Preprojective Modules and Auslander-Reiten Components

Shiping Liu

In [2], Auslander and Smalø introduced and studied extensively preprojective modules and preinjective modules over an artin algebra. We now call a module *hereditarily preprojective* or *hereditarily preinjective* if its submodules are all preprojective or its quotient modules are all preinjective, respectively. In [4], Coelho studied Auslander-Reiten components containing only hereditarily preprojective modules and gave a number of characterizations of such components. We shall study further these modules by using the description of shapes of semi-stable Auslander-Reiten components; see [6, 7]. Our results will imply the result of Coelho [4, (1.2)] and that of Auslander-Smalø [2, (9.16)]. As an application, moreover, we shall show that a stable Auslander-Reiten component with "few" stable maps in TrD-direction is of shape $\mathbb{Z}A_\infty$.

1. Preliminaries on Auslander-Reiten components

Throughout this note, A denotes an artin algebra, $\operatorname{mod} A$ the category of finitely generated right A-modules, and $\operatorname{rad}^\infty(\operatorname{mod} A)$ the infinite radical of $\operatorname{mod} A$. Let Γ_A be the Auslander-Reiten quiver of A which is defined in such a way that its vertices form a complete set of the representatives of isoclasses of the indecomposables of $\operatorname{mod} A$. We denote by τ the Auslander-Reiten translation DTr. The reader is referred to [7] for notions not defined here. We first reformulate a result stated in [7, (2.3)] for later use. Its proof can be found in the proofs of [7, (2.2), (2.3)].

1.1. Proposition. *Let Γ be a left stable component of Γ_A with no τ-periodic module. If Γ contains an oriented cycle, then every module in Γ admits at most two immediate successors in Γ and there exists an infinite sectional path*

$$N_1 \to \cdots \to N_s \to \tau^t N_1 \to \cdots \to \tau^t N_s \to \tau^{2t} N_1 \to \cdots,$$

with $t > 0$ and $\{N_1, \ldots, N_s\}$ a complete set of representatives of τ-orbits in Γ.

1.2. Lemma. *Let X be a module in Γ_A, not lying in any finite τ-periodic stable component. Then there exists some $r \geq 0$ such that $\tau^r X$ lies on an oriented cycle of Γ_A of left stable modules if one of the following holds:*

(1) *$\tau^n X$ lies on an oriented cycle in Γ_A for infinitely many $n > 0$.*

(2) *A module is a predecessor of $\tau^n X$, for infinitely many $n > 0$, in Γ_A.*

Proof. Assume that either (1) or (2) occurs. In particular, X is left stable. It suffices to consider the case where X is not τ-periodic. Then there exists $s \geq 0$ such that the $\tau^n X$ with $n \geq s$ lie in an infinite non τ-periodic left stable component Γ of Γ_A. Suppose that Γ contains no oriented cycle. Then Γ admits a section Δ, and hence Γ is embedded in $\mathbb{Z}\Delta$; see [7, (3.1), (3.4)]. For modules M, N in Γ, there exist at most finitely many $n \geq 0$ such that N is predecessor in Γ of $\tau^n M$. Moreover, applying some power of τ, we may assume that none of the predecessors of Δ in Γ is an immediate successor of a projective in Γ_A. This implies that Δ admits no projective predecessor in Γ_A. Now by (1) or (2), there exist infinitely many $n \geq s$ such that $\tau^n X$ admits a projective predecessor in Γ_A. However, $\tau^n X$ is a predecessor of Δ when n is suffiently large, a contradiction. Hence Γ contains an oriented cycle. The lemma now follows easily from Proposition 1.1. The proof is completed.

1.3. LEMMA. *Let X be a module in Γ_A, lying on an oriented cycle of left stable modules but not in any finite τ-periodic stable component. Then for every sufficient large positive integer n, there exists in Γ_A an infinite sectional path starting with $\tau^n X$ and one ending with $\tau^n X$.*

Proof. It suffices to show that for some $r \geq 0$, $\tau^r X$ is the start-point of an infinite sectional path of left stable modules. By assumption, the left stable component Γ containing X is infinite and has oriented cycles. If Γ is τ-periodic, then it is a stable tube [5]. Hence X is the start-point of an infinite sectional path of τ-periodic modules. Otherwise, the lemma follows easily from Proposition 1.1. The proof is completed.

We now have our main result of this section which generalizes two equivalent conditions stated in [4, (1.2)].

1.4. PROPOSITION. *Let Γ be a connected full subquiver of Γ_A, closed under predecessors. The following are equivalent*:

(1) *Every module in Γ admits only finitely many predecessors in Γ.*

(2) *All but finitely many modules in Γ lie in τ-orbits of projectives and do not lie on oriented cycles in Γ.*

(3) *Γ contains projectives but no infinite sectional path ending with a projective, and every immediate predecessor of a projective in Γ admits a projective predecessor.*

Proof. That (2) implies (1) can be proved by using the argument given in the proof of [4, (4.2)]. Assume now that (1) occurs. Since Γ_A contains no sectional oriented cycle [3], there exists no infinite sectional path ending with some module in Γ. If Γ contains no projective, then Γ is τ-periodic by (1), and hence a connected component of Γ_A since it is closed under predecessors. Therefore Γ is infinite [1], and hence a stable tube [5], a contradiction to (1). Thus Γ contains projectives. Let Y be an immediate predecessor of a projective P in Γ. If Y is τ-periodic, then P is a predecessor of Y. Otherwise, Y lies in the τ-orbit of a projective, that is a predecessor of Y. Thus, (1) implies (3).

Finally assume that (3) holds. We shall prove that (2) is true. First suppose that the left stable part Θ of Γ is infinite. Then Θ has an infinite connected component Σ. Note that Σ is also closed under predecessors in Γ_A. Since Γ contains projectives, Σ contains modules which are immediate predecessors of projectives in Γ, say $Y_1, \ldots, Y_m$ are all such modules. It follows easily from (3) that each Y_i with $1 \leq i \leq m$ has some Y_j with $1 \leq j \leq m$ as a predecessor in Σ. Thus some Y_s with $1 \leq s \leq m$ is on an oriented cycle in Σ. By Lemma 1.3, Σ contains an infinite sectional path ending with $\tau^r Y_s$ for some $r \geq 0$, and hence one ending with some projective, a contradiction. Thus all but finitely many modules in Γ lie in τ-orbits of projectives. Suppose that Γ contains infinitely many modules lying on oriented cycles. Then there exists a right stable projective module Q such that $\tau^n Q$ lies in Γ for all $n \leq 0$ and $\tau^n Q$ lies on an oriented cycle in Γ for infinitely many $n < 0$. Applying first the dual of Lemma 1.2.(1) and then the dual of Lemma 1.3, we infer that there exists some $r \leq 0$ such that $\tau^r Q$ is the end-point of an infinite sectional path in Γ. Thus Γ contains an infinite sectional path ending with a projective, a contradiction. The proof is completed.

2. Hereditarily preprojective modules

It has been shown by Auslander and Smalø [2, (9.3)] that a module M in Γ_A is hereditarily preprojective if and only if $\mathrm{Hom}_A(X, M) = 0$ for all but finitely many modules X in Γ_A. This leads to more characterizations of hereditarily preprojective modules. For doing so, we first fix some terminology. One says that a module M is *generated* by a module N if M is a quotient of a finite direct sum of copies of N and that a projective module P in $\mathrm{mod}\, A$ is a *progenerator* of a family of modules if P is of minimal length such that every module in the family is generated by P. A path $X_0 \to X_1 \to \cdots \to X_n$ of Γ_A is called *nonzero* if there exists an irreducible map $f_i : X_{i-1} \to X_i$ for each $1 \leq i \leq n$ such that $f_1 \cdots f_n$ is nonzero. An infinite path is *nonzero* if every finite subpath is so.

2.1. Lemma. *Let M be a module in Γ_A. The following are equivalent*:
(1) *M is hereditarily preprojective.*
(2) *M is not the end-point of any infinite nonzero path in Γ_A.*
(3) $\mathrm{rad}^\infty(X, M) = 0$ *for all modules X in Γ_A.*
(4) $\mathrm{rad}^\infty(P, M) = 0$ *with P a progenerator of the predecessors of M in Γ_A.*

The proof of the above result is a routine application of the Harada-Sai Lemma [2, (5.12)] and the result of Auslander-Smalø [2, (9.3)]. Being nonzero [8, (13.4)], an infinite sectional path in Γ_A does not end with a hereditarily preprojective or start with a hereditarily preinjective.

2.2. Lemma. *Let M be a module in Γ_A. The following are equivalent*:
(1) *All predecessors of M in Γ_A are preprojective.*
(2) *All predecessors of M in Γ_A are hereditarily preprojective.*
(3) *The number of predecessors of M in Γ_A is finite.*

Proof. That (1) implies (2) can be proved by using the same argument in the the proof of [4, (3.2)], and that (3) implies (2) follows from Lemma 2.1.(2) and the Harada-Sai Lemma. Assume now that the full subquiver Γ of Γ_A generated by the predecessors of M contains only hereditarily preprojective modules. In particular, every module in Γ admits a projective predecessor [2, (8.3)] and Γ has no infinite sectional path ending with some module. By Proposition 1.4, Γ is finite. The proof is completed.

Let Γ be a connected full subquiver of Γ_A. We say that Γ is *generalized standard* if $\mathrm{rad}^\infty(X, Y) = 0$ for modules X, Y in Γ. Assume now that $\{e_1, \ldots, e_n\}$ is a complete set of pairwise orthogonal primitive idempotents of A and r is an integer with $1 \leq r \leq n$ such that $e_i A$ is isomorphic to a module in Γ if and only if $1 \leq i \leq r$. We then let $A(\Gamma)$ be the quotient of A modulo the ideal generated by $1 - e_1 - \cdots - e_r$. Finally, recall that a module M in Γ_A is *directing* if it does not lie on any cycle of nonzero non-isomorphisms between modules in Γ_A.

2.3. THEOREM. *Let Γ be a connected full subquiver of Γ_A which contains projectives and is closed under predecessors. The following are equivalent*:

(1) *Every module in Γ is preprojective.*

(2) *Every projective module in Γ is hereditarily preprojective.*

(3) *All but finitely many modules in Γ are directing and lie in τ-orbits of projectives.*

(4) $\operatorname{rad}^\infty(P, Q) = 0$ *for projectives P, Q in Γ, and the predecessors of the projectives in Γ are generated by these projectives.*

(5) *Γ is a generalized standard full subquiver of the Auslander-Reiten quiver of $A(\Gamma)$ that is closed under predecessors.*

Proof. That (1) implies (2) follows from Lemma 2.2, and that (3) implies (1) is a consequence of Proposition 1.4 and Lemma 2.2.

Assume now that (2) holds. Then Γ contains no infinite sectional path ending with a projective by Lemma 2.1.(2), and every immediate predecessor of a projective in Γ has a projective predecessor in Γ; see [2, (8.3)]. By Proposition 1.4, all but finitely many modules in Γ lie in τ-orbits of projectives and do not lie on oriented cycles. Moreover, every module in Γ has only finitely many predecessors. By Lemma 2.2, every module in Γ is hereditarily preprojective. Now it follows easily from Lemma 2.1.(3) that a module in Γ not lying on any oriented cycle in Γ is directing. This shows that (3) holds, and hence establishes the equivalence of the first three statements.

If (4) holds, then every projective in Γ is hereditarily preprojective by Lemma 2.1.(4). That is, (4) implies (2). Assume now that (5) holds. In particular, $\operatorname{rad}^\infty(\operatorname{mod} A)$ vanishes on the projectives in Γ. Let X be a module in Γ, and let $P = eA$ with e a primitive idempotent be a projective in Γ_A but not in Γ. Then $Xe = 0$ since X is a module over $A(\Gamma)$. Hence $\operatorname{Hom}_A(P, X) = 0$. This shows that X is generated by the projectives in Γ. Thus (5) implies (4).

Finally assume that (1) holds. Let X be a module in Γ. By Lemma 2.2, X is hereditarily preprojective. By Lemma 2.1.(3), $\operatorname{rad}^\infty(Y, X) = 0$ for all modules Y in Γ_A. Now if Q is a projective in Γ_A but not in Γ, then $\operatorname{Hom}_A(Q, X) = \operatorname{rad}^\infty(Q, X) = 0$. So X is a module over $A(\Gamma)$. Let $f : M \to N$ be an irreducible map in $\operatorname{ind} A(\Gamma)$ with N in Γ. Then $f \notin \operatorname{rad}^\infty(\operatorname{mod} A)$, and hence M is in Γ. This proves that (1) implies (5). The proof is completed.

It is now easy to see that Coelho's result [4, (1.2)] is an immediate consequence of Proposition 1.4, Lemma 2.2 and Theorem 2.3 while the result of Auslander-Smalø [2, (9.16)] follows from Theorem 2.3.

2.4. LEMMA. *A module X in Γ_A lies in a finite τ-periodic stable component of Γ_A if one of the following holds*:

(1) *$\tau^n X$ is hereditarily preprojective for infinitely many $n > 0$.*

(2) *$\tau^n X$ is hereditarily preinjective for infinitely many $n < 0$.*

Proof. Let X be a module in $\varGamma_A$ such that $\tau^n X$ is hereditarily preprojective for infinitely many $n > 0$. In particular, X is left stable and $\tau^n X$ admits a projective predecessor in $\varGamma_A$ for infinitely many $n > 0$. Suppose that X does not lie in any finite τ-periodic stable component. Using first Lemma 1.2.(2) and then Lemma 1.3, we conclude that there exists some $r > 0$ such that for all $n \geq r$, $\tau^n X$ is the end-point of an infinite sectional path, a contradiction. The proof of the lemma is completed.

If A is of finite representation type, then every module in $\varGamma_A$ is both hereditarily preprojective and hereditarily preinjective; see [2, (6.1)]. Conversely we have the following result.

2.5. Proposition. (1) *There exist at most finitely many τ-orbits of $\varGamma_A$ which contain a hereditarily preprojective or hereditarily preinjective module.*

(2) *There exist at most finitely many modules in $\varGamma_A$ which are both hereditarily preprojective and hereditarily preinjective.*

Proof. For part (1), it suffices to prove that there exist at most finitely many τ-orbits which contain a hereditarily preprojective module. If this is not true, then there exists a stable component $\varGamma$ of $\varGamma_A$ in which infinitely many τ-orbits contain hereditarily preprojective modules. However, by König's graph lemma, every module X in $\varGamma$ is the end-point of an infinite path

$$\cdots \to X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 = X$$

with the X_i lying in pairwise different τ-orbits. Hence X is not hereditarily preprojective, a contradiction.

Suppose now that there exist infinitely many modules in $\varGamma_A$ which are both hereditarily preprojective and hereditarily preinjective. It follows from (1) that there exists a τ-orbit $\mathcal{O}$ of $\varGamma_A$ containing infinitely many modules which are both hereditarily preprojective and hereditarily preinjective. Let Y be a module in $\mathcal{O}$. Then either $\tau^n Y$ is hereditarily preprojective for infinitely many $n > 0$ or $\tau^m Y$ is hereditarily preinjective for infinitely many $m < 0$. This is contrary to Lemma 2.4. The proof of the proposition is completed.

3. Components with "few" stable maps in TrD-direction

For modules M, N in mod A, we shall denote by $\underline{\mathrm{Hom}}_A(M,N)$ the quotient of $\mathrm{Hom}_A(M,N)$ modulo the subgroup of maps factoring through a projective module. We say that a map $f : M \to N$ is *projectively stable* if it has nonzero image in $\underline{\mathrm{Hom}}_A(M,N)$ and that a path $X_0 \to X_1 \to \cdots \to X_n$ in $\varGamma_A$ is *projectively stable* if there exists an irreducible map $f_i : X_{i-1} \to X_i$ for each $1 \leq i \leq n$ such that $f_1 \cdots f_n$ is projectively stable. An infinite path is *projectively stable* if every finite subpath is so.

3.1. Lemma. *Let M be a module in $\varGamma_A$ with infinitely many predecessors. Then $\varGamma_A$ has an infinite projectively stable path ending with a predecessor of M.*

Proof. By Lemma 2.2, M admits a predecessor X in $\varGamma_A$ which is not preprojective. By [2, (10.2)], there exist infinitely many modules Y in $\varGamma_A$ such that

$\underline{\mathrm{Hom}}_A(Y, X) \neq 0$. Let $f : Y \to X$ be a projectively stable map in $\mathrm{rad}(Y, X)$. If $f \in \mathrm{rad}^\infty(Y, X)$ then, by using well-known properties of almost split sequences, we deduce easily that there exists an infinite chain

$$\cdots \to X_n \xrightarrow{f_n} X_{n-1} \to \cdots \to X_1 \xrightarrow{f_1} X$$

of irreducible maps through modules in Γ_A such that $f_n \cdots f_1$ is projectively stable for all $n \geq 1$. Otherwise, there exists some $r > 0$ such that $f \in \mathrm{rad}^r(Y, X)$ but not in $\mathrm{rad}^{r+1}(Y, X)$. Then there exists a chain of irreducible maps through modules in Γ_A from Y to X of length r such that the composite is projectively stable. This shows that Γ_A contains projectively stable paths ending with X of arbitrary length. Since Γ_A is locally finite, by König's graph lemma, there exists an infinite projectively stable path ending with X. The proof is completed.

3.2. Proposition. *Let Γ be a left stable component of Γ_A, meeting only finitely many τ-orbits of Γ_A. Then there exists a module X in Γ such that $\underline{\mathrm{Hom}}_A(\tau^n X, X)$ is nonzero for infinitely many $n > 0$.*

Proof. We need only to consider the case where Γ is not τ-periodic. Assume first that Γ contains no oriented cycle. Then Γ contains a finite section Δ, which we may assume has no projective predecessor in Γ_A. Thus the predeccessors of Δ in Γ_A all lie in Γ. Now every module in Δ admits infinitely many predecessors in Γ. By Lemma 3.1, Γ contains an infinite projectively stable path

$$\cdots \to X_n \to X_{n-1} \to \cdots \to X_1 \to X_0$$

with X_0 a predecessor of Δ in Γ. Since Δ is finite, there exists a sequence $0 \leq n_0 < n_1 < n_2 < \cdots < n_i < \cdots$ such that the X_{n_i} with $i \geq 1$ are pairwise distinct but belong to the same τ-orbit. Hence for each $i \geq 1$, $X_{n_i} = \tau^{m_i} X_{n_0}$ with m_i a nonzero integer. Since Γ is embedded in $\mathbb{Z}\Delta$, we have $m_i < m_j$ whenever $i < j$. Now the proposition holds since $\underline{\mathrm{Hom}}_A(X_{n_i}, X_{n_0}) \neq 0$ for all $i \geq 1$. Assume now that Γ contains oriented cycles. Let

$$N_1 \to \cdots \to N_s \to \tau^t N_1 \to \cdots \to \tau^t N_s \to \tau^{2t} N_1 \to \cdots$$

be an infinite path as stated in Proposition 1.1. Applying some power of τ, we may assume that for all $i \geq 0$ and $1 \leq j \leq s$, $\tau^i N_j$ is not an immediate predecessor of a projective in Γ_A. Setting $\tau^{rt} N_j = N_{rs+j}$ for $r \geq 0$ and $1 \leq j \leq s$, and $M_n = \tau^{n-1} N_n$ for $n \geq 1$, one gets an infinite sectional path

$$\cdots \to M_{2s+1} \to M_{2s} \to \cdots \to M_{s+1} \to M_s \to \cdots \to M_2 \to M_1 = N_1.$$

Note that for all $n > 1$ and $1 \leq i < n$, $\tau^i N_n$ admits exactly two immediate successors $\tau^{i-1} N_{n-1}$ and $\tau^i N_{n+1}$ in Γ. Thus for all $n > 1$, a path in Γ from $M_n = \tau^{n-1} N_n$ to N_j with $j > 1$ is of length at least n. As a consequence, if there exists a path in Γ from M_n to X of length less than n, then $X = \tau^i N_j$ with $i \geq 0$ and $j \geq 1$. By our assumption the N_j with $1 \leq j \leq s$, for all $n > 1$, there exists no path in Γ_A from M_n to a projective of length less than n. We choose an irreducible map $f_{n-1} : M_n \to M_{n-1}$ for each $n > 1$. Then $g_n = f_{n-1} \cdots f_2 f_1 \notin \mathrm{rad}^n(M_n, M_1)$; see [8, (13.4)]. If g_n factors through a projective, then there exists a path in Γ_A from M_n to a projective of length less than n, a contradiction. Hence $\underline{\mathrm{Hom}}_A(M_n, M_1) \neq 0$ for all $n > 1$. In particular, for all $r \geq 1$, $\underline{\mathrm{Hom}}_A(\tau^{r(s+t)}N_1, N_1) = \underline{\mathrm{Hom}}_A(M_{rs+1}, M_1)$ is nonzero. The proof is completed.

Kerner proved in [9, (1.1)] that for an indecomposable regular module X over a wild hereditary algebra H, both $\mathrm{Hom}_H(\tau^n X, X)$ and $\mathrm{Hom}_H(X, \tau^{-n}X)$ vanish for sufficiently large n, by using the fact that a regular component is of shape $\mathbb{Z}A_\infty$. Conversely we have the following result.

3.3. THEOREM. *Let Γ be a left stable component of Γ_A. Assume that for every module X in Γ, $\underline{\mathrm{Hom}}_A(\tau^n X, X)$ vanishes for sufficiently large n. Then Γ has a section of type A_∞.*

Proof. It follows from our assumption and Proposition 3.2 that Γ is not τ-periodic and meets infinitely many τ-orbits of Γ_A. Thus Γ contains no oriented cycle by Proposition 1.1, and hence contains an infinite section Δ with a unique sink X [7, (3.6)]. Applying some power of τ, we may assume that Δ admits no projective predecessor in Γ_A. Now Δ contains an infinite sectional path

$$\cdots \to X_i \to X_{i-1} \to \cdots \to X_2 \to X_1 = X.$$

We fix an integer $i > 1$. Let $m > 0$ be such that $\underline{\mathrm{Hom}}_A(\tau^m X_i, X_i) = 0$. We choose irreducible maps $f_j : \tau^j X_i \to \tau^j X_{i-1}$ and $g_j : \tau^j X_{i-1} \to \tau^{j-1} X_i$ for each $1 \leq j \leq m$. Then $h_m = f_m g_m f_{m-1} \cdots f_1 g_1 \in \mathrm{rad}^\infty(\tau^m X_i, X_i)$ since X_i admits no projective predecessor in Γ_A. Thus one of the f_j, g_j with $1 \leq j \leq m$ is of finite left degree [6, (1.1)]. So the arrow $X_i \to X_{i-1}$ is of finite global left degree [7, section 1]. It follows now from [7, (1.3), (1.4)] that Δ is of the form

$$\cdots \to X_i \to X_{i-1} \to \cdots \to X_2 \to X_1 = X = Y_1 \leftarrow Y_2 \leftarrow \cdots \leftarrow Y_r$$

with $r \geq 1$. That is, Δ is of type A_∞. The proof of the theorem is completed.

ACKNOWLEDGEMENT. The author gratefully acknowledges partial support from NSERC of Canada.

REFERENCES

[1] M. AUSLANDER, "Applications of morphisms determined by objects", *Proceedings, Conference on Representation Theory*, Philadelphia 1976 (Dekker, New York, 1978) 245 - 327.

[2] M. AUSLANDER and S. O. SMALØ, "Preprojective modules over artin algebras", *J. Algebra* **66** (1980) 61 - 122.

[3] R. BAUTISTA and S. O. SMALØ, "Non-existent cycles", *Comm. Algebra* **11** (1983) 1755 - 1767.

[4] F. U. COELHO, "Components of Auslander-Reiten quivers with only preprojective modules", *J. Algebra* **157** (1993) 472 - 488.

[5] D. Happel, U. Preiser, and C. M. Ringel, "Vinberg's characterization of Dynkin diagrams using subadditive functions with applications to DTr-periodic modules", *Lecture Notes in Mathematics* **832** (Springer, Berlin, 1980) 280 - 294.

[6] S. Liu, "Degrees of irreducible maps and the shapes of Auslander-Reiten quivers", *J. London Math. Soc.* (2) **45** (1992) 32 - 54.

[7] S. Liu, "Semi-stable components of an Auslander-Reiten quiver", *J. London Math. Soc.* (2) **47** (1993) 405 - 416.

[8] K. Igusa and G. Todorov, "Radical layers of representable functors", *J. Algebra* **89** (1984) 148 - 177.

[9] O. Kerner, "Tilting wild algebras", *J. London Math. Soc.* (2) **30** (1989) 29 - 47.

Département de mathématiques
Université de Sherbrooke
Sherbrooke, Québec
Canada J1K 2R1
Email: *shiping@dmi.usherb.ca*