q}(\mathrm{Mod}\Lambda)$ and $C^{\uparrow}_{p,q}(\mathrm{Mod}\Lambda)$ modulo null-homotopic morphisms respectively, and write $D^{\downarrow}_{p,q}(\mathrm{Mod}\Lambda)$ and $D^{\uparrow}_{p,q}(\mathrm{Mod}\Lambda)$ for the localizations of $K^{\downarrow}_{p,q}(\mathrm{Mod}\Lambda)$ and $K^{\uparrow}_{p,q}(\mathrm{Mod}\Lambda)$ at quasi-isomorphisms, respectively.

5.3. Theorem. *Let $\Lambda = kQ/R$, where Q is a locally finite gradable quiver and R is a quadratic ideal in kQ. Consider $p,q\in\mathbb{R}$ with $p\geq 1$ and $q\geq 0$. Then*

(1) *the Koszul functor* $F: \mathrm{Mod}\,\Lambda^! \to C(\mathrm{Mod}\,\Lambda)$ *induces a commutative diagram*

$$\begin{array}{ccccc} C^{\downarrow}_{p,q}(\mathrm{Mod}\,\Lambda^!) & \xrightarrow{P_{\Lambda^!}} & K^{\downarrow}_{p,q}(\mathrm{Mod}\,\Lambda^!) & \xrightarrow{L_{\Lambda^!}} & D^{\downarrow}_{p,q}(\mathrm{Mod}\,\Lambda^!) \\ \downarrow F^C_{p,q} & & \downarrow F^K_{p,q} & & \downarrow F^D_{p,q} \\ C^{\uparrow}_{q+1,p-1}(\mathrm{Mod}\,\Lambda) & \xrightarrow{P_{\Lambda}} & K^{\uparrow}_{q+1,p-1}(\mathrm{Mod}\,\Lambda) & \xrightarrow{L_{\Lambda}} & D^{\uparrow}_{q+1,p-1}(\mathrm{Mod}\,\Lambda); \end{array}$$

(2) *the Koszul functor* $G: \mathrm{Mod}\,\Lambda \to C(\mathrm{Mod}\,\Lambda^!)$ *induces a commutative diagram*

$$\begin{array}{ccccc} C^{\uparrow}_{q+1,p-1}(\mathrm{Mod}\,\Lambda) & \xrightarrow{P_{\Lambda}} & K^{\uparrow}_{q+1,p-1}(\mathrm{Mod}\,\Lambda) & \xrightarrow{L_{\Lambda}} & D^{\uparrow}_{q+1,p-1}(\mathrm{Mod}\,\Lambda) \\ \downarrow G^C_{p,q} & & \downarrow G^K_{p,q} & & \downarrow G^D_{p,q} \\ C^{\downarrow}_{p,q}(\mathrm{Mod}\,\Lambda^!) & \xrightarrow{P_{\Lambda^!}} & K^{\downarrow}_{p,q}(\mathrm{Mod}\,\Lambda^!) & \xrightarrow{L_{\Lambda^!}} & D^{\downarrow}_{p,q}(\mathrm{Mod}\,\Lambda^!), \end{array}$$

where F^D *and* G^D *are triangle-exact, called the* derived Koszul functors.

Proof. Consider the two complex Koszul functors $F^C C(\mathrm{Mod}\,\Lambda^!) \to C(\mathrm{Mod}\,\Lambda)$ and $G^C : C(\mathrm{Mod}\,\Lambda) \to C(\mathrm{Mod}\,\Lambda^!)$. First, let $M^\bullet \in C^{\downarrow}_{p,q}(\mathrm{Mod}\,\Lambda^!)$. We claim that $F^C(M^\bullet)$ belongs to $C^{\uparrow}_{q+1,p-1}(\mathrm{Mod}\,\Lambda)$. Indeed, by definition, there exist s, t such that $M^i(x) = 0$, for $x \in (Q^{\mathrm{o}})^j$ with $i + pj > s$ or $i - qj < t$. Fix $n, m \in \mathbb{Z}$. Given any $y \in Q^m$, we obtain

$$F^C(M^\bullet)^n(y) = \oplus_{i\in\mathbb{Z};\, x\in(Q^{\mathrm{o}})^{n-i}} P_x(y) \otimes M^i(x) = \oplus_{i\le n+m;\, x\in Q^{i-n}} P_x(y) \otimes M^i(x).$$

Let $i \le n+m$. If $n + (q+1)m < s$, then $i - q(n-i) < s$; and if $n - (p-1)m > t$, then $i + p(n-i) > t$. In either case, $M^i(x) = 0$ for all $x \in (Q^{\mathrm{o}})^{n-i}$. Therefore, $F^C(M^\bullet)^n_m(y) = 0$ if $n + (q+1)m < s$ or $n - (p-1)m > t$. This establishes our claim. Hence, F^C restricts to a functor $F^C_{p,q} : C^{\downarrow}_{p,q}(\mathrm{Mod}\,\Lambda^!) \to C^{\uparrow}_{q+1,p-1}(\mathrm{Mod}\,\Lambda)$.

Fix again $n \in \mathbb{Z}$. Then, $F(M^i)^{n-i} = \oplus_{x\in(Q^{\mathrm{o}})^{n-i}} P_x \otimes M^i(x)$ with $i \in \mathbb{Z}$ form the n-diagonal of $F(M^\bullet)^\bullet$. By the assumption, $M^i(x) = 0$ for $x \in (Q^{\mathrm{o}})^{n-i}$ with $i < (nq+t)(1+q)^{-1}$. Hence, $F(M^\bullet)^\bullet$ is diagonally bounded-below. By Theorem 4.9, we see that $F^C_{p,q}$ induces a commutative diagram as stated in Statement (1).

Next, using a similar argument, we can verify that G^C restricts to a functor $G^C_{p,q} : C^{\uparrow}_{q+1,p-1}(\mathrm{Mod}\,\Lambda) \to C^{\downarrow}_{p,q}(\mathrm{Mod}\,\Lambda^!)$. Let $N^\bullet \in C(\mathrm{Mod}\,\Lambda)$ be acyclic. We shall show that $G^C(N^\bullet)$ is acyclic, or equivalently, $G^C(N^\bullet)(x)$ is acyclic for all $x \in Q_0$. Indeed, fix $x \in Q^s$ for some $s \in \mathbb{Z}$. By definition, $G^C(N^\bullet) = \mathbb{T}(G(N^\bullet)^\bullet)$, and hence, $G^C(N^\bullet)(x) = \mathbb{T}(G(N^\bullet)^\bullet(x))$. Since G is exact, $G(N^\bullet)^\bullet$ has acyclic rows, and so does $G(N^\bullet)^\bullet(x)$. Given any $n \in \mathbb{Z}$, the n-diagonal of $G(N^\bullet)^\bullet(x)$ consists of $G(N^i)^{n-i}(x) = \oplus_{y\in Q^{n-i}} I^!_y(x) \otimes N^i(y), i \in \mathbb{Z}$. If $i < n-s$ and $y \in Q^{n-i}$, since Q contains no path from y to x, we have $I^!_y(x) = D(e_x(\Lambda^!)^{\mathrm{o}}e_y) = 0$, and consequently, $G(N^i)^{n-i}(x) = 0$. Thus, $G(N^\bullet)^\bullet(x)$ is diagonally bounded-below, and by Proposition 4.3, $G^C(N^\bullet)(x)$ is indeed acyclic. Now, we deduce from Theorem 4.9(2) that $G^C_{p,q}$ induces a commutative diagram as stated in Statement (2). The proof of the theorem is completed.

Remark. The case $p = 1$ and $q = 0$ of Theorem 5.3 has been established for quadratic positively graded categories; see [25, Proposition 20].

In case Λ is Koszul, we shall show that the derived Koszul functors F^D and G^D are mutually quasi-inverse. For this purpose, given a simple module S, we shall denote by $\mathcal{P}^{\bullet}_S$ its minimal projective resolution and by $\mathcal{I}^{\bullet}_S$ its minimal injective co-resolution. They can be explicitly described as below; compare [5, (1.2.6)]

5.4. LEMMA. *Let $\Lambda = kQ/R$ be a Koszul algebra, where Q is locally finite with a grading $Q_0 = \cup_{n\in\mathbb{Z}} Q^n$. If $a \in Q^s$, then $F(I^!_a)^{\bullet} \cong \mathcal{P}^{\bullet}_{S_a}[s]$ and $G(P_a)^{\bullet} \cong \mathcal{I}^{\bullet}_{S^!_a}[-s]$.*

Proof. Fix $a \in Q^s$. By Theorem 3.4 and Lemma 3.9, $\mathcal{P}^{\bullet}_{S_a}$ is isomorphic to

$$L^{\bullet}: \ \cdots \longrightarrow L^{-i} \xrightarrow{d^{-i}} L^{-i+1} \longrightarrow \cdots \longrightarrow L^{-1} \xrightarrow{d^{-1}} L^0 \longrightarrow 0 \longrightarrow \cdots,$$

where $L^{-i} = \oplus_{x\in Q_0} P_x \otimes D(e_a\Lambda^!_i e_x)$ and $d^{-i} = (d^{-i}(y,x))_{(y,x)\in Q_0\times Q_0}$ with

$$d^{-i}(y,x) = \textstyle\sum_{\alpha\in Q_1(y,x)} P[\bar{\alpha}] \otimes DP[\alpha^!] : P_x \otimes D(e_a\Lambda^!_i e_x) \to P_y \otimes D(e_a\Lambda^!_{i-1} e_y).$$

Fix an integer $n \geq 0$. Observe that $e_a\Lambda^!_n e_x = 0$ in case $x \notin Q^{n+s}$, and otherwise, $e_a\Lambda^!_n e_x = e_a\Lambda^! e_x$. Therefore, $L^{-n} = \oplus_{x\in Q^{n+s}} P_x \otimes D(e_a\Lambda^! e_x)$. Moreover, the k-linear isomorphism $e_x(\Lambda^!)^{\rm o} e_a \to e_a\Lambda^! e_x$ induces a k-linear isomorphism $\theta_{a,x} : D(e_a\Lambda^! e_x) \to D(e_x(\Lambda^!)^{\rm o} e_a) = I^!_a(x)$ such that the diagram

$$\begin{array}{ccc}
\oplus_{x\in Q^{n+s}} P_x \otimes D(e_a\Lambda^! e_x) & \xrightarrow{\sum_{\alpha\in Q_1(y,x)} P[\bar{\alpha}]\otimes DP[\alpha^!]} & \oplus_{y\in Q^{n+s-1}} P_y \otimes D(e_a\Lambda^! e_y) \\
\downarrow{\scriptstyle \oplus(1\otimes\theta_{a,x})} & & \downarrow{\scriptstyle \oplus(1\otimes\theta_{a,y})} \\
\oplus_{x\in Q^{n+s}} P_x \otimes I^!_a(x) & \xrightarrow{\sum_{\alpha\in Q_1(y,x)} P[\bar{\alpha}]\otimes I^!_a(\alpha^{\rm o})} & \oplus_{y\in Q^{n+s-1}} P_y \otimes I^!_a(y)
\end{array}$$

commutates with vertical isomorphisms. Since $F(I^!_a)^{-n-s} = 0$ for $n < 0$, we see that $L^{\bullet} \cong \mathfrak{t}^s(F(I^!_a)[-s]) \cong F(I^!_a)[-s]$. This establishes the first part of the lemma.

Next, by Theorem 3.10 and Proposition 3.8, $\Lambda^!$ is Koszul with $(\Lambda^!)^! = \Lambda$. In view of Theorem 3.13(3), we see that $\mathcal{I}^{\bullet}_{S^!_a}$ is isomorphic to

$$T^{\bullet}: \quad 0 \longrightarrow T^0 \xrightarrow{d^0} T^1 \longrightarrow \cdots \longrightarrow T^n \xrightarrow{d^n} T^{n+1} \longrightarrow \cdots$$

where $T^n = \oplus_{x\in Q_0} I^!_x \otimes e_x\Lambda_n e_a$ and $d^n = (d^n(y,x))_{(y,x)\in Q_0\times Q_0} : T^n \to T^{n+1}$ with $d^n(y,x) = \sum_{\alpha\in Q_1(x,y)} I[\alpha^!] \otimes P_a(\alpha) : I^!_x \otimes e_x\Lambda_n e_a \to I^!_y \otimes e_y\Lambda_{n+1} e_a$, for $n \geq 0$. Fix an integer $n \geq 0$. Note that $e_x\Lambda_n e_a = 0$ in case $x \notin Q^{n+s}$; and otherwise, $e_x\Lambda_n e_a = e_x\Lambda e_a$. Thus, $T^n = \oplus_{x\in Q^{n+s}} I^!_x \otimes e_x\Lambda e_a = G(P_a)^{n+s}$ and $d^n = d^{n+s}_{G(P_a)}$, for $n \geq 0$. Since $G(P_a)^{n+s} = 0$ for $n < 0$, we see that $\mathcal{I}^{\bullet}_{S^!_a} \cong \mathfrak{t}^s(G(P_a)^{\bullet}[s]) \cong G(P_a)^{\bullet}[s]$. The proof of the lemma is completed.

The following statement describes in particular a projective resolution for every module over a Koszul algebra.

5.5. PROPOSITION. *Let $\Lambda = kQ/R$ be a Koszul algebra, where Q is a locally finite gradable quiver. If $M \in {\rm Mod}\Lambda$, then there exists a natural quasi-isomorphism $\eta^{\bullet}_M : (F^C \circ G)(M)^{\bullet} \to M$.*

Proof. Fix $M \in {\rm Mod}\Lambda$. By definition, $(F^C \circ G)(M)^{\bullet} = \mathbb{T}(F(G(M)^{\bullet})^{\bullet})$. For $n \in \mathbb{Z}$, we obtain $(F^C\circ G)(M)^n = \oplus_{i\in\mathbb{Z}} F(G(M)^i)^{n-i} = \oplus_{i\in\mathbb{Z};\, a\in Q^{i-n}} P_a \otimes G(M)^i(a)$, where $G(M)^i = \oplus_{x\in Q^i} I^!_x \otimes M(x)$. Therefore,

$$(F^C\circ G)(M)^n = \oplus_{i\in\mathbb{Z};\, a\in Q^{i-n};\, x\in Q^i} P_a \otimes I^!_x(a) \otimes M(x).$$

Suppose that $n > 0$. For any $a \in Q^{i-n}$ and $x \in Q^i$, since Q has no path from x to a, we see that $I^!_x(a) = 0$. Thus, $(F^C\circ G)(M)^n = 0$.

Suppose that $n < 0$. We claim that $\mathrm{H}^n((F^C \circ G)(M)^{\bullet}) = 0$, or equivalently, $\mathrm{H}^n((F^C \circ G)(M)^{\bullet}(y)) = 0$, for $y \in Q^p$ with $p \in \mathbb{Z}$. Indeed, $(F^C \circ G)(M)^{\bullet}(y)$ is the total complex of the double complex $F(G(M)^{\bullet})^{\bullet}(y)$, whose n-diagonal consists of

$$F(G(M)^i)^{n-i}(y) = \oplus_{a \in Q^{i-n}; x \in Q^i} P_a(y) \otimes I^!_x(a) \otimes M(x),\ i \in \mathbb{Z}.$$

If $i > n+p$, then $P_a(y) = 0$ for all $a \in Q^{i-n}$. Hence, $F(G(M)^i)^{n-i}(y) = 0$. That is, $F(G(M)^{\bullet})^{\bullet}(y)$ is n-diagonally bounded-above. Given $i \in \mathbb{Z}$, the i-th column of $F(G(M)^{\bullet})^{\bullet}$ is the complex $\mathfrak{t}^i(F(G(M)^i)^{\bullet}) = \oplus_{x \in Q^i} \mathfrak{t}^i(F(I^!_x)^{\bullet}) \otimes M(x)$, where $F(I^!_x)^{\bullet} \cong \mathcal{P}^{\bullet}_{S_x}[i]$; see (5.4). Thus,

$$\mathrm{H}^{n-i}(\mathfrak{t}^i(F(G(M)^i)^{\bullet})) \cong \oplus_{x \in Q^i} \mathrm{H}^{n-i}(\mathcal{P}^{\bullet}_{S_x}[i]) \otimes M(x) = \oplus_{x \in Q^i} \mathrm{H}^n(\mathcal{P}^{\bullet}_{S_x}) \otimes M(x) = 0.$$

Hence, $\mathrm{H}^{n-i}(\mathfrak{t}^i(F(G(M)^i)^{\bullet})(y)) = 0$, for all $i \in \mathbb{Z}$. In view of Lemma 4.2(2), we conclude that $\mathrm{H}^n((F^C \circ G)(M)^{\bullet}(y)) = 0$.

It remains to show that $\mathrm{H}^0((F^C \circ G)(M)^{\bullet})$ is naturally isomorphic to M. For this purpose, observing that the 1-diagonal of the double complex $F(G(M)^{\bullet})^{\bullet}$ is zero, we illustrate its (-1)-diagonal and 0-diagonal as follows:

$$\begin{array}{ccc} \oplus_{b \in Q^i} P_b \otimes I^!_b(b) \otimes M(b) & & \\ \Big\uparrow {\scriptstyle v^{i,-i-1}} & & \\ \oplus_{(a,x) \in Q^{i+1} \times Q^i} P_a \otimes I^!_x(a) \otimes M(x) & \xrightarrow{h^{i,-i-1}} & \oplus_{c \in Q^{i+1}} P_c \otimes I^!_c(c) \otimes M(c), \end{array}$$

where $v^{i,-i-1} = (v^{i,-i-1}(b,a,x))_{(b,a,x) \in Q^i \times Q^{i+1} \times Q^i}$, with

$$v^{i,-i-1}(b,a,x) = \begin{cases} \sum_{\alpha \in Q_1(x,a)} (-1)^i P[\bar{\alpha}] \otimes I^!_x(\alpha^{\mathrm{o}}) \otimes \mathbf{1}_{M(x)}, & \text{if } b = x; \\ 0, & \text{if } b \neq x, \end{cases}$$

and $h^{i,-i-1} = (h^{i,-i-1}(c,a,x))_{(c,a,x) \in Q^{i+1} \times Q^{i+1} \times Q^i}$, with

$$h^{i,-i-1}(c,a,x) = \begin{cases} \sum_{\alpha \in Q_1(x,a)} \mathbf{1}_{P_a} \otimes I[\alpha^!]_a \otimes M(\alpha), & \text{if } c = a; \\ 0, & \text{if } c \neq a. \end{cases}$$

We recall that $(\Lambda^!)^{\mathrm{o}} = kQ/(R^!)^{\mathrm{o}} = \{\hat{\gamma} \mid \gamma \in kQ\}$, where $\hat{\gamma} = \gamma + (R^!)^{\mathrm{o}}$. Given $(x,y) \in Q^i \times Q^{i+1}$ with $i \in \mathbb{Z}$, in view of Lemma 2.7, $I^!_x(x)$ has a k-basis $\{\hat{e}^{\star}_x\}$, while $I^!_x(y)$ has a k-basis $\{\hat{\alpha}^{\star} \mid \alpha \in Q_1(x,y)\}$.

Sublemma. *Let d^{-1} be the differential of degree -1 of $(F^C \circ G)(M)^{\bullet}$. Consider $(x,a) \in Q^i \times Q^{i+1}$ for some $i \in \mathbb{Z}$. If $\bar{\gamma} \in P_a$, $\beta \in Q_1(x,a)$ and $u \in M(x)$, then*

$$d^{-1}(\bar{\gamma} \otimes \hat{\beta}^{\star} \otimes u) = (-1)^i\, \bar{\gamma}\bar{\beta} \otimes \hat{e}^{\star}_x \otimes u + \bar{\gamma} \otimes \hat{e}^{\star}_a \otimes \bar{\beta}u.$$

Proof. Given $\alpha \in Q_1(x,a)$, we see that $I^!_x(\alpha^{\mathrm{o}})(\hat{\beta}^{\star}) = 0$ if $\alpha \neq \beta$, and otherwise, $I^!_x(\alpha^{\mathrm{o}})(\hat{\beta}^{\star}) = \hat{e}^{\star}_x$. On the other hand, $I[\alpha^!]_a(\hat{\beta}^{\star}) = 0$ if $\alpha \neq \beta$, and otherwise, $I[\alpha^!]_a(\hat{\beta}^{*}) = \hat{e}^{\star}_a$. This yields

$$\begin{array}{rcl} d^{-1}(\bar{\gamma} \otimes \hat{\beta}^{\star} \otimes u) & = & (-1)^i \sum_{\alpha \in Q_1(x,a)} (P[\bar{\alpha}] \otimes I^!_x(\alpha^{\mathrm{o}}) \otimes \mathbf{1}_{M(x)})(\bar{\gamma} \otimes \hat{\beta}^{\star} \otimes u) \\ & & + \sum_{\alpha \in Q_1(x,a)} (\mathbf{1}_{P_a} \otimes I[\alpha^!]_a \otimes M(\alpha))(\bar{\gamma} \otimes \hat{\beta}^{\star} \otimes u) \\ & = & (-1)^i \bar{\gamma}\bar{\beta} \otimes \hat{e}^{\star}_x \otimes u + \gamma \otimes \hat{e}^{\star}_a \otimes \bar{\beta}u. \end{array}$$

This establishes the sublemma. Next, we clearly have a natural Λ-linear map

$$\eta^0_{_M} : (F^C \circ G)(M)^0 \to M : \sum_{(i,x) \in \mathbb{Z} \times Q^i} \bar{\gamma}_x \otimes \hat{e}^{\star}_x \otimes u_x \mapsto \sum_{(i,x) \in \mathbb{Z} \times Q^i} (-1)^{\frac{i(i+1)}{2}} \bar{\gamma}_x u_x,$$

where $\bar{\gamma}_x \in P_x$ and $u_x \in M(x)$. We claim that $\eta^0_{_M} \circ d^{-1} = 0$. Indeed, consider an element $\omega \in (F^C \circ G)(M)^{-1}$. We may assume that $\omega \in P_a \otimes I^!_x(a) \otimes M(x)$, for some $(a, x) \in Q^{i+1} \times Q^i$ with $i \in \mathbb{Z}$. In this case, we may assume further that $\omega = \bar{\gamma} \otimes \hat{\beta}^\star \otimes u$, where $\bar{\gamma} \in P_a$, $\beta \in Q_1(x, a)$, and $u \in M(x)$. In view of the sublemma, we obtain

$$\begin{aligned}(\eta^0_{_M} \circ d^{-1})(\omega) &= \eta^0_{_M}\left((-1)^i \bar{\gamma}\bar{\beta} \otimes \hat{e}^\star_x \otimes u + \bar{\gamma} \otimes \hat{e}^\star_a \otimes \bar{\beta}u\right) \\ &= (-1)^{\frac{i(i+1)}{2}+i}(\bar{\gamma}\bar{\beta}u) + (-1)^{\frac{(i+1)(i+2)}{2}}(\bar{\gamma}\bar{\beta}u) \\ &= 0.\end{aligned}$$

Given $\omega \in \mathrm{Ker}(\eta^0_{_M})$, we shall define an integer n_ω as follows. If $\omega = 0$, set $n_\omega = 0$; and in this case, $\omega \in \mathrm{Im}(d^{-1})$. Otherwise, let n_ω be minimal for which $\omega = \sum_{i=1}^s \bar{\gamma}_i \otimes \hat{e}^\star_{x_i} \otimes u_i$, where $x_i \in Q_0$; $\gamma_i \in kQ_{\le n_\omega}(x_i, -)$; the u_i are linearly independent in $M(x_i)$. For $1 \le i \le s$, write $\gamma_i = \lambda_i \varepsilon_{x_i} + \sigma_{i1}\alpha_{i1} + \cdots + \sigma_{i,t_i}\alpha_{i,t_i}$, where $\lambda_i \in k$; $\alpha_{ij} \in Q_1(x_i, a_{ij})$; $\sigma_{ij} \in kQ_{\le n_\omega - 1}(a_{ij}, -)$. Setting $|x| = i$ for $x \in Q^i$, we obtain $\sum_{i=1}^s (-1)^{\frac{|x_i|(|x_i|+1)}{2}} \bar{\gamma}_i u_i = 0$. Then, $\sum_{i=1}^s \lambda_i u_i = 0$, and hence, $\lambda_i = 0$, that is, $\gamma_i = \sigma_{i1}\alpha_{i1} + \cdots + \sigma_{i,t_i}\alpha_{i,t_i}$, for $i = 1, \ldots, s$. Setting

$$\sigma = \sum_{i=1}^s \sum_{j=1}^{t_i} (-1)^{|x_i|}\, \bar{\sigma}_{i,j} \otimes \hat{\alpha}^\star_{ij} \otimes u_i \in (F^C \circ G)(M)^{-1},$$

we deduce from the sublemma that

$$\begin{aligned}d^{-1}(\sigma) &= \textstyle\sum_{i=1}^s \sum_{j=1}^{t_i} (\bar{\sigma}_{ij}\,\bar{\alpha}_{ij} \otimes \hat{e}^\star_{x_i} \otimes u_i + (-1)^{|x_i|}\,\bar{\sigma}_{ij} \otimes \hat{e}^\star_{a_{ij}} \otimes \bar{\alpha}_{ij}\,u_i) \\ &= \textstyle\sum_{i=1}^s (\bar{\gamma}_i \otimes \hat{e}^\star_{x_i} \otimes u_i + \sum_{j=1}^{t_i} (-1)^{|x_i|}\,\bar{\sigma}_{ij} \otimes \hat{e}^\star_{a_{ij}} \otimes \bar{\alpha}_{ij}\,u_i) \\ &= \omega + \omega',\end{aligned}$$

where $\omega' = \sum_{i=1}^s \sum_{j=1}^{t_i} (-1)^{|x_i|}\,\bar{\sigma}_{ij} \otimes \hat{e}^\star_{a_{ij}} \otimes \bar{\alpha}_{ij}\,u_i$. By definition, $n_{\omega'} < n_\omega$, and

$$\eta^0_{_M}(\omega') = \sum_{i=1}^s \sum_{j=1}^{t_i} (-1)^{|x_i| + \frac{|x_i|(x_i|+1)}{2}} \bar{\sigma}_{ij}\bar{\alpha}_{ij}\,u_i = -\sum_{i=1}^s (-1)^{\frac{|x_i|(x_i|+1)}{2}} \bar{\gamma}_i\,u_i = 0.$$

By induction, $\omega \in \mathrm{Im}(d^{-1})$. Thus, $\mathrm{Im}(d^{-1}) = \mathrm{Ker}(\eta^0_{_M})$. This yields a natural quasi-isomorphism $\eta^\bullet_{_M} : (F \circ G)(M)^\bullet \to M$. The proof of the proposition is completed.

The following statement describes in particular an injective co-resolution for every bounded-above module over the Koszul dual.

5.6. Proposition. *Let $\Lambda = kQ/R$ be a Koszul algebra, where Q is a locally finite gradable quiver. If $N \in \mathrm{Mod}^- \Lambda^!$, then there exists a natural quasi-isomorphism $\theta^\bullet_{_N} : N \to (G^C \circ F)(N)^\bullet$.*

Proof. Fix $N \in \mathrm{Mod}^- \Lambda^!$. Let r be such that $N(a) = 0$ for $a \in Q^{-i}$ with $i > r$. By definition, $(G^C \circ F)(N)^\bullet = \mathbb{T}(G(F(N)^\bullet)^\bullet)$. We split our proof into several statements.

Statement 1. *Given $i \in \mathbb{Z}$, the i-th column of $G(F(N)^\bullet)^\bullet$ is*

$$\mathfrak{t}^i(G(F(N)^i)^\bullet) = \oplus_{a \in Q^{-i}}\, \mathfrak{t}^i(G(P_a)^\bullet) \otimes N(a) \cong \oplus_{a \in Q^{-i}}\, \mathfrak{t}^i(\mathcal{I}^\bullet_{S^!_a}[i]) \otimes N(a).$$

Indeed, $F(N)^i = \oplus_{a \in Q^{-i}} P_a \otimes N(a)$. By Lemma 5.4, $G(P_a)^\bullet \cong \mathcal{I}^\bullet_{S^!_a}[i]$ for $a \in Q^{-i}$.

Statement 2. *Given $n \in \mathbb{Z}$, we obtain $(G^C \circ F)(N)^n = 0$ in case $n < 0$; and $\mathrm{H}^n((G^C \circ F)(N)^\bullet) = 0$ in case $n > 0$.*

Indeed, given any $n \in \mathbb{Z}$, we obtain $(G^C \circ F)(N)^n = \oplus_{i \in \mathbb{Z}} G(F(N)^i)^{n-i}$, where

$$G(F(N)^i)^{n-i} = \oplus_{x \in Q^{n-i}} I^!_x \otimes F(N)^i(x) = \oplus_{x \in Q^{n-i};\, a \in Q^{-i}}\, I^!_x \otimes P_a(x) \otimes N(a).$$

If $n < 0$, then $P_a(x) = 0$ for $(x, a) \in Q^{n-i} \times Q^{-i}$ with $i \in \mathbb{Z}$, and therefore, $(G^C \circ F)(N)^n = 0$. Suppose that $n > 0$. Since $N(a) = 0$ for $a \in Q^{-i}$ with $i > r$, we see that $G(F(N)^{\bullet})^{\bullet}$ is n-diagonally bounded-above. And by Statement 1, the $(n-i)$-th cohomology of the i-th column of $G(F(N)^{\bullet})^{\bullet}$ is given by

$$\mathrm{H}^{n-i}(\mathfrak{t}^i(G(F(N)^i)^{\bullet})) \cong \oplus_{a\in Q^{-i}} \mathrm{H}^{n-i}(\mathcal{I}^{\bullet}_{S^!_a}[i]) \otimes N(a) = \oplus_{a\in Q^{-i}} \mathrm{H}^{n}(\mathcal{I}^{\bullet}_{S^!_a}) \otimes N(a) = 0.$$

In view of Lemma 4.2(2), we see that $\mathrm{H}^n((G^C \circ F)(N)^{\bullet}) = 0$.

It remains to construct a natural isomorphism $N \to \mathrm{H}^0((G^C \circ F)(N)^{\bullet})$. Indeed, the 0-diagonal of $G(F(N)^{\bullet})^{\bullet}$ consists of

$$G(F(N)^i)^{-i} = \oplus_{x\in Q^{n-i};\, a\in Q^{-i}}\; I^!_x \otimes P_a(x) \otimes N(a), i \in \mathbb{Z}.$$

We recall that $\Lambda^! = kQ^{\rm o}/R^! = \{\gamma^! \mid \gamma \in kQ\}$, where $\gamma^! = \gamma^{\rm o} + R^!$, while $(\Lambda^!)^{\rm o} = kQ/(R^!)^{\rm o} = \{\hat{\gamma} \mid \gamma \in kQ\}$, where $\hat{\gamma} = \gamma + (R^!)^{\rm o}$. Given $a, y \in Q_0$, there exists a k-linear map

$$N_{a,y} : N(y) \to \mathrm{Hom}_k(e_y(\Lambda^!)^{\rm o}e_a, P_a(a) \otimes N(a)) : u \mapsto N_{a,y}(u),$$

where $N_{a,y}(u)$ maps $\hat{\gamma}$ to $e_a \otimes \gamma^! u$, for all $\gamma \in kQ(a, y)$. By Corollary 1.2, there exists a k-linear isomorphism

$$\theta_{a,y} : \mathrm{Hom}_k(e_y(\Lambda^!)^{\rm o}e_a, k) \otimes P_a(a) \otimes N(a) \to \mathrm{Hom}_k(e_y(\Lambda^!)^{\rm o}e_a, P_a(a) \otimes N(a)).$$

This yields a k-linear map $f^a_y = \theta^{-1}_{a,y} \circ N_{a,y} : N(y) \to I^!_a(y) \otimes P_a(a) \otimes N(a)$.

STATEMENT 3. *If $\{\hat{\gamma}_1, \cdots, \hat{\gamma}_s\}$ is a basis of $e_y(\Lambda^!)^{\rm o}e_a$ with dual basis $\{\hat{\gamma}^\star_1, \cdots, \hat{\gamma}^\star_s\}$, then $f^a_y(u) = \sum_{i=1}^s \hat{\gamma}^\star_i \otimes e_a \otimes \gamma^!_i u$, for all $u \in N(y)$.*

Indeed, every $\hat{\gamma} \in e_z(\Lambda^!)^{\rm o}e_a$ is written as $\hat{\gamma} = \sum_{j=1}^s \lambda_j \hat{\gamma}_j$, for some $\lambda_j \in k$. Given $u \in N(z)$, by the definition of $\theta_{a,y}$, we obtain

$$\begin{aligned} \theta_{a,y}(\textstyle\sum_{i=1}^s \hat{\gamma}^\star_i \otimes e_a \otimes \gamma^!_i u)(\hat{\gamma}) &= \textstyle\sum_{1\le i,j\le s}(\lambda_j \hat{\gamma}^\star_i(\hat{\gamma}_j))(e_a \otimes \gamma^!_i u) \\ &= e_a \otimes \gamma^! u \\ &= N_{a,y}(u)(\hat{\gamma}). \end{aligned}$$

Thus, $\theta_{a,z}(\sum_{i=1}^s \hat{\gamma}^\star_i \otimes e_a \otimes \gamma^!_i u) = N_{a,y}(u)$, and hence, $f^a_y(u) = \sum_{i=1}^s \hat{\gamma}^\star_i \otimes e_a \otimes \gamma^!_i u$.

STATEMENT 4. *Given any $a \in Q_0$, there exists a natural $\Lambda^!$-linear morphism $f^a = (f^a_y)_{y\in Q_0} : N \to I^!_a \otimes P_a(a) \otimes N(a)$.*

Indeed, for any $\alpha : z \to y$ in Q_1, it is easy to verify that commutativity of

$$\begin{array}{ccccc} N(y) & \xrightarrow{N_{a,y}} & \mathrm{Hom}_k(P^{!,\rm o}_a(y), P_a(a) \otimes N(a)) & \xleftarrow{\theta_{a,y}} & I^!_a(y) \otimes P_a(a) \otimes N(a) \\ \downarrow{\scriptstyle N(\alpha^{\rm o})} & & \downarrow{\scriptstyle \mathrm{Hom}(P^{!,\rm o}_a(\alpha), P_a(a)\otimes N(a))} & & \downarrow{\scriptstyle I^!_a(\alpha^{\rm o})\otimes 1\otimes 1} \\ N(z) & \xrightarrow{N_{a,z}} & \mathrm{Hom}_k(P^{!,\rm o}_a(z), P_a(a) \otimes N(a)) & \xleftarrow{\theta_{a,z}} & I^!_a(z) \otimes P_a(a) \otimes N(a). \end{array}$$

Thus, f^a is $\Lambda^!$-linear. Given a $\Lambda^!$-linear morphism $g : N \to M$, we have a diagram

$$\begin{array}{ccccc} N(y) & \xrightarrow{N_{a,y}} & \mathrm{Hom}_k(P^{!,\rm o}_a(y), P_a(a) \otimes N(a)) & \xleftarrow{\theta_{a,y}} & I^!_a(y) \otimes P_a(a) \otimes N(a) \\ \downarrow{\scriptstyle g_y} & & \downarrow{\scriptstyle \mathrm{Hom}(P^{!,\rm o}_a(y), 1\otimes g_a)} & & \downarrow{\scriptstyle 1\otimes 1\otimes g_a} \\ M(y) & \xrightarrow{M_{a,y}} & \mathrm{Hom}_k(P^{!,\rm o}_a(y), P_a(a) \otimes M(a)) & \xleftarrow{\theta_{a,y}} & I^!_a(y) \otimes P_a(a) \otimes M(a), \end{array}$$

where the left square is easily verified to be commutative, while the commutativity of the right square follows from the naturality stated in Lemma 1.2(1).

Given $a \in Q^{-i}$, in view of Statement (4), we obtain a natural $\Lambda^!$-linear morphism $g^a = (g^a_y)_{y\in Q_0} : N \to I^!_a \otimes P_a(a) \otimes N(a)$, where $g^a_y = (-1)^{\frac{(i-1)i}{2}} f^a_y$.

STATEMENT 5. *Setting* $g = (g^a)_{a\in Q_0}$*, we obtain a natural* $\Lambda^!$*-linear monomorphism* $g : N \to (G^C \circ F)(N)^0$.

Indeed, g is a $\Lambda^!$-linear monomorphism if and only if, for any $y \in Q_0$, the linear morphism $g_y = (g^a_y) : N(y) \to (G^C \circ F)(N)^0 = \oplus_{a \in Q_0} I^!_a(y) \otimes P_a(a) \otimes N(a)$ is injective. Assume that $g_y(u) = 0$, for some $u \in N(y)$. Then $g^a_y(u) = 0$, for every $a \in Q_0$. In particular, $g^y_y(u) = 0$, and hence, $f^y_y(u) = 0$. Since $\{e_y\}$ is a basis of $e_y(\Lambda)^{\rm o}e_y$, by Statement 3, we have $e^\star_y \otimes e_y \otimes u = 0$, and hence, $u = 0$. This establishes Statement 5.

For the rest of the proof, observing that the (-1)-diagonal of $G(F(M)^\bullet)^\bullet$ contains only zero objects, we illustrate its 0-diagonal and 1-diagonal as follows:

$$\begin{array}{ccc} \oplus_{b \in Q^{-i}} I^!_b \otimes P_b(b) \otimes N(b) & \xrightarrow{h^{i,-i}} & \oplus_{(a,x) \in Q^{-i-1}\times Q^{-i}} I^!_x \otimes P_a(x) \otimes N(a) \\ & & \uparrow{\scriptstyle v^{i+1,-i-1}} \\ & & \oplus_{c \in Q^{-i-1}} I^!_c \otimes P_c(c) \otimes N(c), \end{array}$$

where $h^{i,-i} = (h^{i,-i}(a,x,b))_{(a,x,b)\in Q^{-i}\times Q^{-i-1}\times Q^{-i}}$, with

$$h^{i,-i}(a,x,b) = \begin{cases} \sum_{\alpha\in Q_1(a,x)} \mathbf{1}_{I^!_x} \otimes P[\bar\alpha] \otimes N(\alpha^{\rm o}), & \text{if } b = x; \\ 0, & \text{if } b \neq x, \end{cases}$$

and $v^{i+1,-i-1} = (v^{i+1,-i-1}(a,x,c))_{(a,x,c)\in Q^{-i}\times Q^{-i-1}\times Q^{-i-1}}$ with

$$v^{i+1,-i-1}(a,x,c) = \begin{cases} \sum_{\alpha\in Q_1(a,x)} (-1)^{i+1} I[\alpha^!] \otimes P_a(\alpha) \otimes \mathbf{1}_{N(a)}, & \text{if } c = a; \\ 0, & \text{if } c \neq a. \end{cases}$$

STATEMENT 6. *If* d^0 *is the* 0*-degree differential of* $(G^C \circ F)(N)^\bullet$*, then* $d^0 \circ g = 0$.

Indeed, it amounts to show, for any $p \in \mathbb{Z}$, that the diagram

$$\begin{array}{ccc} \oplus_{x \in Q^{-p}} I^!_x \otimes P_x(x) \otimes N(x) & \xrightarrow{\oplus h^{p,-p}(a,x,x)} & \oplus_{a\in Q^{-p-1};x\in Q^{-p}} I^!_x \otimes P_a(x) \otimes N(a) \\ {\scriptstyle (g^x)_{x\in Q^{-p}}}\uparrow & & \uparrow{\scriptstyle \oplus v^{p+1,-p-1}(a,x,a)} \\ N & \xrightarrow{(g^a)_{a\in Q^{-p-1}}} & \oplus_{a \in Q^{-p-1}} I^!_a \otimes P_a(a) \otimes N(a), \end{array}$$

is anti-commutative, or equivalently, we have an anti-commutative diagram

$$\begin{array}{ccc} \oplus_{x\in Q^{-p}} I^!_x(y) \otimes P_x(x) \otimes N(x) & \xrightarrow{\oplus h^{p,-p}(a,x,x)(y)} & \oplus_{a\in Q^{-p-1};x\in Q^{-p}} I^!_x(y) \otimes P_a(x) \otimes N(a) \\ {\scriptstyle (g^x_y)_{x\in Q^{-p}}}\uparrow & & \uparrow{\scriptstyle \oplus v^{p+1,-p-1}(a,x,a)(y)} \\ N(y) & \xrightarrow{(g^a_y)_{a\in Q^{-p-1}}} & \oplus_{a \in Q^{-p-1}} I^!_a(y) \otimes P_a(a) \otimes N(a), \end{array}$$

for all $y \in Q_0$. Fix $u \in N(y)$ for some $y \in Q_0$. Consider $\alpha \in Q_1(a,x)$ with $(a,x) \in Q^{-p-1} \times Q^{-p}$. Choosing a k-basis $\{\hat\delta_1, \ldots, \hat\delta_s\}$ of $e_y(\Lambda^!)^{\rm o}e_x$, since $x \in Q^{-p}$, we deduce from Statement 3 that

$$(\mathbf{1} \otimes P[\bar\alpha] \otimes N(\alpha^{\rm o}))\left(g^x_y(u)\right) = (-1)^{\frac{(p-1)p}{2}} \sum_{i=1}^s \hat\delta^\star_i \otimes \bar\alpha \otimes \alpha^! \delta^!_i u.$$

For any k-basis $\{\hat\gamma_1, \ldots, \hat\gamma_t\}$ of $e_y(\Lambda^!)^{\rm o}e_a$, since $a \in Q^{-p-1}$, we obtain

$$(I[\alpha^!]\otimes P_a(\alpha)\otimes \mathbf{1})\left(g_y^a(u)\right)=(-1)^{\frac{p(p+1)}{2}}\sum_{i=1}^t I[\alpha^!](\hat\gamma_i^\star)\otimes\bar\alpha\otimes\gamma_i^!u.$$

Let $\theta: I_x^!(y)\otimes P_a(x)\otimes N(a)\to \mathrm{Hom}_k(e_y(\Lambda^!)^{\rm o}e_x, P_a(x)\otimes N(a))$ be a k-linear isomorphism stated in Corollary 1.2. Given any $1\le j\le s$, it is easy to see that

$$\theta[(\mathbf{1}\otimes P[\bar\alpha]\otimes N(\alpha^{\rm o}))\left(g_y^x(u)\right)](\hat\delta_j)=(-1)^{\frac{(p-1)p}{2}}(\bar\alpha\otimes\alpha^!\delta_j^!u),$$

and

$$\theta[(I[\alpha^!]\otimes P_a(\alpha)\otimes\mathbf{1})\left(g_y^a(u)\right)](\hat\delta_j)=(-1)^{\frac{p(p+1)}{2}}\sum_{i=1}^t\hat\gamma_i^\star(\hat\delta_j\hat\alpha)\left(\bar\alpha\otimes\gamma_i^!u\right).$$

Fix some $1\le j\le s$. If $\hat\delta_j\,\hat\alpha=0$, then $\alpha^!\delta_j^!=0$, and hence,

$$\theta[(I[\alpha^!]\otimes P_a(\alpha)\otimes\mathbf{1})\left(g_y^a(u)\right)](\hat\delta_j)=0=(-1)^p\theta[(\mathbf{1}\otimes P[\bar\alpha]\otimes N(\alpha^{\rm o}))\left(g_y^x(u)\right)](\hat\delta_j).$$

If $\hat\delta_j\,\hat\alpha\ne 0$, then it extends to a k-basis $\{\hat\gamma_1,\ldots,\hat\gamma_t\}$ with $\hat\gamma_1=\hat\delta_j\,\hat\alpha$ of $e_y(\Lambda^!)^{\rm o}e_a$. Under this assumption, we obtain

$$\begin{aligned}\theta[(I[\alpha^!]\otimes P_a(\alpha)\otimes\mathbf{1})\left(g_y^a(u)\right)](\hat\delta_j) &= (-1)^{\frac{p(p+1)}{2}}\sum_{i=1}^t\hat\gamma_i^\star(\hat\gamma_1)(\bar\alpha\otimes\gamma_i^!u)\\ &= (-1)^{\frac{p(p+1)}{2}}(\bar\alpha\otimes\gamma_1^!u)\\ &= (-1)^{\frac{p(p+1)}{2}}(\bar\alpha\otimes\hat\delta_j\hat\alpha\,u)\\ &= (-1)^p\theta[(\mathbf{1}\otimes P[\bar\alpha]\otimes N(\alpha^{\rm o}))\left(g_y^x(u)\right)](\hat\delta_j).\end{aligned}$$

Thus, $\theta[(I[\alpha^!]\otimes P_a(\alpha)\otimes\mathbf{1})\left(g_y^a(u)\right)]=(-1)^p\theta[(\mathbf{1}\otimes P[\bar\alpha]\otimes N(\alpha^{\rm o}))\left(g_y^x(u)\right)]$. Then,

$$(I[\alpha^!]\otimes P_a(\alpha)\otimes\mathbf{1})\left(g_y^a(u)\right)=(-1)^p(\mathbf{1}\otimes P[\bar\alpha]\otimes N(\alpha^{\rm o}))\left(g_y^x(u)\right).$$

Therefore,

$$(h^{p,-p}(a,x,x)(y)\circ g_y^x)(u)+(v^{p+1,-p-1}(a,x,a)(y)\circ g_y^a)(u)=0,$$

and hence,

$$h^{p,-p}(a,x,x)(y)\circ g_y^x+v^{p+1,-p-1}(a,x,a)(y)\circ g_y^a=0.$$

This in turn implies the required anti-commutativity.

We are ready to conclude our proof by claiming that $\mathrm{Ker}(d^0)\subseteq\mathrm{Im}(g)$. Indeed, given any element $\omega=(\omega^i)_{i\in\mathbb{Z}}\in\mathrm{Ker}(d^0)$, where

$$\omega^i\in G(F(N)^i)^{-i}=\oplus_{a\in Q^{-i}}I_a^!\otimes P_a(a)\otimes N(a),$$

observing that $G(F(N)^i)^{-i}=0$ for $i>r$, we define an integer $n_\omega(\le r)$ as follows: if $\omega=0$, then $n_\omega=r$; and otherwise, n_ω is minimal such that $w^{n_\omega}\ne 0$.

If $n_\omega=r$, then $\omega\in\mathrm{Im}(g)$. Assume that $n_\omega<r$. Since $\omega\in\mathrm{Ker}(d^0)$, we see that

$$v^{n_\omega,-n_\omega}(\omega^{n_\omega})=-h^{n_\omega-1,1-n_\omega}(\omega^{n_\omega-1})=0.$$

By Statement 1, the n_ω-th column of the double complex $G(F(N)^\bullet)^\bullet$ is, up to a twist, the shift by n_ω of the minimal injective co-resolution of the module $\oplus_{a\in Q^{-n_\omega}}S_a^!\otimes P_a(a)\otimes N(a)$. Thus, $w^{n_\omega}\in S_J(\oplus_{a\in Q^{-n_\omega}}I_a^!\otimes P_a(a)\otimes N(a))$, and by Lemma 2.7, $\omega^{n_\omega}=\sum_{a\in Q^{-n_\omega}}\hat e_a^\star\otimes e_a\otimes u_a$, where $u_a\in N(a)$. Now, by Statement 3,

$$g(\textstyle\sum_{a\in Q^{-n_\omega}}u_a)=\sum_{a\in Q^{-n_\omega}}\hat e_a^\star\otimes e_a\otimes u_a=\omega^{n_\omega},$$

and by Statement 6, $\nu=\omega-g(\sum_{a\in Q^{-n_\omega}}u_a)\in\mathrm{Ker}(d^0)$. Writing $\nu=(\nu^i)_{i\in\mathbb{Z}}$ with $\nu^i\in G(F(N)^i)^{-i}$, we see that $\nu^{n_\omega}=\omega^{n_\omega}-g(\sum_{a\in Q^{-n_\omega}}u_a)=0$, and $\nu^i=\omega^i=0$ for all $i<n_\omega$. Therefore, $n_\nu<n_\omega$. Assuming inductively that $\nu\in\mathrm{Im}(g)$, we obtain $\omega\in\mathrm{Im}(g)$. Therefore, $\mathrm{Ker}(d^0)=\mathrm{Im}(g)$. Setting $\theta_N^0=g$, and $\theta_N^i=0$ for all

$i \neq 0$, we obtain a quasi-isomorphism $\theta^{\bullet}_N : N \to (G^C \circ F)(N)^{\bullet}$ which, by Statement 4, is natural in N. The proof of the proposition is completed.

We are ready to obtain our promised Koszul duality as follows.

5.7. THEOREM. *Let $\Lambda = kQ/R$ be a Koszul algebra, where Q is a locally finite gradable quiver. If $p, q \in \mathbb{R}$ with $p \geq 1$ and $q \geq 0$, then we obtain two mutual quasi-inverse triangle equivalences*

$$F^D_{p,q} : D^{\downarrow}_{p,q}(\mathrm{Mod}\,\Lambda^!) \to D^{\uparrow}_{q+1,p-1}(\mathrm{Mod}\,\Lambda)$$

and

$$G^D_{p,q} : D^{\uparrow}_{q+1,p-1}(\mathrm{Mod}\,\Lambda) \to D^{\downarrow}_{p,q}(\mathrm{Mod}\,\Lambda^!).$$

Proof. We shall make use of the Koszul functors $F : \mathrm{Mod}\,\Lambda^! \to C(\mathrm{Mod}\,\Lambda)$ and $G : \mathrm{Mod}\,\Lambda \to C(\mathrm{Mod}\,\Lambda^!)$, the complex Koszul functors $F^C : C(\mathrm{Mod}\,\Lambda^!) \to C(\mathrm{Mod}\,\Lambda)$ and $G^C : C(\mathrm{Mod}\,\Lambda) \to C(\mathrm{Mod}\,\Lambda^!)$ and two commutative diagrams in Theorem 5.3.

Let $p, q \in \mathbb{R}$ with $p \geq 1$ and $q \geq 0$. We first claim that the identity functor of $D^{\downarrow}_{p,q}(\mathrm{Mod}\,\Lambda^!)$ is isomorphic to $G^D_{p,q} \circ F^D_{p,q}$. Consider the embedding functor $\kappa : \mathrm{Mod}^{-}\Lambda^! \to C(\mathrm{Mod}\,\Lambda^!)$ and the functor $G^C \circ F : \mathrm{Mod}^{-}\Lambda^! \to C(\mathrm{Mod}\,\Lambda^!)$. By Proposition 5.6, we obtain a functorial morphism $\theta = (\theta^{\bullet}_{N})_{N \in \mathrm{Mod}^{-}\Lambda^!} : \kappa \to G^C \circ F$, and by Lemma 4.11, it induces a functorial morphism $\theta^C : \mathbf{1}_{\mathrm{Mod}\,\Lambda} = \kappa^C \to (G^C \circ F)^C$.

Let $N^{\bullet} \in C^{\downarrow}_{p,q}(\mathrm{Mod}\,\Lambda^!)$. Since $N^{\bullet} \in C(\mathrm{Mod}^{-}\Lambda^!)$, we obtain $\theta^C_{N^{\bullet}} = \mathbb{T}(\theta^{\bullet}_{N^{\bullet}})$, where $\theta^{\bullet}_{N^{\bullet}} = (\theta^j_{N^i})_{i,j \in \mathbb{Z}} : \kappa(N^{\bullet})^{\bullet} \to (G^C \circ F)(N^{\bullet})^{\bullet}$. We claim that $\theta^C_{N^{\bullet}}$ is a quasi-isomorphism. Indeed, by Lemma 5.6, $\eta^{\bullet}_{N^i} : \kappa(N^i)^{\bullet} \to (G^C \circ F)(N^i)^{\bullet}$ is a quasi-isomorphism, and so is $\theta^{\bullet}_{M^i} : \mathfrak{t}^i(\kappa(N^i)^{\bullet}) \to \mathfrak{t}^i((G^C \circ F)(N^i)^{\bullet})$, for every $i \in \mathbb{Z}$. Moreover, given any $n \in \mathbb{Z}$, the n-diagonal of $(G^C \circ F)(N^{\bullet})^{\bullet}$ consists of

$$(G^C \circ F)(N^i)^{n-i} = \oplus_{j \in \mathbb{Z}; x \in Q^{-j}; y \in Q^{n-i-j}} I^!_y \otimes P_x(y) \otimes N^i(x);\ i \in \mathbb{Z}.$$

If $i > n$, then $P_x(y) = e_y \Lambda e_x = 0$, for any $x \in Q^{-j}$ and $y \in Q^{n-i-j}$ with $j \in \mathbb{Z}$, and therefore, $(G^C \circ F)(N^i)^{n-i} = 0$. That is, $(G^C \circ F)(N^{\bullet})^{\bullet}$ is diagonally bounded-above. Since $\kappa(N^{\bullet})^{\bullet}$ clearly is diagonally bounded-above, by Lemma 4.6, $\mathbb{T}(\theta^{\bullet}_{N^{\bullet}})$, that is $\theta^C_{N^{\bullet}} : N^{\bullet} \to (G^C \circ F)^C(N^{\bullet})$, is a quasi-isomorphism. Since $(G^C \circ F)^C = G^C \circ F^C$; see (4.10), we obtain a natural quasi-isomorphism $\theta^C_{N^{\bullet}} : N^{\bullet} \to (G^C_{p,q} \circ F^C_{p,q})(N^{\bullet})$, for $N^{\bullet} \in C^{\downarrow}_{p,q}(\mathrm{Mod}\,\Lambda^!)$. As a consequence, $\theta^D_{N^{\bullet}} = L_{\Lambda^!}(P_{\Lambda^!}(\theta^C_{N^{\bullet}})) : N^{\bullet} \to (G^D_{p,q} \circ F^D_{p,q})(N^{\bullet})$ is a natural isomorphism, for $N^{\bullet} \in D^{\downarrow}_{p,q}(\mathrm{Mod}\,\Lambda^!)$. This establishes our first claim.

To show that $F^D_{p,q} \circ G^D_{p,q}$ is isomorphic to the identity functor of $D^{\uparrow}_{q+1,p-1}(\mathrm{Mod}\,\Lambda^!)$, we consider the functor $F^C \circ G : \mathrm{Mod}\,\Lambda \to C(\mathrm{Mod}\,\Lambda)$ and the embedding functor $\kappa : \mathrm{Mod}\,\Lambda \to C(\mathrm{Mod}\,\Lambda)$. In view of Lemma 5.5, we obtain a functorial morphism $\eta = (\eta^{\bullet}_M)_{M \in \mathrm{Mod}\,\Lambda} : F^C \circ G \to \kappa$, and by Lemma 4.11, it induces a functorial morphism $\eta^C : (F^C \circ G)^C \to \kappa^C = \mathbf{1}_{C(\mathrm{Mod}\,\Lambda)}$.

Let $M^{\bullet} \in C^{\uparrow}_{q+1,p-1}(\mathrm{Mod}\,\Lambda)$. We claim that $\eta^C_{M^{\bullet}} : (F^C \circ G)^C(M^{\bullet}) \to M^{\bullet}$ is a quasi-isomorphism, that is, $\eta^C_{M^{\bullet}}(z) : (F^C \circ G)^C(M^{\bullet})(z) \to M^{\bullet}(z)$ is a quasi-isomorphism, for all $z \in Q_0$. Let $z \in Q^s$ for some $s \in \mathbb{Z}$. By definition, $\eta^C_{M^{\bullet}}(z) = \mathbb{T}(\eta^{\bullet}_{M^{\bullet}}(z))$, where $\eta^{\bullet}_{M^{\bullet}}(z) = (\eta^j_{M^i}(z))_{i,j \in \mathbb{Z}} : (F^C \circ G)(M^{\bullet})^{\bullet}(z) \to \kappa(M^{\bullet})^{\bullet}(z)$.

Given $i \in \mathbb{Z}$, by Lemma 5.5, $\eta^{\bullet}_{M^i} : \mathfrak{t}^i((F^C \circ G)(M^i)^{\bullet}) \to \mathfrak{t}^i(\kappa(M^i)^{\bullet})$ is a quasi-isomorphism, and so is $\eta^{\bullet}_{M^i}(z) : \mathrm{t}^i((F^C \circ G)(M^i)^{\bullet})(z) \to \mathrm{t}^i(\kappa(M^i)^{\bullet})(z)$. On the other hand, given any $n \in \mathbb{Z}$, the n-diagonal of $(F^C \circ G)(M^{\bullet})^{\bullet}(z)$ consists of

$$\begin{aligned}(F^C \circ G)(M^i)^{n-i}(z) &= \oplus_{j\in\mathbb{Z};\, x\in Q^j;\, y\in Q^{i+j-n}} P_y(z) \otimes I^!_x(y) \otimes M^i(x)\\ &= \oplus_{j\le n+s-i;\, x\in Q^j;\, y\in Q^{i+j-n}} P_y(z) \otimes I^!_x(y) \otimes M^i(x),\ i \in \mathbb{Z}.\end{aligned}$$

Since $M^{\bullet} \in C^{\uparrow}_{q+1,p-1}(\mathrm{Mod}\,\Lambda)$, there exists t such that $M^i(x) = 0$ for $x \in Q^j$ with $i - (p-1)j > t$. Let $x \in Q^j$ with $j \le n+s-i$. If $pi > (p-1)(n+s)+t$, then $i-(p-1)j \ge i-(p-1)(n+s-i) = pi-(p-1)(n+s) > t$, and consequently, $M^i(x) = 0$. That is, $(F^C \circ G)(M^{\bullet})^{\bullet}(z)$ is diagonally bounded-above. By Lemma 4.6, $\mathbb{T}(\eta^{\bullet}_{M^{\bullet}}(z))$, that is $\eta^C_{M^{\bullet}}(z)$, is a quasi-isomorphism. Our second claim is established.

Now, since $(F^C \circ G)^C = F^C \circ G^C$; see (4.10), we obtain a natural quasi-isomorphism $\eta^C_{M^{\bullet}} : (F^C_{p,q} \circ G^C_{p,q})(M^{\bullet}) \to M^{\bullet}$, for $M^{\bullet} \in C^{\uparrow}_{q+1,p-1}(\mathrm{Mod}\,\Lambda)$. This induces, as has been seen above, a functorial isomorphism from $F^D_{p,q} \circ G^D_{p,q}$ to the identity functor of $D^{\uparrow}_{q+1,p-1}(\mathrm{Mod}\,\Lambda^!)$. The proof of the theorem is completed.

Remark. The case $p = 1$ and $q = 0$ of Theorem 5.7 has been established for a left finite Koszul algebra; see [5, (2.12.1)] and for a positively graded Koszul category; see [25, Theorem 30].

Specializing to the locally bounded case, we get the following result; see [2, (3.9)].

5.8. Theorem. *Let $\Lambda = kQ/R$ be a Koszul algebra, where Q is a locally finite gradable quiver. If Λ is right (or left) locally bounded and $\Lambda^!$ is left (or right) locally bounded, then $D^b(\mathrm{Mod}^b\Lambda^!) \cong D^b(\mathrm{Mod}^b\Lambda)$ and $D^b(\mathrm{mod}^b\Lambda^!) \cong D^b(\mathrm{mod}^b\Lambda)$.*

Proof. First, assume that Λ is right locally bounded and $\Lambda^!$ is left locally bounded. Then, $P_a \in \mathrm{mod}^b\Lambda$ and $I^!_a \in \mathrm{mod}^b\Lambda^!$, for every $a \in Q_0$. Therefore, the Koszul functors restrict to functors $F : \mathrm{Mod}^b\Lambda^! \to C^b(\mathrm{Mod}^b\Lambda)$ and $G : \mathrm{Mod}^b\Lambda \to C^b(\mathrm{Mod}^b\Lambda^!)$.

Given $M^{\bullet} \in C^b(\mathrm{Mod}^b\Lambda^!)$ and $N^{\bullet} \in C^b(\mathrm{Mod}^b\Lambda)$, the double complexes $F(M^{\bullet})^{\bullet}$ and $G(N^{\bullet})^{\bullet}$ are bounded. Therefore, the complex Koszul functors restrict to functors $F^C : C^b(\mathrm{Mod}^b\Lambda^!) \to C^b(\mathrm{Mod}^b\Lambda)$ and $G^C : C^b(\mathrm{Mod}^b\Lambda) \to C^b(\mathrm{Mod}^b\Lambda^!)$.

Consider $F^C \circ G : \mathrm{Mod}^b\Lambda \to C^b(\mathrm{Mod}^b\Lambda)$ and $G^C \circ F : \mathrm{Mod}^b\Lambda^! \to C^b(\mathrm{Mod}^b\Lambda^!)$. In view of Propositions 5.5 and 5.6, we obtain two natural quasi-isomorphisms $\theta^C_{N^{\bullet}} : N^{\bullet} \to (F^C \circ G)(N^{\bullet})$ and $\eta^C_{M^{\bullet}} : M^{\bullet} \to (F^C \circ G)(M^{\bullet})$, for $N^{\bullet} \in C^b(\mathrm{Mod}^b\Lambda)$ and $M^{\bullet} \in C^b(\mathrm{Mod}^b\Lambda^!)$. As have argued in the proof of Theorem 5.7, we see that the functors F^C and G^C descend to two mutually quasi-inverse triangle equivalences $F^D : D^b(\mathrm{Mod}^b\Lambda^!) \to D^b(\mathrm{Mod}^b\Lambda)$ and $G^D : D^b(\mathrm{Mod}^b\Lambda) \to D^b(\mathrm{Mod}^b\Lambda^!)$.

Next we can show, in the same way, that $D^b(\mathrm{mod}^b\Lambda) \cong D^b(\mathrm{mod}^b\Lambda^!)$. Finally, suppose that Λ is left locally bounded and $\Lambda^!$ is right locally bounded. Since $\Lambda^!$ is a Koszul algebra with $(\Lambda^!)^! = \Lambda$, as has been seen, $D^b(\mathrm{Mod}^b\Lambda) \cong D^b(\mathrm{Mod}^b\Lambda^!)$ and $D^b(\mathrm{mod}^b\Lambda) \cong D^b(\mathrm{mod}^b\Lambda^!)$. The proof of the theorem is completed.

Remark. In case Λ is of finite length and $\Lambda^!$ is left noetherian, Beilinson, Ginzburg and Soergel proved the graded version of the second part of Theorem 5.8 with a rather sophisticated proof; see [5, (2.12.6)], and also, [25, Theorem 35].

Example. (1) Theorem 5.8 holds in case Q has no right infinite path or no left infinite path. Indeed, if this is the case, then Q^{o} has no left infinite path or no right infinite path, and consequently, Λ is right or left locally bounded and $\Lambda^!$ is left or right locally bounded, respectively.

(2) Let $\Lambda = kQ/(kQ^+)^2$, where Q is a locally finitely gradable having some right infinite paths. Then Λ is locally bounded, but $\Lambda^! = kQ^{\rm o}$ is not left locally bounded. In this case, $D^b(\mathrm{Mod}^b\Lambda) \cong D^b(\mathrm{Rep}^-(Q^{\rm o}))$; see [2, (3.9)], where $\mathrm{Rep}^-(Q^{\rm o})$ denotes the category of almost finitely co-presented representations, which is substantially larger than the category of finitely supported representations; [3, (1.12)].

(3) Let Λ be the k-algebra defined by the gradable quiver

$$\cdots \xrightarrow{\gamma_{-4}} -3 \underset{\beta_{-3}}{\overset{\alpha_{-3}}{\rightrightarrows}} -2 \xrightarrow{\gamma_{-2}} -1 \xrightarrow{\gamma_{-1}} 0 \underset{\beta_0}{\overset{\alpha_0}{\rightrightarrows}} 1 \xrightarrow{\gamma_1} 2 \xrightarrow{\gamma_2} 3 \underset{\beta_3}{\overset{\alpha_3}{\rightrightarrows}} 4 \xrightarrow{\gamma_4} \cdots$$

with relations $\alpha_{3n}\gamma_{3n-1}, \beta_{3n}\gamma_{3n-1}$, $n \in \mathbb{Z}$. Then Λ is Koszul and $\Lambda^!$ is defined by

$$\cdots \xleftarrow{\gamma'_{-4}} -3 \underset{\beta'_{-3}}{\overset{\alpha'_{-3}}{\leftleftarrows}} -2 \xleftarrow{\gamma'_{-2}} -1 \xleftarrow{\gamma'_{-1}} 0 \underset{\beta'_0}{\overset{\alpha'_0}{\leftleftarrows}} 1 \xleftarrow{\gamma'_1} 2 \xleftarrow{\gamma'_2} 3 \underset{\beta'_3}{\overset{\alpha'_3}{\leftleftarrows}} 4 \xleftarrow{\gamma'_4} \cdots$$

with relations $\alpha'_{3n}\gamma'_{3n+1}, \beta'_{3n}\gamma'_{3n+1}$, $\alpha'_{3n+2}\gamma'_{3n+1}$, $n \in \mathbb{Z}$. By Theorem 5.8, we obtain $D^b(\mathrm{Mod}\,\Lambda) \cong D^b(\mathrm{Mod}\,\Lambda^!)$ and $D^b(\mathrm{mod}^b\Lambda) \cong D^b(\mathrm{mod}^b\Lambda^!)$. Note that none of the results stated in [2], [5] or [25] applies in this situation.

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References

[1] R. Bautista and S. Liu, "Covering theory for linear categories with application to derived categories", J. Algebra 406 (2014) 173 - 225.

[2] R. Bautista and S. Liu, "The bounded derived categories of an algebra with radical squared zero", J. Algebra 482 (2017) 303 - 345.

[3] R. Bautista, S. Liu and C. Paquette, "Representation theory of strongly locally finite quivers", Proc. London Math. Soc. 106 (2013) 97 - 162.

[4] I. Bernstein, I. Gelfand, and S. Gelfand, "Algebraic bundles over $\mathbb{P}^n$ and problems of linear algebras", Funkts. Anal. Prilozh. 12 (1978); English translation in Functional Analysis and its Applications 12 (1918) 212 - 214.

[5] A. Beilinson, V. Ginzburg and W. Soergel, "Koszul duality patterns in representation theory", J. Amer. Math. Soc. 9 (1996) 473 - 527.

[6] K. Bongartz and P. Gabriel, "Covering spaces in representation theory", Invent. Math. 65 (1982) 331 - 378.

[7] A. K. Bousfield, E. B. Curtis, D. M. Kan, D. G. Quillen, D. L. Rector and J. W. Schlesinger, "The mod-p lower central series and the Adams spectral sequence", Topology 5 (1966) 331 - 342.

[8] H. Cartan and S. Eilenberg, "Homological Algebra", Princeton University Press, Princeton, New Jersey, 1956.

[9] D. Eisenbud, "Commutative Algebra with a view toward algebraic geometry", Springer-Verlarg, New York, 1995.

[10] G. Fløystad, "Koszul Duality and Equivalences of Categories", Trans. Amer. Math. Soc. 358 (2006) 2373 - 2398.

[11] E.L. Green and R. Martinez-Villa, "Koszul and Yoneda algebras," Canad. Math. Soc. Conf. Proc. 18 (1996) 247 - 298.

[12] E.L. Green and R. Martinez-Villa, "Koszul and Yoneda algebras II," Canad. Math. Soc. Conf. Proc. 24 (1998) 227 - 244.

[13] E. L. Green, Ø. Solberg and D. Zacharia, "Minimal projective resolutions," Trans. Amer. Math. Soc. 353 (2001) 2915 - 2939.

[14] M. Goresky, R. Kottwitz and R. MacPherson, "Equivariant cohomology, Koszul duality, and the localization theorem", Invent. Math. 131 (1998) 25 - 83.

[15] B. Keller, "Koszul duality and coderived categories (after K. Lefèvre)", Preprint 2003.

[16] B. Keller, "On the construction of triangle equivalences", Lecture Notes in Math. 1685 (1998) 155 - 176.

[17] S. Liu and J.-P. Morin, "The strong no loop conjecture for special biserial algebras", Proc. Amer. Math. Soc. 132 (2004) 3513 - 3523.

[18] S. MacLane, "Homology", Springer-Verlag, Berline, Heidelberg, New York, 1971.

[19] YU. I. MANIN, "Some remarks on Koszul algebras and quantum groups", Ann. Inst. Fourier 37 (1987) 191 - 205.

[20] R. Martinez-Villa, "Introduction to Koszul algebras," Rev. Un. Mat. Argentina 48 (2007) 67 - 95.

[21] R. Martinez-Villa, "Applications of Koszul algebras: The preprojective algebra," Canad. Math. Soc. Conf. Proc. 18 (1996) 487 - 504.

[22] R. Martinez-Villa and Ø. Solberg, "Graded and Koszul categories", Appl. Categ. Struct. 18 (2010) 615 - 652.

[23] R. Martinez-Villa and M. Saorin, "Koszul equivalences and dualities", Pacific J. Math. 214 (2004) 359 - 378.

[24] J. P. May, "The cohomology of restricted Lie algebras and of Hopf algebras", J. Algebra 3 (1966) 123 - 146.

[25] V. Mazorchuk, S. Ovsienko, and C. Stroppel, "Quadratic duals, Koszul dual functors, and applications", Trans. Amer. Math. Soc. 361(3) (2009) 1129 - 1172.

[26] D. Milicic, "Lectures on derived categories", www.math.utah.edu/~milicic/Eprints/dercat.pdf (preprint).

[27] S. B. Priddy, "Koszul resolutions and the Steenrod algebra", Bull. Amer. Math. Soc. 76 (1970) 834 - 839.

[28] S. B. Priddy, "Koszul resolutions", Trans. Amer. Math. Soc. 152 (1970) 39 - 60.

[29] J. Rickard, "Morita theory for derived categories", J. London Math. Soc. (2) 39 (1989) 436 - 456.

[30] S. Ryom-Hansen, "Koszul duality of translation and Zuckermann functor", J. Lie Theory 14 (2004) 151 - 163.

[31] C. A. Weibel, "An introduction to homological algebra", Cambridge Studies in Advanced Mathematics 38 (Cambridge University Press, Cambridge, 1994).

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