A NOTE ON THE RADICAL OF A MODULE CATEGORY

CLAUDIA CHAIO AND SHIPING LIU

Abstract. We characterize the finiteness of the representation type of an artin algebra in terms of the behavior of the projective covers and the injective envelopes of the simple modules with respect to the infinite radical of the module category. In case the algebra is representation-finite, we show that the nilpotency of the radical of the module category is the maximal depth of the composites of these maps, which is independent from the maximal length of the indecomposable modules.

1. Introduction

Throughout this paper, A stands for an artin algebra, and $\mathrm{mod}\,A$ for the category of finitely generated left A-modules. One of the fundamental tasks of the representation theory of A is to describe the maps in $\mathrm{mod}\,A$. The aim of this paper is to show that the behavior of the maps in $\mathrm{mod}\,A$ with respect to the radical series is somehow controlled by the behavior of the projective covers and the injective envelopes of the simple modules.

More precisely, suppose that A is of finite representation type. By the Harada-Sai lemma, the radical of $\mathrm{mod}\,A$ is nilpotent with a nilpotency bounded by $2^b - 1$, where b is the maximal length of the indecomposable modules; see [7]. A sharper bound is given in [6] which, however, also depends on the maximal length of the indecomposable modules. In case A is a finite dimensional algebra over an algebraically closed field, this nilpotency is described explicitly in terms of the degrees of finitely many irreducible maps associated to the simple modules; see [3]. In this paper, we shall extend this result to an artin algebra and prove that the nilpotency is the maximal depth of the composites of the projective covers and the injective envelopes of the simples modules.

On the other hand, a well known result of Auslander's says that A is of finite representation type provided that the radical of $\mathrm{mod}\,A$ is nilpotent, or equivalently, the infinite radical of $\mathrm{mod}\,A$ vanishes; see [11, (1.1)] and [10, (1.8)], and the latter condition is shown to be equivalent to the vanishing of the square of the infinite radical; see [5]. In this paper, we shall strengthen these two results by showing that A is of finite representation type provided that the projective covers of the simple modules do not lie in the infinite radical, or the injective envelopes of the simple modules do not lie in the infinite radical, or the composites of these maps do not lie in the square of the infinite radical.

2000 *Mathematics Subject Classification.* 16G70, 16G20, 16E10.

Key words and phrases. Artin algebras; finite representation type; radical of a module category; Harada-Sai Lemma; depth of a morphism; projective cover; injective envelope.

This research was carried out when the first named author visited the Université de Sherbrooke, and the second named author is supported in part by the Natural Sciences and Engineering Research Council of Canada.

2. The results

Recall that the *Jacobson radical* of mod$A$, written as rad(mod$A$), is the ideal generated by the non-invertible maps between the indecomposable modules, and the *infinite radical* of mod$A$, written as $\mathrm{rad}^{\infty}(\mathrm{mod}A)$, is the intersection of all the powers $\mathrm{rad}^{n}(\mathrm{mod}A)$ with $n \geq 1$. A map in mod$A$ is called *radical* if it belongs to rad(mod$A$). Throughout, for each simple module S, we fix a projective cover $\pi_S : P_S \to S$ and an injective envelope $\iota_S : S \to I_S$, and put $\theta_S = \iota_S \pi_S$. We shall need some basic notions and results of the Auslander-Reiten theory on irreducible maps and almost split sequences in mod$A$, for which the reader is referred to [1].

The following notion is convenient for us to formulate our results.

2.1. Definition. Let $f : M \to N$ be a map in mod$A$. We define the *depth* of f, written as $\mathrm{dp}(f)$, to be infinity in case $f \in \mathrm{rad}^{\infty}(M, N)$; and otherwise, to be the integer $n \geq 0$ for which $f \in \mathrm{rad}^{n}(M, N)$ but $f \notin \mathrm{rad}^{n+1}(M, N)$.

The following result is crucial in our investigation.

2.2. Lemma. *Let S be a simple module in* mod$A$. *If $f \in \mathrm{Hom}(P_S, I_S)$ is non-zero, then there exist $u \in \mathrm{End}(P_S)$ and $v \in \mathrm{End}(I_S)$ such that $vfu = \theta_S$.*

Proof. Let $f : P_S \to I_S$ be a non-zero map in mod$A$. By the Harada-Sai lemma, there exists a maximal integer $r \geq 0$ for which one can find a chain of radical maps

$$I_S = I_{S_0} \xrightarrow{g_1} I_{S_1} \longrightarrow \cdots \longrightarrow I_{S_{r-1}} \xrightarrow{g_r} I_{S_r}$$

such that $g_r \cdots g_1 f \neq 0$, where the S_i are simple modules. Set $v = g_r \cdots g_1$ if $r > 0$ and $v = \mathbf{1}$ otherwise. Then $vf \neq 0$. Now, consider the short exact sequence

$$0 \longrightarrow S_r \xrightarrow{\iota_{S_r}} I_{S_r} \xrightarrow{p} I_{S_r}/S_r \longrightarrow 0.$$

If $pvf \neq 0$, passing through the injective envelope of I_{S_r}/S_r, we can find a simple module S_{r+1} and a map $q : I_{S_r}/S_r \to I_{S_{r+1}}$ such that $(qp)vf \neq 0$. Since p is irreducible, qp is a radical map, which contradicts the maximality of r. Thus, $pvf = 0$, and consequently, $vf = \iota_{S_r} g$ for some map $g : P_S \to S_r$. Since $g \neq 0$, we see that $S_r = S$ and g is an epimorphism. Therefore, $\pi_S = gu$ for some $u \in \mathrm{End}(P_S)$. This yields $vfu = \iota_S \pi_S = \theta_S$. The proof of the lemma is completed.

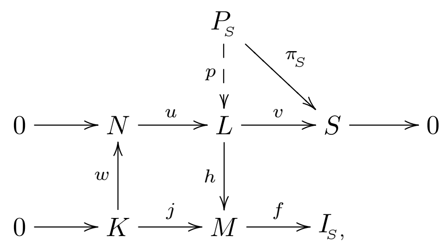
Remark. The above lemma says that $\mathrm{Hom}(P_S, I_S)$, as a $\mathrm{End}(I_S)$-$\mathrm{End}(P_S)$-bimodule, has a simple socle generated by θ_S.

2.3. Lemma. *Let S be a simple module in* mod$A$. *If $f : M \to I_S$ is a non-zero map, then there exists a map $g : P_S \to M$ such that $fg \neq 0$.*

Proof. Let $f : M \to I_S$ be a non-zero map with kernel $j : K \to M$. Consider the short exact sequence

$$0 \longrightarrow K \xrightarrow{j} M \xrightarrow{c} M/K \longrightarrow 0.$$

Since f factors through c, we see that $\mathrm{Hom}_A(M/K, I_S) \neq 0$. In particular, S is a composition factor of M/K. Thus, M has submodules N, L with $K \subseteq N \subset L \subseteq M$ such that $L/N \simeq S$. Since P_S is projective, we have a commutative diagram with exact rows



where w, u, h are all inclusion maps. Suppose that $fhp = 0$. Then, $hp = jq = huwq$ for some map $q : P_S \to K$. Since h is a monomorphism, $p = uwq$, and hence, $\pi_S = vp = vuwq = 0$, a contradiction. The proof of the lemma is completed.

2.4. Proposition. *Let $f : M \to N$ be a map in* mod$A$. *If S is a composition factor of* Im(f), *then there exist maps $g : P_S \to M$ and $h : N \to I_S$ such that $hfg = \theta_S$.*

Proof. Setting $L = \mathrm{Im}(f)$, we have a factorization $f = qp$, where $p : M \to L$ is an epimorphism and $q : L \to N$ is a monomorphism. Suppose that S is a composition factor of L. Then we have a non-zero map $u : L \to I_S$. Since I_S is injective, $u = vq$ for some map $v : N \to I_S$. This yields $vf = vqp = up \neq 0$. By Lemma 2.3, there exists a map $w : P_S \to M$ such that $vfw \neq 0$. Now, applying Lemma 2.2, we get two maps g, h such that $hfg = \theta_S$. The proof of the proposition is completed.

The following statement answers, to certain extent, a problem posed in [6].

2.5. Corollary. *Let a chain of radical maps between indecomposable modules*

$$M_0 \xrightarrow{f_1} M_1 \longrightarrow \cdots \longrightarrow M_{n-1} \xrightarrow{f_n} M_n$$

in mod$A$. *If $n > \mathrm{dp}(\theta_S)$ for every simple submodule S of M_n, then $f_n \cdots f_1 = 0$.*

Proof. Suppose that $f = f_n \cdots f_1 \neq 0$. Then Im$(f)$ contains a simple submodule S of M_n. By Proposition 2.4, $\theta_S = hfg$ for some maps $g : P_S \to M$ and $h : N \to I_S$. Hence, $\mathrm{dp}(\theta_S) \geq \mathrm{dp}(f) \geq n$. The proof of the corollary is completed.

Applying Proposition 2.4 to an identity map, we get another interesting consequence as follows.

2.6. Corollary. *Let M be a module in* mod$A$. *If S is a composition factor of M, then θ_S factors through M.*

We are ready to state the main result, in which the equivalence of the first three statements for the algebraically closed case has been essentially obtained in [4].

2.7. Theorem. *Let A be an artin algebra. The following statements are equivalent.*

(1) *The representation type of A is finite.*
(2) *The depth of ι_S is finite, for every simple module S.*
(3) *The depth of π_S is finite, for every simple module S.*
(4) *The map θ_S does not lie in $(\mathrm{rad}^\infty(\mathrm{mod}A))^2$, for every simple module S.*

Moreover, in this case, the nilpotency of rad(mod$A$) *is $m + 1$, where m is the maximal depth of the θ_S with S ranging over the simple modules.*

Proof. Suppose first that A is of representation-finite. Then $\text{rad}^{\infty}(\text{mod}A) = 0$; see [11, (1.1)]. In particular, Statements (2), (3), and (4) hold trivially. Let m be the maximal depth of the θ_S with S ranging over the simple modules. By definition, $\text{rad}^m(\text{mod}A) \neq 0$. Moreover, by Corollary 2.5, every chain of radical maps between indecomposable modules of length greater than m has zero composite. That is, $\text{rad}^{m+1}(\text{mod}A) = 0$.

Conversely, suppose that Statement (2) holds. Denote by r the maximal depth of the ι_S with S ranging over the simple modules. In order to show Statement (1), we may assume that A is connected. Choose an Auslander-Reiten component $\varGamma$ of A, and consider an arbitrary module M lying in $\varGamma$. Let S be a simple submodule of M with inclusion map $q : S \to M$. Since I_S is injective, there exists some map $p : M \to I_S$ such that $\iota_S = pq$. As a consequence, $\text{dp}(p) \leq \text{dp}(\iota_S) \leq r$. Therefore, $\varGamma$ has a path from M to I_S of length less than or equal to r. Being locally finite and containing at most finitely many indecomposable injective modules, $\varGamma$ is finite. Therefore, A is of finite representation type; see, for example, [1, (VI.1.4)]. Dually, Statement (3) implies Statement (1).

Finally, suppose that Statement (4) holds. Assume that $(\text{rad}^{\infty}(\text{mod}A))^2$ has a non-zero map $f : M \to N$. Let S be a composition factor of $\text{Im}(f)$. We then deduce from Proposition 2.4 that $\theta_S \in (\text{rad}^{\infty}(\text{mod}A))^2$. This contradiction shows that $(\text{rad}^{\infty}(\text{mod}A))^2 = 0$, and hence, A is of finite representation type; see [5]. The proof of the theorem is completed.

We illustrate the above theorem by the following example. For this purpose, we shall need the notion of degrees of an irreducible map; see [9, (1.1)].

Example. Let k be a field and consider the Nakayama algebra $A = k[x]/{<}x^n{>}$ with $n \geq 2$. It is well known that the non-isomorphic indecomposable modules in $\text{mod}A$ are $M_i = {<}x^i{>}/{<}x^n{>}$, for $i = 0, 1, \cdots, n-1$, and the Auslander-Reiten quiver $\varGamma_A$ of A is of the following shape:

$$M_0 \rightleftarrows M_1 \rightleftarrows \cdots \rightleftarrows M_{n-1}$$

with $\tau M_i = M_i$, for $i = 1, \cdots, n-1$. Note that M_{n-1} is the unique simple module. For each $1 \leq i < n-1$, let $f_i : M_{i-1} \to M_i$ be the multiplication by $\bar{x}$; and for each $0 < i \leq n-1$, let $g_i : M_i \to M_{i-1}$ be the inclusion map. Then the f_i, g_i are irreducible maps such that $\pi = f_{n-1} \cdots f_1 : M_0 \to M_{n-1}$ is the projective cover, while $\iota = g_1 \cdots g_{n-1} : M_{n-1} \to M_0$ is the injective envelope of M_{n-1}. Being the composite of a sectional chain of $n-1$ irreducible maps, both ι and π are of depth $n-1$; see [2, 8]. Moreover, since $g_1 : M_1 \to M_0$ has a projective co-domain, its left degree is infinite. Now, consider $g_i : M_i \to M_{i-1}$ with $1 < i < n$. Observe that

$$M_0 \longrightarrow M_1 \longrightarrow \cdots \longrightarrow M_{i-2} \longrightarrow M_{i-1}$$

is a sectional path in $\varGamma_A$ such that $M_i \oplus M_{i-2}$ is the middle term of the almost split sequence ending with M_{i-1}. Since M_0 is projective, the left degree of g_i is infinite; see [9, (1.2), (1.6)]. As a consequence, $\iota\pi = g_1 \cdots g_{n-1}\pi \notin \text{rad}^{2(n-1)+1}(\text{mod}A)$. This implies that $\text{dp}(\iota\pi) = 2(n-1)$. By Theorem 2.7, the nilpotency of $\text{rad}(\text{mod}A)$ is $2n-1$. In particular, every chain of radical maps between indecomposable modules of length greater than $2n-1$ has zero composite. On the other hand, we note that the Harada-Sai bound, as well as the bound given in [6], for the nilpotency of $\text{rad}(\text{mod}A)$ is $2^n - 1$.

REMARK. Let A be a finite dimensional algebra over an algebraically closed field. If A is of finite representation type, then $\mathrm{dp}(\theta_S) = \mathrm{dp}(\iota_S) + \mathrm{dp}(\pi_S)$ for every simple module S; see [3]. However, we do not know if this is still the case for artin algebras.

References

[1] M. Auslander, I. Reiten and S. Smalø, "Representation Theory of Artin Algebras", Cambridge Studies in Advanced Mathematics 36 (Cambridge University Press, Cambridge, 1995).

[2] K. Bongartz, "On a result of Bautista and Smalø", Comm. Algebra 11 (1983) 2133 - 2124.

[3] C. Chaio, "On the Harada and Sai bound", Bull. London Math. Soc. (to appear).

[4] C. Chaio, P. Le Meur, and S. Trepode, "Degrees of irreducible morphisms and finite representation type", J. London Math. Soc. (2) 84 (2011) 35 - 57.

[5] F.U. Coelho, E.N. Marcos, H. A. Merklen and A. Skowroński, "Module categories with infinite radical square zero are of finite type", Comm. Algebra 22 (1994) 4511 - 4517.

[6] D. Eisenbud and J. A. de la Peña, "Chains of maps between indecomposable modules", J. Reine Angew. Math. 504 (1998) 29 - 35.

[7] M. Harada and Y. Sai, "On categories of indecomposable modules I", Osaka J. Math. 7 (1970) 323 - 344.

[8] K. Igusa and G. Todorov, "Radical layers of representable functors", J. Algebra 89 (1984) 105 - 147.

[9] S. Liu, "Degrees of irreducible maps and the shapes of Auslander-Reiten quivers," J. London Math. Soc. 45 (1992) 32 - 54.

[10] O. Kerner and A. Skowroński, "On module categories with nilpotent infinite radical", Compositio Math. 77 (1991) 313 - 333.

[11] A. Skowroński, "Cycles in module categories", Proceedings of Annual Canadian Mathematical Society / NATO Advanced Reserch Workshop (Ottawa, 1992); NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. 424 (Kluwer Acad. Publ., Dordrecht, 1994) 309 - 345.

Claudia Chaio, Departamento de Matemática, Facultad de Ciencias Exactas y Naturales, Funes 3350, Universidad Nacional de Mar del Plata, 7600 Mar del Plata, Argentina

E-mail address: claudia.chaio@gmail.com

Shiping Liu, Département de mathématiques, Université de Sherbrooke, Sherbrooke, Québec, Canada

E-mail address: shiping.liu@usherbrooke.ca