REPRESENTATION THEORY OF HEREDITARY ARTIN ALGEBRAS OF FINITE REPRESENTATION TYPE

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ABSTRACT. Let H be a hereditary artin algebra of finite representation type. We first determine all hammocks in the Auslander-Reiten quiver $\Gamma_{\mathrm{mod}H}$ of $\mathrm{mod}H$, the category of finitely generated left H-modules. This enables us to obtain an effective method to construct $\Gamma_{\mathrm{mod}H}$ by simply viewing the ext-quiver of H. As easy applications, we compute the numbers of non-isomorphic indecomposable objects in $\mathrm{mod}H$ and the associated cluster category $\mathscr{C}_H$, as well as the nilpotencies of the radicals of $\mathrm{mod}H$, $D^b(\mathrm{mod}H)$ and $\mathscr{C}_H$.

INTRODUCTION

Since Gabriel's pivotal work on representations of quivers; see [17], the representation theory of hereditary artin algebras has been extensively studied over the past fifty years; see, for example, [13, 14, 29, 32, 35]. The representation type of such an algebra is finite precisely when its ext-quiver is of Dynkin type; see [2, 14, 17]. In this case, the information on the module category is encoded in its Auslander-Reiten quiver. In the linearly oriented $\mathbb{A}_n$-case, the Auslander-Reiten quiver has a wing shape; see [28, (2.6)] and [32, (3.3)]. In all other cases, the Auslander-Reiten quiver has only been roughly described; see [2, (VIII.1.15)] and [28, (1.13)].

The main objective of this paper is to provide an effective method to construct the Auslander-Reiten quiver simply by viewing the ext-quiver of the algebra. This enables us to compute the number of non-isomorphic indecomposable objects in the module category and the associated cluster category, and also the nilpotency of the radicals of the module category, its bounded derived category and the associated cluster category. More details are outlined below.

In this introduction, H denotes a hereditary artin algebra with a Dynkin extension quiver Q_H; see (1.6.1). Let $\mathrm{mod}H$ be the category of finitely generated left H-modules and $\Gamma_{\mathrm{mod}H}$ be its Auslander-Reiten quiver; see (1.5). It is well-known that every τ-orbit in $\Gamma_{\mathrm{mod}H}$ is finite and contains a projective module and an injective module; see [29, (1.8)], and also [14, (1.9)]. Moreover, $\Gamma_{\mathrm{mod}H}$ embeds as a convex valued translation quiver in the repetitive quiver $\mathbb{Z}\Delta_H$, where Δ_H is the full valued subquiver of $\Gamma_{\mathrm{mod}H}$ generated by the projective modules; see [28, (1.13)], and also [2, (VIII.1.15)]. Thus, in order to describe the precise shape of $\Gamma_{\mathrm{mod}H}$, it suffices to determine the injective module and the number of modules in the τ-orbit of any projective module.

For giving a combinatorial criterion for a finite translation quiver to be the Auslander-Reiten quiver of a finite dimensional algebra over an algebraically closed field, Brenner introduced the notions of hammocks and hammock functions; see [10]. We adapt and extend these notions to the artin setting; see (1.7.2) and (4.2.1). After a thorough study of the valued translation quivers of tree type in Section 2, we describe all the hammocks in $\Gamma_{\mathrm{mod}H}$; see (4.2.4). This allows us to determine which injective module lies in the τ-orbit of any given projective module; see, for example, (4.2.5), (4.2.6) and (4.2.7).

To study representations of a valued graph, Dlab and Ringel defined a Coxeter transformations as a product of all reflections in the corresponding rational vector space; see [13, Page 8]. Alternatively, for any hereditary artin algebra, Auslander and Platzeck defined the Coxeter transformation as an special automorphism of the Grothendieck group of the module category; see (3.3.1) and [29, Section 2]. Since Q_H is of Dynkin type, the Coxeter transformation C_H for H is of finite order $|C_H|$; see [29, (4.1)], called the *Coxeter order*. The Coxeter order has been computed for each Dynkin diagram with a particular

orientation; see [2, Pages 289, 290]. We show that $|C_H|$ is independent of the orientation of Q_H, so it is explicitly given no matter how Q_H is oriented; see (4.4.2). Further, we shall pair the τ-orbits of projective modules in $\Gamma_{\mathrm{mod}H}$ in such a way that the sum of the numbers of modules in the paired τ-orbits is equal to $|C_H|$; see (4.4.1), while the difference equals the difference of the numbers of arrows in two reduced walks in Q_H between the corresponding vertices; see (4.3.2). So, the number of modules in any τ-orbit is expressed in terms of $|C_H|$ and the number of arrows in a specific reduced walk in Q_H. This ultimately completes the picture of $\Gamma_{\mathrm{mod}H}$ in terms of Q_H; see (4.5.1).

Since H is of finite representation type, it is interesting to compute the number of non-isomorphic indecomposable H-modules. Since the center $\Bbbk$ of H is a field; see [29, (3.1)], $\mathrm{mod}\,H$ is equivalent to the representation category of a $\Bbbk$-linear species of Q_H; see [13, Theorem C]. Hence, this number coincides with the number of positive roots of Q_H; see [14, 17], which can be found in [12]. However, we directly obtain this number from our results, that is, half of $|C_H|$ multiplied by the number of non-isomorphic simple H-modules; see (4.6.1).

A well-known result of Auslander says that an artin algebra Λ is representation-finite if and only if the radical of $\mathrm{mod}\,\Lambda$ is nilpotent; see [2, (V.7.7)] and [34, (1.1)]. In this case, it is interesting to compute the nilpotency of $\mathrm{rad}(\mathrm{mod}\,\Lambda)$, namely the least integer r for which $\mathrm{rad}^r(\mathrm{mod}\,\Lambda) = 0$. Using preprojective partitions, Zacharia proved that $|C_H| - 2$ is the maximal length of chains of irreducible maps in $\mathrm{mod}\,\Lambda$ with a non-zero composite; see [35, (4.11)], and consequently, $\mathrm{rad}(\mathrm{mod}\,H)$ is of nilpotency $|C_H| - 1$. Alternatively, using some nice properties of $\Gamma_{\mathrm{mod}H}$; see (3.2.1), we show that the radicals of $\mathrm{mod}\,H$ and $D^b(\mathrm{mod}\,H)$ have the same nilpotency $|C_H| - 1$; see (4.6.4) and (4.6.5).

In order to categorify Fomin and Zelevinsky's cluster algebra associated with a finite acyclic unvalued quiver [16], Buan, Marsh, Reineke, Reiten and Todorov introduced the associated cluster category, which is an orbit category of the bounded derived category of finite dimensional representations of the quiver; see [7]. The same construction yields the cluster category $\mathscr{C}_H$ associated with H. We show that the radical of $\mathscr{C}_H$ is also of nilpotency $|C_H| - 1$, and the number of non-isomorphic indecomposable objects in $\mathscr{C}_H$ is half of $|C_H| + 2$ multiplied by the number of non-isomorphic simple H-modules; see (4.6.6), which coincides with the number of cluster variables of the cluster algebra associated with Q_H; see [15, (5.9.1)]. Finally, we want to express our gratitude to Platzeck and Zacharia for some helpful discussions.

1. PRELIMINARIES

The objective of this section is to lay the foundation for this paper. Besides fixing some terminology and notation, we shall collect and also prove some results in the general context, which are needed later. Throughout this paper, R denotes a commutative artinian ring. All algebras are R-algebras, and all categories are additive R-categories whose morphisms are composed from right to left.

1.1. QUIVERS. We start with laying the combinatorial foundation. Let $Q = (Q_0, Q_1)$ be a quiver, where Q_0 is the set of vertices and Q_1 is the set of arrows from a vertex, called the *start point*, to another vertex, called the *end point*. An arrow α from x to y is usually represented graphically as $x \to y$, and we write $x = s(\alpha)$ and $y = e(\alpha)$. Given $x \in Q_0$, write x^+ for the set of vertices y such that there exists an arrow $x \to y$, and x^- for the set of vertices z such that there exists an arrow $z \to x$. We call x a *source* or *sink* if x^- or x^+ is empty, respectively. With each vertex $x \in Q_0$, one associates a *trivial path* ε_x of length 0 with $s(\varepsilon_x) = e(\varepsilon_x) = x$. A *path* of length $r > 0$ is a sequence $\eta = \alpha_r \cdots \alpha_1$, where $\alpha_i \in Q_1$ such that $e(\alpha_i) = s(\alpha_{i+1})$ for $1 \le i < r$; and in this case, write $s(\eta) = s(\alpha_1)$ and $e(\eta) = e(\alpha_r)$. In the sequel, the length of a path η will be written as $l(\eta)$. A non-trivial path η is called an *oriented cycle* if $s(\eta) = e(\eta)$, and an oriented cycle of length 2 is called a *2-cycle*. Two paths η, ζ are called *parallel* if $s(\eta) = s(\zeta)$ and $e(\eta) = e(\zeta)$. A subquiver Q' of Q is called *full* if it contains all arrows $x \to y$ in Q with $x, y \in Q'$, and *convex* if it contains all paths $x \rightsquigarrow y$ in Q with $x, y \in Q'$.

For each arrow $\alpha : x \to y$ in Q, one introduces its *formal inverse* α^{-1} with $s(\alpha^{-1}) = e(\alpha)$ and $e(\alpha^{-1}) = s(\alpha)$. The formal inverses of arrows will be called *inverse arrows*. A *trivial walk* of length 0 is a trivial path. A *walk* of length $t(> 0)$ is a sequence $w = c_t c_{t-1} \cdots c_1$, where c_i is an arrow or an inverse arrow such that $e(c_i) = s(c_{i+1})$ for $1 \le i < t$. In this case, we write $s(w) = s(c_1)$ and $e(w) = e(c_t)$, and call w a *walk from $s(w)$ to $e(w)$*. Note that $w^{-1} = c_1^{-1} \cdots c_{t-1}^{-1} c_t^{-1}$ is a walk from y to x, and the

number of arrows in w^{-1} is the number of inverse arrows in w, while the number of inverse arrows in w^{-1} is the number of arrows in w. A walk in w in Q is called *reduced* if w is trivial or $w = c_t c_{t-1} \ldots c_1$ such that $c_{i+1} \neq c_i^{-1}$ for all $1 \leq i < t$. One says that Q is a *tree* if, for any $x, y \in Q_0$, there exists at most one reduced walk from x to y. The following notation is important for later purposes.

Notation 1.1.1. Let Q be a connected tree with $x, y \in Q_0$. In case $x = y$, put $\mathfrak{a}^+(x, y) = \mathfrak{a}^-(x, y) = 0$. Otherwise, there exists a unique reduced $w = c_t \cdots c_1$ from x to y, where each c_i is an arrow or an inverse arrow in Q. Then we denote by $\mathfrak{a}^+(x, y)$ the number of indices i such that c_i is an arrow; and by $\mathfrak{a}^-(x, y)$ the number of indices j such that c_j is an inverse arrow.

We define the *opposite quiver* Q^{op} of Q by $(Q^{\mathrm{op}})_0 = Q_0$ and $(Q^{\mathrm{op}})_1 = \{\alpha^{\mathrm{o}} : y \to x \mid \alpha : x \to y \in Q_1\}$. The trivial paths in Q are identified with the trivial paths in Q^{op}. And a non-trivial walk w in Q from x to y induces a non-trivial walk w^{o} in Q^{op} from x to y so that the number of arrows in w^{o} is the number of inverse arrows in w, and the number of inverse arrows in w^{o} is the number of arrows in w.

1.2. VALUED QUIVERS. Let $\mathbb{Z}^+$ denote the set of positive integers. A *valued quiver* is a pair (Q, v), where $Q = (Q_0, Q_1)$ is a quiver without multiple arrows and v is the *valuation*, that is a map

$$v : Q_1 \to \mathbb{Z}^+ \times \mathbb{Z}^+ : \alpha \mapsto v(\alpha) := (v_\alpha, v'_\alpha);$$

see, for example, [2, Page 69], and compare [19, Page 287]. The valuation $v(\alpha)$ of an arrow α is called *trivial* if $v(\alpha) = (1, 1)$. In a graphic representation of a valued quiver, we will omit the trivial valuations. In case Q has no loop, we will write $v(\alpha) = (v_{xy}, v'_{xy})$ for an arrow $\alpha : x \to y$. And in case Q has no 2-cycle, we will write $v(\alpha) = (v_{xy}, v_{yx})$ for an arrow $\alpha : x \to y$. A *full valued subquiver* of a valued quiver (Q, v) is a full subquiver of Q with the valuation obtained by restricting v. In order to define the opposite valued quiver, we write $(a, b)^\circ = (b, a)$ for any pair $(a, b) \in \mathbb{Z}^+ \times \mathbb{Z}^+$.

Definition 1.2.1. Let (Q, v) be a valued quiver. Its *opposite valued quiver* $(Q^{\mathrm{op}}, v^{\mathrm{o}})$ is defined by

(1) Q^{op} is the opposite quiver of Q;
(2) v^{o} is defined by $v^{\mathrm{o}}(\alpha^\circ) = v(\alpha)^\circ$, for any arrow α in Q.

1.3. VALUED GRAPHS. We denote by $\mathbb{N}$ the set of non-negative integers. The following definition is slightly modified from the one given in [2, Page 241]; compare [14, Page 1].

Definition 1.3.1. A *valued graph* is a triplet (Δ_0, Δ_1, v), where

(1) Δ_0 is a set of vertices;
(2) Δ_1 is a set of edges between vertices, containing no loop or multiple edge;
(3) v is the *valuation*, that is a function

$$v : \Delta_0 \times \Delta_0 \to \mathbb{N} : (x, y) \mapsto v(x, y) := v_{xy}$$

such that $v_{xy} \neq 0$ if and only if $v_{yx} \neq 0$ if and only if there exists an edge between x and y.

In the sequel, a valued graph (Δ_0, Δ_1, v) will be written as (Δ, v), where $\Delta = (\Delta_0, \Delta_1)$; or simply as Δ. If e is an edge between x and y such that $v_{xy} = v_{yx} = 1$, then we say that e is *trivially valued* with *trivial valuation* $(1, 1)$. One says that Δ is *simply laced* if every edge in Δ is trivially valued. In this paper, we shall only consider valued graphs with a planar representation defined as follows.

Notation 1.3.2. Let (Δ, v) be a planar valued graph such that every vertex is incident to at most one non-trivially valued edge. Consider an edge e between vertices x and y. If e is trivially valued, then it is represented by a blank line between x and y. Otherwise, e is represented by a horizontal line $i \text{---} j$ labeled with the pair (v_{xy}, v_{yx}) or $y \text{---} x$ labeled with the pair (v_{yx}, v_{xy}). In either case, the pair is called the *valuation* of e in the respective representation.

Remark 1.3.3. In contrast to the valuation of an arrow in a valued quiver, the valuation of a non-trivially valued edge in a valued graph depends on how the edge is written, namely, which vertex is on the left and which is on the right.

Now, we introduce valued graph isomorphisms, which we cannot find in any existing literature.

Definition 1.3.4. Let (Δ, v) and (Δ', v') be valued graphs. We define a *valued graph isomorphism* $\varphi : (\Delta, v) \to (\Delta', v')$ to be a bijection $\varphi : \Delta_0 \to \Delta'_0$ such that $v'_{\varphi(x)\varphi(y)} = v_{xy}$, for all $x, y \in \Delta_0$.

Remark 1.3.5. Let $\varphi : (\Delta, v) \to (\Delta', v')$ be a valued graph isomorphism. Then, for $x, y \in \Delta_0$, there exists an edge e between x and y in Δ if and only if there exists an edge e' between $\varphi(x)$ and $\varphi(y)$ in Δ'. And if e is written as $x \text{---} y$ with valuation (a, a'), then e' can be written as $\varphi(x) \text{---} \varphi(y)$ with valuation (a, a') or as $\varphi(y) \text{---} \varphi(x)$ with valuation (a', a).

Definition 1.3.6. Let (Δ, v) be a valued graph. The *weight* of $x \in \Delta_0$ is defined by $\mathrm{w}(x) := \sum_{y \in \Delta_0} v_{xy}$.

The following statement follows immediately from the definition a valued graph isomorphism.

Lemma 1.3.7. *Let* $\varphi : (\Delta, v) \to (\Delta', v')$ *be a valued graph isomorphism. Given any vertex x in Δ, we have* $\mathrm{w}(x) = \mathrm{w}(\varphi(x))$.

For later reference, we introduce the following definition; see [2, Page 242].

Definition 1.3.8. A valued graph is called a *Dynkin diagram* if it is isomorphic to one of the following *canonical Dynkin diagrams*:

$$\mathbb{A}_n : \quad 1 \text{---} 2 \text{---} \cdots \text{---} n, \ \text{ where } n \geq 1.$$

$$\mathbb{B}_n : \quad 1 \xrightarrow{(1,2)} 2 \text{---} 3 \text{---} \cdots \text{---} n, \ \text{ where } n \geq 2.$$

$$\mathbb{C}_n : \quad 1 \xrightarrow{(2,1)} 2 \text{---} 3 \text{---} \cdots \text{---} n, \ \text{ where } n \geq 3.$$

$$\mathbb{D}_n : \quad \begin{array}{c} 2 \\ | \\ 1 \text{---} 3 \text{---} 4 \text{---} \cdots \text{---} n, \end{array} \ \text{ where } n \geq 4.$$

$$\mathbb{E}_n : \quad \begin{array}{c} 4 \\ | \\ 1 \text{---} 2 \text{---} 3 \text{---} 5 \text{---} 6 \text{---} \cdots \text{---} n, \end{array} \ \text{ where } n = 6, 7, 8.$$

$$\mathbb{F}_4 : \quad 1 \text{---} 2 \xrightarrow{(1,2)} 3 \text{---} 4.$$

$$\mathbb{G}_2 : \quad 1 \xrightarrow{(1,3)} 2.$$

It is important for us to relate the valued quivers without 2-cycles to the valued graphs.

Definition 1.3.9. Let (Q, v) be a valued quiver without 2-cycles. The *underlying valued graph* $(\overline{Q}, \overline{v})$ of (Q, v) is a valued graph defined in the following way.
(1) The graph $\overline{Q}$ is obtained by forgetting the orientation of the arrows in Q.
(2) Given vertices x, y in $\overline{Q}$, we define the valuation $\overline{v}_{xy}$ by

$$\overline{v}_{xy} = \begin{cases} 0, & \text{if } Q \text{ has no arrow } x \to y \text{ or } y \to x; \\ v_{xy}, & \text{if } Q \text{ contains an arrow from } x \text{ to } y \text{ with valuation } (v_{xy}, v_{yx}); \\ v_{xy}, & \text{if } Q \text{ contains an arrow from } y \text{ to } x \text{ with valuation } (v_{yx}, v_{xy}). \end{cases}$$

Remark 1.3.10. Let Q be a valued quiver without 2-cycles. An arrow $x \to y$ with valuation (a, b) in Q induces an edge $x \text{---} y$ with valuation (a, b), or equivalently, $y \text{---} x$ with valuation (b, a) in $\overline{Q}$.

Example 1.3.11. If Q is $1 \xrightarrow{(1,3)} 2 \leftarrow 3$, then $\overline{Q}$ is $1 \xrightarrow{(1,3)} 2 \text{---} 3$, or equivalently, $3 \text{---} 2 \xrightarrow{(3,1)} 1$.

The following statement is evident.

Lemma 1.3.12. *Let (Q, v) be a valued quiver without 2-cycles. Then (Q, v) and $(Q^{\mathrm{op}}, v^{\circ})$ have the same underlying valued graph.*

In the sequel, a valued quiver will be called a (*canonical*) *Dynkin quiver* if it admits no 2-cycle and its underlying valued graph is a (canonical) Dynkin diagram.

1.4. VALUED TRANSLATION QUIVERS. A *valued translation quiver* is a triplet (Γ, v, τ), where (Γ, v) is a valued quiver and τ is a bijection, called the *translation*, from a subset Γ_0' of Γ_0 to another Γ_0'' such, for any $x \in \Gamma_0'$ and any arrow $y \to x$ with valuation (v_{yx}, v_{yx}'), that there exists an arrow $\tau x \to y$ with valuation (v_{yx}', v_{yx}). In this case, we shall write τ^- for the inverse of τ. The τ-*orbit* of a vertex x is the set of vertices $\tau^s x$ with $s \in \mathbb{Z}$. A path $x \rightsquigarrow y$ in Γ is called *sectional* if it does not contain any subpath of the form $\tau a \to b \to a$; and in this case, we call x a *sectional predecessor* of y, and y a *sectional successor* of x. Note that a trivial path in Γ is sectional. A sectional path is called *strictly sectional* if it meets any τ-orbit in Γ at most once. The following easy statement follows immediately from the definition of a valued translation quiver, which will be used frequently.

Lemma 1.4.1. *Let (Γ, v, τ) be a valued translation quiver. Let $x \to y$ be an arrow in Γ with valuation (a, a'). If $\tau^s x$ and $\tau^s y$ are defined, then Γ contains an arrow $\tau^s x \to \tau^s y$ with valuation (a, a').*

1.5. KRULL-SCHMIDT CATEGORIES. Let $\mathcal{A}$ be a Hom-finite Krull-Schmidt R-category. We write $\mathrm{rad}\mathcal{A}$ for the Jaconson radical of $\mathcal{A}$, and $\mathrm{rad}^s\mathcal{A}$ for the s-th power of $\mathrm{rad}\mathcal{A}$ for all $s \geq 0$. A subcategory $\mathcal{B}$ of $\mathcal{A}$ is called *convex* provided that any sequence of morphisms $X_0 \to X_1 \to \cdots \to X_{r-1} \to X_r$ between indecomposable objects in $\mathcal{A}$ lies entirely in $\mathcal{B}$ whenever $X_0, X_r \in \mathcal{B}$.

For our later purpose, we briefly recall the Auslander-Reiten theory in this general setting; see [3] and [26]. Given indecomposable objects X, Y in $\mathcal{A}$, we put $k_X = \mathrm{End}_{\mathcal{A}}(X)/\mathrm{rad}(\mathrm{End}_{\mathcal{A}}(X))$, that is a division R-algebra; and $\mathrm{irr}(X, Y) = \mathrm{rad}_{\mathcal{A}}(X, Y)/\mathrm{rad}_{\mathcal{A}}^2(X, Y)$, that is a k_Y-k_X-bimodule. It is well-known that a morphism $f : X \to Y$ is irreducible if and only if $f \in \mathrm{rad}_{\mathcal{A}}(X, Y)\backslash\mathrm{rad}_{\mathcal{A}}^2(X, Y)$. Write $d_{XY} = \dim_{k_Y}\mathrm{irr}(X, Y)$ and $d_{XY}' = \dim \mathrm{irr}(X, Y)_{k_X}$, which are related by Bautista to minimal left almost morphism and minimal right almost split morphisms as follows; see [3, (3.3), (3.4)].

Proposition 1.5.1. *Let $\mathcal{A}$ be a Hom-finite Krull-Schmidt R-category, and let $f : X \to Y$ be an irreducible morphism between indecomposable objects in $\mathcal{A}$.*

(1) *If $\mathcal{A}$ has a minimal left almost split morphism $g : X \to M$, then d_{XY} is the multiplicity of Y as a direct summand of M.*

(2) *If $\mathcal{A}$ has a minimal right almost split morphism $h : N \to Y$, then d_{XY}' is the multiplicity of X as a direct summand of N.*

The following definition unifies the notions of almost split sequences in abelian categories and almost split triangles in triangulated categories; see [26, (1.3)], and compare [21, (2.7)].

Definition 1.5.2. Let $\mathcal{A}$ be a Hom-finite Krull-Schmidt R-category. A sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ of morphisms in $\mathcal{A}$ with $Y \neq 0$ is called *almost split* if f is minimal left almost split and a pseudo kernel of g, while g is minimal right almost split and a pseudo cokernel of f.

The *Auslander-Reiten quiver* of $\mathcal{A}$ is a valued translation quiver $\Gamma_{\mathcal{A}}$ defined as follows; [26, (2.1)]. The vertex set of $\Gamma_{\mathcal{A}}$ is a complete set of representatives of isomorphism classes of indecomposable objects in $\mathcal{A}$. Given vertices X, Y in $\Gamma_{\mathcal{A}}$, there exists an arrow $X \to Y$ in $\Gamma_{\mathcal{A}}$ if and only if there exists an irreducible morphism $f : X \to Y$ in $\mathcal{A}$; and in this case, the valuation of $X \to Y$ is $(d_{XY}, d_{X,Y}')$. The translation $\tau_{\mathcal{A}}$, called the *Auslander-Reiten translation*, is defined in such a way that $X = \tau_{\mathcal{A}} Z$ if and only if $\mathcal{A}$ has an almost split sequence $X \longrightarrow Y \longrightarrow Z$.

To conclude this subsection, we recall the construction of an orbit category of $\mathcal{A}$; compare [1, (2.1)].

Definition 1.5.3. Let $\mathcal{A}$ be a Hom-finite Krull-Schmidt R-category with an action by a group G. The *orbit category* $\mathcal{A}/G$ is defined as follows.

(1) The objects of $\mathcal{A}/G$ are those of $\mathcal{A}$.

(2) Given objects X, Y, one has $\mathrm{Hom}_{\mathcal{A}/G}(X, Y) = \oplus_{g \in G} \mathrm{Hom}_{\mathcal{A}}(X, g \cdot Y)$.

(3) Given morphisms $u = (u_g)_{g \in G} : X \to Y$ and $v = (v_g)_{g \in G} : Y \to Z$ in $\mathcal{A}/G$, where $u_g : X \to g \cdot Y$ and $v_g : Y \to g \cdot Z$ are morphisms in $\mathcal{A}$, one has $vu = (w_g)_{g \in G}$, where $w_g = \sum_{fh=g}(f \cdot v_h)u_f$.

The action of a group G on $\mathcal{A}$ is called *free* provided that $g \cdot X \not\cong X$ for any non-identity $g \in G$ and any indecomposable X in $\mathcal{A}$; and *locally bounded* provided that $\mathrm{Hom}_{\mathcal{A}}(X, g \cdot Y) \neq 0$ for all but finitely many $g \in G$ and all objects X, Y in $\mathcal{A}$; see [4, (1.3)].

Proposition 1.5.4. *Let $\mathcal{A}$ be a Hom-finite Krull-Schmidt R-category with a free and locally bounded action by a group G. Then $\mathcal{A}/G$ is a Hom-finite Krull-Schmidt R-category, whose indecomposable objects are those of $\mathcal{A}$. Moreover, if $X, Y \in \mathcal{A}$ are indecomposable, then*

(1) $\mathrm{rad}^{s}_{\mathcal{A}/G}(X, Y) = \oplus_{g \in G} \mathrm{rad}^{s}_{\mathcal{A}}(X, g \cdot Y)$, *for $s \geq 0$;*

(2) $X \cong Y$ *in $\mathcal{A}/G$ if and only if $X \cong g \cdot Y$ in $\mathcal{A}$ for some $g \in G$.*

Proof. First of all, a direct sum in $\mathcal{A}$ is a direct sum in $\mathcal{A}/G$. And since the action of G is locally bounded, $\mathcal{A}/G$ is a Hom-finite additive R-category. Let $X, Y \in \mathcal{A}$ be indecomposable. Consider a morphism $u = (u_g)_{g \in G} : X \to Y$ in $\mathcal{A}/G$, where $u_g \in \mathrm{Hom}_{\mathcal{A}}(X, g \cdot Y)$. Suppose that $u_f \notin \mathrm{rad}_{\mathcal{A}}(X, f \cdot Y)$ for some $f \in G$. Then, u has an inverse $(u_g^{-1})_{g \in G} : Y \to X$ in $\mathcal{A}/G$, where $u_g^{-1} = 0$ for all $g \, (\neq f^{-1}) \in G$, and $u_{f^{-1}}^{-1} = f^{-1} \cdot u_f^{-1} : Y \to f^{-1} \cdot X$. Suppose now that $u_g \in \mathrm{rad}_{\mathcal{A}}(X, g \cdot Y)$ for all $g \in G$. Given any $v = (v_g)_{g \in G} : Y \to X$ in $\mathcal{A}/G$, we have $vu = ((vu)_g)_{g \in G}$, where $(vu)_g = \sum_{fh=g}(f \cdot v_h)u_f \in \mathrm{rad}_{\mathcal{A}}(X, g \cdot X)$ for all $g \in G$. Let e be the identity of G. Then, $1_X - vu = (w_g)_{g \in G}$, where $w_e = 1_X - (vu)_e$, and $w_g = -(vu)_g$ for all $g \, (\neq e) \in G$. Since w_e is invertible, as seen previously, so is $1_X - vu$. Hence, $u \in \mathrm{rad}_{\mathcal{A}/G}(X, Y)$. This shows that $\mathrm{rad}_{\mathcal{A}/G}(X, Y) = \oplus_{g \in G} \mathrm{rad}_{\mathcal{A}}(X, g \cdot Y)$.

Since G acts freely on $\mathcal{A}$, $\mathrm{rad}_{\mathcal{A}/G}(X, X) = \mathrm{rad}_{\mathcal{A}}(X, X) \oplus (\oplus_{g \, (\neq e) \in G} \mathrm{Hom}_{\mathcal{A}}(X, g \cdot X))$. So, $\mathrm{End}_{\mathcal{A}/G}(X)$ is local. Since a non-zero object in $\mathcal{A}$ is non-zero in $\mathcal{A}/G$, an object is indecomposable in $\mathcal{A}$ if and only if it is indecomposable in $\mathcal{A}/G$, and consequently, $\mathcal{A}/G$ is Krull-Schmidt. Now, by induction, we easily see that Statement (1) for all $s \geq 0$. Finally, $X \cong Y$ in $\mathcal{A}/G$ if and only if $\mathrm{Hom}_{\mathcal{A}/G}(X, Y) \neq \mathrm{rad}_{\mathcal{A}/G}(X, Y)$, if and only if $\mathrm{Hom}_{\mathcal{A}}(X, g \cdot Y) \neq \mathrm{rad}_{\mathcal{A}}(X, g \cdot Y)$ for some $g \in G$, if and only if $X \cong g \cdot Y$ in $\mathcal{A}$ for some $g \in G$. The proof of the proposition is completed.

1.6. MODULE CATEGORY. Let Λ be an artin algebra. We denote by $\mathrm{mod}\Lambda$ the category of finitely generated left Λ-modules, and by $\mathrm{rad}(\mathrm{mod}\Lambda)$ the Jacobson radical of $\mathrm{mod}\Lambda$. The maps in $\mathrm{rad}(\mathrm{mod}\Lambda)$ are called *radical maps*. Given a module M in $\mathrm{mod}\Lambda$, we shall write $\mathrm{rad}M$, $\mathrm{top}M$ and $\mathrm{soc}M$ for the radical, the top and the socle of M respectively. The Auslander-Reiten quiver $\Gamma_{\mathrm{mod}\Lambda}$ of $\mathrm{mod}\Lambda$ carries the essential information of the finite powers of $\mathrm{rad}(\mathrm{mod}\Lambda)$. The Auslander-Reiten translations τ_{Λ} and τ_{Λ}^{-} of $\Gamma_{\mathrm{mod}\Lambda}$ are given by $D\mathrm{Tr}$ and $\mathrm{Tr}D$, respectively; see [2, Page 22]. Besides the Auslander-Reiten quiver, the ext-quiver of Λ; see [2, Page 69] also plays an important role in our study.

Definition 1.6.1. Let Λ be an artin algebra, and let $S_1, \dots, S_n$ be the non-isomorphisc simple modules in $\mathrm{mod}\Lambda$. The *ext-quiver* of Λ is a valued quiver Q_Λ defined in the following way.

(1) The vertex set is $(Q_\Lambda)_0 = \{1, \dots, n\}$.

(2) The arrow set is $(Q_\Lambda)_1 = \{i \to j \mid \mathrm{Ext}^1_\Lambda(S_i, S_j) \neq 0\}$.

(3) The valuation for an arrow $i \to j$ in Q_Λ is the pair (d_{ij}, d'_{ij}), where $d_{ij} = \dim_{\mathrm{End}_\Lambda(S_j)} \mathrm{Ext}^1_\Lambda(S_i, S_j)$ and $d'_{ij} = \dim \mathrm{Ext}^1_\Lambda(S_i, S_j)_{\mathrm{End}_\Lambda(S_i)}$.

In the study of the Auslander-Reiten quiver $\Gamma_{\mathrm{mod}\Lambda}$, sectional paths play an important role; see [6, 23, 24, 20]. The following statement tells us when we can extend a sectional path in $\Gamma_{\mathrm{mod}\Lambda}$.

Proposition 1.6.2. *Let Λ be an artin algebra, and let $X_0 \to X_1 \to \cdots \to X_{s-1} \to X_s$ be a non-trivial sectional path in the Auslande-Reiten quiver $\Gamma_{\mathrm{mod}\Lambda}$.*

(1) *Suppose that X_0 is projective. If $Y \to X_s$ is an arrow in $\Gamma_{\mathrm{mod}\Lambda}$ with $Y \neq X_{s-1}$, then Y is not injective, and consequently, $\Gamma_{\mathrm{mod}\Lambda}$ contains a sectional path $X_0 \to \cdots \to X_{s-1} \to X_s \to \tau_\Lambda^- Y$.*

(2) *Suppose that X_s is injective. If $X_0 \to Y$ is an arrow in $\Gamma_{\mathrm{mod}\Lambda}$ with $Y \neq X_1$, then Y is not projective, and consequently, $\Gamma_{\mathrm{mod}\Lambda}$ contains a sectional path $\tau_\Lambda Y \to X_0 \to X_1 \to \cdots \to X_s$.*

Proof. We shall only prove Statement (1). Let $Y \to X_s$ be an arrow in $\Gamma_{\mathrm{mod}\Lambda}$ with $Y \neq X_{s-1}$. Suppose that Y is injective. Then, we have an irreducible epimorphism $f_s : Y \to X_s$. Since $Y \neq X_{s-1}$, there exists an irreducible map $(f_s, g) : Y \oplus X_{s-1} \to X_s$; see [3, (3.2)]. So, $\mathrm{mod}\Lambda$ has an almost split sequence

$$0 \longrightarrow \tau_{\Lambda} X_s \xrightarrow{(h, u, f_{s-1})} Y \oplus Z \oplus X_{s-1} \xrightarrow{\begin{pmatrix} f_s \\ w \\ g \end{pmatrix}} X_s \longrightarrow 0.$$

Since f_s is an irreducible epimorphism, so is $f_{s-1} : \tau_{\Lambda} X_s \to X_{s-1}$. If $s > 1$, then $\tau_{\Lambda} X_s \neq X_{s-2}$, and we similarly obtain an irreducible epimorphism $f_{s-2} : \tau_{\Lambda} X_{s-1} \to X_{s-2}$. Continuing this process, we obtain an irreducible epimorphism $f_1 : \tau_{\Lambda} X_1 \to X_0$, absurd. The proof of the proposition is completed.

It is well-known that A is representation-finite if and only if rad(modΛ) is nilpotent; see [2, (V.7.7)] and [34, (1.1)]. To compute the nilpotency of rad(modΛ), one introduces the notion of depth for maps in terms of the radical series of modΛ.

Definition 1.6.3. Let Λ be an artin algebra. The *depth* of a map $f : M \to N$ in modΛ is defined by

$$\mathrm{dp}(f) := \sup\{s \in \mathbb{N} \mid f \in \mathrm{rad}^s(M, N)\}.$$

Remark 1.6.4. If $M, N \in \mathrm{mod}\Lambda$ are indecomposable, then a map $f : M \to N$ is irreducible if and only if $\mathrm{dp}(f) = 1$.

The following statement relates maps of finite depth in modΛ to paths in $\Gamma_{\mathrm{mod}\Lambda}$.

Lemma 1.6.5. *Let Λ be an artin algebra. Consider a radical map $f : M \to N$ between indecomposable modules in* modΛ. *If* $\mathrm{dp}(f) = t$, *then there exists a chain of irreducible maps between indecomposable modules* $M \xrightarrow{f_1} M_1 \to \cdots \to M_{t-1} \xrightarrow{f_t} N$ *in* modΛ *such that* $\mathrm{dp}(f_t \cdots f_1) = t$.
Proof. Assume that $\mathrm{dp}(f) = t > 0$. Since $f \in \mathrm{rad}^t(M, N)$, we may write $f = \sum_{i=1}^{p} f_{it} \cdots f_{i1}$, where the f_{ij} are radical maps between indecomposable modules in modΛ. Since $f \notin \mathrm{rad}^{t+1}(M, N)$, there exists some $1 \le s \le p$ such that $f_{st} \cdots f_{s1} \notin \mathrm{rad}^{t+1}(M, N)$. Thus, $f_{s1}, \ldots, f_{st}$ are irreducible such that $\mathrm{dp}(f_{st} \cdots f_{s1}) = t$. The proof of the lemma is completed.

1.7. HAMMOCKS. Let Λ be an artin algebra with S a simple module in modΛ. For any module M in mod$A$, we write $\ell_S(M)$ for the multiplicity of S as a composition factor of M. For convenience of reference, we state the following well-known statement; see, for example, [2, Page 45].

Lemma 1.7.1. *Let Λ be an artin algebra. Consider a simple module in* modΛ *with projective cover P and injective envelope I. If $M \in \mathrm{mod}\Lambda$, then $\ell_S(M)$ is equal to the length of the right $\mathrm{End}_\Lambda(P)$-module* $\mathrm{Hom}_\Lambda(P, M)$, *as well as the length of the left $\mathrm{End}_\Lambda(I)$-module* $\mathrm{Hom}_\Lambda(M, I)$.

Recall that Λ is of *directed representation type* if $\Gamma_{\mathrm{mod}\Lambda}$ is finite and contains no oriented cycle. The following definition is due to Brenner; see [10].

Definition 1.7.2. Let Λ be an artin algebra of directed representation type. Consider a simple module S in the Auslander-Reiten quiver $\Gamma_{\mathrm{mod}\Lambda}$.
(1) The *hammock* associated with S is the full valued translation subquiver $\mathcal{H}_S$ of $\Gamma_{\mathrm{mod}\Lambda}$ generated by the modules M of which S is a composition factor.
(2) The *canonical hammock function* associated with S is the function $h_S : \mathcal{H}_S \to \mathbb{N} : M \mapsto \ell_S(M)$.

The following statement explains in particular the terminology of a hammock.

Proposition 1.7.3. *Let Λ be an artin algebra of directed representation type. Consider a simple module S with projective cover P and injective envelope I, where $S, P, I \in \Gamma_{\mathrm{mod}\Lambda}$.*
(1) *Every module in the hammock $\mathcal{H}_S$ is a successor of P and a predecessor of I.*
(2) *If M is a sectional successor of P or a sectional predecessor of I in $\Gamma_{\mathrm{mod}\Lambda}$, then $M \in \mathcal{H}_S$.*
Proof. It is clear that $P, I \in \mathcal{H}_S$. Consider a module $M \in \mathcal{H}_S$. By Lemma 1.7.1, we have non-zero maps $f : P \to M$ and $g : M \to I$. Since Λ is representation-finite, $\Gamma_{\mathrm{mod}\Lambda}$ contains paths $P \rightsquigarrow M$ and $M \rightsquigarrow I$; see [2, (V.7.8)]. We claim that $\mathcal{H}_S$ contains a path $P \rightsquigarrow M$. Since $\Gamma_{\mathrm{mod}H}$ is finite without oriented cycles, we have a maximal length l_M of paths $P \rightsquigarrow M$ in $\Gamma_{\mathrm{mod}H}$. We may assume that $l_M > 0$. Then, f factors through the minimal right almost split map for M. Thus, $\mathcal{H}_S$ contains an arrow $N \to M$. Since

$l_N < l_M$, by the induction hypothesis, $\mathcal{H}_S$ contains a path $P \rightsquigarrow N$, and hence, a path $P \rightsquigarrow M$. Our claim holds. Dually, $\mathcal{H}_S$ contains a path $M \rightsquigarrow I$. This establishes Statement (1). Next, if M is sectional successor of P or a sectional predecessor of I in $\Gamma_{\mathrm{mod}\Lambda}$, then $\mathrm{Hom}_\Lambda(P, M) \neq 0$ or $\mathrm{Hom}_\Lambda(M, I) \neq 0$; see [20, (13.4)] and also [2, (VII.2.4)]. So, $M \in \mathcal{H}_S$; see (1.7.1). The proof of the proposition is completed.

2. Valued translation quivers with sections

The notion of sections is useful in describing Auslander-Reiten components without oriented cycles for artin algebras; see [24, 25]. The objective of this section is to further investigate valued translation quivers with sections, by studying the repetitive quiver $\mathbb{Z}\Delta$ of a valued quiver Δ without oriented cycles. The sectional paths in $\mathbb{Z}\Delta$ are precisely described in terms of the reduced walks in Δ; see (2.1.4) and (2.1.6). In case Δ is a tree, a sectional path in $\mathbb{Z}\Delta$ is the only path between its end-points, and any two parallel paths have the same length; see (2.1.8). Finally, we shall consider additive functions on stable valued translation quivers with sections. These results will be applied to preprojective and preinjective components of hereditary artin algebras in the next two sections.

2.1. REPETITIVE QUIVER. A valued translation quiver (Γ, v, τ) is called *stable* if τx and $\tau^- x$ are defined for all $x \in \Gamma_0$. In her classification of self-injective algebras of finite representation type, Riedtmann introduced a canonical way to construct a stable translation quiver by "knitting" repetitively a tree; see [31]. This has been generalized to "knit" any valued quivers without oriented cycles.

Definition 2.1.1. Let (Δ, v) be a valued quiver without oriented cycles. The *repetitive quiver* $\mathbb{Z}\Delta$ of Δ is a stable valued translation quiver defined as follows:

(1) The vertex set is $(\mathbb{Z}\Delta)_0 = \{(s, x) \mid s \in \mathbb{Z}; \ x \in \Delta_0\}$.

(2) The arrow set is $(\mathbb{Z}\Delta)_1 = \{(s, x) \xrightarrow{(s,\alpha)} (s, y) \text{ and } (s, y) \xrightarrow{(s,\alpha)^*} (s+1, x) \mid s \in \mathbb{Z}; \ \alpha : x \to y \in \Delta_1\}$.

(3) The translation $\tau : (\mathbb{Z}\Delta)_0 \to (\mathbb{Z}\Delta)_0$ is defined by $\tau(s, x) = (s-1, x)$, for all $(s, x) \in \mathbb{Z} \times \Delta_0$.

(4) The valuation $\tilde{v}$ is such that $\tilde{v}((s, \alpha)) = v(\alpha)$ and $\tilde{v}((s, \alpha)^*) = v(\alpha)^\circ$, for all $\alpha \in \Delta_1$ and $s \in \mathbb{Z}$.

Remark 2.1.2. Since Δ contains no oriented cycles, it is evident that $\mathbb{Z}\Delta$ contains no oriented cycles.

We denote by $\mathcal{P}(\mathbb{Z}\Delta)$ the set of paths in $\mathbb{Z}\Delta$, and by $\mathcal{W}(\Delta)$ the set of walks in Δ.

Proposition 2.1.3. *Let $\mathbb{Z}\Delta$ be the repetitive quiver of a valued quiver Δ without oriented cycles. Setting $\pi(s, \alpha) = \alpha$ and $\pi(s, \alpha)^* = \alpha^{-1}$, we obtain a length-preserving map $\pi : \mathcal{P}(\mathbb{Z}\Delta) \to \mathcal{W}(\Delta)$ such that $\pi(\eta \cdot \xi) = \pi(\eta) \cdot \pi(\xi)$, for all $\eta, \xi \in \mathcal{P}(\mathbb{Z}\Delta)$.*

Proof. Given a trivial path $\varepsilon_{(s,x)}$ with $(s, x) \in (\mathbb{Z}\Delta)_0$, we set $\pi(\varepsilon_{(s,x)}) = x$. Consider a path $\eta = \delta\gamma$, where γ, δ are arrows in $\mathbb{Z}\Delta$. Assume that $\gamma = (s, \alpha)$ for some $s \in \mathbb{Z}$ and $\alpha : x \to y$ in Δ_1. Then $\delta = (s, \alpha)^*$ or $\delta = (s, \beta)$ for some $\beta : y \to z$ in Δ_1. This yields a walk $\pi(\delta) \cdot \pi(\gamma) = \alpha^{-1}\alpha$ in the first case and a path $\pi(\delta) \cdot \pi(\gamma) = \beta\alpha$ in the second case. In any case, we set $\pi(\eta) = \pi(\delta) \cdot \pi(\gamma) \in \mathcal{W}(\Delta)$. Similarly in case $\gamma = (s, \alpha)^*$, we set $\pi(\eta) = \pi(\delta) \cdot \pi(\gamma) \in \mathcal{W}(\Delta)$. By induction, π extends to a map $\pi : \mathcal{P}(\mathbb{Z}\Delta) \to \mathcal{W}(\Delta)$. It is easy to see that $\pi(\eta \cdot \xi) = \pi(\eta) \cdot \pi(\xi)$, for all $\eta, \xi \in \mathcal{P}(\mathbb{Z}\Delta)$. The proof of the proposition is completed.

In the sequel, the map $\pi : \mathcal{P}(\mathbb{Z}\Delta) \to \mathcal{W}(\Delta)$ defined above will be called the *covering map*, which can be used to determine sectional paths in $\mathbb{Z}\Delta$ as shown below.

Lemma 2.1.4. *Let $\mathbb{Z}\Delta$ be the repetitive quiver of a valued quiver Δ without oriented cycles. Then a path η in $\mathbb{Z}\Delta$ is sectional if and only if $\pi(\eta)$ is a reduced walk in Δ, where π is the covering map.*

Proof. Let $\eta \in \mathcal{P}(\mathbb{Z}\Delta)$. If η is not sectional, then it contains a subpath $(s-1, \alpha)^*(s, \alpha)$ or $(s, \alpha)(s, \alpha)^*$, for some $s \in \mathbb{Z}$ and $\alpha \in \Delta_1$. This yields a subwalk $\alpha^{-1}\alpha$ or $\alpha\alpha^{-1}$ of $\pi(\eta)$. So, $\pi(\eta)$ is not reduced. If $\pi(\eta)$ is not reduced, then we similarly show that η is not sectional. The proof of the lemma is completed.

The following statement says that π is indeed a covering map, which we do not rigorously define.

Lemma 2.1.5. *Let $\mathbb{Z}\Delta$ be the repetitive quiver of a valued quiver Δ without oriented cycles, with paths η, ξ such that $\pi(\eta) = \pi(\xi)$, where π is the covering map. If $s(\eta) = s(\xi)$ or $e(\eta) = e(\xi)$, then $\eta = \xi$.*

Proof. Consider only the case where $s(\eta) = s(\xi) = (t, x) =: \tilde{x}$, for some $t \in \mathbb{Z}$ and $x \in \Delta_0$. Since π preserves the length, it is easy to see that the statement holds if $l(\eta) = 0$ or 1. Suppose that $l(\eta) \geq 2$. Write $\eta = \eta'\tilde{\gamma}$ and $\xi = \xi'\tilde{\delta}$, for some arrows $\tilde{\gamma} : \tilde{x} \to \tilde{y}$ and $\tilde{\delta} : \tilde{x} \to \tilde{z}$, and some paths η', ξ' such that $s(\eta') = \tilde{y}$ and $s(\xi') = \tilde{z}$. Then, $\pi(\eta') \cdot \pi(\tilde{\gamma}) = \pi(\eta) = \pi(\xi) = \pi(\xi') \cdot \pi(\tilde{\delta})$. Since $\pi(\tilde{\gamma})$ and $\pi(\tilde{\delta})$ are arrows or inverse arrows in Δ, we have $\pi(\tilde{\gamma}) = \pi(\tilde{\delta})$ and $\pi(\eta') = \pi(\xi')$. As seen above, $\tilde{\gamma} = \tilde{\delta}$. Then $s(\eta') = s(\xi')$. By the induction hypothesis, $\eta' = \xi'$. Hence, $\eta = \xi$. The proof of the lemma is completed.

We precisely describe the sectional paths in $\mathbb{Z}\Delta$ in terms of reduced walks in Δ as follows.

Lemma 2.1.6. *Let Δ be a valued quiver without oriented cycles, and let $w = c_t \cdots c_2 c_1$ be a reduced walk in Δ, where c_i is an arrow or an inverse arrow from x_{i-1} to x_i, such that the edge $x_{i-1} \text{---} x_i$ in $\overline{\Delta}$ has valuation (v_i, v_i'), for $i = 1, \ldots, t$. Then, $\mathbb{Z}\Delta$ contains a sectional path*

$$(0, x_0) \longrightarrow (r_1, x_1) \longrightarrow \cdots \longrightarrow (r_t, x_t),$$

where r_i is the number of inverse arrows in the subwalk of w from x_0 to x_i, and $(r_{i-1}, x_{i-1}) \to (r_i, x_i)$ has valuation (v_i, v_i'), for $i = 1, \ldots, t$.

Proof. Observe that $r_0 = 0$. We may assume that $t > 0$ and that $\mathbb{Z}\Delta$ contains a desired sectional path $(x_0, r_0) \to \cdots \to (r_{t-1}, x_{t-1})$. Suppose first that Δ has an arrow $x_{t-1} \to x_t$. Then, its valuation is (v_t, v_t'); see (1.3.10) and $r_t = r_{t-1}$. By definition, $\mathbb{Z}\Delta$ has an arrow $(r_{t-1}, x_{t-1}) \to (r_t, x_t)$ with valuation (v_t, v_t'). Suppose now that Δ has an arrow $x_{t-1} \leftarrow x_t$. Then, its valuation is (v_t', v_t); see (1.3.10) and $r_t = r_{t-1} + 1$. Thus, $\mathbb{Z}\Delta$ has an arrow $(r_{t-1}, x_{t-1}) \to (r_t, x_t)$ with valuation (v_t, v_t'). Since $x_t \neq x_{t-2}$, we obtain a desired sectional path as stated in the lemma. The proof of the lemma is completed.

The following easy statement is useful for inductive arguments.

Lemma 2.1.7. *Let (Γ, v, τ) be a stable valued translation quiver. If Γ contains a non-sectional path $\zeta : x \rightsquigarrow y$, then it contains a path $\xi : x \rightsquigarrow \tau y$ with $l(\xi) = l(\zeta) - 2$.*

Now, we state some nice properties of repetitive quivers of valued trees.

Proposition 2.1.8. *Let $\mathbb{Z}\Delta$ be the repetitive quiver of a valued tree Δ with translation τ.*

(1) *If $\eta : \tilde{x} \rightsquigarrow \tilde{y}$ is a sectional path in $\mathbb{Z}\Delta$, then it is the only path from $\tilde{x}$ to $\tilde{y}$.*

(2) *Any two parallel paths $\eta : \tilde{x} \rightsquigarrow \tilde{y}$ and $\zeta : \tilde{x} \rightsquigarrow \tilde{y}$ in $\mathbb{Z}\Delta$ have the same length.*

Proof. Consider the covering map $\pi : \mathcal{P}(\mathbb{Z}\Delta) \to \mathcal{W}(\Delta)$. Let $\eta : \tilde{x} \rightsquigarrow \tilde{y}$ and $\zeta : \tilde{x} \rightsquigarrow \tilde{y}$ be paths in $\mathbb{Z}\Delta$.

(1) If $\ell(\eta) = 0$, then clearly $\xi = \eta$. Assume that $\ell(\eta) > 0$. Write $\eta = \tilde{\alpha}\eta'$, where $\tilde{\alpha} : \tilde{z} \to \tilde{y}$ is an arrow and $\eta' : \tilde{x} \rightsquigarrow \tilde{z}$ is a sectional path with $l(\eta') = l(\eta) - 1$. Consider the arrow $\tilde{\beta} : \tau\tilde{y} \to \tilde{z}$ in $\mathbb{Z}\Delta$. If ξ is not sectional then, by Lemma 2.1.7, $\mathbb{Z}\Delta$ contains a path $\xi' : \tilde{x} \rightsquigarrow \tau\tilde{y}$, and a path $\tilde{\beta}\xi' : \tilde{x} \rightsquigarrow \tau\tilde{y} \to \tilde{z}$. By the induction hypothesis, $\eta' = \tilde{\beta}\xi'$, and consequently, $\eta = \tilde{\alpha}\tilde{\beta}\xi'$, a contradiction to η being sectional. Thus, ξ is sectional. By Lemma 2.1.4, both $\pi(\eta)$ and $\pi(\xi)$ are reduced walks in Δ from $\pi(\tilde{x})$ to $\pi(\tilde{y})$. Since Δ is tree, $\pi(\eta) = \pi(\xi)$, and by Proposition 2.1.5, $\eta = \xi$.

(2) By Statement (1), we may assume that η and η' are non-sectional. By Lemma 2.1.7, there exist paths $\xi : \tilde{x} \rightsquigarrow \tau\tilde{y}$ and $\xi' : \tilde{x} \rightsquigarrow \tau\tilde{y}$, where τ is the translation of $\mathbb{Z}\Delta$, such that $l(\xi) = l(\eta) - 2$ and $l(\xi') = l(\eta') - 2$. By the induction hypothesis, $l(\xi) = l(\xi')$. Therefore, $l(\eta) = l(\eta')$. The proof of the lemma is completed.

2.2. SECTIONS. In the existing literature, sections are defined only for connected valued translation quivers; see [25, (2.1)]. We shall drop this restriction in order to include the Auslander-Reiten quiver of the bounded derived category of a simple artin algebra; see (4.1.2). Throughout this subsection, Γ denotes a valued translation quiver with translation τ.

Definition 2.2.1. Let Γ be a valued translation quiver with translation τ. A valued subquiver Δ of Γ is called a *section* if the following conditions are satisfied:

(1) Δ is connected and convex in Γ.

(2) Δ contains no oriented cycle.

(3) Δ meets every τ-orbit in Γ exactly once.

The following statement says that $\mathbb{Z}\mathbb{A}_1$ is essentially the only valued translation quiver which is not connected and contains sections.

Lemma 2.2.2. *Let Γ be a valued translation quiver with translation τ, containing a non-trivial section Δ. Then, x^+ and x^- are non-empty for all $x \in \Gamma_0$. And consequently, Γ is connected.*

Proof. Let $x \in \Gamma_0$. Then $x = \tau^s y$ for some $y \in \Delta_0$ and $s \in \mathbb{Z}$. Since Δ is non-trivial and connected, we may assume that it contains an arrow $y \to z$. Being stable, Γ contains arrows $\tau^s y \to \tau^s z$ and $\tau^{s+1} z \to \tau^s y$. Thus, x^+ and x^- are non-empty. The proof of the lemma is completed.

Let Δ be a section in Γ. By definition, every vertex in Γ is uniquely written as $\tau^{-r} x$ with $r \in \mathbb{Z}$ and $x \in \Delta_0$. We shall say that Δ is a *leftmost section* if all vertices in Γ are of the form $\tau^{-r} x$ with $r \geq 0$ and $x \in \Delta_0$; and a *rightmost section* if all vertices in Γ are of the form $\tau^r x$ with $r \geq 0$ and $x \in \Delta_0$.

Proposition 2.2.3. *Let Γ be a valued translation quiver with translation τ, containing a section Δ.*

(1) *If $x \to y$ is an arrow in Γ, then $y \in \Delta$ implies x or $\tau^- x \in \Delta$; and $x \in \Delta$ implies y or $\tau y \in \Delta$.*
(2) *There exists a valued translation quiver embedding $\phi : \Gamma \to \mathbb{Z}\Delta$, sending $\tau^s x$ to $(-s, x)$.*
(3) *If Δ is leftmost or rightmost, then it is the unique leftmost or rightmost section in Γ, respectively.*

Proof. Statements (1) and (2) are quoted from [25, (2.2), (2.3)], and Statement (2) follows immediately from the definitions. The proof of the proposition is completed.

In view of Proposition 2.2.3(2), we will see that valued translation quivers with sections inherits many nice properties of repetitive quivers. We will also need the following notion.

Definition 2.2.4. Let Γ be a valued translation quiver with a vertex x. A section Δ in Γ is called an *x-source section* if every vertex in Δ is a successor of x in Δ; and an *x-sink section* if every vertex in Δ is a predecessor of x in Δ.

Given a vertex x in Γ, we denote by $\mathrm{Suc}(x)$ and $\mathrm{Pred}(x)$ the full valued translation subquivers of Γ generated by the successors of x and by the predecessors of x, respectively.

Lemma 2.2.5. *Let Γ be a valued translation quiver with a vertex x.*

(1) *If Δ is an x-source section in Γ, then it is unique in Γ and a leftmost section in $\mathrm{Suc}(x)$.*
(2) *If Σ is an x-sink section in Γ, then it is unique in Γ and a rightmost section in $\mathrm{Pred}(x)$.*

Proof. We shall only prove Statement (1). Let Δ be an x-source section in Γ. Then, $\Delta \subseteq \mathrm{Suc}(x)$. Thus, Δ is a section in $\mathrm{Suc}(x)$. Suppose that Γ contains a path $x \rightsquigarrow y$. Then, $y = \tau^r z$ for some $r \in \mathbb{Z}$ and $z \in \Delta_0$. If $r < 0$, then Γ contains a path $x \rightsquigarrow y \rightsquigarrow z$. Since Δ is convex in Γ, both y and $\tau^r y$ belong to Δ, a contradiction. Thus, $r \geq 0$. That is, Δ is a leftmost section in $\mathrm{Suc}(x)$, which is unique by Lemma 2.2.3. Thus, Δ is the unique x-source section in Γ. The proof of the lemma is completed.

The following statement is important for our investigation in the next section.

Proposition 2.2.6. *Let $\mathbb{Z}\Delta$ be the repetitive quiver of a connected valued tree Δ with a vertex (r, x). Then the sectional successors of (r, x) in $\mathbb{Z}\Delta$ generate an (r, x)-source section in $\mathbb{Z}\Delta$; and the sectional predecessors of (r, x) in $\mathbb{Z}\Delta$ generate an (r, x)-sink section in $\mathbb{Z}\Delta$.*

Proof. We shall only prove the first part of the statement. Let Ω be the full valued subquiver of $\mathbb{Z}\Delta$ generated by the sectional successors of (r, x) in $\mathbb{Z}\Delta$. It is easy to see that every vertex in Ω is a successor of (r, x) in Ω. In particular, Ω is connected and contains no oriented cycle. Now, suppose that $\mathbb{Z}\Delta$ contains a path $(s, y) \rightsquigarrow (s', y') \rightsquigarrow (t, z)$, where $(s, y), (t, z) \in \Omega_0$. Then, $\mathbb{Z}\Delta$ contains sectional paths $\zeta : (r, x) \rightsquigarrow (s, y)$ and $\xi : (r, x) \rightsquigarrow (t, z)$. By Lemma 2.1.8, $(r, x) \rightsquigarrow (s, y) \rightsquigarrow (s', y') \rightsquigarrow (t, z)$ coincides with ξ. Thus, $(r, x) \rightsquigarrow (s, y) \rightsquigarrow (s', y')$ is sectional, and $(s', y') \in \Omega$. So, Ω is convex in $\mathbb{Z}\Delta$.

Consider finally any $y \in \Delta_0$. Since Δ is connected, by Lemma 2.1.6, $\mathbb{Z}\Delta$ has a sectional path $(r, x) \rightsquigarrow (s_0, y)$ with $s_0 \in \mathbb{Z}$. So, $(s_0, y) \in \Omega$. Suppose that $(t_0, y) \in \mathbb{Z}\Delta$ for another $t_0 \in \mathbb{Z}$, say $s_0 \leq t_0$. Then, $\mathbb{Z}\Delta$ contains a path $(s_0, y) \rightsquigarrow (t_0, y)$ and a sectional path $(r, x) \rightsquigarrow (t_0, y)$. By Lemma 2.1.8, the path $(r, x) \rightsquigarrow (s_0, y) \rightsquigarrow (t_0, y)$ is sectional. So, $s_0 = t_0$. Therefore, Ω meets every τ-orbit in $\mathbb{Z}\Delta$ exactly once. Thus, Ω is a (r, x)-source section in $\mathbb{Z}\Delta$. The proof of the proposition is completed.

2.3. ADDITIVE FUNCTIONS. In the study of Auslander-Reiten components of artin algebras, a powerful tool is additive functions on translation quivers; see, for example, [10, 19, 33]. We shall need it to describe hammocks for hereditary artin algebras of finite representation type.

Definition 2.3.1. Let (Γ, τ, v) be a valued translation quiver. A function $f : \Gamma_0 \to \mathbb{Z}$ is called *additive* on Γ provided, for any $x \in \Gamma_0$ with τx defined, that $f(\tau x) + f(x) = \sum_{y \in x^-} v'_{yx} f(y)$, where the sum is zero in case x^- is empty.

Proposition 2.3.2. *Let (Γ, τ, v) be a stable valued translation quiver with a finite section Δ. Given $r_x \in \mathbb{Z}$ with $x \in \Delta_0$, there exists a unique additive function f on Γ such that $f(x) = r_x$ for all $x \in \Delta_0$.*
Proof. It suffices to establish the existence of f. We start with setting $f(x) = r_x \in \mathbb{Z}$, for all $x \in \Delta_0$. Given $s \in \mathbb{Z}$, the vertices $\tau^s x$ with $x \in \Delta_0$ generate a section $\tau^s \Delta$ in Γ. We shall first define f on $\tau \Delta$. Being finite without oriented cycles, Δ contains a sink vertex x_1. By Lemma 2.2.3(1), $x_1^- \subseteq \Delta_0$. Put $f(\tau x_1) = \sum_{y \in x_1^-} v'_{x_1, y} f(y) - f(x_1)$. Clearly, the vertices in $\{\tau x_1\} \cup (\Delta_0 \backslash \{x_1\})$ generate a section $\Delta^{(1)}$, on which f is defined. Suppose that f is defined on a section $\Delta^{(i)}$ generated by the vertices in $\{\tau x_1, \ldots, \tau x_i\} \cup (\Delta_0 \backslash \{x_1, \ldots, x_i\})$ for some $i \geq 1$. If $\Delta_0 = \{x_1, \ldots, x_i\}$, then $\Delta^{(i)} = \tau \Delta$. Otherwise, Δ is non-trivial and there exists $x_{i+1} \in \Delta_0 \backslash \{x_1, \ldots, x_i\}$ such that $\Delta^{(i)}$ contains no arrow $x_{i+1} \to y$ with $y \in \Delta \backslash \{x_1, \ldots, x_i\}$. Suppose that $\Delta^{(i)}$ contains an arrow $x_{i+1} \to \tau x_j$ for some $1 \leq j \leq i$. Since $x_j^- \neq \emptyset$; see (2.2.2), Γ contains a path $x_{i+1} \to \tau x_j \rightsquigarrow x_j$, and hence, $\tau x_j \in \Delta$, absurd. Thus, x_{i+1} is a sink in $\Delta^{(i)}$. By Lemma 2.2.3(1), $x_{i+1}^- \subseteq \Delta^{(i)}$. Define $f(\tau x_{i+1}) = \sum_{y \in x_{i+1}^-} v'_{x_{i+1}, y} f(y) - f(x_{i+1})$. Then, f is defined on the section $\Delta^{(i+1)}$ generated by the vertices in $\{\tau x_1, \ldots, \tau x_{i+1}\} \cup (\Delta_0 \backslash \{x_1, \ldots, x_{i+1}\})$. By induction, we may define f on $\tau \Delta$. Repeating this process, we may define f on $\tau^s \Delta$ for all $s > 0$. Dually, considering sources in sections, we define f on $\tau^{-s} \Delta$ for all $s > 0$. This yields a desired additive function f on Γ. The proof of the proposition is completed.

3. HEREDITARY ARTIN ALGEBRAS

The main objective of this section is to study the preprojective and preinjective components of a connected hereditary artin algebra. Although they have already been well studied; see [2, Section VIII.1] and [32, Pages 80 - 81], we shall further study them in terms of the ext-quiver of the algebra. In case the ext-quiver is a tree, a sectional path in the preprojective component is the unique path between the end-points, two parallel paths have the same length, and every projective module is the source of a source section, which is explicitly described in terms of reduced walks in the ext-quiver; see (3.1.5) and (3.2.1). Finally, we shall show that the Coxeter transformation is compatible with the derived Auslander-Reiten translation; see (3.3.2).

Throughout this section, H stands for a connected hereditary artin algebra with ext-quiver Q_H, and $\mathrm{mod}\,H$ for the category of finitely generated left H-modules. We simply write τ for the Auslander-Reiten translation τ_H of the Auslander-Reiten quiver $\Gamma_{\mathrm{mod}H}$. And we associate a simple module S_i, a projective module P_i and an injective module I_i in $\Gamma_{\mathrm{mod}H}$ with each $i \in (Q_H)_0$ such that $\mathrm{top}P_i \cong S_i \cong \mathrm{soc}I_i$.

3.1. PREPROJECTIVE AND PREINJECTIVE COMPONENTS. The following result is probably well-known. We sketch a proof for the first statement and refer to [2, Page 267] for a proof for the second.

Lemma 3.1.1. *Let H be a hereditary artin algebra with ext-quiver Q_H. Consider the simple modules S_i, the projective modules P_i and the injective modules I_i in $\Gamma_{\mathrm{mod}H}$ associated with $i \in Q_H$.*
(1) If i is a vertex in Q_H, then $\mathrm{End}_H(P_i) \cong \mathrm{End}_H(I_i) \cong \mathrm{End}_H(S_i)$ and $\ell_{S_i}(P_i) = \ell_{S_i}(I_i) = 1$.
(2) There exists an arrow $i \to j$ in Q_H if and only if there exists an irreducible map $f : P_j \to P_i$, or equivalently, an irreducible map $g : I_j \to I_i$, in $\mathrm{mod}H$.
Proof. Fix $i \in (Q_H)_0$. Since I_i is not a direct summand of the injective module I_i/S_i, we have $\mathrm{Hom}_H(S_i, I_i/S_i) = 0$. Thus, we deduce from Lemma 1.7.1 that $\ell_{S_i}(I_i) = 1$. Since $S_i \cong \mathrm{soc}I_i$, every non-zero map in $\mathrm{End}_H(S_i)$ induces a non-zero map in $\mathrm{End}_H(I_i)$. Consider a non-zero map $f : I_i \to I_i$. Since $\mathrm{Im}(f)$ is injective, f is invertible. Since $\mathrm{Hom}_H(S_i, I_i/S_i) = 0$, we see that f is induced from a non-zero map $g : S_i \to S_i$. Thus, $\mathrm{End}_H(I_i) \cong \mathrm{End}_H(S_i)$. The proof of the lemma is completed.

Since Q_H is connected, by Lemma 3.1.1(2), the projective modules in $\Gamma_{\mathrm{mod}H}$ lie in the same connected component $\mathscr{P}_H$, called the *preprojective component*; and the injective modules lie in the same connected component $\mathscr{I}_H$, called the *preinjective component*. The following statement is reformulated from Proposition 1.15 in [2, Chapter VIII] and its dual statement. Since our valuation for $\Gamma_{\mathrm{mod}H}$ is different from that given in [2, Page 225], we include a detailed proof for the preinjective component.

Theorem 3.1.2. *Let H be a connected hereditary artin R-algebra with ext-quiver Q_H. Consider the projective modules P_i and the injective modules I_i in $\Gamma_{\mathrm{mod}H}$ associated with $i \in Q_H$.*

(1) *The projective modules P_i with $i \in Q_H$ generate a leftmost section Δ_H in the preprojective component $\mathscr{P}_H$. Moreover, there exists a valued quiver isomorphism $Q_H^{\mathrm{op}} \to \Delta_H$, sending i to P_i.*

(2) *The injective modules I_i with $i \in Q_H$ generate a rightmost section Σ_H in the preinjective component $\mathscr{I}_H$. Moreover, there exists a valued quiver isomorphism $Q_H^{\mathrm{op}} \to \Sigma_H$, sending i to I_i.*

Proof. We shall only prove Statement (2). We write $\ell_R(V)$ for the R-length of an R-module V, and put $k_M = \mathrm{End}_H(M)/\mathrm{rad}(\mathrm{End}_H(M))$ for $M \in \Gamma_{\mathrm{mod}H}$. Let Σ_H be the full valued subquiver of $\mathscr{I}_H$ generated by the I_i with $i \in Q_H$. Set $S_i = \mathrm{soc}I_i$ for $i \in Q_H$. By Lemma 3.1.1(2), Q_H contains an arrow $i \to j$ if and only if Σ_H contains an arrow $I_j \to I_i$; and in this case, the valuation for $i \to j$ is (d_{ij}, d'_{ij}), where $d_{ij} = \dim_{\mathrm{End}_H(S_j)} \mathrm{Ext}_H^1(S_i, S_j)$ and $d'_{ij} = \dim \mathrm{Ext}_H^1(S_i, S_j)_{\mathrm{End}_H(S_i)}$. Thus,

$$d'_{ij} \cdot \ell_R(\mathrm{End}_H(S_i)) = \ell_R(\mathrm{Ext}_H^1(S_i, S_j)) = d_{ij} \cdot \ell_R(\mathrm{End}_H(S_j)).$$

On the other hand, we see from Proposition 1.5.1 that $I_j \to I_i$ has valuation $(d_{I_j, I_i}, d'_{I_j, I_i})$, where $d_{I_j, I_i} = \dim {}_{k_{I_i}} \mathrm{irr}(I_j, I_i)$ and $d'_{I_j, I_i} = \dim \mathrm{irr}(I_j, I_i)_{k_{I_j}}$. Since $k_{I_j} \cong \mathrm{End}_H(S_j)$ and $k_{I_i} \cong \mathrm{End}_H(S_i)$ by Lemma 3.1.1(1), $d'_{I_j, I_i} \cdot \ell_R(\mathrm{End}_H(S_j)) = \ell_R(\mathrm{Irr}(I_j, I_i)) = d_{I_j, I_i} \cdot \ell_R(\mathrm{End}_H(S_i))$.

Now since I_j/S_j is injective, by Proposition 1.15(b) in [2, (III.1.15)], d'_{ij} is the multiplicity of I_i as a direct summand of I_j/S_j. And since the canonical projection $\varphi : I_j \to I_j/S_j$ is minimal left almost split, d_{I_j, I_i} is the multiplicity of I_i as a direct summand of I_j/S_j; see (1.5.1). So, $d'_{ij} = d_{I_j, I_i}$. Hence,

$$d'_{I_j, I_i} \cdot \ell_R(\mathrm{End}_H(S_j)) = d_{I_j, I_i} \cdot \ell_R(\mathrm{End}_H(S_i)) = d'_{ij} \cdot \ell_R(\mathrm{End}_H(S_i)) = d_{ij} \cdot \ell_R(\mathrm{End}_H(S_j)).$$

Thus, $d'_{I_j, I_i} = d_{ij}$. This proves the second part of Statement (2). In particular, Σ_H is connected. Being injective, the I_i lie in pairwise distinct τ-orbits in $\Gamma_{\mathrm{mod}H}$. Note that the irreducible maps between the modules I_i are all epimorphisms. Thus, since $\mathrm{End}_H(I_i)$ is divisible; see (3.1.1), Σ_H has no oriented cycle. Hence, Σ_H is a section in $\mathscr{I}_H$, which is clearly rightmost. The proof of the theorem is completed.

In the sequel, we shall call the underlying valued graph $\overline{Q}_H$ of Q_H the *ext-graph* of H. Moreover, Δ_H will be called the *projective section* in $\mathscr{P}_H$, and Σ_H called the *injective section* in $\mathscr{I}_H$. The following well-known statement is a consequence of Theorem 3.1.2 and well-known; see [2, (V.7.5), (VIII.1.9)].

Corollary 3.1.3. *Let H be a connected hereditary artin algebra. Consider modules M, N in $\Gamma_{\mathrm{mod}H}$ with M in the preinjective component $\mathscr{I}_H$ or N in the preprojective component $\mathscr{P}_H$. Then, every non-zero radical map $f : M \to N$ is a sum of composites of irreducible maps between modules in $\Gamma_{\mathrm{mod}H}$.*

We shall need the following easy statement.

Lemma 3.1.4. *Let H be a hereditary artin algebra. Consider an arrow $M \to N$ in $\Gamma_{\mathrm{mod}H}$.*

(1) *If $\Gamma_{\mathrm{mod}H}$ contains $\tau^r M$ for some $r \geq 0$, then it contains an arrow $\tau^r M \to \tau^r N$.*

(2) *If $\Gamma_{\mathrm{mod}H}$ contains $\tau^{-r} N$ for some $r \geq 0$, then it contains an arrow $\tau^{-r} M \to \tau^{-r} N$.*

Proof. We shall only prove Statement (1). Assume that $\tau^r M \in \Gamma_{\mathrm{mod}H}$ for some $r > 0$, but $\tau^r N \notin \Gamma_{\mathrm{mod}H}$. Then, $\tau^s N$ is projective for some $0 \leq s < r$. Since $\tau^s M \in \Gamma_{\mathrm{mod}H}$, by Lemma 1.4.1, $\Gamma_{\mathrm{mod}H}$ contains an arrow $\tau^s M \to \tau^s N$. Since H is hereditary, $\tau^s M$ is projective, and hence, $\tau^r M \notin \Gamma_{\mathrm{mod}H}$, absurd. Thus, $\tau^r N \in \Gamma_{\mathrm{mod}H}$. So, $\Gamma_{\mathrm{mod}H}$ contains an arrow $\tau^r M \to \tau^r N$. The proof of the lemma is completed.

The following statement describes in terms of Q_H the sectional paths in $\Gamma_{\mathrm{mod}H}$, which start with a projective module or end with an injective module.

Lemma 3.1.5. *Let H be a hereditary artin algebra with ext-quiver Q_H. Consider the projective modules P_i and the injective modules I_i in $\Gamma_{\mathrm{mod}H}$ associated with $i \in Q_H$. And let $w = c_t \cdots c_2 c_1$ be a non-trivial reduced walk in Q_H, where c_j is an arrow or inverse arrow from i_{j-1} to i_j, for $j = 1, \ldots, t$.*

(1) *The preprojective component $\mathscr{P}_H$ of $\Gamma_{\mathrm{mod}H}$ contains a sectional path*

$$P_{i_0} \longrightarrow \tau^{-r_1}P_{i_1} \longrightarrow \cdots \longrightarrow \tau^{-r_{t-1}}P_{i_{t-1}} \longrightarrow \tau^{-r_t}P_{i_t},$$

where r_j is the number of arrows in the subwalk of w from i_0 to i_j, for $j = 1, \ldots, t$.

(2) *The preinjective component $\mathscr{I}_H$ of $\Gamma_{\mathrm{mod}H}$ contains a sectional path*

$$\tau^{s_0}I_{i_0} \longrightarrow \tau^{s_1}I_{i_1} \longrightarrow \cdots \longrightarrow \tau^{s_{t-1}}I_{i_{t-1}} \longrightarrow I_{i_t},$$

where s_j is the number of arrows in the subwalk of w from i_j to i_t, for $j = 0, 1, \ldots, t-1$.

Proof. We shall only prove Statement (1). Since w is reduced, the P_{i_j} with $0 \le j \le t$ lie in pairwise distinct τ-orbits in $\mathscr{P}_H$. Set $r_0 = 0$. Assume that $\mathscr{P}_H$ contains a desired sectional path $P_{i_0} \longrightarrow \cdots \longrightarrow \tau^{-r_{t-1}}P_{i_{t-1}}$. Assume first that Q_H contains an arrow $i_{t-1} \to i_t$. Then $r_t = r_{t-1} + 1$ and by Theorem 3.1.2(1), $\mathscr{P}_H$ contains an arrow $P_{j_t} \to P_{j_{t-1}}$. By Lemma 3.1.4(2), $\mathscr{P}_H$ contains an arrow $\tau^{-r_{t-1}}P_{i_t} \to \tau^{-r_{t-1}}P_{i_{t-1}}$. Since $r_{t-1} + 1 = r_t$, it follows from Lemma 1.6.2(1) that $\mathscr{P}_H$ contains a sectional path $P_{i_0} \longrightarrow \cdots \longrightarrow \tau^{-r_{t-1}}P_{i_{t-1}} \longrightarrow \tau^{-r_t}P_{i_t}$.

Assume now that Q_H contains an arrow $i_{t-1} \leftarrow i_t$. Then $r_t = r_{t-1}$ and $\mathscr{P}_H$ contains an arrow $P_{i_{t-1}} \to P_{i_t}$. We claim that $\mathscr{P}_H$ contains an arrow $\tau^{-r_{t-1}}P_{i_{t-1}} \to \tau^{-r_t}P_{i_t}$. This is the case if $r_{t-1} = 0$. Otherwise, $\mathscr{P}_H$ has an arrow $P_{i_t} \to \tau^-P_{i_{t-1}}$. Since $\tau^{1-r_{t-1}}(\tau^-P_{i_{t-1}}) = \tau^{-r_{t-1}}P_{i_{t-1}}$, by Lemma 3.1.4(1), $\mathscr{P}_H$ contains an arrow $\tau^{1-r_{t-1}}P_{i_t} \to \tau^{-r_{t-1}}P_{i_{t-1}}$. By Lemma 1.6.2(1), $\tau^{1-r_{t-1}}P_{i_t}$ is not injective. Since $r_t = r_{t-1}$, there exists an arrow $\tau^{-r_{t-1}}P_{i_{t-1}} \to \tau^{-r_t}P_{i_t}$ in $\mathscr{P}_H$. This establishes our claim. Hence, $\mathscr{P}_H$ contains a sectional path $P_{i_0} \longrightarrow \cdots \longrightarrow \tau^{-r_{t-1}}P_{i_{t-1}} \longrightarrow \tau^{-r_t}P_{i_t}$. The proof of the lemma is completed.

The following statement is crucial for us to study hammocks in the next section.

Proposition 3.1.6. *Let H be a connected hereditary artin algebra.*

(1) *The preprojective component $\mathscr{P}_H$ contains a P-source section for every projective module P in $\mathscr{P}_H$.*
(2) *The preinjective component $\mathscr{I}_H$ contains an I-sink section for every injective module I in $\mathscr{I}_H$.*

Proof. We shall only prove Statement (1). Consider a module P in the projective section Δ_H in $\mathscr{P}_H$. Let Σ be the full valued subquiver of $\mathscr{P}_H$ generated by the successors M of P in $\mathscr{P}_H$ such that τM is not a succesor of P. Then, Σ is contains no oriented cycle and meets any τ-orbit in $\mathscr{P}_H$ at most once. Given $P' \in \Delta_H$, since the ext-quiver of H is connected, we deduce from Lemma 3.1.5(1) that there exists a minimal $s \ge 0$ such that $\tau^{-s}P'$ is a successor of P in $\mathscr{P}_H$. Then $\tau^{-s}P' \in \Sigma$. Hence, Σ meets every τ-orbit in $\mathscr{P}_H$ exactly once. Finally, suppose that $\mathscr{P}_H$ contains a path $\eta : M \rightsquigarrow L \rightsquigarrow N$ with $M, N \in \Sigma$. Then, $\mathscr{P}_H$ contains a path $P \rightsquigarrow M \rightsquigarrow L$. Assume that $L \notin \Sigma$. Then, $\mathscr{P}_H$ contains a path $P \rightsquigarrow \tau L$. Since H is hereditary, $L \rightsquigarrow N$ contains no projective module. Hence, $\mathscr{P}_H$ contains a path $P \rightsquigarrow \tau L \rightsquigarrow \tau N$, a contradiction. Therefore, Σ is convex in $\mathscr{P}_H$. In particular, every module in Σ is a succesor of P in Σ. Hence, Σ is a P-source section in $\mathscr{P}_H$. The proof of the proposition is completed.

3.2. Tree type. We shall say that H is of *tree type* or *Dynkin type* if Q_H is a tree or a Dynkin quiver, respectively. These hereditary algebras have the following important properties.

Proposition 3.2.1. *Let H be a connected hereditary artin algebra of tree type. Consider the preprojective component $\mathscr{P}_H$ and the preinjective component $\mathscr{I}_H$ of $\Gamma_{\mathrm{mod}H}$.*

(1) *If η, ζ are parallel paths in $\mathscr{P}_H$ or $\mathscr{I}_H$, then $l(\eta) = l(\zeta)$; and $\eta = \zeta$ in case η is sectional.*
(2) *The P-source section in $\mathscr{P}_H$ with P projective is generated by the sectional successors of P in $\mathscr{P}_H$.*
(3) *The I-sink section in $\mathscr{I}_H$ with I injective is generated by the sectional predecessors of I in $\mathscr{I}_H$.*

Proof. (1) Let Δ_H be the projective section in $\mathscr{P}_H$. By Proposition 2.2.3(2), $\mathscr{P}_H$ embeds in $\mathbb{Z}\Delta_H$ as a full valued translation subquiver. In particular, a (sectional) path in $\mathscr{P}_H$ gives rise to a (sectional) path in $\mathbb{Z}\Delta_H$. Since Δ_H is a tree; see (3.1.2), we see that Statement (1) follows from Proposition 2.1.8.

(2) Given $P \in \Delta_H$, by Proposition 3.1.6, $\mathscr{P}_H$ contains a P-source section Σ. Clearly, every module in Σ is a sectional successor P. Suppose that $\mathscr{P}_H$ contains a sectional path $\eta : P \rightsquigarrow M$. By Lemma 2.2.5, $M = \tau^{-s}N$ for some $N \in \Sigma$. Then, $\mathscr{P}_H$ contains a path $\zeta : P \rightsquigarrow N \rightsquigarrow M$. By Statement (1), $\zeta = \eta$. So, ζ is sectional, and hence, $M = N \in \Sigma$. The proof of the proposition is completed.

The following statement will enable us to define extended hammocks in the next section.

Lemma 3.2.2. *Let H be a connected hereditary artin algebra of tree type. Consider a sectional path $P = M_0 \longrightarrow M_1 \longrightarrow \cdots \longrightarrow M_{t-1} \longrightarrow M_t$ in $\Gamma_{\mathrm{mod}H}$, where P is a projective module with top S, and $M_{i-1} \to M_i$ with $1 \leq i \leq t$ has valuation (d_i, d_i'). Then $\ell_S(M_i) = d_i' \cdots d_1'$, for $i = 1, \ldots, t$.*

Proof. By Lemma 3.1.1(1), $\ell_S(M_0) = 1$. Suppose that $t > 0$. Consider a minimal right almost split map $f : L \to M_t$. Since $P \not\cong M_t$, the $\mathrm{End}_H(P)$-linear map $f_* : \mathrm{Hom}_H(P, L) \to \mathrm{Hom}_H(P, M_t)$ is an epimorphism. We claim that it is an isomorphism. If M_t is projective, then f is a monomorphism, and so is f_*. Otherwise, we have an exact sequence

$$0 \longrightarrow \mathrm{Hom}_H(P, \tau M_t) \longrightarrow \mathrm{Hom}_H(P, L) \xrightarrow{\ f_*\ } \mathrm{Hom}_H(P, M_t) \longrightarrow 0.$$

Since the path stated in the lemma is sectional, by Lemma 3.2.1(1), $\mathscr{P}_H$ contains no path from P to τM_t. Hence, $\mathrm{Hom}_H(P, \tau M_t) = 0$; see (3.1.3). This establishes our claim. Therefore, $\ell_S(M_t) = \ell_S(L)$; see (1.7.1). On the other hand, by Proposition 1.5.1, $L = M_{t-1}^{d_t'} \oplus L_1 \cdots \oplus L_r$, where $L_1 \ldots, L_r$ with $r \geq 0$ are modules in $\mathscr{P}_H$ different from M_{t-1}. By Lemma 3.2.1(1), $\mathscr{P}_H$ contains no path from P to L_j, and hence, $\mathrm{Hom}_H(P, L_j) = 0$ and $\ell_S(L_j) = 0$, for $j = 1, \ldots, r$. By the induction hypothesis, we see that $\ell_S(M_t) = \ell_S(L) = d_t' \ell_S(M_{t-1}) = d_t' \cdots d_1'$. The proof of the lemma is completed.

3.3. COXETER TRANSFORMATION. Let us recall Auslander and Platzeck's Coxeter transformation of the Grothendieck group $K_0(\mathrm{mod}H)$. Consider the simple modules S_i, the projective modules P_i and the injective mdoules in $\Gamma_{\mathrm{mod}H}$ associated with $i \in (Q_H)_0 = \{1, \ldots, n\}$. For any module M in $\mathrm{mod}H$, one defines its *dimension vector* to be $\underline{\dim}M := (\ell_{S_1}(M), \ldots, \ell_{S_n}(M)) \in K_0(\mathrm{mod}H)$. Since $\{\underline{\dim}P_1, \ldots, \underline{\dim}P_n\}$ and $\{\underline{\dim}I_1, \ldots, \underline{\dim}I_n\}$ are bases for $K_0(\mathrm{mod}H)$; see [29, (2.1)], one may introduce the following definition; see [29, Section 2], and compare [13, Page 8].

Definition 3.3.1. Let H be a hereditary artin algebra. The *Coxeter transformation* of the Grothendieck group $K_0(\mathrm{mod}H)$ is the unique automorphism

$$C_H : K_0(\mathrm{mod}H) \to K_0(\mathrm{mod}H) \text{ defined by } C_H(\underline{\dim}P_i) = -\underline{\dim}I_i, \text{ for } i = 1, \ldots, n.$$

An important property of the Coxeter transformation C_H of $K_0(\mathrm{mod}H)$ is its compatibility with the Auslander-Reiten translation τ of $\Gamma_{\mathrm{mod}H}$; see [29, (2.2)], and also [2, (VIII.2.2)]. We shall extend this to the bounded derived category $D^b(\mathrm{mod}H)$ of $\mathrm{mod}H$, which is a Hom-finite Krull-Scmidt R-category. It is well-known; see [5, (7.3)], [18, (3.6)] and [30, (I.3.3)] that $D^b(\mathrm{mod}H)$ has almost split triangles as defined in [18, (3.1)]. Note that a sequence of morphisms $L^\bullet \longrightarrow M^\bullet \longrightarrow N^\bullet$ in $D^b(\mathrm{mod}H)$ with $M^\bullet \neq 0$ is an almost split sequence if and only if it embeds in an almost split triangle $L^\bullet \longrightarrow M^\bullet \longrightarrow N^\bullet \longrightarrow L^\bullet[1]$; see [26, (6.1)]. Thus, the Auslander-Reiten quiver $\Gamma_{D^b(\mathrm{mod}H)}$ of $D^b(\mathrm{mod}H)$ as defined in Subsection 1.5 coincides with that defined by Happel in the algebraically closed setting; see [18, (3.7)].

Given $M \in \mathrm{mod}H$ and $s \in \mathbb{Z}$, we write $M[s]$ for the stalk complex concentrated in degree $-s$ where the component is M. Then, the indecomposable objects in $D^b(\mathrm{mod}H)$ are the stalk complexes $M[s]$, where $s \in \mathbb{Z}$ and M is an indecomposable module in $\mathrm{mod}H$; see [22, (3.1)]. Thus, we may choose the vertices in $\Gamma_{D^b(\mathrm{mod}H)}$ to be the stalk complexes $M[s]$ with $M \in \Gamma_{\mathrm{mod}H}$ and $s \in \mathbb{Z}$. We write τ_D for the Auslander-Reiten translation of $\Gamma_{D^b(\mathrm{mod}H)}$. On the other hand, given a complex $M^\bullet$ in $D^b(\mathrm{mod}H)$, Happel defined its *dimension vector* by

$$\underline{\dim}M^\bullet := \sum_{s \in \mathbb{Z}} (-1)^s \underline{\dim}M^s \in K_0(\mathrm{mod}H),$$

which is invariant on isomorphism classes in $D^b(\mathrm{mod}H)$; see [18, (2.2)]. As shown below, the Coxeter transformation of $K_0(\mathrm{mod}H)$ is compatible with the Auslander-Reiten translation of $\Gamma_{D^b(\mathrm{mod}H)}$.

Proposition 3.3.2. *Let H be a hereditary artin algebra. Consider the Coxeter transformation C_H of $K_0(\mathrm{mod}H)$ and the Auslander-Reiten translation τ_D of $\Gamma_{D^b(\mathrm{mod}H)}$. Then, $\underline{\dim}\tau_D^t(M[s]) = C_H^t(\underline{\dim}M[s])$, for all $M \in \Gamma_{\mathrm{mod}H}$ and $s, t \in \mathbb{Z}$.*

Proof. We shall only show that $\underline{\dim}\tau_D(M[s]) = C_H(\underline{\dim}M[s])$, for $M \in \Gamma_{\mathrm{mod}H}$ and $s \in \mathbb{Z}$. Suppose first that M is not projective. Then, $\underline{\dim}\tau M = C_H(\underline{\dim}M)$; see [29, (2.2)], and also [2, (VIII.2.2)]. Since τ_D commutes with the shift functor [1], we have $\tau_D(M[s]) = (\tau_D M)[s] = (\tau M)[s]$; see [5, (7.2)], and also [18]. By definition, $\underline{\dim}N[s] = (-1)^s \underline{\dim}N$ for any $N \in \mathrm{mod}H$. Thus,

$$\underline{\dim}\,\tau_D(M[s]) = \underline{\dim}(\tau M)[s] = (-1)^s\underline{\dim}\,\tau M = (-1)^s C_H(\underline{\dim}M) = C_H((-1)^s\underline{\dim}M) = C_H(\underline{\dim}M[s]).$$

Suppose now that $M = P$, the projective cover of a simple module S in $\Gamma_{\mathrm{mod}H}$. By definition, $C_H(\underline{\dim}P) = -\underline{\dim}I$, where I is the injective envelope of S. Moreover, $\tau_D(P[0]) = I[-1]$; see [5, (7.2)], and also [18]. This yields

$$\underline{\dim}\,\tau_D(P[s]) = \underline{\dim}(\tau_D P[0])[s] = \underline{\dim}I[s-1] = (-1)^s C_H(\underline{\dim}P) = C_H((-1)^s\underline{\dim}P) = C_H(\underline{\dim}P[s]).$$

The proof of the proposition is completed.

4. Main Results

The objective of this section is to study the module category of a hereditary artin algebra of finite representation type with a connection to the derived category and the associated cluster category. We first determine all the hammocks in the Auslander-Reiten quiver; see (4.2.4). This leads to a description of the precise shape of the Auslander-Reiten quiver in terms of the ext-quiver of the algebra; see (4.5.1). As applications, we obtain the number of non-isomorphic indecomposable objects in the module category and the associated cluster category; see (4.6.1) and (4.6.6). Moreover, the radicals of the module category, the bounded derived category and the associated cluster category all have the same nilpotency; see (4.6.4), (4.6.5), and (4.6.6).

It is well-known that a hereditary artin algebra is connected of finite representation type if and only if its ext-graph is a Dynkin diagram; see [2, (VIII.5.4)]. In the algebraically closed case, this is equivalent to the ext-graph being $\mathbb{A}_n(n \geq 1)$, $\mathbb{D}_n(n \geq 4)$ or $\mathbb{E}_n(n = 6, 7, 8)$; see [2, (VIII.5.5)].

Throughout this section, H stands for a hereditary artin algebbra with a Dynkin ext-quiver Q_H, and $\mathrm{mod}H$ for the category of finitely generated left H-modules. Let $\Gamma_{\mathrm{mod}H}$ and $\Gamma_{D^b(\mathrm{mod}H)}$ be the Auslander-Reiten quivers of $\mathrm{mod}H$ and $D^b(\mathrm{mod}H)$, respectively. We shall simply write τ for the Auslander-Reiten translation τ_H of $\Gamma_{\mathrm{mod}H}$, and τ_D for the Auslander-Reiten translation of $\Gamma_{D^b(\mathrm{mod}H)}$. With each vertex $i \in Q_H$, we associate a simple module S_i, a projective module P_i and an injective module I_i in $\Gamma_{\mathrm{mod}H}$ such that $\mathrm{top}P_i \cong S_i \cong \mathrm{soc}I_i$.

4.1. The canonical embedding. Since H is of finite representation type, it is well-known that $\Gamma_{\mathrm{mod}H}$ coincides with its preprojective component $\mathscr{P}_H$, and its preinjective component $\mathscr{I}_H$; see, for example, [2, (VIII.3.13)]. The following statement provides a rough description of the shape of $\Gamma_{\mathrm{mod}H}$.

Proposition 4.1.1. *Let H be a hereditary artin algebra with a Dynkin ext-quiver Q_H. Consider the projective modules P_i in $\Gamma_{\mathrm{mod}H}$ associated with vertices $i \in Q_H$. Then there exists a valued translation quiver embedding $\varphi : \Gamma_{\mathrm{mod}H} \to \mathbb{Z}Q_H^{\mathrm{op}} : \tau^{-r}P_i \mapsto (r,i)$, with a convex image in $\mathbb{Z}Q_H^{\mathrm{op}}$. Moreover, φ maps the P_i-source section in $\Gamma_{\mathrm{mod}H}$ onto the $(0,i)$-source section in $\mathbb{Z}Q_H^{\mathrm{op}}$, for every vertex $i \in Q_H$.*

Proof. Let Δ_H be the projective section in $\Gamma_{\mathrm{mod}H}$. Then, there exists valued translation quiver embedding $\psi : \Gamma_{\mathrm{mod}H} \to \mathbb{Z}\Delta_H : \tau^{-r}P_i \mapsto (r,P_i)$, whose image is convex in $\mathbb{Z}\Delta_H$; see [28, (1.13)]. Since $\Delta_H \cong Q_H^{\mathrm{op}}$; see (3.1.2), we have a valued quiver isomorphism $\theta : \mathbb{Z}\Delta_H \to \mathbb{Z}Q_H^{\mathrm{op}} : (r,P_i) \mapsto (r,i)$. This yields an embedding $\varphi = \theta \circ \psi : \Gamma_{\mathrm{mod}H} \to \mathbb{Z}Q_H^{\mathrm{op}} : \tau^{-r}P_i \mapsto (r,i)$, whose image is convex in $\mathbb{Z}Q_H^{\mathrm{op}}$.

Fix $i \in (Q_H)_0$. The P_i-source section Δ_i in $\Gamma_{\mathrm{mod}H}$ is generated by the sectional successors of P_i in $\Gamma_{\mathrm{mod}H}$; see (3.2.1), and the $(0,i)$-source section Ω_i in $\mathbb{Z}Q_H^{\mathrm{op}}$ is generated by the sectional successors of $(0,i)$ in $\mathbb{Z}Q_H^{\mathrm{op}}$; see (2.2.6). In particular, φ maps to Δ_i into Ω_i. Let $(r,j) \in \Omega_i$. Since Σ_i is a section in $\Gamma_{\mathrm{mod}H}$, there exists some $s \geq 0$ such that $\tau^{-s}P_j \in \Delta_i$. Thus, $(s,j) \in \Omega_i$. Since Ω_i is a section in $\mathbb{Z}Q_H^{\mathrm{op}}$, we have $s = r$. So, $\tau^{-r}P_j$ is a preimage of (r,j) in Δ_i. The proof of the proposition is completed.

The following statement is due to Happel in the algebraically closed case; see [18, (4.5)].

Theorem 4.1.2. *Let H be a hereditary artin algebra of Dynkin type. Then*
(1) $\Gamma_{\mathrm{mod}H}$ embeds in $\Gamma_{D^b(\mathrm{mod}H)}$ as a convex valued translation subquiver;
(2) $\Gamma_{D^b(\mathrm{mod}H)} \cong \mathbb{Z}\Delta_H$, where Δ_H is the projective section in $\Gamma_{\mathrm{mod}H}$.

Proof. It is well-known that there exists a full convex embedding of $\mathrm{mod}H$ in $D^b(\mathrm{mod}H)$, sending a module M to the stalk complex $M[0]$. By Theorem 7.2(1) in [5], this induces a convex valued translation quiver embedding of $\Gamma_{\mathrm{mod}H}$ in $\Gamma_{D^b(\mathrm{mod}H)}$. Next, in view of Theorem 7.2 in [5], we may apply Happel's argument in [18, (4.5)] to show that $\Gamma_{D^b(\mathrm{mod}H)} \cong \mathbb{Z}\Delta_H$. The proof of the theorem is completed.

4.2. HAMMOCKS. Since $\Gamma_{\mathrm{mod}H}$ is finite and contains no oriented cycle, we may study hammocks in $\Gamma_{\mathrm{mod}H}$; see (1.7.2). Given a vertex (s, k) in $\mathbb{Z}Q_H^{\mathrm{op}}$, by Proposition 2.2.6, $\mathbb{Z}Q_H^{\mathrm{op}}$ contains an (s, k)-source section $\Delta_{s,k}$ generated by the sectional successors of (s, k). And by Lemma 2.2.5, $\Delta_{s,k}$ is a leftmost section in $\mathrm{Suc}(s, k)$, the full valued translation subquiver of $\mathbb{Z}Q_H^{\mathrm{op}}$ generated by the successors of (s, k). By Proposition 2.3.2, we may introduce the following definition.

Definition 4.2.1. Let H be a hereditary artin algebra with a Dynkin ext-quiver Q_H. Consider a vertex $k \in Q_H$ and the $(0, k)$-source section Δ in $\mathbb{Z}Q_H^{\mathrm{op}}$. The *extended hammock function* h_k associated with $k \in Q_H$ is the unique additive function on $\mathrm{Suc}(0, k)$ such, for any vertex $(r, i) \in \Delta$, that

(1) $h_k(r, i) = 1$ if $(r, i) = (0, k)$;

(2) $h_k(r, i) = d_1' \cdots d_t'$ if Δ contains a path $(0, k) = (r_0, i_0) \longrightarrow (r_1, i_1) \longrightarrow \cdots \longrightarrow (r_t, i_t) = (r, i)$, where the arrow $(r_{j-1}, i_{j-1}) \longrightarrow (r_j, i_j)$ has valuation (d_j, d_j'), for $j = 1, \ldots, t$.

The following statement says in particular that an extended hammock function is indeed an extension of Brenner's canonical hammock function; see (1.7.2).

Lemma 4.2.2. *Let H be a hereditary artin algebra with a Dynkin ext-quiver Q_H. Consider the projective modules P_i and the simple modules S_i in $\Gamma_{\mathrm{mod}H}$ associated with $i \in Q_H$. Let h_k be the extended hammock function associated with some $k \in Q_H$. If $(r, i) \in \mathrm{Suc}(0, k)$ such that $\tau^{-r}P_i \in \Gamma_{\mathrm{mod}H}$, then*

$$h_k(s, j) = \ell_{S_k}(\tau^{-s}P_j),$$

for any predecessor (s, j) of (r, i) in $\mathrm{Suc}(0, k)$.

Proof. By Proposition 4.1.1, we have a canonical embedding $\varphi : \Gamma_{\mathrm{mod}H} \to \mathbb{Z}Q_H^{\mathrm{op}} : \tau^{-r}P_i \mapsto (r, i)$. Recall that φ has a convex image Γ in $\mathbb{Z}Q_H^{\mathrm{op}}$, and maps the P_k-source section Σ in $\Gamma_{\mathrm{mod}H}$ onto the $(0, k)$-source section Δ in $\mathbb{Z}Q_H^{\mathrm{op}}$. Assume that $(r, i) \in \mathrm{Suc}(0, k)$ such that $\tau^{-r}P_i \in \Gamma_{\mathrm{mod}H}$, that is, $(r, i) \in \Gamma$. Since Γ is convex in $\mathbb{Z}Q_H^{\mathrm{op}}$, we see that $\tau^{-s}P_j \in \Gamma_{\mathrm{mod}H}$, for any predecessor (s, j) of (r, i) in $\mathrm{Suc}(0, k)$. Therefore, it suffices to show that $h_k(r, i) = \ell_{S_k}(\tau^{-r}P_i)$.

By Proposition 2.1.8, the paths in $\mathbb{Z}Q_H^{\mathrm{op}}$ from $(0, k)$ to (r, i) have the same length, written as $l_{r,i}$. Since $h_k(0, k) = 1 = \ell_{S_k}(P_k)$; see (3.1.1), we may assume that $l_{r,i} > 0$. Suppose first that $(r, i) \in \Delta$. Then, Δ contains a non-trivial sectional path $(0, k) = (r_0, i_0) \longrightarrow (r_1, i_1) \longrightarrow \cdots \longrightarrow (r_t, i_t) = (r, i)$, where $(r_{j-1}, i_{j-1}) \to (r_j, i_j)$ has valuation, say (d_j, d_j'), for $j = 1, \ldots, t$. Since $\varphi(\Sigma) = \Delta$, we have a sectional path $P_k = \tau^{-r_0}P_{i_0} \longrightarrow \tau^{-r_1}P_{i_1} \longrightarrow \cdots \longrightarrow \tau^{-r_t}P_{i_t} = \tau^{-r}P_i$ in Σ, where the arrow $\tau^{-r_{j-1}}P_{i_{j-1}} \to \tau^{-r_j}P_{i_j}$ also has valuation (d_j, d_j'), for $j = 1, \ldots, t$. In view of Lemma 3.2.2 and Definition 4.2.1, we see that $\ell_S(\tau^{-r}P_i) = d_1' \cdots d_t' = h_i(r, i)$.

Assume now that $(r, i) \notin \Delta$. Since Δ is a leftmost section in $\mathrm{Suc}(0, k)$; see (2.2.5), we see that $(r - 1, i) \in \mathrm{Suc}(0, k)$. Let $(s_j, k_j) \to (r, i)$ with valuation (n_j, n_j'), $j = 1, \ldots, p$, be the arrows in $\mathbb{Z}Q_H^{\mathrm{op}}$ ending with (r, i). Since Γ is convex in $\mathbb{Z}Q_H^{\mathrm{op}}$, we see that $(r - 1, i), (s_j, k_j) \in \mathrm{Suc}(0, k) \cap \Gamma$. Since $l_{r-1,i} < l_{s_j,k_j} < l_{r,i}$ for $1 \le j \le p$, by the induction hypothesis, $h_k(r - 1, i) = \ell_{S_k}(\tau^{1-r}P_i)$ and $h_k(s_j, k_j) = \ell_{S_k}(\tau^{-s_j}P_{k_j})$ for $1 \le j \le p$. Since $\tau^{-s_j}P_{k_j} \to \tau^{-r}P_i$ and $(s_j, k_j) \to (r, i)$ have the same valuation (n_j, n_j'), for $j = 1, \ldots, p$, we deduce from Proposition 1.5.1 an almost split sequence

$$0 \longrightarrow \tau^{1-r}P_i \longrightarrow \oplus_{j=1}^{p}(\tau^{-s_j}P_{k_j})^{n_j'} \longrightarrow \tau^{-r}P_i \longrightarrow 0$$

in $\mathrm{mod}H$. Since ℓ_{S_k} and h_k are additive, we obtain

$$\ell_{S_k}(\tau^{-r}P_i) = \sum_{j=1}^{p} n_j' \ell_{S_k}(\tau^{-s_j}P_{k_j}) - \ell_{S_k}(\tau^{1-r}P_i) = \sum_{j=1}^{p} n_j' h_k(s_j, k_j) - h_k(r - 1, i) = h_k(r, i).$$

The proof of the lemma is completed.

Given a vertex k in Q_H, we shall write $\mathcal{H}_k$ for the hammock $\mathcal{H}_{S_k}$, where S_k is the simple module in $\Gamma_{\mathrm{mod}H}$ associated with k.

Lemma 4.2.3. *Let H be a hereditary artin algebra with a Dynkin ext-quiver Q_H. Consider the projective modules P_i and the injective modules I_i in $\Gamma_{\mathrm{mod}H}$ associated with $i \in Q_H$. Let $\mathcal{H}_k$ be the hammock and h_k the extended hammock function associated with some $k \in Q_H$. Assume that $(r, i) \in \mathrm{Suc}(0, k)$ such that $h_k(s, j) \ge 0$ for all proper predecessors (s, j) of (r, i) in $\mathrm{Suc}(0, k)$.*

(1) If $h_k(r,i) > 0$, then $\tau^{-r}P_i \in \mathcal{H}_k$.
(2) If $(r-1,i) \in \mathrm{Suc}(0,k)$ such that $\tau^{1-r}P_i = I_l$ for some vertex $l \in Q_H$, then $h_k(r,i) = -1$ in case $l = k$, and otherwise, $h_k(r,i) = 0$.

Proof. Consider the canonical embedding $\varphi : \Gamma_{\mathrm{mod}H} \to \mathbb{Z}Q_H^{\mathrm{op}}$, which has a convex image Γ in $\mathbb{Z}Q_H^{\mathrm{op}}$. Let Σ be the P_k-source section in $\Gamma_{\mathrm{mod}H}$, and Δ the $(0,k)$-source section in $\mathbb{Z}Q_H^{\mathrm{op}}$. Furthermore, let $(s_j, k_j) \to (r,i)$ with valuation (n_j, n_j'), $j = 1, \ldots, p$, be the arrows in $\mathbb{Z}Q_H^{\mathrm{op}}$ ending with (r,i). Given a vertex $(s,j) \in \mathrm{Suc}(0,k)$, denote by $l_{s,j}$ the length of paths in $\mathbb{Z}Q_H^{\mathrm{op}}$ from $(0,k)$ to (s,j). In case $l_{r,i} \leq 1$, we have $(r,i) \in \Delta$, and consequently, $\tau^{-r}P_i \in \mathcal{H}_k$; see (4.1.1) and $(r-1,i) \notin \mathrm{Suc}(0,k)$; see (2.2.5). Thus, the two statements hold in this case. Consider the case where $l_{r,i} > 1$.

Suppose first that $(r-1,i) \in \mathrm{Suc}(0,k)$ such that $\tau^{1-r}P_i = I_l$, for some vertex $l \in Q_H$. Writing $S_l = \mathrm{soc}I_l$, we have a short exact sequence

$$0 \longrightarrow S_l \longrightarrow \tau^{1-r}P_i \overset{g}{\longrightarrow} M \longrightarrow 0,$$

where g is minimal left almost split. By the assumption on (r,i) stated in the lemma, $h_k(s_j, k_j) \geq 0$, for all $1 \leq j \leq p$. We may assume that there exists some $0 \leq t \leq p$ such that $(s_j, k_j) \in \Gamma$ if and only if $1 \leq j \leq t$. Given $t < j \leq p$, in view of the induction hypothesis, we see that $h_k(s_j, k_j) = 0$. Since $\tau^{1-r}P_i \to \tau^{-s_j}P_{k_j}$ and $(r-1,i) \to (s_j, k_j)$ have the same valuation (n_j', n_j) for $1 \leq j \leq t$, by Proposition 1.5.1(1), $M \cong \oplus_{j=1}^{t}(\tau^{-s_j}P_{k_j})^{n_j'}$. Since h_k and ℓ_{S_k} are additive, by Lemma 4.2.2, we obtain $h_k(r,i) = \sum_{j=1}^{t} n_j' h_k(s_j, k_j) - h(1-r,i) = \sum_{j=1}^{t} n_j' \ell_{S_k}(\tau^{-s_j}P_{k_j}) - \ell_{S_k}(\tau^{1-r}P_i) = \ell_{S_k}(M) - \ell_{S_k}(\tau^{1-r}P_i)$.

If $l = k$, then $\ell_{S_k}(\ell_{S_k}(\tau^{1-r}P_i) = 1$; see (3.1.1). Thus $\ell_{S_k}(M) = 0$, and hence, $h_k(r,i) = -1$. If $l \neq k$, then $\ell_{S_k}(S_l) = 0$. Thus $\ell_{S_k}(M) = \ell_{S_k}(\tau^{1-r}P_i)$, hence, $h_k(r,i) = 0$. Statement (2) holds in this case.

Suppose now that $h_k(r,i) > 0$. If $(r,i) \in \Delta$, then $\tau^{-r}P_i \in \Sigma \subseteq \mathcal{H}_k$; see (1.7.3). Otherwise, since Δ is a leftmost section of $\mathrm{Suc}(0,k)$; see (2.2.5), $(r-1,i) \in \mathrm{Suc}(0,k)$ and $h_k(r-1,i) \geq 0$ by the assumption on (r,i). Since h_k is additive and $h_k(r,i) > 0$, we may assume that $h_k(s_1, k_1) > 0$. By the induction hypothesis, $\tau^{-s_1}P_{k_1} \in \mathcal{H}_k$. Since Γ is convex in $\mathbb{Z}Q_H^{\mathrm{op}}$, we see that $\tau^{1-r}P_i \in \Gamma_{\mathrm{mod}H}$. Since $h_k(r,i) > 0$, as has been shown, $\tau^{1-r}P_i$ is not injective. Then, $\tau^{-r}P_i \in \Gamma_{\mathrm{mod}H}$ with $h_k(r,i) > 0$. By Lemma 4.2.2, $\tau^{-r}P_i \in \mathcal{H}_k$. Statement (1) holds in this case. The proof of the lemma is completed.

The following statement describes all the hammocks in $\Gamma_{\mathrm{mod}H}$ and tells us an injective module lies in the τ-orbit of which projective module.

Theorem 4.2.4. *Let H be a hereditary artin algebra with a Dynkin ext-quiver Q_H. Consider the projective modules P_i and the injective modules I_i in $\Gamma_{\mathrm{mod}H}$ associated with vertices $i \in Q_H$. Let $\mathcal{H}_k$ be the hammock and h_k the extended hammock function associated with some vertex $k \in Q_H$. Then,*
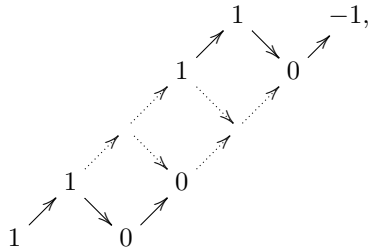(1) *$I_k = \tau^{1-s_k}P_{i_k}$, where $(s_k, i_k) \in \mathrm{Suc}(0,k)$ such that $h_k(s_k, i_k) = -1$, and $h_k(s,j) \geq 0$ for all proper predecessors (s,j) of (s_k, i_k) in $\mathrm{Suc}(0,k)$;*
(2) *$\mathcal{H}_k$ is generated by the modules $\tau^{-r}P_i \in \Gamma_{\mathrm{mod}H}$, where (r,i) lies in the convex hull of $(0,k)$ and (s_k-1, i_k) in $\mathbb{Z}Q_H^{\mathrm{op}}$ and $h_k(r,i) > 0$.*

Proof. Consider the canonical embedding $\varphi : \Gamma_{\mathrm{mod}H} \to \mathbb{Z}Q_H^{\mathrm{op}}$, which has a convex image Γ in $\mathbb{Z}Q_H^{\mathrm{op}}$. By Proposition 1.7.3(1) and Lemma 4.2.2, we easily deduce Statement (2) from Statement (1). So, we only prove Statement (1). Suppose first that $(s_k, i_k) \in \mathrm{Suc}(0,k)$ such that $h_k(s_k, i_k) = -1$ and $h_k(s,j) \geq 0$ for all proper predecessors (s,j) of (s_k, i_k) in $\mathrm{Suc}(0,k)$. By Proposition 4.1.1 and Lemma 4.2.2, (s_k, i_k) is not in the $(0,k)$-source section Δ in $\mathbb{Z}Q_H^{\mathrm{op}}$. By Lemma 2.2.5(1), $(s_k-1, i_k) \in \mathrm{Suc}(0,k)$. Since h_k is additive, we deduce from the assumption on (s_k, i_k) that $h(s_k-1, i_k) > 0$. By Lemma 4.2.3(1), $\tau^{1-s_k}P_{i_k} \in \mathcal{H}_k$. If $\tau^{1-s_k}P_{i_k}$ is not injective, then $\tau^{-s_k}P_{i_k} \in \Gamma_{\mathrm{mod}H}$, and hence, $h_k(s_k, i_k) \geq 0$; see (4.2.2), a contradiction. Thus, $\tau^{1-s_k}P_{i_k}$ is injective, and by Lemma 4.2.3(2), $\tau^{1-s_k}P_{i_k} = I_k$.

Suppose conversely that $I_k = \tau^{-r_k}P_{i_k}$, where $(r_k, i_k) \in \mathbb{Z} \times (Q_H)_0$. Since I_k is a succesor of P_k in $\Gamma_{\mathrm{mod}H}$ by Proposition 1.7.3(1), $(r_k, i_k) \in \mathrm{Suc}(0,k) \cap \Gamma$. Put $s_k = r_k + 1$. We claim $h_k(s,j) \geq 0$, for any proper predecessor (s,j) of (s_k, i_k) in $\mathrm{Suc}(0,k)$. Indeed, if (s,j) is a predecessor of (r_k, i_k) in $\mathrm{Suc}(0,k)$, then $h_k(s,j) \geq 0$ by Lemma 4.2.2. Otherwise, $(s-1,j)$ is a predecessor of (r_k, i_k) in $\mathrm{Suc}(0,k)$. By Lemma 4.2.2, $\tau^{1-s}P_j \in \Gamma_{\mathrm{mod}H}$. If $\tau^{1-s}P_j$ is not injective, then $\tau^{-s}P_j \in \Gamma_{\mathrm{mod}H}$, and by Lemma 4.2.2, $h_k(s,j) \geq 0$. Suppose that $\tau^{1-s}P_j = I_l$ for some $l \in (Q_H)_0$. Since $(s,j) \neq (s_k, i_k)$, we

have $(s-1, j) \neq (r_k, i_k)$. Thus, $l \neq k$, and by Lemma 4.2.3(2), $h_k(s, j) = 0$. This establishes our claim. In particular, by Lemma 4.2.3(2), $h_k(s_k, i_k) = -1$. The proof of the theorem is completed.

Theorem 4.2.4 allows us to determine an injective module lies in which τ-orbit of a projective module. For our later purpose, we shall provide an example for each of the types $\mathbb{A}_n$, $\mathbb{D}_n$ and $\mathbb{E}_6$; see (1.3.8).

Lemma 4.2.5. *Let H be a hereditary artin algebra with a Dynkin ext-quiver Q_H. Consider the projective modules P_i and the injective modules I_i in $\Gamma_{\mathrm{mod}H}$ associated with $i \in Q_H$. If $\bar{Q}_H = \mathbb{A}_n$ with $n \geq 1$, then $I_1 = \tau^{-r_{1,n}} P_n$, where $r_{1,n}$ is the number of arrows in the reduced walk in Q_H from 1 to n.*
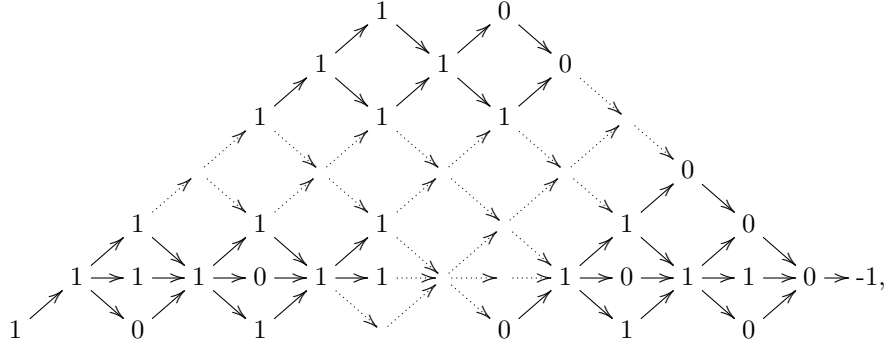
Proof. Let $\bar{Q}_H = \mathbb{A}_n$ with $n \geq 1$. Given $i \in (Q_H)_0$, write $r_{1,i} = \mathfrak{a}(1, i)$, the number of arrows in the reduced walk in Q_H from 1 to i, that is the number of inverse arrows in the reduced walk in Q_H^{op} from 1 to i. In particular, $r_{1,1} = 0$. If $n = 1$, then $I_1 = \tau^{-r_{1,1}} P_1$. Suppose that $n \geq 2$. By Lemma 2.1.6(1) and Proposition 2.2.6, the $(0, 1)$-source section in $\mathbb{Z}Q_H^{\mathrm{op}}$ is $(r_{1,1}, 1) \longrightarrow (r_{1,2}, 2) \longrightarrow \cdots \longrightarrow (r_{1,n}, n)$. Consider the extended hammock function h_1 associated with the vertex 1. Since $\mathbb{Z}Q_H^{\mathrm{op}}$ is trivially valued, by Definition 4.2.1, $h_1(r_{1,i}, i) = 1$ for $i = 1, \ldots, n$. Since h_1 is additive, we can depict its valuation on the convex hull of $(0, 1)$ and $(r_{1,n} + 1, n)$ in $\mathbb{Z}Q_H^{\mathrm{op}}$ as follows:

where the source is $(0, 1)$ and the sink is $(r_{1,n} + 1, n)$. By Theorem 4.2.4(1), $I_1 = \tau^{-r_{1,n}} P_n$. The proof of the lemma is completed.

Lemma 4.2.6. *Let H be a hereditary artin algebra with a Dynkin ext-quiver Q_H. Consider the projective modules P_i and the injective modules I_i in $\Gamma_{\mathrm{mod}H}$ associated with $i \in Q_H$. If $\bar{Q}_H = \mathbb{D}_n$ with $n \geq 4$, then $I_1 = \tau^{2-n} P_1$ in case n is even; and $I_1 = \tau^{3-n-r_{1,2}} P_2$ in case n is odd, where $r_{1,2}$ is the number of arrows in the reduced walk in Q_H from 1 to 2.*

Proof. Assume that $\bar{Q}_H = \mathbb{D}_n$ with $n \geq 4$. For $i \in (Q_H)_0$, write $r_{1,i} = \mathfrak{a}^+(1, i)$, the number of arrows in the reduced walk in Q_H, that is the number of inverse arrows in the reduced walk in Q_H^{op}, from 1 to i. In particular, $r_{1,1} = 0$. By Lemma 2.1.6(1) and Proposition 2.2.6, the $(0, 1)$-source section in $\mathbb{Z}Q_H^{\mathrm{op}}$ is

$$(r_{1,1}, 1) \longrightarrow (r_{1,3}, 3) \longrightarrow (r_{1,4}, 4) \longrightarrow \cdots \longrightarrow (r_{1,n}, n).$$
$$\downarrow$$
$$(r_{1,2}, 2)$$

Consider the extended hammock function h_1 associated with the vertex 1. Since $\mathbb{Z}Q_H^{\mathrm{op}}$ is trivially valued, $h_1(r_{1,i}, i) = 1$ for $i = 1, 2, \ldots, n$. Consider first the case where n is even. Since h_1 is additive, its valuation on the convex hull of $(0, 1)$ and $(n - 1, 1)$ in $\mathbb{Z}Q_H^{\mathrm{op}}$ can be depicted as follows:

where the source is $(0,1)$ and the sink is $(n-1,1)$. By Theorem 4.2.4(1), $I_1 = \tau^{2-n}P_1$. Suppose next that n is odd. Since h_1 additive, the valuation of h_1 on the convex hull of $(0,1)$ and $(r_2 + n - 2, 2)$ in $\mathbb{Z}Q_H^{\mathrm{op}}$ can be depicted as follows:

$$
\begin{array}{c}
\text{(hammock diagram with entries: } 1,\ 0 \text{ at top row; } 1,1,0 \text{; } 1,1,1,0 \text{; } 1,1,1,1,0 \text{;} \\
1 \to 1 \to 1 \to 0 \to 1 \to 1 \dashrightarrow 1 \to 0 \to 1 \to 1 \to 0 \to -1, \\
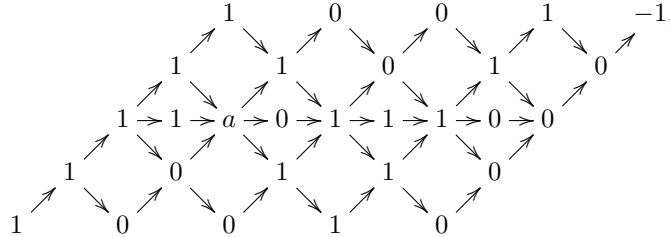1,\ 0,\ 1,\ 0,\ 1,\ 0 \text{ bottom row)}
\end{array}
$$

where the source is $(0,1)$ and the sink is $(r_{1,2} + n - 2, 2)$. By Theorem 4.2.4(1), $I_1 = \tau^{-r_{1,2}-n+3}P_2$. The proof of the lemma is completed.

Lemma 4.2.7. *Let H be a hereditary artin algebra with a Dynkin ext-quiver Q_H. Consider the projective modules P_i and the injective modules I_i in $\Gamma_{\mathrm{mod}H}$ associated with $i \in Q_H$. If $\overline{Q}_H = \mathbb{E}_6$, then $I_1 = \tau^{-r_{1,6}-3}P_6$, where $r_{1,6}$ is the number of arrows in the reduced walk in Q_H from 1 to 6.*

Proof. Assume that $\overline{Q}_H = \mathbb{E}_6$. Given $i \in (Q_H)_0$, let $r_{1,i}$ be the number of arrows in the reduced walk in Q_H, that is, the number of inverse arrows in the reduced walk in Q_H^{op}, from 1 to i. In particular, $r_{1,1} = 0$. By Lemma 2.1.6(1) and Proposition 2.2.6, the $(0,1)$-sourced section in $\mathbb{Z}Q_H^{\mathrm{op}}$ is

$$(r_{1,1}, 1) \longrightarrow (r_{1,2}, 2) \longrightarrow (r_{1,3}, 3) \longrightarrow (r_{1,5}, 5) \longrightarrow (r_{1,6}, 6).$$
$$\downarrow$$
$$(r_{1,4}, 4)$$

Consider the extended hammock function h_1 associated with the vertex 1. Since $\mathbb{Z}Q_H^{\mathrm{op}}$ is trivially valued, $h_1(r_{1,i}, i) = 1$ for $i = 1, \ldots, 6$. So, its valuation on the convex hull of $(0,1)$ and $(r_6 + 4, 6)$ in $\mathbb{Z}Q_H^{\mathrm{op}}$ can be depicted as follows:

$$
\begin{array}{c}
\text{(hammock diagram with entries: } 1,\ 0,\ 0,\ 1,\ -1 \text{ at top row;} \\
1,1,0,1,0 \text{; } 1 \to 1 \to a \to 0 \to 1 \to 1 \to 1 \to 0 \to 0 \text{;} \\
1,0,1,1,0 \text{; } 1,0,0,1,0 \text{ bottom rows)}
\end{array}
$$

where the source is $(0,1)$ and the sink is $(r_{1,6} + 4, 6)$. By Theorem 4.2.4(1), $I_1 = \tau^{-r_{1,6}-3}P_6$. The proof of the lemma is completed.

4.3. PI-PERMUTATION AND PI-INDICES. By Proposition 4.1.1, $\Gamma_{\mathrm{mod}H}$ is finite and convexly embeds in $\mathbb{Z}Q_H^{\mathrm{op}}$ in such a way that every module lies in the τ-orbit of a projective module. We need to determine the number of modules and the injective module lying in the τ-orbit of a projective module. For this purpose, we introduce the notions of the pi-permutation and the pi-index function in the following well-known statement; see [29, (1.8)].

Lemma 4.3.1. *Let H be a hereditary artin algebra with a Dynkin ext-quiver Q_H. Consider the projective modules and the injective modules I_i in $\Gamma_{\mathrm{mod}H}$ associated with vertices $i \in Q_H$. Then there exists a permutation ρ of $(Q_H)_0$, called the pi-permutation, and a function $m : (Q_H)_0 \to \mathbb{N}$, called the pi-index function, such that $\tau^{-m(i)}P_i = I_{\rho(i)}$, for every vertex i in Q_H.*

The following statement is important in our investigation, where $\mathfrak{a}^+(i,j)$ denotes the number of arrows in the reduced walk in Q_H from i to j; see (1.1.1).

Lemma 4.3.2. *Let H be a hereditary artin algebra with a Dynkin ext-quiver Q_H. Consider the pi-permutation ρ and the pi-index function m for $(Q_H)_0$. If i, j are vertices in $\overline{Q}_H$, then*

$$m(i) - m(j) = \mathfrak{a}^+(\rho(i), \rho(j)) - \mathfrak{a}^+(i,j).$$

Proof. Consider the projective modules P_i and the injective modules I_i in $\Gamma_{\mathrm{mod}H}$ associated with $i \in Q_H$. By Proposition 4.1.1, we have a canonical embedding $\varphi : \Gamma_{\mathrm{mod}H} \to \mathbb{Z}Q_H^{\mathrm{op}}$, sending $\tau^{-r}P_i$ to (r, i). Write $r_{p,q} = \mathfrak{a}^+(p,q)$ for $p, q \in (Q_H)_0$. Fix some vertices $i, j \in Q_H$. Observe that $r_{i,j}$ is the number of inverse arrows in the reduced walk in Q_H^{op} from i to j. By Lemma 2.1.6(2), $\mathbb{Z}Q_H^{\mathrm{op}}$ contains a sectional path from $(0, i)$ to $(r_{i,j}, j)$, and hence, a sectional path from $(m(j) - r_{i,j}, i)$ to $(m(j), j)$.

On the other hand, since $r_{\rho(i),\rho(j)}$ is the number of arrows in the reduced walk in Q_H from $\rho(i)$ to $\rho(j)$, by Lemma 3.1.5(2), $\Gamma_{\mathrm{mod}H}$ contains a sectional path from $\tau^{r_{\rho(i),\rho(j)}} I_{\rho(i)}$ to $I_{\rho(j)}$. By definition, $I_{\rho(i)} = \tau^{-m(i)} P_i$ and $I_{\rho(j)} = \tau^{-m(j)} P_j$. Thus, $\Gamma_{\mathrm{mod}H}$ contains a sectional path from $\tau^{r_{\rho(i),\rho(j)} - m(i)} P_i$ to $\tau^{-m(j)} P_j$, which induces a sectional path in $\mathbb{Z}Q_H^{\mathrm{op}}$ from $(m(i) - r_{\rho(i),\rho(j)}, i)$ to $(m(j), j)$. By Proposition 2.2.6, both $(m(j) - r_{i,j}, i)$ and $(m(i) - r_{\rho(i),\rho(j)}, i)$ belong to the $(m(j), j)$-sink section in $\mathbb{Z}Q_H^{\mathrm{op}}$, and consequently, $m(i) - r_{\rho(i),\rho(j)} = m(j) - r_{i,j}$. The proof of the lemma is completed.

Next, we shall show that the pi-permutation of $(Q_H)_0$ is of order at most two. For doing this, we shall extend it to an automorphism of the ext-graph $\overline{Q}_H$ in the following statement.

Proposition 4.3.3. *Let H be a hereditary artin algebra with a Dynkin ext-quiver Q_H. Then the pi-permutation ρ of $(Q_H)_0$ induces a valued graph automorphism of $\overline{Q}_H$, called the pi-automorphism and denoted again by ρ.*

Proof. Let ρ be the pi-permutation of $(Q_H)_0$ and m the pi-index function on $(Q_H)_0$. Consider the projective modules P_i and the injective modules I_i in $\Gamma_{\mathrm{mod}H}$ associated with $i \in (Q_H)_0$. Let Δ_H be the projective section generated by the P_i, and Σ_H the injective section generated by the I_i, in $\Gamma_{\mathrm{mod}H}$. We first construct a valued graph isomorphism $\rho^* : \overline{\Delta}_H \to \overline{\Sigma}_H$, sending P_i to $I_{\rho(i)}$.

Fix $i, j \in (Q_H)_0$. Suppose that $\overline{\Delta}_H$ contains an edge $P_i \text{ ---- } P_j$ with valuation (d, d'). We claim that $\overline{\Sigma}_H$ contains an edge $I_{\rho(i)} \text{ ---- } I_{\rho(j)}$ with valuation (d, d'). First, assume that Δ_H contains an arrow $P_i \to P_j$, whose valuation is then (d, d'); see (1.3.10). Note that $\tau^{-m(i)} P_i = I_{\rho(i)}$ and $\tau^{-m(j)} P_j = I_{\rho(j)}$. By Lemma 3.1.4(2), $\Gamma_{\mathrm{mod}H}$ contains an arrow $\tau^{-m(j)} P_i \to I_{\rho(j)}$, whose valuation is (d, d'); see (1.4.1). So, $m(j) \le m(i)$. If $m(j) = m(i)$, then Σ_H contains an arrow $I_{\rho(i)} \to I_{\rho(j)}$ with valuation (d, d'), and hence, $\overline{\Sigma}_H$ contains an edge $I_{\rho(i)} \text{ ---- } I_{\rho(j)}$ with valuation (d, d'). If $m(j) < m(i)$, then $\Gamma_{\mathrm{mod}H}$ contains an arrow $I_{\rho(j)} \to \tau^{-m(j)-1} P_i$ with valuation (d', d). Since H is hereditary, $\tau^{-m(j)-1} P_i$ is injective, and hence, $\tau^{-m(j)-1} P_i = I_{\rho(i)}$. By definition, $m(i) = m(j) + 1$, and hence, Σ_H has an arrow $I_{\rho(j)} \to I_{\rho(i)}$ with valuation (d', d). So, $\overline{\Sigma}_H$ contains an edge $I_{\rho(j)} \text{ ---- } I_{\rho(i)}$ with valuation (d', d), that is, an edge $I_{\rho(i)} \text{ ---- } I_{\rho(j)}$ with valuation (d, d'). Next, assume that Δ_H contains an arrow $P_j \to P_i$, whose valuation is then (d', d); see (1.3.10). As has been shown, $\overline{\Sigma}_H$ contains an edge $I_{\rho(j)} \text{ ---- } I_{\rho(i)}$ with valuation (d', d), that is, an edge $I_{\rho(i)} \text{ ---- } I_{\rho(j)}$ with valuation (d, d'). This establishes our claim.

Suppose, on the other hand, that $\overline{\Sigma}_H$ contains an edge $I_{\rho(i)} \text{ ---- } I_{\rho(j)}$. We shall show that $\overline{\Delta}_H$ contains an edge $P_i \text{ ---- } P_j$. Note that $P_i = \tau^{m(i)} I_{\rho(i)}$ and $P_j = \tau^{m(j)} I_{\rho(j)}$. Assume first that Σ_H contains an arrow $I_{\rho(i)} \to I_{\rho(j)}$. By Lemma 3.1.4(2), $\Gamma_{\mathrm{mod}H}$ contains an arrow $P_i \to \tau^{m(i)} I_{\rho(j)}$. So, $m(i) \le m(j)$. If $m(i) = m(j)$, then Σ_H contains an arrow $P_i \to P_j$, and hence, $\overline{\Delta}_H$ contains an edge $P_i \text{ ---- } P_j$. If $m(i) < m(j)$, then $\Gamma_{\mathrm{mod}H}$ contains an arrow $\tau^{m(i)+1} I_{\rho(j)} \to P_i$. Since H is hereditary, $\tau^{m(i)+1} I_{\rho(j)}$ is projective. Then $m(j) = m(i) + 1$, and hence, Σ_H contains an arrow $P_j \to P_i$. So, $\overline{\Delta}_H$ contains an edge $P_i \text{ ---- } P_j$. In case Σ_H contains an arrow $I_{\rho(j)} \to I_{\rho(i)}$, by a similar argument, $\overline{\Delta}_H$ contains an edge $P_i \text{ ---- } P_j$. Thus, we have a valued graph isomorphism $\rho^* : \overline{\Delta}_H \to \overline{\Sigma}_H$, sending P_i to $I_{\rho(i)}$.

Finally, the valued quiver isomorphisms stated in Theorem 3.1.2 induce a valued graph isomorphism $\overline{Q}_H^{\mathrm{op}} \to \overline{\Delta}_H$ sending i to P_i, and a valued graph isomorphism $\overline{\Sigma}_H \to \overline{Q}_H^{\mathrm{op}}$ sending I_i to i. Composing

these two isomorphisms with ρ^*, we obtain a valued graph automorphism $\overline{Q}_H^{\text{op}} \to \overline{Q}_H^{\text{op}}$, sending i to $\rho(i)$. This induces a desired valued graph automorphism $\rho : \overline{Q}_H \to \overline{Q}_H$, sending i to $\rho(i)$. The proof of the proposition is completed.

We collect some easy properties of the pi-automorphism in the following statement.

Lemma 4.3.4. *Let H be a hereditary artin algebra with a Dynkin ext-quiver Q_H, and let ρ be the pi-automorphism of $\overline{Q}_H$.*

(1) *If i is a vertex in Q_H of weight 3, then $\rho(i) = i$.*

(2) *If $i \mathbin{\rule[0.5ex]{2em}{0.4pt}} j$ is a non-trivially valued edge in $\overline{Q}_H$, then $\rho(i) = i$ and $\rho(j) = j$.*

(3) *If $i \mathbin{\rule[0.5ex]{2em}{0.4pt}} j \mathbin{\rule[0.5ex]{2em}{0.4pt}} k$ is a reduced walk in $\overline{Q}_H$, where j has only two neighbors i and k, such that $\rho(i) = i$ and $\rho(j) = j$, then $\rho(k) = k$.*

Proof. (1) Being a Dynkin diagram, $\overline{Q}_H$ has at most one vertex of weight 3; see (1.3.8). And by Lemma 1.3.7, $\text{w}(i) = \text{w}(\rho(i))$ for any vertex i in Q_H. Thus, if $\text{w}(i) = 3$, then $\rho(i) = i$.

(2) Let $i \mathbin{\rule[0.5ex]{2em}{0.4pt}} j$ be an edge in $\overline{Q}_H$ with a non-trivial valuation (d_{ij}, d_{ji}). Since $\overline{Q}_H$ is a Dynkin digram, this is the only non-trivially valued edge, and moreover, $d_{ij} \neq d_{ji}$; see (1.3.8). Since ρ is a valuded graph automorphism, $\rho(i), \rho(j) \in \{i, j\}$ and $d_{\rho(i)\rho(j)} = d_{ij}$. If $\rho(i) = j$, then $\rho(j) = i$. So, $d_{ji} = d_{\rho(i)\rho(j)} = d_{ij}$, a contradiction. Thus, $\rho(i) = i$ and $\rho(j) = j$.

(3) Let $i \mathbin{\rule[0.5ex]{2em}{0.4pt}} j \mathbin{\rule[0.5ex]{2em}{0.4pt}} k$ be a reduced walk in $\overline{Q}_H$, where j has only two neighbors i and k. If $\rho(j) = j$, then $j \mathbin{\rule[0.5ex]{2em}{0.4pt}} \rho(k)$ is an edge in $\overline{Q}_H$, and hence, $\rho(k) \in \{i, k\}$. If $\rho(i) = i$, then $\rho(k) = k$. The proof of the lemma is completed.

We are ready to determine the order of the pi-automorphism for every Dynkin type. In particular, the pi-automorphism is the identity for all non-simply laced types.

Theorem 4.3.5. *Let H be a hereditary artin algebra of Dynkin type, and let ρ be the pi-automorphism ρ of the ext-graph $\overline{Q}_H$.*

(1) *If $\overline{Q}_H$ is of type $\mathbb{A}_1$, $\mathbb{G}_2$, $\mathbb{F}_4$, $\mathbb{E}_7$, $\mathbb{E}_8$, $\mathbb{B}_n(n \geq 2)$, $\mathbb{C}_n(n \geq 3)$ or $\mathbb{D}_n(n \geq 4 \text{ even})$, then ρ is the identity.*

(2) *If $\overline{Q}_H$ is of type $\mathbb{A}_n(n \geq 2)$, $\mathbb{D}_n(n \geq 5 \text{ odd})$ or $\mathbb{E}_6$, then ρ is of order 2.*

Proof. We may assume that $\overline{Q}_H$ is a canonical Dynkin diagram as stated in Definition 1.3.8. The theorem holds trivially in case $\overline{Q}_H = \mathbb{A}_1$. We shall proceed case by case for all other cases.

1) Suppose that $\overline{Q}_H = \mathbb{G}_2$ or $\mathbb{B}_2$. Then it contains only one edge, which is non-trivially valued. By Lemma 4.3.4(2), ρ is the identity.

2) Suppose that $\overline{Q}_H = \mathbb{F}_4$. By Lemma 4.3.4(2), $\rho(2) = 2$ and $\rho(3) = 3$. And by Lemma 4.3.4(3), $\rho(4) = 4$. Thus, $\rho(1) = 1$. Therefore, ρ is the identity.

3) Suppose that $\overline{Q}_H$ is $\mathbb{E}_7$ or $\mathbb{E}_8$. Since 3 is the only vertex of weight 3, by Lemma 4.3.4(1), $\rho(3) = 3$. Thus, $\{\rho(2), \rho(4), \rho(5)\} \subseteq \{2, 4, 5\}$. Since $\text{w}(4) = 1$, we have $\rho(4) = 4$. If $\rho(2) = 5$, then $\rho(1) = 6$, a contradiction to Lemma 1.3.7 for $\rho(6) = 2$. Thus, $\rho(2) = 2$, and consequently, $\rho(1) = 1$ and $\rho(5) = 5$. Since each of the vertices $5, 6, 7$ has at most two neighbors, we deduce from 4.3.4(3) that $\rho(i) = i$ for all vertices $i > 5$. So, ρ is the identity.

4) Suppose that $\overline{Q}_H = \mathbb{B}_n$ or $\mathbb{C}_n$ with $n \geq 3$. By Lemma 4.3.4(2), $\rho(1) = 1$ and $\rho(2) = 2$. Since every vertex i with $1 < i < n$ has exactly two neighbors, we easily deduce from Lemma 4.3.4(3) that $\rho(i) = i$ for all $3 \leq i \leq n$. So, ρ is the identity.

5) Suppose that $\overline{Q}_H = \mathbb{D}_n$ with $n \geq 4$ even. By Lemma 4.2.6, $I_1 = \tau^{2-n}P_1$. Using the same argument there, we can show that $I_2 = \tau^{2-n}P_2$. Thus, $\rho(1) = 1$ and $\rho(2) = 2$. Since 3 is of weight 3, by Lemma 4.3.4(1), $\rho(3) = 3$. Then, $\rho(4) = 4$. Since every vertex i with $4 \leq i < n$ has exactly two neighbors, by Lemma 4.3.4(3), $\rho(i) = i$ for all $4 < i \leq n$. So, ρ is the identity.

6) Suppose that $\overline{Q}_H = \mathbb{A}_n$ with $n \geq 2$. By Lemma 4.2.5, $I_1 = \tau^{-r_{1,n}}P_n$ with $r_{1,n} \geq 0$. So, $\rho(n) = 1$. Since every vertex i with $1 < i < n$ has two neighbors, $\rho(1) = n$. Since $n - 1$ is the only neighbor of n, we have $\rho(2) = n - 1$. Similarly, $\rho(i) = n + 1 - i$ for all $1 \leq i \leq n$. Hence, ρ is of order 2.

7) Suppose that $\overline{Q}_H = \mathbb{D}_n$ with $n \geq 5$ odd. by Lemma 4.3.4(1), $\rho(3) = 3$. Since $\tau^{-s}P_2 = I_1$ for some $s \geq 0$; see (4.2.6), $\rho(2) = 1$. Since the vertex 4 has two neighbors, $\rho(2) = 1$, and hence, $\rho(4) = 4$.

Since every vertex i with $3 < i < n$ has only two neighbors, by Lemma 4.3.4(3), $\rho(i) = i$ for $5 \le i \le n$. So, ρ is of order 2.

8) Suppose $\overline{Q}_H = \mathbb{E}_6$. By Lemma 4.3.4(1), $\rho(3) = 3$. Since the vertex 4 has only one neighbors, $\rho(4) = 4$. By Lemma 4.2.7, $\tau^{-t}P_6 = I_1$ for some $t \ge 0$. So, $\rho(6) = 1$. Then $\rho(1) = 6$. Now, $\rho(2) = 5$ and $\rho(5) = 2$. So ρ is of order 2. The proof of the theorem is completed.

4.4. COXETER ORDER. In case H is of finite representation type, it is well-known that the Coxeter transformation C_H of $K_0(\mathrm{mod}H)$ is of finite order; see [29, (4.1)]. We call this finite order the *Coxeter order* and write as $|C_H|$. The following statement reinforces this fact in particular,.

Proposition 4.4.1. *Let H be a hereditary artin algebra with a Dynkin ext-quiver Q_H. Consider the Auslander-Reiten translation τ_D of $\Gamma_{D^b(\mathrm{mod}H)}$, the pi-permutation ρ and the pi-index function m for $(\overline{Q}_H)_0$. Then, the Coxeter transformation C_H for H is of finite order $|C_H|$ such, for any $i \in Q_H$, that*

$$|C_H| = m(i) + m(\rho(i)) + 2 \quad and \quad \tau_D^{-|C_H|}P_i = P_i[2].$$

Proof. First, we claim that $a := m_i + m_{\rho(i)}$ is a constant, for all $i \in (Q_H)_0$. Fix $i, j \in (Q_H)_0$. By Lemma 4.3.2, $m(i) = m(j) - r_{i,j} + r_{\rho(i),\rho(j)}$, where $r_{p,q}$ is the the number of arrows in the reduced walk in Q_H from p to q. Since $\rho^2 = \mathrm{id}$ by Theorem 4.3.5, $m(\rho(i)) = m(\rho(j)) - r_{\rho(i),\rho(j)} + r_{i,j}$. Thus, $m(i) + m(\rho(i)) = m(j) + m(\rho(j))$. This establishes our claim.

Now, since $\tau^{-m(i)}P_i = I_{\rho(i)}$ by definition and $\tau_D^{-}I_{\rho(i)} = P_{\rho(i)}[1]$; see [18, (4.3)] and [5, (7.2)], we obtain $\tau_D^{-(m(i)+1)}P_i = \tau_D^{-}(\tau^{-(m(i)}P_i) = P_{\rho(i)}[1]$. This yields

$$\tau_D^{-(a+2)}P_i = \tau_D^{-(m(\rho(i))+1)}(\tau_D^{-(m(i)+1)}P_i) = \tau_D^{-(m(\rho(i))+1)}(P_{\rho(i)}[1]) = P_{\rho^2(i)}[1][1] = P_i[2].$$

It remains to show that C_H is of order $a + 2$. Given $i \in (Q_H)_0$, from Proposition 3.3.2 we see that

$$C_H^{-(a+2)}(\underline{\dim}P_i) = \underline{\dim}\tau_D^{-(a+2)}P_i = \underline{\dim}P_i[2] = (-1)^2\underline{\dim}P_i = \underline{\dim}P_i.$$

Since $\{\underline{\dim}P_1, \dots, \underline{\dim}P_n\}$ is a basis for $K_0(\mathrm{mod}H)$, $C_H^{-(a+2)} = \mathrm{id}$. Consider now an integer t with $1 \le t \le a + 1 = m(1) + m(\rho(1)) + 1$. Suppose first that $t \le m(1)$. Then, $\tau_D^{-t}P_1 = \tau^{-t}P_1 \in \Gamma_{\mathrm{mod}H}$ with $\tau^{-t}P_1 \not\cong P_1$. Since the modules in $\Gamma_{\mathrm{mod}H}$ are determined by their composition factors; see [2, (IX.2.3)], $\underline{\dim}\tau^{-t}P_1 \ne \underline{\dim}P_1$. That is, $C_H^{-t}(\underline{\dim}P_1) \ne \underline{\dim}P_1$. Suppose now that $m(1) + 1 \le t$. Then $0 \le t - (m(1) + 1) \le m(\rho(1))$. Therefore,

$$\tau_D^{-t}P_1 = \tau_D^{-(t-(m(1)+1))}(\tau_D^{-(m(1)+1)}P_1) = \tau_D^{-(t-(m(1)+1))}(P_{\rho(1)}[1]) = M[1],$$

where $M = \tau^{-(t-(m(1)+1))}P_{\rho(1)} \in \Gamma_{\mathrm{mod}H}$. Applying again Proposition 3.3.2, we see that

$$C_H^{-t}(\underline{\dim}P_1) = \underline{\dim}\,\tau_D^{-t}P_1 = \underline{\dim}\,M[1] = -\underline{\dim}M \ne \underline{\dim}P_1.$$

So, C_H is indeed of order $a + 2$. The proof of the proposition is completed.

The Coxeter orders are given for each canonical Dynkin diagram with a particular orientation in [2, Pages 289-290]. The following statement says that they are independent of the orientation.

Proposition 4.4.2. *Let H be a hereditary artin algebra with a Dynkin ext-quiver Q_H. Then the Coxeter order $|C_H|$ for H is independent of the orientation of Q_H and given by the following table:*

$\overline{Q}_H$	$\mathbb{A}_n$	$\mathbb{B}_n$	$\mathbb{C}_n$	$\mathbb{D}_n$	$\mathbb{E}_6$	$\mathbb{E}_7$	$\mathbb{E}_8$	$\mathbb{F}_4$	$\mathbb{G}_2$
$|C_H|$	$n+1$	$2n$	$2n$	$2(n-1)$	12	18	30	12	6

Proof. Assume that $\overline{Q}_H$ is a canonical Dynkin diagram; see (1.3.8). Consider the pi-permutation ρ and the mi-index function m for $(Q_H)_0$. Suppose first that $\overline{Q}_H = \mathbb{A}_n$ with $n \ge 1$. In view of Lemma 4.2.5, we see that $\tau^{-r_{1,n}}P_n = I_1$ and $\tau^{-r_{n,1}}P_1 = I_n$, where $r_{i,j}$ is the number of arrows in the reduced walk in Q_H from i to j. Thus, $m(n) = r_{1,n}$ and $m(1) = r_{n.1}$. Since $r_{n,1}$ is also the number of inverse arrows in the reduced walk in Q_H from 1 to n, we have $m(1) + m(n) = n - 1$. Since $\rho(1) = n$, from Propsoition 4.4.1 we deduce that $|C_H| = (n - 1) + 2 = n + 1$.

Suppose now that $\overline{Q}_H \ne \mathbb{A}_n$. Then, $\overline{Q}_H$ contains a vertex s, which is of weight 3 or incident to a non-trivially valued edge. By Lemma 4.3.4, $\rho(s) = s$, and by Proposition 4.4.1, $|C_H| = 2m(s) + 2$.

Let h_s be the extended hammock function on $\mathrm{Suc}(0, s)$. By Theorem 4.2.4(1), $m(s)$ is such that $h_s(m(s) + 1, s) = -1$ and $h_s(s, j) \geq 0$ for all proper predecessors (s, j) of $(m(s) + 1, s)$ in $\mathrm{Suc}(0, s)$.

Let H' be a hereditary artin algebra with $\overline{Q}_{H'} = \overline{Q}_H$. Consider the pi-permutation ρ' and the pi-index function m' for $(Q_{H'})_0 = (Q_H)_0$. Let h'_s be the extended hammock function on $\mathrm{Suc}_{H'}(0, s)$, the full subquiver of $\mathbb{Z}Q_{H'}^{\mathrm{op}}$ generated by the successors of $(0, s)$. Since $\rho'(s) = s$, as previously seen, $|C_{H'}| = 2m'(s) + 2$, where $m'(s)$ is such that $h'_s(m'(s) + 1, s) = -1$ and $h'_s(s, j) \geq 0$ for all proper predecessors (s, j) of $(m'(s) + 1, s)$ in $\mathrm{Suc}_{H'}(0, s)$.

By Proposition 2.2.6 and Lemma 2.1.6, the vertices of the $(0, s)$-source section Δ in $\mathbb{Z}Q_H^{\mathrm{op}}$ are (r_i, i), where $i \in (Q_H)_0$, and r_i is the number of inverse arrows in the reduced walk in Q_H^{op}, that is, the number of arrows in the reduced walk in Q_H, from s to i. Since Δ is a leftmost section in $\mathrm{Suc}(0, s)$; see (2.2.5), the vertices of $\mathrm{Suc}(0, s)$ are $(r_i + p, i)$ with $p \geq 0$ and $i \in (Q_H)_0$. Similarly, the vertices of the $(0, s)$-source section Δ' in $\mathbb{Z}Q_{H'}^{\mathrm{op}}$ are (r'_i, i) with $i \in (\overline{Q}_{H'})_0 = (\overline{Q}_H)_0$ and r'_i the number of arrows in the reduced walk in $Q_{H'}$ from s to i, and those of $\mathrm{Suc}_{H'}(0, s)$ are $(r'_i + p, i)$ with $i \in (\overline{Q}_H)_0$ and $p \geq 0$.

Fix $i, j \in \overline{Q}_H$. By Lemma 2.1.6, Δ contains an arrow $(r_i, i) \to (r_j, j)$ if and only if $\overline{Q}_H$ (that is $\overline{Q}_{H'}$) contains an edge $i \;\text{---}\; j$ if and only if Δ' contains an arrow $(r'_i, i) \to (r'_j, j)$; and in this case, the arrows $(r_i, i) \to (r_j, j)$ and $(r'_i, i) \to (r'_j, j)$ have the same valuation as $i \;\text{---}\; j$. So, we have a valued quiver isomorphism $f : \Delta \to \Delta'$, sending (r_i, i) to (r'_i, i). Clearly, f extends to a valued translation quiver isomorphism $f : \mathrm{Suc}(0, s) \to \mathrm{Suc}_{H'}(0, s)$, sending $(r_i + p, i)$ to $(r'_i + p, i)$, for all $p \geq 0$. Since $r_s = r'_s = 0$, we have $f(p, s) = (p, s)$ for all $p \geq 0$.

By Definition 4.2.1, $h'_s(r'_i, i) = h'_s(f(r_i, i)) = h_s(r_i, i)$ for all $(r_i, i) \in \Delta_0$. Since h_s and h'_s are additive, $h'_s(r'_i + p) = h_s(r_i + p, i)$ for all $p \geq 0$ and $i \in (\overline{Q}_H)_0$. In particular, $h'_s(m(s)+1, s) = h_s(m(s)+1, s) = -1$. If $(r'_i + p, i)$ is a proper predecessor of $(m_i(s) + 1, s)$ in $\mathbb{Z}Q_H^{\mathrm{op}}$, then $(r_i + p, i)$ is a proper predecessor of $(m_i(s) + 1, s)$ in $\mathbb{Z}Q_H^{\mathrm{op}}$, and hence, $h'_s(r'_i + p, i) = h_s(r_i + p, i) \geq 0$. Therefore, $m'(s) = m(s)$, and consequently, $|C_{H'}| = |C_H|$. Thus, $|C_H|$ is independent of the orientation of Q_H. Using the Coxeter orders given in [2, Pages 289-290], we obtain the table stated in the proposition. The proof of the proposition is completed.

Remark 4.4.3. As in the first part of the proof of Proposition 4.4.2, one can also use hammocks and hammock functions to explicitly compute the Coxeter order case by case.

4.5. SHAPE OF THE AUSLANDER-REITEN QUIVER. Applying our previous results, we can describe the precise shape of $\Gamma_{\mathrm{mod}H}$ in terms of Q_H in the following statement.

Theorem 4.5.1. *Let H be a hereditary artin algebra with a canonical Dynkin ext-quiver Q_H. Let P_i be the projective modules and I_i are the injective modules in $\Gamma_{\mathrm{mod}H}$ associated with $i \in (Q_H)_0$. Then $\tau^{-m(i)}P_i = I_{\rho(i)}$, and there exists a convex embedding $\Gamma_{\mathrm{mod}H} \to \mathbb{Z}Q_H^{\mathrm{op}}$, sending $\tau^{-r}P_i$ to (r, i), where ρ is the pi-permutation of $(Q_H)_0$ and m is the pi-index function on $(Q_H)_0$ given case by case as follows.*
(1) *If $\overline{Q}_H = \mathbb{A}_n$ with $n \geq 1$, then $\rho(i) = n + 1 - i$ and $m(i) = \mathfrak{a}^+(1, i) + \mathfrak{a}^-(1, n + 1 - i)$, for $i \in (Q_H)_0$.*
(2) *If $\overline{Q}_H = \mathbb{E}_6$, then $\rho = (16)(25)$ and $m(i) = 5 - \mathfrak{a}^+(i, 3) + \mathfrak{a}^+(\rho(i), 3)$ for $i \in (Q_H)_0$.*
(3) *If $\overline{Q}_H = \mathbb{D}_n$ ($n \geq 5$ odd), then $\rho = (12)$ and $m(i) = n - 2$ for all $3 \leq i \leq n$. Moreover, $m(1) = (n - 2) - \mathfrak{a}^+(1, 3) + \mathfrak{a}^+(2, 3)$ and $m(2) = (n - 2) + \mathfrak{a}^+(1, 3) - \mathfrak{a}^+(2, 3)$.*
(4) *If $\overline{Q}_H = \mathbb{G}_2, \mathbb{F}_4, \mathbb{E}_7, \mathbb{E}_8, \mathbb{B}_n$ ($n \geq 2$), $\mathbb{C}_n$ ($n \geq 3$) or $\mathbb{D}_n$ ($n \geq 4$ even), then we have $\rho(i) = i$ and $m(i) = \frac{1}{2}|C_H| - 1$ for all $i \in Q_H$, where $|C_H|$ is the Coxeter order for H.*

Proof. By Proposition 4.1.1, we have a canonical embedding $\varphi : \Gamma_{\mathrm{mod}H} \to \mathbb{Z}Q_H^{\mathrm{op}} : \tau^{-r}P_i \mapsto (r, i)$, which has a convex image in $\mathbb{Z}Q_H^{\mathrm{op}}$. By Lemma 4.3.1, $\tau^{-m(i)}P_i = I_{\rho(i)}$ for all $i \in (Q_H)_0$. Write $r_{i,j} = \mathfrak{a}^+(i, j)$, for $i, j \in (Q_H)_0$.

(1) Suppose that $\overline{Q}_H = \mathbb{A}_n$ with $n \geq 1$. As seen in the proof of Theorem 4.3.5, $\rho(i) = n + 1 - i$ for all $\in (Q_H)_0$. Moreover, by Lemma 4.2.5, $\tau^{-r_{1,n}}P_n = I_1$. Thus, $m(n) = r_{1,n}$. Given $i \in (Q_H)_0$, in view of Lemma 4.3.2, we see that $m(i) = m(n) - r_{i,n} + r_{n+1-i,1} = r_{1,n} - r_{i,n} + r_{n+1-i,n} = r_{1,i} + r_{n+1-i,1}$.

(2) Suppose that $\overline{Q}_H = \mathbb{E}_6$. In view of the proof of Theorem 4.3.5, we see that $\rho = (16)(25)$. In particular, $\rho(3) = 3$. Since $|C_H| = 12$; see (4.4.2), we deduce from Proposition 4.4.1 that $m(3) = 5$. Given $i \in Q_H$, by Lemma 4.3.2, $m(i) = 5 - r_{i,3} + r_{\rho(i),3}$.
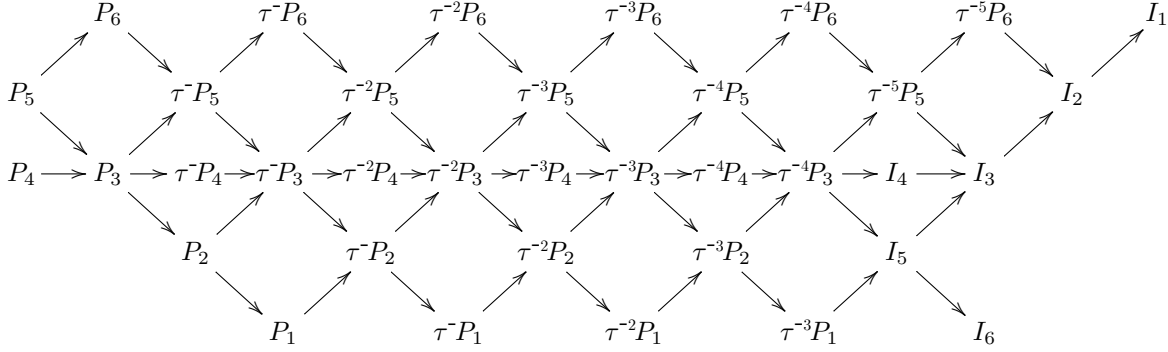
(3) Suppose that $\overline{Q}_H = \mathbb{D}_n$ with $n \geq 5$ odd. In view of the proof of Theorem 4.3.5, we see that $\rho = (12)$. In particular, $\rho(i) = i$ for all $3 \leq i \leq n$. Since $|C_H| = 2(n-1)$; see (4.4.2), we deduce from Proposition 4.4.1 that $m(i) = n-2$, for $3 \leq i \leq n$. Moreover, by Lemma 4.3.2, $m(1) = (n-2)-r_{1,3}+r_{2,3}$ and $m(2) = (n-2)+r_{1,3}-r_{2,3}$.

(4) Suppose that $\overline{Q}_H = \mathbb{G}_2, \mathbb{F}_4, \mathbb{E}_7, \mathbb{E}_8, \mathbb{B}_n(n \geq 2), \mathbb{C}_n(n \geq 3)$ or $\mathbb{D}_n(n \geq 4$ even$)$. By Theorem 4.3.5(1), we have $\rho(i) = i$, and by Proposition 4.4.1, $|C_H| = 2m(i) + 2$, for all $i \in Q_H$. The proof of the theorem is completed.
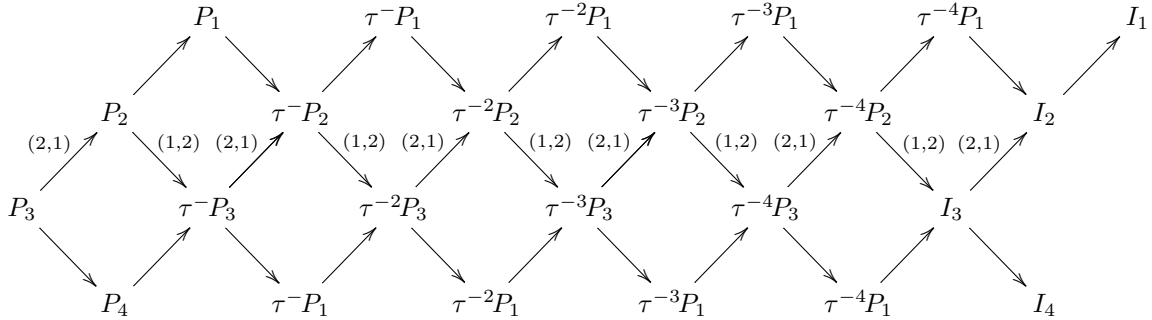
Example 4.5.2. Let H be a hereditary artin algebre with ext-quiver

$$Q_H: \qquad 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 5 \longleftarrow 6.$$
with 4 above 3 (arrow $4 \to 3$).

Since Q_H is of type $\mathbb{E}_6$, by Theorem 4.5.1(2), we have $\rho(1) = 6$; $\rho(2) = 5$; $\rho(3) = 3$; $\rho(4) = 4$; $\rho(5) = 2$ and $\rho(6) = 1$. Moreover, $m(1) = m(2) = 4$; $m(3) = m(4) = 5$; $m(5) = m(6) = 6$. Thus, $\Gamma_{\mathrm{mod}H}$ is of the following shape:

Example 4.5.3. Let H be a hereditary artin algebra with ext-quiver $Q_H: \; 1 \longrightarrow 2 \xrightarrow{(1,2)} 3 \longleftarrow 4.$ Since $\overline{Q}_H = \mathbb{F}_4$, by Theorem 4.5.1(4), $\rho(i) = i$ and $m(i) = 5$, for all $1 \leq i \leq 4$. Thus, $\Gamma_{\mathrm{mod}H}$ is of the following shape

The following statement is interesting in its own right.

Proposition 4.5.4. *Let H be a hereditary artin algebra of Dynkin type. If P is a projective module and I is an injective module in $\Gamma_{\mathrm{mod}H}$, then I is a successor of P in $\Gamma_{\mathrm{mod}H}$.*

Proof. Let P be a projective module and I an injective module in $\Gamma_{\mathrm{mod}H}$. By Proposition 3.1.6, $\Gamma_{\mathrm{mod}H}$ contains a P-source section Δ. Then $I = \tau^s M$ for some $M \in \Delta$ and $s \in \mathbb{Z}$. If $s > 0$, then $\tau^- I = \tau^{s-1}M \in \Gamma_{\mathrm{mod}H}$, which gives a contradiction since $\tau^- I = 0$. Thus, $s \leq 0$, and consequently, I is a successor of P. The proof of the proposition is completed.

4.6. APPLICATIONS. Our first application is to compute the number of non-isomorphic indecomposable modules in $\text{mod}\,H$. Applying a series of results in [29] and [13], we see that this number was given case by case in [13] and [17], and also by the number of positive roots of Dynkin diagrams; see [14, 17], which was given in [12]. However, using our previous results, we obtain this number in a direct and easy way.

Theorem 4.6.1. *Let H be a hereditary artin algebra with a Dynkin ext-quiver Q_H and Coxeter order $|C_H|$. Then, the number of non-isomorphic indecomposable modules in $\text{mod}\,H$ is equal to*

$$\frac{1}{2}n|C_H|,$$

where n is the number of non-isomorphic simple modules in $\text{mod}\,H$.

Proof. Consider the pi-permutation ρ and the pi-index function m for $(Q_H)_0 = \{1, \ldots, n\}$. Let P_i be the projective module and I_i the injective module in $\Gamma_{\text{mod}\,H}$ associated with $i \in (Q_H)_0$. Since $\tau^{-m(i)} = I_{\rho(i)}$, the τ-orbit of P_i contains exactly $m(i) + 1$ modules. Let t be the number of modules in $\Gamma_{\text{mod}\,H}$. Since the P_i generate a section in $\Gamma_{\text{mod}\,H}$, we see that $t = \sum_{i=1}^{n}(m(i) + 1)$. And since ρ is a permutation, $t = \sum_{i=1}^{n}(m(\rho(i)) + 1)$. Applying Proposition 4.4.1 yields $2t = \sum_{i=1}^{n}(m(i) + m(\rho(i)) + 2) = n|C_H|$. The proof of the theorem is completed.

Now, we shall compute the nilpotency of the radical of $\text{mod}\,H$. This has been essentially done by Zacharia using preprojective partitions; see [35, Section 4]. However, we shall provide an alternative approach. Since Q_H is a Dynkin diagram, by Propositions 2.1.8 and 4.1.1, any two parallel paths in $\Gamma_{\text{mod}\,H}$ and $\Gamma_{D^b(\text{mod}\,H)}$ have the same length. So we may introduce the following definition.

Definition 4.6.2. Let H be a hereditary artin algebra of Dynkin type.
(1) If $\Gamma_{\text{mod}\,H}$ contains a path from M to N, then we define the *distance* between M and N, written as $\text{dist}(M, N)$, to be the length of any path from M to N in $\Gamma_{\text{mod}\,H}$.
(2) If $\Gamma_{D^b(\text{mod}\,H)}$ contains a path from $M^\bullet$ to $N^\bullet$, then we define the *distance* between $M^\bullet$ and $N^\bullet$, written as $\text{dist}(M^\bullet, N^\bullet)$, to be the length of any path from $M^\bullet$ to $N^\bullet$ in $\Gamma_{D^b(\text{mod}\,H)}$.

The following statement allows us to easily determine the depth of any map in $\text{mod}\,H$.

Lemma 4.6.3. *Let H be a hereditary artin algebra of Dynkin type. If $f : M \to N$ is a non-zero map with $M, N \in \Gamma_{\text{mod}\,H}$, then $\text{dp}(f) = \text{dist}(M, N)$.*

Proof. Let $f : M \to N$ be a non-zero map with $M, N \in \Gamma_{\text{mod}\,H}$. Since H is representation-finite, $f \notin \text{rad}^s(M, N)$ for some $s > 0$; see [2, (V.7.6)]. Thus, $\text{dp}(f) = t < \infty$. If $t = 0$, then $M = N$ and $\text{dist}(M, N) = 0$. Otherwise, by Lemma 1.6.5, $\Gamma_{\text{mod}\,H}$ contains a path from M to N of length t. Hence, $\text{dist}(M, N) = t$. The proof of the lemma is completed.

We are ready to provide a new proof for Zacharia's theorem as follows; see [35, (4.11)].

Theorem 4.6.4. *Let H be a hereditary artin algebra with a Dynkin ext-quiver Q_H and Coxeter order $|C_H|$. Then, the radical of $\text{mod}\,H$ is nilpotent of nilpotency $|C_H| - 1$.*

Proof. Given $i \in (Q_H)_0 = \{1, \ldots, n\}$, consider the simple module $S_i \in \Gamma_{\text{mod}\,H}$ with a projective cover $\pi_i : P_i \to S_i$ and an injective envelope $\iota_i : S_i \to I_i$, where $P_i, I_i \in \Gamma_{\text{mod}\,H}$. Applying Theorem 2.7 in [11] followed by Lemma 4.6.3, we see that the nilpotency of $\text{rad}(\text{mod}\,H)$ is equal to

$$\sup\{\text{dp}(\iota_1\pi_1), \ldots, \text{dp}(\iota_n\pi_n)\} + 1 = \sup\{\text{dist}(P_1, I_1), \ldots, \text{dist}P_n, I_n)\} + 1.$$

Thus, it suffices to show that $\text{dist}(P_i, I_i) = |C_H| - 2$, for any $i \in (Q_H)_0$. In fact, since $\Gamma_{\text{mod}\,H}$ is a convex valued translation subquiver of $\Gamma_{D^b(\text{mod}\,H)}$; see (4.1.2), we have $\text{dist}(M[0], N[0]) = \text{dist}(M, N)$, for all $M, N \in \Gamma_{\text{mod}\,H}$. And since $\tau_D^- I_i[0] = P_i[1]$; see [5, (7.2)], we deduce

$$\text{dist}(P_i[0], P_i[1]) = \text{dist}(P_i[0], I_i[0]) + \text{dist}(I_i[0], P_i[1]) = \text{dist}(P_i, I_i) + 2.$$

As a consequence, we obtain

$$\text{dist}(P_i[0], P_i[2]) = \text{dist}(P_i[0], P_i[1]) + \text{dist}(P_i[1], P_i[2]) = 2\text{dist}(P_i[0], P_i[1]) = 2(\text{dist}(P_i, I_i) + 2).$$

On the other hand, by Proposition 4.4.1, $\text{dist}(P_i[0], P_i[2]) = \text{dist}(P_i[0], \tau_D^{-|C_H|} P_i[0]) = 2|C_H|$. That is, $|C_H| = \text{dist}(P_i, I_i) + 2$. The proof of the theorem is completed.

Next, we shall study the radical of the bounded derived category $D^b(\mathrm{mod}\,H)$. Being a triangulated category, $D^b(\mathrm{mod}\,H)$ coincides with its projectively stable category and its injectively stable category; see [27, (2.4)]. Since $D^b(\mathrm{mod}\,H)$ has almost split triangles, the Auslander-Reiten translation τ_D of $\Gamma_{D^b(\mathrm{mod}\,H)}$ induces an auto-equivalence $\tau_D : D^b(\mathrm{mod}\,H) \to D^b(\mathrm{mod}\,H)$; see [27, (4.10)].

Theorem 4.6.5. *Let H be a hereditary algebra of Dynkin type with Coxeter order $|C_H|$. Then the radical of $D^b(\mathrm{mod}\,H)$ is nilpotent of nilpotency $|C_H| - 1$.*

Proof. Write $\mathscr{D} = D^b(\mathrm{mod}\,H)$. Since $\mathrm{mod}\,H$ is a convex subcategory in $\mathscr{D}$, by Theorem 4.6.4, $\mathrm{rad}^{|C_H|-2}(\mathscr{D}) \neq 0$. Suppose that $\mathrm{rad}^r(\mathscr{D}) \neq 0$ with $r \geq 0$. Then, $\mathrm{rad}^r_{\mathscr{D}}(M, N[s]) \neq 0$, for some $M, N \in \Gamma_{\mathrm{mod}\,H}$ and $s \in \mathbb{Z}$. Write $M = \tau^{-t}P$ for some $t \geq 0$, where P is a projective module in $\Gamma_{\mathrm{mod}\,H}$. Applying the equivalence τ_D, we obtain $\mathrm{rad}^r_{\mathscr{D}}(P[0], (\tau_D N[0])[s]) \neq 0$. Since P is projective, $(\tau_D N[0])[s] = L[0]$ for some $L \in \Gamma_{\mathrm{mod}\,H}$. Then, $\mathrm{rad}^r(P, L) \neq 0$, and by Theorem 4.6.4, $r \leq |C_H| - 2$. So, $\mathrm{rad}(\mathscr{D})$ is nilpotent of nilpotency $|C_H| - 1$. The proof of the theorem is completed.

We conclude with the cluster category associated with H. Let $\mathscr{D}^b(H)$ be a skeleton of $D^b(\mathrm{mod}\,H)$, containing the complexes in $\Gamma_{D^b(\mathrm{mod}\,H)}$. Then, $\mathscr{D}^b(H)$ is a Hom-finite Krull-Schmidt R-category, which has almost split sequences. Note that the Auslander-Reiten quiver $\Gamma_{\mathscr{D}^b(H)}$ of $\mathscr{D}^b(H)$ coincides with $\Gamma_{D^b(\mathrm{mod}\,H)}$, and its Auslander-Reiten translation $\tau_{\mathscr{D}}$ extends to an automorphism of $\mathscr{D}^b(H)$. Thus, we have an automorphism $F = \tau_{\mathscr{D}}^{-1} \circ [1]$ of $\mathscr{D}^b(H)$ such, for any indecomposable objects $M^{\bullet}, N^{\bullet} \in \mathscr{D}^b(H)$, that $F^p(M^{\bullet}) \not\cong M^{\bullet}$ for all $p \neq 0$ and $\mathrm{Hom}_{\mathscr{D}^b(H)}(M^{\bullet}, F^p(N^{\bullet})) = 0$ for all but finitely many $p \in \mathbb{Z}$. Thus, the action of the group $\mathcal{F}$ generated by F on $\mathscr{D}^b(H)$ is free and locally bounded. As did in [7], we define the *cluster category* associated with H to be the orbit category

$$\mathscr{D}^b(H)/\mathcal{F} =: \mathscr{C}_H,$$

which is Hom-finite and Krull-Schmidt; see 1.5.4.

Theorem 4.6.6. *Let H be a hereditary artin algebra of Dynkin type with Coxeter order $|C_H|$, and let $\mathscr{C}_H$ be the cluster category associated with H.*
(1) The radical of $\mathscr{C}_H$ is nilpotent of nilpotency $|C_H| - 1$.
(2) The number of non-isomorphich indecomposable objects in $\mathscr{C}_H$ is equal to

$$\frac{1}{2}n(|C_H| + 2),$$

where n is the number of non-isomorphic simple modules in $\mathrm{mod}\,H$.

Proof. First, Statement (1) follows immediately from Proposition 1.5.4(1) and Theorem 4.6.5. Let $P_1, \ldots, P_n$ be the projective modules and $I_1, \ldots, I_n$ the injective modules in $\Gamma_{\mathrm{mod}\,H}$ with $\mathrm{top}\,P_i = \mathrm{soc}\,I_i$. Recall that $F = \tau_{\mathscr{D}}^{-1} \circ [1]$, where $\tau_{\mathscr{D}}$ is the Auslander-Reiten translation for $\mathscr{D}^b(H)$. Given $p \in \mathbb{Z}$, write $\mathcal{S}^{(p)} = \{M[p] \mid M \in \Gamma_{\mathrm{mod}\,H}\} \cup \{P_1[p+1], \ldots, P_n[p+1]\}$ and $\mathcal{T}^{(p)} = F^p(\mathcal{S}^{(0)})$. Since $\tau_{\mathscr{D}}^- I_i = P_i[1]$, we obtain $F(I_i) = P_i[2]$ for $1 \leq i \leq n$. Thus, $\mathcal{T}^{(1)} = F(\mathcal{S}^{(0)})$. In general, $\mathcal{T}^{(p)} = F^p(\mathcal{S}^{(0)})$ for all $p \in \mathbb{Z}$. Since $\mathcal{F} = \{F^p \mid p \in \mathbb{Z}\}$, we see that $\mathcal{S}^{(0)}$ is a complete set of representatives of the $\mathcal{F}$-orbits of non-isomorphic indecomposable objects in $\mathscr{D}^b(H)$. Now, Statement (2) follows from Proposition 1.5.4 and Theorem 4.6.1. The proof of the theorem is completed.

Remark 4.6.7. The number given in Theorem 4.6.6(2) coincides with the number cluster variables in the cluster algebra associated with Q_H; see [15, (5.9.1)].

References

[1] H. Assashiba, *A generalization of Gabriel's Galois covering functors and derived equivalences,* J. Algebra 334 (2011) 109 - 149.

[2] M. Auslander, I. Reiten and S. O. Smalø, *Representation Theory of Artin Algebras,* Cambridge Studies in Advanced Mathematics 36 (Cambridge University Press, Cambridge, 1995).

[3] R. Bautista, *Irreducible morphisms and the radical of a category,* Ann. Inst. Math. Univ. Nac. Auto. Mex. 22 (1982) 83 - 135.

[4] R. Bautista and S. Liu, *The bounded derived categories of an algebra with radical squared zero*, J. Algebra 482 (2017) 303 - 345.

[5] R. Bautista, S. Liu, and C. Paquette, *Representation theory of strongly locally finite quivers*, Proc. London Math. Soc. 106 (2013) 97 - 162.

[6] R. Bautista and S. O. Smalø, *Nonexistent cycles*, Comm. Algebra 11 (1983) 1755 - 1767.

[7] A. B. Buan, R. Marsh, M. Reineke, I. Reiten and G. Todorov, *Tilting theory and cluster combinatorics,* Adv. Math. 204 (2006) 572 - 618.

[8] K. Bongartz and P. Gabriel, *Coverings spaces in representation-theory*, Invent. Math. 65 (1982) 331 - 378.

[9] A. Borel, P.-P. Grivel, B. Kaup, A. Haefliger, B. Malgrange and F. Ehlers, *Algebraic D-Modules*, Perspectives in Mathematics 2 (Academic Press, Boston, 1987).

[10] S. BRENNER, *A combinatorial characterisation of finite Auslander-Reiten quivers*, Lecture Notes in Mathematics 1177 (Springer, Berlin, 1986) 13 - 49.

[11] C. Chaio and S. Liu, *A note on the radical of a module category*, Comm. Algebra 41 (2013) 4419 - 4424.

[12] H. S. M. Coxeter, *The product of the generators of a finite group generated by reflections*, Duke Math. J. 18 (1951) 765 - 782.

[13] V. Dlab and C. M. Ringel, *On algebras of finite representation type*, J. Algebra 33 (1975) 306 - 394.

[14] V. Dlab and C. M. Ringel, *Indecomposable representations of graphs and algebras*, Mem. Amer. Math. Soc. 173 (1976).

[15] S. Fomin, L. Williams and A. Zelevinsky, *Introduction to Cluster Algebras* (preliminary version), arXiv: 1608.05735v4.

[16] S. Fomin and A. Zelevinsky, *Cluster algebras I: Foundations*, J. Amer. Math. Soc. 15 (2002) 497 - 529.

[17] P. Gabriel, *Unzerlegbare Darstellungen I*, Manuscripta Math. 6 (1972) 71 - 103.

[18] D. Happel, *On the derived category of a finite-dimensional algebra*, Comment. Math. Helv. 62 (1987) 339 - 389.

[19] D. Happel, U. Preiser and C. M. Ringel, *Vinberg's characterization of Dynkin diagrams using subadditive functions with applications to DTr-periodic modules*, Lecture Notes in Mathematics 832 (Springer, Berlin, 1980) 280-294.

[20] K. Igusa and G. Todorov, *A characterization of finite Auslander-Reiten quivers*, J. Algebra 89 (1984) 148 - 177.

[21] O. Iyama, H. Nakaoka and Y. Palu, *Auslander-Reiten theory in extriangulated categories*, arXiv: 1805.03776v3 (2023).

[22] H. Lenzing, *Hereditary Categories,* Handbook of Tilting Theory, London Mathematical Society Lectures Note Series 332 (Cambridge University Press, Cambridge, 2007) 105 - 146.

[23] S. Liu, *The degrees of irreducible maps and the shapes of Auslander-Reiten quivers*, J. London Math. Soc. (2) 45 (1992) 32 - 54.

[24] S. Liu, *Semi-stable components of an Auslander-Reiten quiver*, J. London Math. Soc. (2) 47 (1993) 405 - 416.

[25] S. Liu, *Shapes of connected components of the Auslander-Reiten quivers of artin algebras*, Canad. Math. Soc. Conf. Proc. 19 (1995) 109 - 137.

[26] S. Liu, *Auslander-Reiten theory in a Krull-Schmidt category*, Sao Paulo J. Math. Sci. 4 (2010) 425 - 472.

[27] S. Liu and H. Niu, *Almost split sequences in tri-exact categories*, J. Pure Appl. Algebra 226 (2022) 1 - 31.

[28] S. Liu and Y. Yin, *Module categories of small radical nilpotency*, Algebr. Represent. Theor. (2023) 333 - 361.

[29] M. I. Platzeck and M. Auslander, *Representation theory of hereditary Artin algebras*, Lecture Notes in Pure and Applied Mathematics 37 (Dekker, New York, 1978) 389 - 424.

[30] I. Reiten and M. Van den Bergh, *Noetherian hereditary abelian categories satisfying Serre duality*, J. Amer. Math. Soc. 15 (2002) 295 - 366.

[31] Chr. Riedtmann, *Algebren, Darstellungsköcher, Überlagerungen und Zurück*, Comment. Math. Helv. 55 (1980) 199 - 224.

[32] C. M. Ringel, *Tame algebras and Integral Quadratic Forms*, Lecture Notes in Mathematics 1099 (Springer-Verlag, Berlin, 1984).

[33] C. M. Ringel, *The regular components of the Auslander-Reiten quiver of a tilted algebra*, Chinese Ann. Math. Ser B9 (1988) 1 - 18.

[34] A. Skowroński, *Cycles in module categories*, Proceedings of Annual Canadian Mathematical Society/NATO Advanced Reserch Workshop (Ottawa, 1992); NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. 424 (Kluwer Acad. Publ., Dordrecht, 1994) 309 - 345

[35] D. Zacharia, *The preprojective partition for hereditary Artin algebras*, Trans. Amer. Math. Soc. 274 (1982) 327 - 343.

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