Hochschild Cohomology and Representation-finite Algebras

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Dedicated to Idun Reiten to mark her sixtieth birthday

Introduction

Hochschild cohomology is a subtle invariant of associative algebras; see, for example, [13], [16], [21]. The lower dimensional Hochschild cohomology groups have well known interpretations. Indeed, the first, the second, and the third Hochschild cohomology groups control the infinitesimal deformation theory; see, for example, [16]. Recall that the first Hochschild cohomology group can be interpreted as the group of classes of outer derivations, and here we are mainly concerned with this group, in particular, we wish to understand when and why it may vanish. In more concrete terms, the content of the paper is as follows.

We first approach the vanishing of Hochschild cohomology via semicontinuity. As recently pointed out by Hartshorne, [20], Grothendieck's classical semicontinuity result [15] on the variation of cohomological functors in a flat family admits a very simple and elegant extension to half-exact coherent functors, a framework introduced by M. Auslander in his fundamental paper [3]. Using these tools we derive two semicontinuity results: the first one applies to all Hochschild cohomology groups of an algebra that is finitely generated projective over its base ring, whereas the second one pertains to homogeneous components of the first Hochschild cohomology group of graded algebras and is tailored towards its application to mesh algebras.

Indeed, we will show that a translation quiver is simply connected if and only if its mesh algebra over a domain admits no outer derivation if and only if its mesh algebra over every commutative ring admits no outer derivation.

We then move on to investigate how the Hochschild cohomology of an algebra relates to that of the endomorphism algebra of a module over it. In the most general context we obtain an exact sequence relating first Hochschild cohomology groups. This enables us, in particular, to show that the first Hochschild cohomology group of the Auslander algebra of a representation-finite artin algebra always embeds into that of the algebra. In the more restrictive situation where both algebras are projective over the base ring, we will deduce from a classical pair of spectral sequences the invariance of Hochschild cohomology under pseudo-tilting, a notion that includes tilting, co-tilting, and thus, Morita equivalence.

Finally we apply the results obtained so far to investigate the vanishing of the first Hochschild cohomology group of a finite dimensional algebra. Our main result in this direction establishes the equivalence of the following conditions for a finite dimensional algebra A of finite representation type over an algebraically closed field and Λ its Auslander algebra:

(1) A admits no outer derivation;
(2) Λ admits no outer derivation;
(3) A is simply connected;
(4) Λ is strongly simply connected.

Note that the representation theory of a simply connected algebra is well understood, and in most cases, for example, if the base field is of characteristic different from two, one can use covering techniques to reduce the representation theory of an algebra of finite representation type to that of a simply connected algebra [9].

To prove the equivalences just stated, we first reduce to standard algebras and then apply our result on general mesh algebras. Some of the implications in question were already known and some have at least been claimed to be true under varying additional hypotheses. To be more specific, let us recall briefly some of the history. First, Happel proved in [17, (5.5)] the equivalence of (1) and (3) for A of directed representation type. Further, the equivalence of (3) and (4) is due to Assem-Brown [1]. Moreover, as pointed out by Skowroński, in case A is standard, one deduces that (1) implies (3) from [22, Theorem 1], [11, (1.2)], and [9, (4.7)]. Finally, the equivalence of (2) and (3) is claimed in [18, Section 4] for a base field of characteristic zero. However, in the proof given there, it is assumed implicitly that A is tilted. Unfortunately, not only is the given argument thus incomplete, but it has also been widely misquoted; see, for example, [1], [14].

To conclude this introduction, we draw attention to the long standing question as to whether vanishing of the first Hochschild cohomology of an algebra precludes the existence of an oriented cycle in its ordinary quiver. We showed earlier that the answer is negative in general [12]. However, it would still be interesting to know for which classes of algebras the answer is affirmative. Our results show that algebras of finite representation type, Auslander algebras, and mesh algebras each form such a class.

1. *Coherent functors and semicontinuity*

The main objective of this section is to derive two semicontinuity results on Hochschild cohomology. The first one is a direct application of Grothendieck's semicontinuity theorem [15, (7.6.9)] for homological functors arising from complexes of finitely generated projective modules, whereas the second one needs more care and is crucial for our characterization of (strongly) simply connected (Auslander) algebras. We use Hartshorne's reworking of the semicontinuity theorem as it saves some work and makes the proof more transparent.

All rings and algebras in this paper are associative with unit and all modules are unital. Throughout, R denotes a commutative ring and unadorned tensor products are taken over R. Let $\mathrm{Mod}R$ denote the category of R-modules and $\mathrm{mod}R$ its full subcategory of finitely generated modules.

Let F be an endofunctor on Mod$R$, that is, an R-linear covariant functor from Mod$R$ to itself. Recall that F is *half-exact on* mod$R$ if for any exact sequence $0 \to M \to N \to Q \to 0$ in mod$R$, the sequence $F(M) \to F(N) \to F(Q)$ is exact. Following Auslander [3], we say that F is *coherent on* mod$R$ if there exists an exact sequence of functors

$$\mathrm{Hom}_R(M,\ -) \longrightarrow \mathrm{Hom}_R(N,\ -) \longrightarrow F \longrightarrow 0\,,$$

with M, N in mod$R$. Such an exact sequence is called a *coherent presentation* of F. If F is coherent on mod$R$, then F clearly commutes with inductive limits in mod$R$. In particular, for any field L that is an R-algebra, $F(L)$ is a finite dimensional vector space over L. We first reformulate a result of Hartshorne, [20, (4.6)], that characterizes half-exact coherent functors. For convenience, we call a sequence

$$\mathbf{M}:\ M_0 \xrightarrow{f} M_1 \xrightarrow{g} M_2$$

of morphisms in an abelian category with $gf = 0$ a *short complex* and denote by $\mathcal{H}(\mathbf{M})$ the unique homology group of $\mathbf{M}$.

1.1. Proposition (Hartshorne). *Let R be a noetherian commutative ring.*
(1) *An endofunctor F of* Mod$R$ *admits a coherent presentation*

$$\mathrm{Hom}_R(P,\ -) \longrightarrow \mathrm{Hom}_R(N,\ -) \longrightarrow F \longrightarrow 0$$

with P projective if and only if there exists a short complex $\mathbf{P}$ of finitely generated projective R-modules such that $F \cong \mathcal{H}(\mathbf{P} \otimes_R -)$.

(2) *An endofunctor of* Mod$R$ *is half-exact coherent on* mod$R$ *if and only if it is a direct summand of an endofunctor satisfying the conditions stated in* (1).

In [15, Section 7], Grothendieck discusses the behaviour under base change of cohomological functors that satisfy the conditions stated in Proposition 1.1.(1). As Hartshorne observed, the key semicontinuity theorem [15, (7.6.9)] can be formulated just as well for half-exact coherent functors. To do so, we denote as usual by $k(p)$ the residue field of the localization R_p of a commutative ring R at a prime ideal p.

1.2. Theorem (Grothendieck). *Let R be a noetherian commutative ring and F an endofunctor on* Mod$R$ *that is half-exact and coherent on* mod$R$*. The dimension function*

$$p \mapsto \dim_{k(p)} F(k(p))$$

is then upper semicontinuous on Spec(R) *and takes on only finitely many values.*

To apply this result to the Hochschild cohomology of algebras, let us briefly recall the definition; see [13] or [21] for more details. From now on, A denotes an R-algebra. Let $A^{\mathrm{o}} = \{a^{\mathrm{o}} \mid a \in A\}$ be the *opposite algebra* and $A^{\mathrm{e}} = A^{\mathrm{o}} \otimes A$ the *enveloping algebra* of A over R. An A-*bimodule* X will always be assumed to be *symmetric* as R-module, whence it becomes a right A^{e}-module via $x \cdot (a^{\mathrm{o}} \otimes b) = axb$. This action does not interfere with the left R-module structure on X, whence for any R-module M, the tensor product $M \otimes X$ over R inherits a right A^{e}-module structure through that on X, as well as a compatible left R-module structure through that on M.

The *bar resolution* of A over R is given by the complex:

$$\cdots \to A^{\otimes(i+2)} \xrightarrow{b_i} A^{\otimes(i+1)} \to \cdots \to A^{\otimes 3} \xrightarrow{b_1} A^{\otimes 2} \xrightarrow{\mu_A} A \to 0\,,$$

where μ_A is the multiplication map on A, and b_i is the map given by

$$b_i(a_0 \otimes a_1 \otimes \cdots \otimes a_{i+1}) = \sum\nolimits_{j=0}^{i} (-1)^j a_0 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_{i+1}\,.$$

Each term in the bar resolution of A over R is naturally an A-bimodule and the differentials respect that structure, whence the bar resolution can be viewed as a complex of right A^{e}-modules. Let $\mathbf{B}$ denote the truncated bar resolution of A over R. The i-th cohomology group of the complex $\mathrm{Hom}_{A^{\mathrm{e}}}(\mathbf{B},\ X)$ is called the *i-th Hochschild cohomology group* of A over R with coefficients in X and is denoted by $\mathrm{HH}^i_R(A,X)$. In case $X = A$, we write $\mathrm{HH}^i_R(A) = \mathrm{HH}^i_R(A,A)$. If R is understood, we shall simply write $\mathrm{HH}^i(A,X)$ for $\mathrm{HH}^i_R(A,X)$ and $\mathrm{HH}^i(A)$ for $\mathrm{HH}^i_R(A)$.

The following semicontinuity result on Hochschild cohomology is a straightforward application of Theorem 1.2.

1.3. Proposition. *Let R be a noetherian commutative ring. Let A be an R-algebra and X an A-bimodule, and assume that both A and X are finitely generated projective as R-modules. For each $i \geq 0$, the dimension function*

$$p \mapsto \dim_{k(p)} \mathrm{HH}^i_{k(p)}(k(p) \otimes_R A, k(p) \otimes_R X)$$

is then upper semicontinuous on $\mathrm{Spec}(R)$ *and takes on only finitely many values.*

Proof. As A is finitely generated projective over R, so is $A^{\otimes i}$ for each $i \geq 0$. There is thus an isomorphism of functors on $\mathrm{Mod} R$ as follows:

$$- \otimes X \otimes \mathrm{Hom}_R(A^{\otimes i},\ R) \cong \mathrm{Hom}_{A^{\mathrm{e}}}(A^{\otimes(i+2)},\ - \otimes X)\,,$$

where the right A^{e}-module structure on $- \otimes X$ is inherited from the one on X. Consequently, for each R-module M, the complex $\mathrm{Hom}_{A^{\mathrm{e}}}(\mathbf{B},\ M \otimes X)$ is isomorphic to a complex $\mathbf{P}^{\cdot}$ of the following form:

$$0 \longrightarrow M \otimes X \otimes \mathrm{Hom}_R(R,\ R) \longrightarrow \cdots \longrightarrow M \otimes X \otimes \mathrm{Hom}_R(A^{\otimes i},\ R) \longrightarrow \cdots\,,$$

and so $\mathrm{HH}^i_R(A,\ M \otimes X) \cong \mathrm{H}^i(\mathbf{P}^{\cdot})$. As X is assumed to be finitely generated projective over R, the same holds for $X \otimes \mathrm{Hom}_R(A^{\otimes i},\ R)$, whence the functor $\mathrm{HH}^i_R(A,\ - \otimes X)$ satisfies the assumptions of (1.2) for each $i \geq 0$.

To conclude, let S be a commutative R-algebra and consider the S-algebra $A_S = S \otimes A$. With $\mathbf{B}$ the truncated bar resolution of A over R, the truncated bar resolution of A_S over S is isomorphic to $\mathbf{B}_S = S \otimes \mathbf{B}$. Moreover, $X_S = S \otimes X$ is naturally an A_S-bimodule. Now adjunction gives rise to the following isomorphism of complexes:

$$\mathrm{Hom}_{(A_S)^{\mathrm{e}}}(\mathbf{B}_S, X_S) \cong \mathrm{Hom}_{A^{\mathrm{e}}}(\mathbf{B}, X_S)\,,$$

whence $\mathrm{HH}^i_S(A_S, X_S) \cong \mathrm{HH}^i_R(A, X_S)$ for each $i \geq 0$. Applying this isomorphism to $S = k(p)$ for $p \in \mathrm{Spec}(R)$ completes the proof.

Now we turn our attention to the first Hochschild cohomology group. Note that $\mathrm{HH}^1_R(A,\ X) \cong \mathrm{Ext}^1_{A^{\mathrm{e}}}(A,\ X)$ for any R-algebra A and any A-bimodule X. As usual, it is useful to interpret this group through (classes of) outer derivations. To this end, recall that an R-*derivation* on A with values in X is an R-linear map $\delta : A \to X$ such that $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in A$. For each $x \in X$, the map

$$\mathrm{ad}(x) = [x, -] : A \to X : a \mapsto [x, a] = xa - ax$$

is an R-derivation. A derivation of this form is called *inner*, whereas the others are called *outer*. Denoting by $\mathrm{Der}_R(A, X)$ the R-module of R-derivations on A with values in X, and by $\mathrm{Inn}_R(A, X)$ its submodule of inner derivations, the first Hochschild cohomology group can be identified as $\mathrm{HH}^1_R(A, X) \cong \mathrm{Der}_R(A, X)/\mathrm{Inn}_R(A, X)$, whence it can be defined through the exact sequence of R-modules

$$X \xrightarrow{\mathrm{ad}} \mathrm{Der}_R(A, X) \longrightarrow \mathrm{HH}^1_R(A, X) \to 0\,. \tag{1}$$

In case $X = A$, we shall simply write $\mathrm{Der}_R(A) = \mathrm{Der}_R(A, A)$ and $\mathrm{Inn}_R(A) = \mathrm{Inn}_R(A, A)$.

Assume now that we are given a second R-algebra B and a homomorphism $B \to A$ of R-algebras. For each A-bimodule X we have then an R-linear restriction map $\mathrm{Der}_R(A, X) \to \mathrm{Der}_R(B, X)$ whose kernel we denote $\mathrm{Der}^B_R(A, X)$ and call the B-*normalized* derivations on A. Recall that an R-algebra B is *separable* if B is projective as (right) B^{e}-module. In particular, for every B-bimodule Y, any R-derivation on B with values in Y is inner. One may exploit this as follows.

1.4. Lemma. *Assume that the structure map of the R-algebra A factors through a separable R-algebra B. For an A-bimodule X, set*

$$X^B = \{x \in X | xb = bx \text{ for each } b \in B\}\,,$$

the B invariants of X. The following sequence of R-modules is then exact:

$$X^B \xrightarrow{\mathrm{ad}} \mathrm{Der}^B_R(A, X) \longrightarrow \mathrm{HH}^1_R(A, X) \to 0\,, \tag{2}$$

equivalently, every derivation on A is the sum of a B-normalized derivation and an inner one. Moreover, X^B is a direct summand of X as R-module.

Proof: For the first statement, just apply the Ker-Coker-Lemma to the maps $X \xrightarrow{\mathrm{ad}} \mathrm{Der}_R(A, X) \to \mathrm{Der}_R(B, X)$ and observe that the composition is surjective as B is separable. For the final statement, note that $X^B = Xe$, where $e \in B^{\mathrm{e}}$ is a separating idempotent for the separable algebra B. The lemma is thus established.

We will use the following standard application of this result. Let U be a complete set of pairwise orthogonal idempotents of the R-algebra A. Then $B = \oplus_{e \in U} Re$ is an R-subalgebra of A that is separable over R. An R-derivation $\delta : A \to X$ vanishes on B if and only if it vanishes on each idempotent in U, whence such a derivation is also called U-normalized, or simply *normalized* in case U is understood. Let $\mathrm{Der}^U_R(A, X)$ denote the R-module of U-normalized derivations and $\mathrm{Inn}^U_R(A, X)$ its submodule of U-normalized inner derivations. For each A-bimodule X, its B-invariants are easy to describe:

$$X^B = \oplus_{e \in U} eXe\,. \tag{3}$$

Let now $A = \oplus_{i \geq 0} A_i$ be a positively graded R-algebra and $X = \oplus_{j \in \mathbb{Z}} X_j$ a graded A-bimodule. An R-derivation $\delta : A \to X$ is *of degree d* if $\delta(A_i) \subseteq X_{i+d}$ for all $i \geq 0$. Let $\mathrm{Der}^U_R(A, X)_d$ denote the R-module of U-normalized derivations of degree d and $\mathrm{Inn}_R^U(A,X)_d$ that of U-normalized inner derivations of degree d. The R-module $\mathrm{HH}_R^1(A,X)$ contains then

$$\mathrm{HH}_R^1(A,X)_d = \mathrm{Der}_R^U(A,\ X)_d/\mathrm{Inn}_R^U(A,\ X)_d\,,$$

the group of outer derivations of degree d, as a submodule. Repeating the arguments above in the graded context yields the following result.

1.5. Lemma. *Let $A = \oplus_{i\ge 0}A_i$ be a graded R-algebra and $X = \oplus_{j\in \mathbf{Z}}X_j$ a graded A-bimodule. Let U be a complete set of pairwise orthogonal idempotents of A. There exists an exact sequence of R-modules as follows*:

$$\oplus_{e\in U} eX_d e \xrightarrow{\mathrm{ad}} \mathrm{Der}_R^U(A,\ X)_d \longrightarrow \mathrm{HH}_R^1(A,X)_d \longrightarrow 0\,. \tag{4}$$

Proof. We need to show that any normalized inner derivation of degree d is of the form $[x,-]$ with $x\in\oplus_e eX_de$. Clearly $[x,-]\in\mathrm{Inn}_R^U(A,X)_d$ if $x\in\sum_{e\in U} eX_de$. Conversely, let $x=\sum_j x_j$ with $x_j\in X_j$ be such that $[x,\,-]$ is normalized of degree d. For each $i\ge 0$ and any j, one has $[x_j,\,A_i]\subseteq X_{j+i}$. Hence $[x_j,\,A_i]=0$ for all $i\ge 0$ and each $j\ne d$, whence $[x,\,-]=[x_d,\,-]$. As $[x_d,-]$ is then normalized, it is an element of degree d in $\oplus_e eXe$, that is, in $\oplus_e eX_de$ as each idempotent is of degree zero. The proof of the lemma is completed.

The graded R-algebra A is called *finitely generated in degrees* 0 *and* 1 if there exists a naturally graded tensor algebra $T=\oplus_{i\ge 0}T_i$, where T_0 and T_1 are finitely presented over R and T_i with $i\ge 2$ is the i-fold tensor product of T_1 with itself over T_0, and a homogeneous ideal I contained in $\oplus_{i\ge 2}T_i$ such that $A\cong T/I$. Note that the homogeneous components of T are finitely presented over R and those of A are finitely generated over R.

The following is the semicontinuity result on the first Hochschild cohomology group that we alluded to before.

1.6. Theorem. *Let R be a commutative ring, A a graded R-algebra finitely generated in degrees* 0 *and* 1, *and $X=\oplus_{j\in\mathbf{Z}}X_j$ a graded A-bimodule. Let $A\cong T/I$ be a presentation as above such that I is finitely generated as ideal and $T_0=\oplus_{e\in U}Re$ with U a complete set of pairwise orthogonal idempotents of T. Denote by $\Delta(I)$ the set of degrees of a finite set of homogeneous generators of I.*

(1) *For any commutative R-algebra S and any integer d, there is a natural isomorphism $\mathrm{HH}_R^1(A,\,S\otimes_R X)_d\cong\mathrm{HH}_S^1(S\otimes_R A, S\otimes_R X)_d$.*

(2) *If the R-algebra S is flat over R, then for any integer d there is an isomorphism, natural in X, as follows*:

$$S\otimes_R \mathrm{HH}_R^1(A,X)_d\cong\mathrm{HH}_S^1(S\otimes_R A, S\otimes_R X)_d\,.$$

(3) *Let R be noetherian. If d is an integer such that $\oplus_{e\in U}\, e\,X_d\,e$ is a finitely generated R-module and X_{i+d} is finitely generated projective over R for each $i\in\{1\}\cup\Delta(I)$, then the dimension function*

$$p\mapsto \dim_{k(p)}\mathrm{HH}^1_{k(p)}(k(p)\otimes_R A,\ k(p)\otimes_R X)_d$$

is upper semicontinuous on $\mathrm{Spec}(R)$.

Proof. The set U is finite, say $U=\{e_1,\dots,e_n\}$. Then $V=\{\bar e=e+I\mid e\in U\}$ is a complete set of pairwise orthogonal idempotents of A.

Since T is freely generated as an R-algebra by T_0 and T_1, every R-derivation is uniquely determined by its values on these R-modules. Each $\delta \in \mathrm{Der}_R^U(T, X)_d$ vanishes on T_0 and determines R-linear maps from eT_1e' to $eX_{d+1}e'$ for each pair $e, e' \in U$. As T_1 generates T freely over T_0, each such family of maps determines conversely such a derivation. Thus, $\mathrm{Der}_R^U(T, X)_d \cong \oplus_{i,j}\mathrm{Hom}_R(e_iT_1e_j, e_iX_{d+1}e_j)$. Moreover, a derivation δ in $\mathrm{Der}_R^U(T, X)_d$ induces a derivation in $\mathrm{Der}_R^V(A, X)_d$ if and only if $\delta(I) = 0$, equivalently, $\delta(I_i) = 0$ for all $i \in \Delta(I)$. Write $T_{ij} = e_iT_1e_j$, $I_{mij} = e_iI_me_j$ and $X_{mij} = e_iX_{d+m}e_j$, for $1 \le i, j \le n$ and $m \in \Delta(I)$. Thus, we obtain an exact sequence of R-modules

$$(*) \qquad 0 \to \mathrm{Der}_R^V(A, X)_d \to \oplus_{i,j}\mathrm{Hom}_R(T_{ij}, X_{1ij}) \to \oplus_{m,i,j}\,\mathrm{Hom}_R(I_{mij}, X_{mij})\,,$$

where i, j range over $\{1, \dots, n\}$ and m over $\Delta(I)$.

Now let S be a commutative R-algebra and set $T_S = S \otimes T$, $I_S = S \otimes I$ and $A_S = S \otimes A = T_S/I_S$. Clearly, $V_S = \{1_S \otimes (e_i + I) \mid 1 \le i \le n\}$ is a complete set of pairwise orthogonal idempotents in A_S and $X_S = S \otimes X = \oplus_{j\in\mathbf{Z}}(S \otimes X_i)$ is a graded A_S-bimodule. Repeating the above argument, we obtain the sequence $(*)$ as well for the corresponding tensored objects.

(1) Apply the exact sequence $(*)$ twice to X_S, first as A-bimodule and then as A_S-bimodule. Using adjunction in the Hom-terms, we conclude that

$$\mathrm{Der}_S^{V_S}(A_S, X_S)_d \cong \mathrm{Der}_R^V(A, S \otimes X)_d.$$

As furthermore $S \otimes (\oplus_e eX_de) \cong \oplus_e e(S \otimes X)_de$, the exact sequence (4) yields

$$\mathrm{HH}_R^1(A,\, S \otimes X)_d \cong \mathrm{HH}_S^1(S \otimes A, S \otimes X)_d.$$

(2) Assume that S is flat as R-module. We use again the exact sequence $(*)$ twice: First, we tensor it with S and secondly apply it to X_S. By assumption, the module T_1 is finitely presented over R and so are then the direct summands T_{ij}. Moreover, as I is a finitely generated ideal, the R-modules I_{mij} are finitely generated over R. This implies that the natural map

$$S \otimes \oplus_{i,j}\mathrm{Hom}_R(T_{ij}, X_{1ij}) \to \oplus_{i,j}\mathrm{Hom}_R(T_{ij}, S \otimes X_{1ij})$$

is an isomorphism, whereas the natural map

$$S \otimes \oplus_{m,i,j}\,\mathrm{Hom}_R(I_{mij}, X_{mij}) \to \oplus_{m,i,j}\,\mathrm{Hom}_R(I_{mij}, S \otimes X_{mij})$$

is at least injective. We infer that $S \otimes \mathrm{Der}_R^V(A, X)_d \cong \mathrm{Der}_R^V(A, S \otimes X)_d$ for any d. Applying the same argument as in (1) to the exact sequence (4), we obtain $S \otimes \mathrm{HH}_R^1(A, X)_d \cong \mathrm{HH}_R^1(A,\, S \otimes X)_d$. Combining this isomorphism with the one from part (1) yields the claim.

(3) By assumption now, X_{d+m} is finitely generated projective over R for each $m \in \{1\} \cup \Delta(I)$. As direct R-summands of X_{d+m}, the X_{mij} are all finitely generated projective R-modules too. Write $(-)^* = \mathrm{Hom}_R(-, R)$ for the R-dual. If P is finitely generated projective over R, then so are P^* and $P^{**} \cong P$. Hence for any R-modules M, N, we obtain first $M \otimes X_{mij} \cong \mathrm{Hom}_R(X_{mij}^*, M)$, and then $\mathrm{Hom}_R(N, M \otimes X_{mij}) \cong \mathrm{Hom}_R(X_{mij}^* \otimes N, M)$ by adjunction. Employing this last isomorphism in the exact sequence $(*)$, we deduce that the functor $\mathrm{Der}_R^V(A, -\otimes X)_d$ is the kernel of a morphism of functors

$$\mathrm{Hom}_R(\oplus_{i,j}X_{1ij}^* \otimes T_{ij}, -) \to \mathrm{Hom}_R(\oplus_{m,i,j}X_{mij}^* \otimes I_{mij}, -)\,.$$

Therefore, $\mathrm{Der}_R^V(A, -\otimes X)_d = \mathrm{Hom}_R(C, -)$, where C is the cokernel of some R-linear map from $\oplus_{m,i,j} X^*_{mij} \otimes I_{mij}$ to $\oplus_{i,j} X^*_{1ij} \otimes T_{ij}$. As $\oplus_{i,j} X^*_{1ij} \otimes T_{ij}$ is a finitely generated R-module, so is C.

To conclude the argument, consider the functor $G(M) = \oplus_e e(M \otimes X)_d e$ on $\mathrm{Mod}R$. Applying $(\ )_d$ and multiplying with $\sum_e \bar{e} \otimes \bar{e} \in A^{\mathrm{e}}$ are exact functors. Thus, $G(-)$ is right exact and so isomorphic to $-\otimes G(R)$. Now $G(R) = \oplus_e e X_d e$ is finitely generated over R and we may choose a surjection from a finitely generated projective R-module Q onto it. But then $-\otimes Q \to -\otimes \oplus_e e X_d e$ is an epimorphism of functors on $\mathrm{Mod}R$. Finally, we may identify $-\otimes Q \cong \mathrm{Hom}_R(Q^*, -)$, to obtain from the above and the exact sequence (4) the coherent presentation

$$\mathrm{Hom}_R(Q^*, -) \to \mathrm{Hom}_R(C, -) \to \mathrm{HH}^1_R(A, -\otimes X)_d \to 0\,.$$

Thus, $\mathrm{HH}^1(A, -\otimes X)_d$ is half-exact and coherent on $\mathrm{mod}R$ by Hartshorne's result. Hence, $\dim_{k(p)} \mathrm{HH}^1_R(A, k(p) \otimes X)_d$, which equals $\dim_{k(p)} \mathrm{HH}^1_{k(p)}(k(p) \otimes A, k(p) \otimes X)_d$ by part (1), varies upper semicontinuously with p on $\mathrm{Spec}(R)$. This finishes the proof of the theorem.

2. *Mesh algebras without outer derivations*

The main objective of this section is to show that a finite translation quiver is simply connected if and only if its mesh algebra over a domain admits no outer derivation, and if this is the case, its mesh algebra over any commutative ring admits no outer derivation.

We begin with some combinatorial considerations on quivers and their underlying graphs. A sequence

$$a_0 \overset{e_1}{\text{---}} a_1 \text{---} \cdots \text{---} a_{r-1} \overset{e_r}{\text{---}} a_r$$

of edges of a graph such that $e_i \neq e_{i+1}$ for all $1 \leq i < r$ is called a *reduced walk*, and the sequence is a *cycle* if $a_r = a_0$ and $e_r \neq e_1$. A vertex of a graph is considered as a *trivial* reduced walk.

Let now Q be a quiver, that is an oriented graph. A reduced walk or a cycle of Q is in fact a reduced walk or a cycle, respectively of its underlying graph of edges. A sequence of arrows $a_0 \to a_1 \to \cdots \to a_n$ with $n \geq 1$ is called a *path of length* n from a_0 to a_n; and *an oriented cycle* if $a_0 = a_n$. Such an oriented cycle is called *simple* if the a_i with $0 \leq i < n$ are pairwise distinct. A vertex a is considered as a *path of length* 0 from a to a. The set of paths of length n is denoted by Q_n. In particular, Q_0 is the set of vertices and Q_1 is that of arrows.

A vertex a of Q is called a *source* if no arrow ends in a, a *sink* if no arrow starts from a. Let Δ be a possibly empty subquiver of Q. We say that Δ is *convex* in Q if a path of Q lies entirely in Δ as soon as its end-points lie in Δ. A convex subquiver in our terminology is thus in particular a full subquiver. We say that Q is a *one-point extention* of Δ by a vertex a, if a is a source of Q and is the only vertex of Q that is not in Δ. In the dual situation, we say that Q is a *one-point co-extension* of Δ. Note that Δ is convex in Q in either situation. The following simple lemma is a reformulation of [2, (2.3)], see also [19, (7.6)].

2.1. Lemma. *Let Q be a finite connected quiver. If Q contains no oriented cycle, then Q is a one-point extension or co-extension of one of its connected subquivers.*

Now fix a commutative ring R and a finite quiver Q. For each $i \geq 0$, let RQ_i be the free R-module with basis Q_i. Then $RQ = \oplus_{i \geq 0} RQ_i$ is a positively graded R-algebra, called the *path algebra* of Q over R, with respect to the multiplication that is induced from the composition of paths. Note that we use the convention to compose paths from the left to the right. The set Q_0 yields a complete set of pairwise orthogonal idempotents of RQ and for $a, b \in Q_0$, the Peirce component $a(RQ)b$ is the free R-module spanned by the paths from a to b.

Two or more paths of Q are *parallel* if they have the same start-point and the same end-point. A *relation* on Q over R is an element $\rho = \sum_{i=1}^{r} \lambda_i p_i \in RQ$, where the $\lambda_i \in R$ are all non-zero and the p_i are parallel paths of length at least two. In this case, we say that $p_1, \ldots, p_r$ are the paths *forming* ρ and that a path *appears* in ρ if it is a subpath of one of the p_i. The relation ρ is called *polynomial* if $r \geq 2$; and *homogeneous* if the p_i are of the same length. Moreover, a relation on Q over $\mathbb{Z}$ with only coefficients 1 or -1 is called *universal*. The point is that an universal relation on Q remains a relation involving the same paths over any commutative ring.

Let Ω be a set of relations on Q over R. The pair (Q, Ω) is called a *bound quiver*, while the quotient $R(Q, \Omega)$ of RQ modulo the ideal generated by Ω is called the *algebra of the bound quiver* (Q, Ω). If Ω contains only homogeneous relations, then $R(Q, \Omega)$ is a graded R-algebra with grading induced from that of RQ. Moreover, $R(Q, \Omega)$ is called *monomial* if Ω contains no polynomial relation. The following easy lemma generalizes slightly a result of Bardzell-Marcos, [5, (2.2)], which states that Q is a tree if $R(Q, \Omega)$ is monomial without outer derivations.

2.2. Lemma. *Let R be a commutative ring, and let Q be a finite quiver with Ω a set of relations on Q over R. If there exists a cycle in Q containing an arrow that appears in no polynomial relation in Ω, then $\mathrm{HH}^1_R(R(Q, \Omega))$ does not vanish.*

Proof. Let α be an arrow of Q. There exists a normalized derivation δ of RQ such that $\delta(\alpha) = \alpha$ and $\delta(\beta) = 0$ for any other arrow β. Assume that α appears in no polynomial relation in Ω. Then δ preserves the ideal I generated by Ω. Thus δ induces a derivation $\bar{\delta}$ of $R(Q, \Omega)$. Suppose that $\bar{\delta}$ is inner. Then there exists $u = \sum_{a \in Q_0} \lambda_a a + w \in RQ$ with $\lambda_a \in R$ and $w \in \oplus_{i \geq 1} RQ_i$ such that $\delta(v) - [u, v] \in I$, for any $v \in RQ$. Suppose further that α is on a cycle

$$a_0 \overset{\alpha_1}{\text{---}} a_1 \text{---} \cdots \text{---} a_{r-1} \overset{\alpha_r}{\text{---}} a_r = a_0,$$

where $\alpha_i : a_i \text{---} a_{i+1}$ denotes an arrow in either direction. We may assume that $\alpha = \alpha_1$ and $\alpha_i \neq \alpha$ for all $1 < i \leq r$. Since $I \subseteq \oplus_{i \geq 2} RQ_i$, evaluting δ on the α_i yields $\lambda_{a_0} = \lambda_{a_1} \pm 1$ and $\lambda_{a_i} = \lambda_{a_{i+1}}$ for all $1 \leq i < r$. This contradiction establishes the lemma.

Applying our semicontinuity result obtained in the previous section for the first Hochschild cohomology group of a graded algebra, we are able to exclude oriented cycles from Q if $R(Q, \Omega)$ is well graded and admits no outer derivation.

2.3. Proposition. *Let Q be a finite quiver and $n \geq 2$ an integer. Assume that Ω is a set of universal relations formed by paths of Q_n such that every path of Q_n appears in at most one relation. Let Σ be a subset of Q_n obtained by removing, for each relation $\rho \in \Omega$, one path from those forming ρ.*

(1) *For any commutative ring R, the component of degree n of $R(Q, \Omega)$ is a free R-module having as basis the set of the classes of the paths of Σ.*

(2) *If R is a domain such that $\mathrm{HH}^1(R(Q, \Omega))$ vanishes, then Q contains no oriented cycle and any two parallel paths have the same length.*

Proof. Let R be a commutative ring. By assumption, the ideal I of RQ generated by Ω is generated by elements of degree n. Write $R(Q, \Omega) = \oplus_{i \geq 0} A_i$ with $A_i = (RQ_i + I)/I$. Statement (1) is obvious.

(2) To simplify the notation, we write $R(Q) = R(Q, \Omega)$. For any commutative R-algebra S, one has clearly $S(Q) \cong S \otimes_R R(Q)$. The general assumptions in Theorem 1.6 are thus satisfied for $A = X = R(Q)$. Let now R be a domain and L its field of fractions. Assume that $\mathrm{HH}^1_R(R(Q)) = 0$. In particular, $\mathrm{HH}^1_R(R(Q), R(Q))_0 = 0$. As L is flat over R, Theorem 1.6.(2) yields

$$0 = L \otimes \mathrm{HH}^1_R(R(Q), R(Q))_0 \cong \mathrm{HH}^1_L(L(Q), L(Q))_0 \,.$$

Suppose that L is of prime characteristic. Let $\chi : \mathbb{Z} \to L$ be the canonical ring homomorphism and let p be its kernel so that $\mathbb{Z}_p = \mathbb{Z}/p$ becomes a subfield of L. Using Theorem 1.6.(2) again, we get

$$L \otimes_{\mathbb{Z}_p} \mathrm{HH}^1_{\mathbb{Z}_p}(\mathbb{Z}_p(Q),\ \mathbb{Z}_p(Q))_0 \cong \mathrm{HH}^1_L(L(Q), L(Q))_0 = 0.$$

As a consequence, $\mathrm{HH}^1_{\mathbb{Z}_p}(\mathbb{Z}_p(Q), \mathbb{Z}_p(Q))_0 = 0$. Now we apply Theorem 1.6.(3) with $R = \mathbb{Z}$ and $d = 0$. The conditions there are satisfied as A_0, A_1 are free of finite rank by definition, and so is A_n by part (1). Semicontinuity then shows that $\mathrm{HH}^1_{\mathbb{Q}}(\mathbb{Q}(Q), \mathbb{Q}(Q)\,)_0 = 0$. Thus we may assume that L is of characteristic zero. Note that the Euler derivation $\mathcal{E}$ of $L(Q)$ is normalized of degree 0, and hence is inner. As already observed by Happel, [17], evaluating $\mathcal{E}$ on an oriented cycle of Q would then give a contradiction. Evaluating it on two parallel paths shows that they are of the same length. The proof of the proposition is completed.

We now turn our attention to *translation quivers*. Let Γ be a translation quiver with translation τ, that is, Γ is a quiver containing neither loops nor multiple arrows and τ is a bijection from a subset of Γ_0 to another one such that, for each $a \in \Gamma_0$ with $\tau(a)$ defined, there exists at least one arrow $\alpha : b \to a$ and any such arrow determines a unique arrow $\sigma(\alpha) : \tau(a) \to b$; see [9, (1.1)]. One defines the *orbit graph* $\mathcal{O}(\Gamma)$ of Γ as follows: the τ-*orbit* of a vertex a is the set $o(a)$ of vertices of the form $\tau^n(a)$ with $n \in \mathbb{Z}$; the vertices of $\mathcal{O}(\Gamma)$ are the τ-orbits of Γ, and there exists an edge $o(a)$ — $o(b)$ in $\mathcal{O}(\Gamma)$ if Γ contains an arrow $x \to y$ or $y \to x$ with $x \in o(a)$ and $y \in o(b)$. Note that $\mathcal{O}(\Gamma)$ contains no multiple edge by definition. If Γ contains no oriented cycle, then $\mathcal{O}(\Gamma)$ is the graph G_Γ defined in [9, (4.2)]. Now we say that Γ is *simply connected* if Γ contains no oriented cycle and $\mathcal{O}(\Gamma)$ is a tree; see [7, 8]. By [9, (1.6), (4.1), (4.2)], this definition is equivalent to that in [9, (1.6)]. Finally, if Δ is a convex subquiver of Γ, then Δ is a translation quiver with respect to the translation induced from that of Γ. In this case, $\mathcal{O}(\Delta)$ is clearly a subgraph of $\mathcal{O}(\Gamma)$. Thus, if Γ is simply connected, then so is every connected convex translation subquiver of Γ.

We shall study some properties of simply connected translation quivers. If a is a vertex of quiver, we shall denote by a^- the set of immediate predecessors and by a^+ that of immediate successors of a.

2.4. Lemma. *Let Γ be a translation quiver, containing no oriented cycle.*

(1) *Let a be a vertex of Γ and $a_1, a_2 \in a^-$ or $a_1, a_2 \in a^+$. Then $o(a_1) = o(a_2)$ if and only if $a_1 = a_2$.*

(2) *If Γ is simply connected, then an arrow $\alpha : a \to b$ is the only path of Γ from a to b.*

Proof. (1) It suffices to consider the case where $a_1, a_2 \in a^-$. Suppose that $o(a_1) = o(a_2)$. We may assume that $a_2 = \tau^r a_1$ for some $r \geq 0$. If $r > 0$, then $a \to \tau^{r-1}a_1 \to \cdots \to a_1 \to a$ would be an oriented cycle in Γ. Hence $r = 0$, that is, $a_1 = a_2$.

(2) Let $\alpha : a \to b$ be an arrow and

$$p:\ a \xrightarrow{\alpha_1} a_1 \to \cdots \to a_{r-1} \longrightarrow a_s = b$$

a different path from a to b. Then $b \neq a_1$ since Γ contains neither multiple arrows nor oriented cycle. Therefore $o(b) \neq o(a_1)$ by (1). In particular, the edges $o(a) — o(b)$ and $o(a) — o(b)$ of $\mathcal{O}(\Gamma)$ are distinct. Now p induces a walk

$$w(p) : o(a) — o(a_1) — \ \cdots\ — o(a_r)$$

in $\mathcal{O}(\Gamma)$ and $w(p)$, in turn, determines a unique reduced walk $w_{\mathrm{red}}(p)$ from $o(a)$ to $o(b)$. We claim that $w_{\mathrm{red}}(p)$ starts with the edge $o(a) — o(a_1)$. This implies that $\mathcal{O}(\Gamma)$ is not a tree, that is, Γ is not simply connected.

To prove our claim, it suffices to show that $o(a) \neq o(a_i)$ for all $1 \leq i \leq s$. If this is not the case, then $a_t = \tau^{-m}a$ for some $1 \leq t \leq s$ and $m \in \mathbb{Z}$. Now $m > 0$ as Γ contains no oriented cycle, and consequently there exists a path from $\tau^- a$ to $\tau^{-m}a$. Further the arrow $a \to b$ gives rise to an arrow $b \to \tau^- a$. Thus we obtain an oriented cycle

$$b_1 \to \tau^- a \to \cdots \to \tau^{-m}a = a_t \to a_{t+1} \to \cdots \to a_{s-1} \to b$$

in Γ. This contradiction completes the proof of the lemma.

Recall that a vertex a of Γ is *projective* or *injective* if $\tau(a)$ or $\tau^-(a)$ is not defined, respectively. The following result demonstrates how to construct inductively simply connected translation quivers.

2.5. Lemma. *Let Γ be a connected translation quiver that is a one-point extension of a simply connected translation subquiver Δ by a vertex a. Then Γ is not simply connected if and only if a is injective and is the start-point of at least two distinct arrows.*

Proof. Suppose that Γ is not simply connected. Then $\mathcal{O}(\Gamma)$ contains a cycle $o(a_0) — o(a_1) — \cdots — o(a_{r-1}) — o(a_0)$. This cycle contains $o(a)$, since $\mathcal{O}(\Delta)$ is a tree by assumption. We may assume that $o(a_0) = o(a)$. If a is not injective, then $b = \tau^-(a) \in \Delta$, and hence $o(a_0) = o(b)$. Therefore the preceding cycle gives rise to a cycle in $\mathcal{O}(\Delta)$. This contradiction shows that a is injective, and hence the only vertex in $o(a)$. Therefore, Γ contains arrows $\alpha : a \to b$ and $\beta : a \to c$ with $b \in o(a_1)$ and $c \in o(a_{r-1})$. Since $\mathcal{O}(\Gamma)$ contains no multiple edge, we have $o(a_1) \neq o(a_{r-1})$. In particular, $b \neq c$. This shows that the conditions on a are necessary.

Conversely suppose that a is injective and that there exist two distinct arrows $\alpha : a \to b$ and $\beta : a \to c$. By Lemma 2.4.(1), $o(b) \neq o(c)$. In particular, $o(a) — o(b)$ and $o(a) — o(c)$ are two distinct edges of $\mathcal{O}(\Gamma)$. Being connected, $\mathcal{O}(\Delta)$ contains a non-trivial reduced walk $o(b) — o(c_1) — \cdots — o(c_{s-1}) — o(c)$, that can be considered as a reduced walk in $\mathcal{O}(\Gamma)$. This gives rise to a cycle

$$o(a) — o(b) — o(c_1) — \cdots — o(c_{s-1}) — o(c) — o(a)$$

in $\mathcal{O}(\Gamma)$, that is, Γ is not simply connected. The proof of the lemma is completed.

Let us now recall the definition of a mesh algebra. Let Γ be a finite translation quiver. A non-projective vertex a of Γ determines a universal relation on Γ, called a *mesh relation*, $m(a) = \sum \sigma(\alpha_i)\alpha_i$, where the sum is taken over the arrows ending in a. It is clear that every path of length two appears in at most one mesh relation on Γ. Let R be a commutative ring. The ideal of the path algebra $R\Gamma$ generated by the mesh relations is called the *mesh ideal*, whereas the quotient $R(\Gamma)$ of $R\Gamma$ modulo the mesh ideal is called the *mesh algebra* of Γ over R.

2.6. Proposition. *Let R be a commutative ring and Γ a finite translation quiver. If Γ is simply connected, then $R(\Gamma)$ admits no outer derivation.*

Proof. Assume that Γ is simply connected. Then an arrow $\alpha : x \to y$ is the only path of Γ from x to y by Lemma 2.4.(2). Hence every normalized derivation of $R(\Gamma)$ is of degree zero. We shall use induction on the number n of vertices of Γ to prove the result. If $n = 1$, then $R(\Gamma) \cong R$ and the result holds trivially. Assume that $n > 1$ and the result holds for $n-1$. By Lemma 2.1, we may assume that Γ is an one-point extension of a connected translation subquiver Δ by a vertex a. Note that a is projective and Δ is simply connected.

Let δ be a normalized derivation of $R(\Gamma)$. For $w \in R\Gamma$, write $\bar{w} = w + I_\Gamma$, where I_Γ is the mesh ideal of $R\Gamma$. Since δ is of degree zero, for each arrow $\alpha \in \Gamma$, $\delta(\bar{\alpha}) = \lambda_\alpha \bar{\alpha}$ for some $\lambda_\alpha \in R$. In particular, $\delta(R(\Delta)) \subseteq R(\Delta)$. Thus the restriction δ_Δ of δ to $R(\Delta)$ is a normalized derivation of $R(\Delta)$. Now δ_Δ is inner by the inductive hypothesis and is of degree zero. Therefore there exists $u = \sum_{x \in \Delta_0} \mu_x \bar{x},\ \mu_x \in R$ such that $\delta_\Delta = [u, -]$. Since Γ is connected, there exist arrows starting from a. Let $\alpha_i : a \to b_i$ with $1 \le i \le r$ be all such arrows. Set $\mu_a = \mu_{b_1} + \lambda_{\alpha_1}$ and $v = \mu_a \bar{a} + u$. Then $\delta(\bar{\alpha}_1) = [v, \bar{\alpha}_1]$. We claim that $\delta(\bar{\alpha}_i) = [v, \bar{\alpha}_i]$, for all $1 \le i \le r$. This is trivial if $r = 1$. Assume now that $r > 1$. Then a is not injective by Lemma 2.5. Therefore, $c = \tau^-(a)$ exists and lies in Δ. Hence Δ admits arrows $\beta_i : b_i \to c$, $i = 1, \ldots, r$. Evaluating δ on the equality $\sum_{i=1}^r \bar{\alpha}_i \bar{\beta}_i = 0$ gives rise to $\sum_{i=2}^r (\lambda_{\alpha_i} + \lambda_{\beta_i} - \lambda_{\alpha_1} - \lambda_{\beta_1}) \bar{\alpha}_i \bar{\beta}_i = 0$. Note that $\bar{\alpha}_2 \bar{\beta}_2, \ldots, \bar{\alpha}_r \bar{\beta}_r$ are linearly independant over R by Proposition 2.3.(1). Hence $\lambda_{\alpha_i} + \lambda_{\beta_i} = \lambda_{\alpha_1} + \lambda_{\beta_1}$ for all $2 \le i \le r$. Moreover, evaluating $\delta_\Delta = [u, -]$ on $\bar{\beta}_i$ results in $\mu_c = \mu_{b_i} - \lambda_{\beta_i}$ for all $1 \le i \le r$, that is $\mu_{b_i} - \lambda_{\beta_i} = \mu_{b_1} - \lambda_{\beta_1}$, for all $1 \le i \le r$. Now one can deduce that $\delta(\bar{\alpha}_i) = [v, \bar{\alpha}_i]$, for all $1 \le i \le r$. This shows that $\delta(\bar{\beta}) = [v, \bar{\beta}]$ for any arrow of Γ, and consequently, $\delta = [v, -]$. The proof of the proposition is completed.

We finally reach the main result of this section.

2.7. Theorem. *Let R be a domain and Γ a finite connected translation quiver. The following are equivalent*:

(1) $\mathrm{HH}^1(R(\Gamma)) = 0$.

(2) *Γ is simply connected.*

(3) $\mathrm{HH}^1(R(\Delta)) = 0$ *for every connected convex translation subquiver* Δ *of* Γ.

Proof. It is trivial that (3) implies (1). Assume that Γ is simply connected. Then every connected convex translation subquiver of Γ is simply connected, and hence its mesh algebra admits no outer derivation by Proposition 2.6. This proves that (2) implies (3).

We shall show by induction on the number n of vertices of Γ that (1) implies (2). This is trivial if $n = 1$. Assume that $n > 1$ and this implication holds for $n-1$. Suppose that $\mathrm{HH}^1(R(\Gamma)) = 0$. By Proposition 2.3.(2), Γ contains no oriented cycle and any two parallel paths are of the same length. In particular, a normalized derivation of $R(\Gamma)$ is of degree zero. By Lemma 2.1, we may assume that Γ is an one-point extension of a connected translation subquiver Δ by a vertex a.

Suppose on the contrary that Γ is not simply connected. First we consider the case where Δ is simply connected. By Lemma 2.5, a is injective and there exist two distinct arrows $\alpha : a \to b$ and $\beta : a \to c$ with $b, c \in \Delta_0$. Since Δ is connected, b and c are connected by a reduced walk in Δ. Therefore, α lies on a cycle of Γ and it appears in no mesh relation on Γ since a is injective. This contradicts Lemma 2.2.

We now turn to the case where Δ is not simply connected. By the inductive hypothesis, there exists a normalized outer derivation $\bar{\delta}$ of $R(\Delta)$ that is of degree zero. Thus for each arrow $\alpha \in \Delta$, we have $\bar{\delta}(\bar{\alpha}) = \lambda_\alpha \bar{\alpha}$ with $\lambda_\alpha \in R$, where $\bar{w}$ denotes the class of $w \in R\Delta$ modulo the mesh ideal I_Δ of $R\Delta$. Let δ be the normalized derivation of $R\Delta$ such that $\delta(\alpha) = \lambda_\alpha \alpha$ for all $\alpha \in \Delta$. Then $\delta(I_\Delta) \subseteq I_\Delta$. Being connected, Γ contains arrows starting with a. Let $\alpha_i : a \to b_i$ with $1 \le i \le s$ be all such arrows. If a is injective, let ∂ be the normalized derivation of $R\Gamma$ such that ∂ coincides with δ on $R\Delta$ and $\partial(\alpha_i) = 0$ for all $1 \le i \le s$. Then ∂ preserves the mesh ideal I_Γ of Γ, since I_Γ is generated by the mesh relations on Δ in this case. Suppose now that a is not injective and let β_i be the arrow from b_i to $\tau^-(a)$ for $1 \le i \le s$. Then $m = \sum_{i=1}^s \alpha_i\beta_i$ is the only mesh relation on Γ that is not in I_Δ. Note that $\beta_i \in \Delta$, and in particular, λ_{β_i} is defined for all $1 \le i \le s$. Let ∂ be the normalized derivation of $R\Gamma$ such that ∂ coincides with δ on $R\Delta$ and $\partial(\alpha_i) = (1 - \lambda_{\beta_i})\alpha_i$ for all $1 \le i \le s$. It is easy to see that in this case ∂ preserves I_Γ as well. Now ∂ induces a normalized derivation $\bar{\partial}$ of $R(\Gamma)$, that coincides with $\bar{\delta}$ on $R(\Delta)$ and is of degree zero. However, $\bar{\partial}$ is inner since $\mathrm{HH}^1(R(\Gamma)) = 0$. Hence its restriction to $R(\Delta)$, that is $\bar{\delta}$, is easily seen to be inner as well. This contradiction shows that (1) implies (2). The proof of the theorem is now completed.

Remark. In case Γ contains no oriented cycle and R is an algebraically closed field, Coelho and Vargas proved in [14] the equivalence of (2) and (3) in terms of strongly simply connected algebras. They claimed as well the equivalence of (1) and (2) when in addition the base field is of characteristic zero. However, the proof for this equivalence given there misquoted the incomplete argument given in the proof of the second theorem in [18, Section 4].

3. *Hochschild cohomology of endomorphism algebras*

The objective of this section is to compare Hochschild cohomology of an algebra and that of the endomorphism algebra of a module. We shall first study the relation between the first Hochschild cohomogy groups in the most general situation and then prove the invariance of Hochschild cohomology in case the algebras involved are projective over the base ring and the module is pseudo-tilting.

As before, let A be an R-algebra and A^{e} its enveloping algebra. Recall that all unadorned tensor products are taken over R. We fix the following extension of A-bimodules:

$$\omega_A : \qquad 0 \to \Omega_A \xrightarrow{j_A} A \otimes A \xrightarrow{\mu_A} A \to 0,$$

where μ_A is the multiplication map of A and j_A is the inclusion map. Each map $f \in \mathrm{Hom}_{A^{\mathrm{e}}}(\Omega_A, X)$ determines an R-derivation $\delta_f : A \to X : a \mapsto f(a \otimes 1 - 1 \otimes a)$. The following well-known facts will be used extensively in our investigation below, whence we state them explicitly for the convenience of the reader. We refer to [10, (AIII.132)] for a proof of the first part.

3.1. Lemma. *Let A be an R-algebra and X an A-bimodule.*
(1) *The following map is an isomorphism of R-modules*:

$$D : \mathrm{Hom}_{A^{\mathrm{e}}}(\Omega_A, X) \to \mathrm{Der}_R(A, X) : \quad f \mapsto \delta_f .$$

(2) *There exists a commutative diagram of R-modules*:

$$\begin{tikzcd} X \arrow[r, "\mathrm{ad}"] \arrow[d, "\cong"'] & \mathrm{Der}_R(A, X) \arrow[d, "\cong"', "D^{-1}"] \\ \mathrm{Hom}_{A^{\mathrm{e}}}(A \otimes A,\ X) \arrow[r, "{(j_A,\ X)}"] & \mathrm{Hom}_{A^{\mathrm{e}}}(\Omega_A,\ X)\,. \end{tikzcd}$$

Now let B be another R-algebra. Any B-A-bimodule X becomes a right $B^{\mathrm{o}} \otimes A$-module through $x \cdot (b^{\mathrm{o}} \otimes a) = bxa$. Let M, N be B-A-bimodules. Forgetting the right A-module structure or the left B-module structure on an extension of B-A-bimodules yields respectively the following maps:

$$f_A : \mathrm{Ext}^1_{B^{\mathrm{o}} \otimes A}(M, N) \to \mathrm{Ext}^1_B(M, N)\,,$$

$$f_B : \mathrm{Ext}^1_{B^{\mathrm{o}} \otimes A}(M, N) \to \mathrm{Ext}^1_A(M, N)\,.$$

The kernels of these maps are identified in the following result.

3.2. Proposition. *Let M, N be B-A-bimodules. The forgetful maps introduced above fit into the following exact sequences*:

(1) $0 \to \mathrm{HH}^1(A, \mathrm{Hom}_B(M, N)) \xrightarrow{g_A} \mathrm{Ext}^1_{B^{\mathrm{o}} \otimes A}(M, N) \xrightarrow{f_A} \mathrm{Ext}^1_B(M, N)\,,$

(2) $0 \to \mathrm{HH}^1(B, \mathrm{Hom}_A(M, N)) \xrightarrow{g_B} \mathrm{Ext}^1_{B^{\mathrm{o}} \otimes A}(M, N) \xrightarrow{f_B} \mathrm{Ext}^1_A(M, N)\,.$

Proof. It suffices to show the first part of the proposition. To this end, we shall define explicitly the map g_A. Applying first Lemma 3.1 to the A-bimodule $\mathrm{Hom}_B(M, N)$ and then using adjunction, we get the following commutative diagram:

$$\begin{array}{ccc}
\mathrm{Hom}_B(M,\ N) & \xrightarrow{\mathrm{ad}} & \mathrm{Der}_R(A,\ \mathrm{Hom}_B(M,N)) \\
\cong\downarrow & & \cong\downarrow \\
\mathrm{Hom}_{A^e}(A\otimes A,\ \mathrm{Hom}_B(M,N)) & \xrightarrow{(j_A,\,(M,N))} & \mathrm{Hom}_{A^e}(\varOmega_A,\ \mathrm{Hom}_B(M,N)) \\
\cong\downarrow & & \cong\downarrow \\
\mathrm{Hom}_{B^\circ\otimes A}(M\otimes A,\ \ N) & \xrightarrow{(M\otimes j_A,\,N)} & \mathrm{Hom}_{B^\circ\otimes A}(M\otimes_A \varOmega_A,\ \ N).
\end{array} \qquad (*)$$

Since ω_A splits as an extension of left A-modules, the sequence

$$M\otimes_A\omega_A:\qquad 0\longrightarrow M\otimes_A\varOmega_A \xrightarrow{M\otimes_A j_A} M\otimes A \xrightarrow{M\otimes_A\mu_A} M\longrightarrow 0$$

is an extension of B-A-bimodules that splits as an extension of left B-modules. For $\delta\in\mathrm{Der}_R(A,\ \mathrm{Hom}_B(M,N))$, let $\tilde\delta\in\mathrm{Hom}_{B^\circ\otimes A}(M\otimes_A\varOmega_A,\ N)$ be the corresponding map under the isomorphisms given in the above diagram $(*)$. Pushing out $M\otimes_A\omega_A$ along the map $\tilde\delta$, we get an exact commutative diagram:

$$\begin{array}{lccccccccc}
M\otimes_A\omega_A: & 0 & \longrightarrow & M\otimes_A\varOmega_A & \xrightarrow{M\otimes_A j_A} & M\otimes A & \xrightarrow{M\otimes_A\mu_A} & M & \longrightarrow & 0 \\
 & & & \tilde\delta\downarrow & & \downarrow & & \Vert & & \\
\tilde\delta_*(M\otimes_A\omega_A): & 0 & \longrightarrow & N & \longrightarrow & E & \longrightarrow & M & \longrightarrow & 0,
\end{array}$$

where the rows are extensions of B-A-bimodules that split as extensions of left B-modules. Now the lower row splits as an extension of B-A-bimodules if and only if $\tilde\delta$ factors through $M\otimes_A j_A$ if and only if $\delta\in\mathrm{Inn}_R(A,\mathrm{Hom}_B(M,N))$. This shows that associating to the class $[\delta]\in\mathrm{HH}^1(A,\mathrm{Hom}_B(M,N))$ of a derivation δ the class

$$g_A([\delta])=[\tilde\delta_*(M\otimes_A\omega_A)]\in\mathrm{Ext}^1_{B^\circ\otimes A}(M,N)$$

yields a well-defined injective map g_A with image contained in the kernel of f_A.

It remains to show that the kernel of f_A is contained in the image of g_A. For this purpose, let $\eta:\ \ 0\to N\to E\xrightarrow{p} M\to 0$ be an extension of B-A-bimodules that splits as an extension of left B-modules. Let $q: M\to E$ be a B-linear map such that $pq=\mathbf{1}_M$, and let $\sigma: M\otimes A\to E$ be the B-A-bilinear map such that $\sigma(m\otimes a)=q(m)a$. These data define an exact commutative diagram of B-A-bimodules as follows:

$$\begin{array}{lccccccccc}
M\otimes_A\omega_A: & 0 & \longrightarrow & M\otimes_A\varOmega_A & \xrightarrow{M\otimes_A j_A} & M\otimes A & \xrightarrow{M\otimes_A\mu_A} & M & \longrightarrow & 0 \\
 & & & \zeta\downarrow & & \sigma\downarrow & & \Vert & & \\
\eta: & 0 & \longrightarrow & N & \longrightarrow & E & \xrightarrow{p} & M & \longrightarrow & 0,
\end{array}$$

where ζ is induced by σ. In view of the isomorphisms given in $(*)$, there exists a derivation $\delta\in\mathrm{Der}_R(A,\ \mathrm{Hom}_B(M,N))$ whose image is ζ. Hence $g_A([\delta])=[\eta]$. This completes the proof of the proposition.

Now we specialize the preceding proposition to the case $M=N$. The canonical algebra anti-homomorphism $\rho: A\to\mathrm{End}_B(M)$ is a homomorphism of A-bimodules.

Thus ρ induces an R-linear map

$$\mathrm{HH}^1(A,\rho):\ \mathrm{HH}^1(A,A)\to \mathrm{HH}^1(A,\mathrm{End}_B(M))\,,$$

which in turn allows us to define a map $\chi_M:\mathrm{HH}^1(A)\to \mathrm{Ext}^1_A(M,M)$ through the following commutative diagram:

$$\begin{tikzcd}
\mathrm{HH}^1(A) \arrow[r, "\chi_M"] \arrow[d, "\mathrm{HH}^1(A,\rho)"'] & \mathrm{Ext}^1_A(M,M) \\
\mathrm{HH}^1(A,\mathrm{End}_B(M)) \arrow[r, "g_A"] & \mathrm{Ext}^1_{B^{\circ}\otimes A}(M,M). \arrow[u, "f_B"]
\end{tikzcd}$$

To give an explicit description of χ_M, consider the commutative diagram

$$\begin{tikzcd}
\mathrm{Der}_R(A,A) \arrow[r, "\rho\circ-"] \arrow[d, "\cong"'] & \mathrm{Der}_R(A,\mathrm{End}_B(M)) \arrow[d, "\cong"'] \\
\mathrm{Hom}_{A^{\mathrm{e}}}(\Omega_A,\ A) \arrow[r, "M\otimes_A-"] & \mathrm{Hom}_{B^{\circ}\otimes A}(M\otimes_A\Omega_A,\ M),
\end{tikzcd}$$

where $\rho\circ$ denotes composition with ρ, the isomorphism on the left comes from Lemma 3.1.(2) while that on the right comes from the diagram $(*)$ in the proof of Proposition 3.2. Let $\delta\in\mathrm{Der}_R(A,A)$ be a derivation and let $f\in\mathrm{Hom}_{A^{\mathrm{e}}}(\Omega_A,A)$ be the corresponding map. Applying $M\otimes_A-$ to the extension ω_A and pushing out $M\otimes_A\omega_A$ along the map $M\otimes_A f$ yields a self-extension $(M\otimes_A f)_*(M\otimes_A\omega_A)$ of the right A-module M. Then

$$\chi_M([\delta])=[(M\otimes_A f)_*(M\otimes_A\omega_A)]$$

by our earlier definition of g_A and f_B.

To study the map χ_M further, we give an alternative and explicit description. To this end, we define the differentiation of a morphism between projective modules of the following form: let

$$\phi:\oplus_{j=1}^n u_jA\to\oplus_{i=1}^m v_iA$$

be an A-linear map with u_j,v_i some idempotents of A. Let $\delta\in\mathrm{Der}_R(A)$ be such that $\delta(v_iAu_j)\subseteq v_iAu_j$ for all i,j. If the matrix of ϕ with respect to the given decomposition is $(a_{ij})_{m\times n}$ with $a_{ij}\in v_iAu_j$, then we call the A-linear map

$$\delta(\phi):\oplus_{j=1}^n u_jA\to\oplus_{i=1}^m v_iA$$

given by the matrix $(\delta(a_{ij}))_{m\times n}$ the *derivative of ϕ along δ.*

3.3. Lemma. *Let M be a B-A-bimodule that is finitely presented as right A-module. Let*

$$P_2\xrightarrow{\psi}\oplus_{j=1}^n u_jA\xrightarrow{\phi}\oplus_{i=1}^m v_iA\xrightarrow{\varepsilon}M\to 0$$

be an exact sequence of right A-modules with P_2 projective and u_j,v_i idempotents in A. Let $\delta\in\mathrm{Der}_R(A)$ be such that $\delta(v_iAu_j)\subseteq v_iAu_j$ for all i,j. Then the image of $[\delta]\in\mathrm{Der}_R(A)/\mathrm{Inn}_R(A)$ under χ_M is the class of the self-extention of M that corresponds to the class $[\varepsilon\delta(\phi)]$ in $\mathrm{Ext}^1_A(M,M)$.

Proof. Let $f\in\mathrm{Hom}_{A^{\mathrm{e}}}(\Omega_A,A)$ be such that $\delta(a)=f(a\otimes 1-1\otimes a)$ for all $a\in A$; see Lemma 3.1.(1). We have seen that $\chi_M([\delta])=[(M\otimes_A f)_*(M\otimes_A\omega_A)]$. We claim that there exist A-linear maps α,β rendering the following diagram commutative:

$$
\begin{tikzcd}
\oplus_{i=1}^{n} u_j A \arrow[r, "\phi"] \arrow[d, "\delta(\phi)"'] \arrow[dr, "\beta"] & \oplus_{j=1}^{m} v_i A \arrow[r, "\varepsilon"] \arrow[dr, "\alpha"] & M \arrow[r] \arrow[dr, equal] & 0 & \\
\oplus_{j=1}^{m} v_i A \arrow[dr, "\varepsilon"] & M \otimes_A \Omega_A \arrow[r, "M\otimes j_A"] \arrow[d, "M\otimes_A f"] & M \otimes A \arrow[r, "M\otimes \mu_A"] & M \arrow[r] & 0 \\
& M. & & &
\end{tikzcd}
$$

In fact, since $\varepsilon(v_i)\otimes v_i = (\varepsilon(v_i)\otimes v_i)v_i \in (M\otimes A)v_i$, there exists a unique A-linear map $\alpha : \oplus_{i=1}^{m} v_i A \to M\otimes A$ such that $\alpha(v_i) = \varepsilon(v_i)\otimes v_i$ for all $1 \le i \le m$. Moreover, assume that the matrix of ϕ is $(a_{ij})_{m\times n}$ with $a_{ij} \in v_i A u_j$. Then $\phi(u_j) = \sum_{i=1}^{m} v_i a_{ij}$ and $\delta(\phi)(u_j) = \sum_{i=1}^{m} v_i \delta(a_{ij})$ for all $1 \le j \le n$. Let

$$x_j = \sum\nolimits_{i=1}^{m} \varepsilon(v_i) \otimes_A (a_{ij} \otimes 1 - 1 \otimes a_{ij}) \in M \otimes_A \Omega_A\,.$$

Note that $a_{ij} = a_{ij}u_j$ and $\sum_{i=1}^{m} \varepsilon(v_i)a_{ij} = \varepsilon(\phi(u_j)) = 0$. This implies that $x_j = x_j u_j \in (M \otimes_A \Omega_A)u_j$. Therefore, there exists a unique A-linear map

$$\beta : \oplus_{j=1}^{n} u_j A \to M \otimes_A \Omega_A$$

such that $\beta(u_j) = x_j$ for all $1 \le j \le n$. It is now easy to verify that the above diagram is commutative. Consider now the following exact commutative diagram:

$$
\begin{tikzcd}
& 0 \arrow[r] & \Omega_M \arrow[r, "j"] \arrow[d, "\gamma"'] & \oplus_{i=1}^{m} v_i A \arrow[r, "\varepsilon"] \arrow[d, "\alpha"'] & M \arrow[r] \arrow[d, equal] & 0 \\
M \otimes_A \omega_A : & 0 \arrow[r] & M \otimes_A \Omega_A \arrow[r, "M\otimes_A j_A"] \arrow[d, "M\otimes_A f"'] & M \otimes A \arrow[r, "M\otimes_A \mu_A"] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\
\eta : & 0 \arrow[r] & M \arrow[r] & E \arrow[r] & M \arrow[r] & 0,
\end{tikzcd}
$$

where j is the inclusion map and γ is induced by α. Let $\tilde{\phi} : \oplus_{j=1}^{n} u_j A \to \Omega_M$ be such that $\phi = j\tilde{\phi}$. Then $\beta = \gamma\tilde{\phi}$, and hence $\varepsilon\delta(\phi) = (M \otimes_A f)\gamma\tilde{\phi}$. Note that $\tilde{\phi}\psi = 0$. This implies $\varepsilon\delta(\phi) \in \mathrm{Ker}(\psi, M)$ and exhibits η as the self-extension of M corresponding to $\varepsilon\delta(\phi)$. The proof of the lemma is completed.

Next, we look more carefully at the image of the map χ_M.

3.4. Lemma. *Let M be a B-A-bimodule with $M = \bigoplus_{i=1}^{n} M_i$ a decomposition of right A-modules. Then χ_M factors as follows :*

$$
\begin{tikzcd}
\mathrm{HH}^1(A) \arrow[r, "\chi_M"] \arrow[dr, "\sum_i \chi_{M_i}"'] & \mathrm{Ext}^1_A(M, M) \\
& \bigoplus_{i=1}^{n} \mathrm{Ext}^1_A(M_i, M_i). \arrow[u, hook]
\end{tikzcd}
$$

Proof. Let $\delta \in \mathrm{Der}_R(A)$ be a derivation and $f \in \mathrm{Hom}_{A^e}(\Omega_A, A)$ the corresponding map; see Lemma 3.1.(1). For each i, the map f defines an exact commutative diagram:

$$\begin{array}{lccccccccc}
M_i \otimes_A \omega_A: & 0 & \longrightarrow & M_i \otimes_A \Omega_A & \xrightarrow{M_i \otimes_A j_A} & M_i \otimes A & \xrightarrow{M_i \otimes_A \mu_A} & M_i & \longrightarrow & 0 \\
 & & & \Big\downarrow{\scriptstyle M_i \otimes_A f} & & \Big\downarrow{\scriptstyle \sigma_i} & & \Big\| & & \\
\eta_i: & 0 & \longrightarrow & M_i & \longrightarrow & E_i & \longrightarrow & M_i & \longrightarrow & 0.
\end{array}$$

Summing up yields the following exact commutative diagram:

$$\begin{array}{lccccccccc}
M \otimes_A \omega_A: & 0 & \longrightarrow & M \otimes_A \Omega_A & \xrightarrow{M \otimes_A j_A} & M \otimes A & \xrightarrow{M \otimes_A \mu_A} & M & \longrightarrow & 0 \\
 & & & \Big\downarrow{\scriptstyle M \otimes_A f} & & \Big\downarrow{\scriptstyle \oplus_i \sigma_i} & & \Big\| & & \\
\oplus_i \eta_i: & 0 & \longrightarrow & M & \longrightarrow & \bigoplus_i E_i & \longrightarrow & M & \longrightarrow & 0.
\end{array}$$

Therefore, $\chi_M([\delta]) = [\oplus_i \zeta_i] \in \bigoplus_{i=1}^n \mathrm{Ext}_A^1(M_i, M_i)$. This completes the proof of the lemma.

In our main application, we shall consider the case where M is a right A-module and $B = \mathrm{End}_A(M)$, whence M is endowed with the canonical B-A-bimodule structure. Moreover, there exists a morphism of B-A-bimodules $\rho : A \to \mathrm{End}_B(M)$, where the map $\rho(a)$ for $a \in A$ is given by $\rho(a)(x) = xa$, for all $x \in M$. Recall that M is a *faithfully balanced* B-A-bimodule if ρ is an isomorphism.

3.5. Theorem. *Let A be an algebra over a commutative ring R. Let M be a right A-module and set $B = \mathrm{End}_A(M)$. Assume that M is a faithfully balanced B-A-bimodule with $\mathrm{Ext}_B^1(M, M) = 0$. Then there exists an exact sequence*

$$0 \longrightarrow \mathrm{HH}^1(B) \longrightarrow \mathrm{HH}^1(A) \xrightarrow{\chi_M} \mathrm{Ext}_A^1(M, M)$$

of R-modules. Moreover, if $M = \oplus_{i=1}^n M_i$ is a decomposition of right A-modules, then the image of χ_M lies in the diagonal part $\oplus_{i=1}^n \mathrm{Ext}_A^1(M_i, M_i)$.

Proof. By hypothesis, $\mathrm{HH}^1(A, \rho) : \mathrm{HH}^1(A) \to \mathrm{HH}^1(A, \mathrm{End}_B(M))$ is an isomorphism. Furthermore, the map g_A in Proposition 3.2.(1) is an isomorphism since $\mathrm{Ext}_B^1(M, M) = 0$. Thus $\gamma = g_A \circ \mathrm{HH}^1(A, \rho)$ is an isomorphism. Set $c = \gamma^{-1} \circ g_B$. It follows from the definition of χ_M that the diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{HH}^1(B) & \xrightarrow{c} & \mathrm{HH}^1(A) & \xrightarrow{\chi_M} & \mathrm{Ext}_A^1(M, M) \\
 & & \Big\| & & \Big\downarrow{\scriptstyle \gamma}\,{\scriptstyle\cong} & & \Big\| \\
0 & \longrightarrow & \mathrm{HH}^1(B) & \xrightarrow{g_B} & \mathrm{Ext}_{B^{\circ} \otimes A}^1(M, M) & \xrightarrow{f_B} & \mathrm{Ext}_A^1(M, M)
\end{array}$$

is commutative. By Proposition 3.2.(2), the lower row is exact, and hence the upper row is an exact sequence as desired. The last part of the theorem follows from Lemma 3.4. This completes the proof.

For a right A-module, denote by $\mathrm{add}(M)$ the full subcategory of the category of right A-modules generated by the direct summands of direct sums of finitely many copies of M. We call M an *A-generator* if A lies in $\mathrm{add}(M)$. We shall now study the behaviour of Hochschild cohomology under (r-)pseudo-tilting, a notion we define as follows.

3.6. Definition. Let A be an algebra over a commutative ring R and let $r \geq 0$ be an integer. A right A-module M is called *r-pseudo-tilting* if it satisfies the following conditions:

(1) $\mathrm{Ext}_A^i(M, M) = 0$ for $1 \leq i \leq r$;

(2) there exists an exact sequence

$$0 \to A \to M_0 \to M_1 \to \cdots \to M_r \to 0$$

of right A-modules, where $M_0, \dots, M_r \in \mathrm{add}(M)$.

Moreover, the module M is called *pseudo-tilting* if $\mathrm{Ext}_A^i(M, M) = 0$ for all $i \geq 1$ and (2) holds for some $r \geq 0$; in other words, M is r-pseudo-tilting for all sufficiently large r.

Note that a 0-pseudo-tilting A-module is nothing but an A-generator. Moreover, a tilting or co-tilting module of projective or injective dimension at most r, respectively, in the sense of [23] is an r-pseudo-tilting module, and consequently, a pseudo-tilting module. We recall now the following result from [23, (1.4)].

3.7. Lemma (Miyashita). *Let A be an R-algebra and $r \geq 0$ an integer. Let M be an r-pseudo-tilting right A-module and set $B = \mathrm{End}_A(M)$. Then M is a faithfully balanced B-A-bimodule and $\mathrm{Ext}_A^i(M, M) = 0$ for all $i \geq 1$.*

As an immediately consequence of Lemma 3.7 and Theorem 3.5, we get the following result.

3.8. Proposition. *Let A be an algebra over a commutative ring R. Let M be an r-pseudo-tilting right A-module and set $B = \mathrm{End}_A(M)$.*

(1) *There exists an exact sequence of R-modules*

$$0 \longrightarrow \mathrm{HH}^1(B) \longrightarrow \mathrm{HH}^1(A) \xrightarrow{\chi_M} \mathrm{Ext}_A^1(M, M)\,.$$

(2) *If $r \geq 1$, then $\mathrm{HH}^1(A) \cong \mathrm{HH}^1(B)$.*

So far we have not put any restriction on the R-algebra structure. As usual in Hochschild theory, much sharper results can be obtained if the algebras involved are projective as R-modules, for example, when R is a field. First we recall the following result from [13, p.346].

3.9. Proposition (Cartan-Eilenberg). *Let A, B be algebras over a commutative ring R, and let M, N be B-A-bimodules. If A and B are projective as R-modules, then there exist spectral sequences*

$$\mathrm{HH}^i(A, \mathrm{Ext}_B^j(M, N)) \Longrightarrow \mathrm{Ext}_{B^o \otimes A}^{i+j}(M, N)$$

and

$$\mathrm{HH}^i(B, \mathrm{Ext}_A^j(M, N)) \Longrightarrow \mathrm{Ext}_{B^o \otimes A}^{i+j}(M, N)\,.$$

Remark. The first spectral sequence yields the following exact sequence of lower terms that extends the exact sequence from Proposition 3.2.(1):

$$\begin{aligned} 0 \ &\to\ \mathrm{HH}^1(A, \mathrm{Hom}_B(M,N)) \xrightarrow{g_A^1} \mathrm{Ext}^1_{B^\circ\otimes A}(M,N) \xrightarrow{f_A^1} \mathrm{HH}^0(A, \mathrm{Ext}^1_B(M,N)) \\ &\to\ \mathrm{HH}^2(A, \mathrm{Hom}_B(M,N)) \xrightarrow{g_A^2} \mathrm{Ext}^2_{B^\circ\otimes A}(M,N). \end{aligned}$$

Applying the preceding proposition to the case where $M = N$ is a (r-)pseudo-tilting A-module and $B = \mathrm{End}_A(M)$, we obtain the following result.

3.10. Theorem. *Let R be a commutative ring, and let A be an R-algebra that is projective as R-module. Let M be a right A-module such that $B = \mathrm{End}_A(M)$ is projective as R-module as well.*

(1) *If M is r-pseudo-tilting, then $\mathrm{HH}^i(B) \cong \mathrm{HH}^i(A)$ for $0 \le i \le r$.*

(2) *If M is pseudo-tilting, then $\mathrm{HH}^i(B) \cong \mathrm{HH}^i(A)$ for all $i \ge 0$.*

Proof. First we note that the edge homomorphisms of the first spectral sequence in Proposition 3.9 become $g_A^i : \mathrm{HH}^i(A) \to \mathrm{Ext}^i_{B^\circ\otimes A}(M,M)$ and

$$f_A^i : \mathrm{Ext}^i_{B^\circ\otimes A}(M,M) \to \mathrm{HH}^0(A, \mathrm{Ext}^i_B(M,M)), \quad \text{for all } \ i \ge 0,$$

whereas those of the second one become $g_B^i : \mathrm{HH}^i(B) \to \mathrm{Ext}^i_{B^\circ\otimes A}(M,M)$ and

$$f_B^i : \mathrm{Ext}^i_{B^\circ\otimes A}(M,M) \to \mathrm{HH}^0(A, \mathrm{Ext}^i_A(M,M)), \quad \text{for all } \ i \ge 0.$$

Assume now that M is r-pseudo-tilting. By Lemma 3.7, $\mathrm{Ext}^i_B(M,M) = 0$ for all $i \ge 1$. So each g_B^i with $i \ge 0$ is an isomorphism. Moreover, since $\mathrm{Ext}^i_A(M,M) = 0$ for $1 \le i \le r$, the g_A^i are isomorphisms for $0 \le i \le r$. These give rise to the desired isomorphisms

$$c^i = (g_A^i)^{-1} g_B^i : \mathrm{HH}^i(B) \to \mathrm{HH}^i(A); \ i = 0, 1, \ldots, r.$$

If M is pseudo-tilting, then the map c^i is defined and an isomorphism for each $i \ge 0$. This completes the proof of the theorem.

Remark. In case A is a finite dimensional algebra over an algebraically closed field and M is a finite dimensional tilting A-module, Happel showed that $\mathrm{HH}^i(A) \cong \mathrm{HH}^i(B)$ for all $i \ge 0$ in [17, (4.2)].

4. *Representation-finite algebras without outer derivations*

The objective of this section is to investigate when an algebra of finite representation type admits no outer derivation. To begin with, let A be an artin algebra. Denote by $\mathrm{mod}\, A$ the category of finitely generated right A-modules and by $\mathrm{ind}\, A$ its full subcategory generated by a chosen complete set of representatives of iso-classes of the indecomposable modules. Let Γ_A denote the Auslander-Reiten quiver of A, which is a translation quiver with respect to the Auslander-Reiten translation DTr. We refer to [4] for general results on almost split sequences and irreducible maps as they pertain to the structure of Auslander-Reiten quivers.

Assume that A is of finite representation type, that is, $\mathrm{ind}\, A$ contains only finitely many objects, say $M_1, \ldots, M_n$. Then $M = \bigoplus_{i=1}^n M_i$ is an A-generator, called a *minimal representation generator* for A. One calls $\Lambda = \mathrm{End}_A(M)$ the *Auslander* *algebra* of A. Applying Lemma 3.7 and Theorem 3.5, one obtains immediately the following exact sequence:

$$0 \to \mathrm{HH}^1(\Lambda) \to \mathrm{HH}^1(A) \xrightarrow{\chi} \bigoplus_{i=1}^{n} \mathrm{Ext}_A^1(M_i, M_i)\,.$$

Thus, $\mathrm{HH}^1(\Lambda)$ is always embedded into $\mathrm{HH}^1(A)$. To investigate when this embedding is an isomorphism, recall that a module in $\mathrm{ind}\,A$ is a *brick* if its endomorphism algebra is a division ring. It is shown in [25, (4.6)] that if $\mathrm{ind}\,A$ contains only bricks, then A is of finite representation type. Conversely, it is well known that if A is of finite representation type and Γ_A contains no oriented cycle, then $\mathrm{ind}\,A$ contains only bricks.

4.1. Proposition. *Let A be an artin algebra such that* $\mathrm{ind}\,A$ *contains only bricks, and let Λ be the Auslander algebra of A. Then* $\mathrm{HH}^1(\Lambda) \cong \mathrm{HH}^1(A)$.

Proof. In view of the exact sequence stated above, we need only to show that $\mathrm{Ext}_A^1(N, N) = 0$ for every module N in $\mathrm{ind}\,A$. Assume on the contrary that this fails for some module M in $\mathrm{ind}\,A$. Then M is non-projective and $\mathrm{Hom}_A(M, \mathrm{DTr}M) \neq 0$; see, for example, [4, p.131]. Let

$$0 \to \mathrm{DTr}M \to E \to M \to 0$$

be an almost split sequence in $\mathrm{mod}A$. Using the fact that an irreducible map is either a monomorphims or an epimorphism, one finds easily that either M or an indecomposable direct summand of E is not a brick. This contradiction completes the proof of the proposition.

From now on, let k be an algebraically closed field and A a finite dimensional k-algebra. It is well known, see [9, (2.1)], that there exists a unique finite quiver Q_A (called the *ordinary quiver* of A) and an admissible ideal I_A in kQ_A such that the basic algebra B of A is isomorphic to kQ_A/I_A. Such an isomorphism $B \cong kQ_A/I_A$ is called a *presentation* of B. Moreover, one says that A is *connected* or *triangular* if Q_A is connected or contains no oriented cycle, respectively. In case A is of finite representation type, one says that A is *standard*, [9, (5.1)], if its Auslander algebra is isomorphic to the mesh algebra $k(\Gamma_A)$ of Γ_A over k. It follows from [11, (3.1)] that this definition is equivalent to that given in [6, (1.11)].

4.2. Lemma. *Let A be of finite representation type, and let Λ be its Auslander algebra. If* $\mathrm{HH}^1(A) = 0$ *or* $\mathrm{HH}^1(\Lambda) = 0$*, then A is standard.*

Proof. It is well known that $\mathrm{HH}^1(A)$ is invariant under Morita equivalence; see also Proposition 3.8. We may hence assume that A is basic. Suppose that A is not standard. It follows from [6, (9.6)] that there exists a presentation $A \cong kQ_A/I_A$ such that the bound quiver (Q_A, I_A) contains a Riedtmann contour C as a full bound subquiver. This means that the ordinary quiver Q_C of C consists of a vertex a, a loop ρ at a, and a simple oriented cycle

$$a = a_0 \xrightarrow{\alpha_1} a_1 \to \cdots \to a_{n-1} \xrightarrow{\alpha_n} a_n = a\,,$$

bound by the following relations:

$$\rho^2 - \alpha_1 \cdots \alpha_n = \alpha_n\alpha_1 - \alpha_n\rho\alpha_1 = \alpha_{i+1} \cdots \alpha_n\rho\alpha_1 \cdots \alpha_{f(i)} = 0,\ i = 1, \ldots, n-1,$$

where $f : \{1, \ldots, n-1\} \to \{1, \ldots, n-1, n\}$ is a non-decreasing function that is not constant with value 1. Furthermore, $\rho^3 \notin I_A$ since ρ^2 is a non-deep path [6, (2.5), (2.7), (6.4), (9.2)]. Using [6, (7.7)] and its dual, we have the following fact:

(1) *If p is a path of Q_A that does not lie completely in Q_C but contains ρ as a subpath, then $p \in I_A$.*

Next, write $\bar{x} = x + I_A \in A$ for $x \in kQ_A$. For simplicity, we identify the vertex set of Q_A with a complete set of orthogonal primitive idempotents of A. Since A is finite dimensional, there exists an integer $m \geq 4$ such that $\bar{\rho}^m = 0$ and $\bar{\rho}^{m-1} \neq 0$. Now

$$\bar{\alpha}_n \bar{\rho}^2 = \bar{\alpha}_n \bar{\alpha}_1 \cdots \bar{\alpha}_n = \bar{\alpha}_n \bar{\rho} \bar{\alpha}_1 \cdots \bar{\alpha}_n = \bar{\alpha}_n \bar{\rho} \bar{\rho}^2 = \bar{\alpha}_n \bar{\rho}^3 = \cdots = \bar{\alpha}_n \bar{\rho}^m = 0\,.$$

Thus, $\alpha_n \rho^2 \in I_A$ and analogously $\rho^2 \alpha_1 \in I_A$. We now show the following:

(2) *If p is a non-trivial path of Q_A different from ρ, then $\rho^{m-2}p$, $p\rho^{m-2} \in I_A$. As a consequence, $\bar{\rho}^{m-1}$ lies in the socle of A.*

In fact, write $p = p_1 \beta$ with β an arrow. If $\beta \neq \rho$, using (1) together with $\alpha_n \rho^2 \in I_A$ and $m \geq 4$, we infer that $\beta \rho^{m-2} \in I_A$. If $\beta = \rho$, then p_1 is non-trivial and we can write $p_1 = p_2 \gamma$, with γ an arrow. Then $\gamma \beta \rho^{m-2} = \gamma \rho^{m-1} \in I_A$, and therefore $p \rho^{m-2} \in I_A$. Similarly, $\rho^{m-2} p \in I_A$, and we have established statement (2).

Now let ∂ be the unique normalized derivation of kQ_A with $\partial(\rho) = \rho^{m-1}$ and $\partial(\alpha) = 0$ for any arrow $\alpha \neq \rho$. Our next claim is:

(3) *If p is a path of Q_A that is different from ρ, then $\partial(p) \in I_A$.*

This is trivially true if p is of length at most one. Assume thus that p is non-trivial and that (3) holds for paths shorter than p. If ρ does not appear in p, then $\partial(p) = 0$. Otherwise $p = u\rho v$, where u, v are paths shorter than p with u or v non-trivial. Now $\partial(p) = \partial(u)\rho v + u\rho^{m-1}v + u\rho\partial(v)$. It follows from (2) that $u\rho^{m-1}v \in I_A$. Moreover, if $u \neq \rho$, then $\partial(u) \in I_A$ by the inductive hypothesis. Otherwise $\partial(u)\rho v = \rho^m v \in I_A$. Similarly, $u\rho\partial(v) \in I_A$. Therefore, $\partial(p) \in I_A$, which proves statement (3).

Since I_A is generated by linear combinations of paths of length at least two, $\partial(I_A) \subseteq I_A$ by (3). Hence ∂ induces a normalized derivation δ of A with $\delta(\bar{y}) = \overline{\partial(y)}$ for all $y \in kQ_A$. We now wish to show that δ is an outer derivation of A. Assume on the contrary that $\delta = [x,\ -]$ for some $x \in A$. Write $x = x_1 + x_2$, where $x_1 \in Aa + aA$ and $x_2 \in a'Aa'$, where $a' = 1 - a$. Then $\delta(\bar{\rho}) = [x_1, \bar{\rho}] = \bar{\rho}^{m-1} \neq 0$. Thus $x_1 \notin aAa$ since $\bar{\rho}$ lies in the center of aAa. Therefore $x_1 = z_1 + z_2 + z$, where $z_1 \in a'Aa,\ z_2 \in aAa',\ z \in aAa$ and $z_1 + z_2 \neq 0$. However, this would imply that $\delta(a) = z_1 - z_2 \neq 0$, contrary to δ being normalized. Hence $[\delta]$ is a non-zero element of $\mathrm{HH}^1(A)$.

Let M be a minimal representation generator of A such that $\Lambda = \mathrm{End}_A(M)$. By Proposition 3.8, there exists an exact sequence

$$0 \to \mathrm{HH}^1(\Lambda) \to \mathrm{HH}^1(A) \xrightarrow{\chi_M} \mathrm{Ext}^1_A(M, M)\,.$$

We want to show that $[\delta] \in \mathrm{Ker}(\chi_M)$. Let $P_1 \xrightarrow{\phi} P_0 \xrightarrow{\varepsilon} M \to 0$ be a minimal projective presentation of M. Let $b_1, \ldots, b_n$ and $c_1, \ldots, c_m$ be vertices of Q_A such that $P_1 = \oplus_{j=1}^n b_j A$ and $P_0 = \oplus_{i=1}^m c_i A$. The matrix of ϕ is then $(x_{ij})_{m\times n}$ with $x_{ij} \in c_i \,\mathrm{rad}(A)\, b_j$. Since δ is normalized, $\delta(c_i A b_j) \subseteq c_i A b_j$. Hence the matrix of $\delta(\phi) : P_1 \to P_0$ is $(\delta(x_{ij}))_{m\times n}$. For each fixed $1 \leq i \leq m$, we consider the i-th row $(x_{i1}, x_{i2}, \ldots, x_{in})$ of the matrix of ϕ. If $c_i \neq a$, then $\delta(x_{ij}) = 0$ for all $1 \leq j \leq n$ by (3). Thus,

$$(\delta(x_{i1}), \delta(x_{i2}), \ldots, \delta(x_{in})) = 0 \cdot (x_{i1}, x_{i2}, \ldots, x_{in})\,.$$

Assume that $c_i = a$. If $b_j \neq a$, then $\delta(x_{ij}) = 0$ by (3) and $\bar{\rho}^{\,m-2} x_{ij} = 0$ by (2), since $x_{ij} \in c_i \operatorname{rad}(A)\, b_j$. If $b_j = a$, then $x_{ij} = \sum_{s=1}^{m-1} \lambda_s \bar{\rho}^{\,s}$ with $\lambda_s \in k$ since $aAa = k[\rho]/(\rho^m)$. Now $\delta(\bar{\rho}) = \bar{\rho}^{\,m-2} \cdot \bar{\rho}$, whereas for $s \geq 2$, both $\delta(\bar{\rho}^{\,s}) = 0$ and $\bar{\rho}^{\,m-2} \cdot \bar{\rho}^{\,s} = 0$. This shows that

$$(\delta(x_{i1}), \delta(x_{i2}), \ldots, \delta(x_{in})) = \bar{\rho}^{\,m-2} \cdot (x_{i1}, x_{i2}, \ldots, x_{in})\,.$$

For $1 \leq i, j \leq m$, let $y_{ij} = 0$ if $i \neq j$; $y_{ii} = 0$ if $c_i \neq a$; and $y_{ii} = \bar{\rho}^{\,m-2}$ if $c_i = a$. Then $(\delta(x_{ij})) = (y_{ij})(x_{ij})$. Let $\zeta : P_0 \to P_0$ be the A-linear map given by the matrix (y_{ij}). Then $\delta(\phi) = \zeta\phi$, and hence $\varepsilon\delta(\phi) = (\varepsilon\zeta)\phi$. This shows that the class $[\varepsilon\delta(\phi)] \in \operatorname{Ext}^1_A(M, M)$ is zero. Now Lemma 3.3 implies that the class $[\delta] \in \operatorname{HH}^1(A)$ lies in the kernel of χ_M. Therefore, $\operatorname{HH}^1(\mathit{\Lambda}) \neq 0$. This completes the proof of the lemma.

If A is of finite representation type, we say that A is *simply connected* if $\mathit{\Gamma}_A$ is a simply connected translation quiver. In general, one says that A is *strongly simply connected* if A is connected and triangular, and its basic algebra B admits a presentation $B \cong kQ_A/I_A$ such that the first Hochschild cohomology group vanishes for every algebra defined by a convex bound subquiver of (Q_A, I_A). It follows from [24, (4.1)] that this definition coincides with the original one given in [24, (2.2)]. Moreover, if A is of finite representation type, then A is simply connected if and only if it is strongly simply connected. We are now ready to establish the main result of this section.

4.3. Theorem. *Let A be a connected finite dimensional algebra over an algebraically closed field k. Assume that A is of finite representation type and let $\mathit{\Lambda}$ be its Auslander algebra. The following statements are equivalent*:

(1) $\operatorname{HH}^1(A) = 0$.

(2) $\operatorname{HH}^1(\mathit{\Lambda}) = 0$.

(3) *A is simply connected.*

(4) *$\mathit{\Lambda}$ is strongly simply connected.*

Proof. First of all, we claim that each of the conditions stated in the theorem implies that A is standard. Indeed, for (1) or (2), this follows from Lemma 4.2. Now each of (3) or (4) implies that $\mathit{\Gamma}_A$ contains no oriented cycle. This in turn implies that the ordinary quiver of A contains no oriented cycle, as A is of finite representation type. Therefore, A is standard by [6, (9.6)].

Thus we may assume that A is standard, that is, $\mathit{\Lambda} \cong k(\mathit{\Gamma}_A)$. Now the equivalence of (2), (3), and (4) follows immediately from Theorem 2.7. Moreover, (1) implies (2) since $\operatorname{HH}^1(\mathit{\Lambda})$ embeds into $\operatorname{HH}^1(A)$. Finally, if (2) holds, then (3) holds as well. In particular, $\operatorname{ind} A$ contains only bricks. By Proposition 4.1, $\operatorname{HH}^1(A) = \operatorname{HH}^1(\mathit{\Lambda}) = 0$. This completes the proof of the theorem.

We conclude this paper with a consequence of Theorem 4.3. Recall that A is an *Auslander algebra* if its global dimension is less than or equal to two while its dominant dimension is greater than or equal to two. Note that A is an Auslander algebra if and only if it is Morita equivalent to the Auslander algebra of an algebra of finite representation type; see, for example, [4].

4.4. Corollary. *Let A be of finite representation type or an Auslander algebra. If* $\mathrm{HH}^1(A) = 0$*, then* $\mathrm{HH}^i(A) = 0$ *for all* $i \geq 1$*. In particular, A is rigid in this case.*

Proof. For the first part, we apply Theorem 4.3 and the two results stated, respectively, in [17, (5.4)] and [18, Section 5]. For the last part, observe that $\mathrm{HH}^2(A) = 0$ implies rigidity, as originally proved by Gerstenhaber [16].

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