COVERING THEORY FOR LINEAR CATEGORIES WITH APPLICATION TO DERIVED CATEGORIES

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Abstract. We extend the Galois covering theory introduced by Bongartz-Gabriel for skeletal linear categories to general linear categories. We show that a Galois covering between Krull-Schmidt categories preserves irreducible morphisms and almost splits sequences. Specializing to derived categories, we study when a Galois covering between locally bounded linear categories induces a Galois covering between the bounded derived categories of finite dimensional modules. As an application, we show that each locally bounded linear category with radical squared zero admits a gradable Galois covering, which induces a Galois covering between the bounded derived categories of finite dimensional modules, and a Galois covering between the Auslander-Reiten quivers of these bounded derived categories. In a future paper, this will enable us to obtain a complete description of the bounded derived category of finite dimensional modules over a finite dimensional algebra with radical squared zero.

Introduction

The covering technique has been playing an important role in the representation theory of finite dimensional algebras; see, for example, [6, 8, 9, 16]. In this connection, algebras are regarded as locally bounded linear categories; see [6]. To each Galois covering between such categories, Bongartz-Gabriel associated a push-down functor between their module categories, which induces a Galois covering between the Auslander-Reiten quivers in the locally representation-finite case; see [6, 8]. This technique was extended later by Asashiba by studying the induced push-down functor between the bounded homotopy categories of finitely generated projective modules; see [1]. Now, the push-down functor also induces an exact functor between the bounded derived categories of finite dimensional modules. It is then natural to ask when this derived push-down functor is a Galois covering. Unfortunately, this question is somehow problematic, since Gabriel's notion of a Galois covering is only for skeletal linear categories. To overcome this difficulty, Asashiba introduced the notion of a precovering and called a dense precovering a *covering*; see [2]. Strengthening this notion of a covering, we obtain the notion of a Galois covering for general linear categories. As an interesting example, the bounded derived category of finite dimensional representations of a finite acyclic quiver is a Galois covering of the corresponding cluster category introduced in [7]. One of the nice properties of such a Galois covering is that it preserves the Auslander-Reiten theory in case

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the categories are Krull-Schmidt. Most importantly, this provides a useful tool for studying the bounded derived category of finite dimensional modules over a locally bounded linear category. To give more details, we outline the content section by section.

In Section 1, we shall deal with the problem as to when the derived category of an abelian category has arbitrary direct sums. For this purpose, we introduce the notion of essential direct sums and show that if an abelain category has essential direct sums, then its derived category has direct sums. In particular, the derived category of all modules over a locally bounded linear category has direct sums.

In Section 2, based on Asashiba's notion of a precovering, we first define the notion of a Galois covering for general linear categories, and then search for conditions for a precovering to be a Galois covering. Moreover, we introduce the notion of a graded adjoint pair between linear categories, and show that restricting such an adjoint pair to appropriate subcategories yields precoverings.

In Sections 3, we show that a Galois covering between two Krull-Schmidt categories preserves irreducible morphisms and almost split sequences. In particular, one of these categories has almost split sequences if and only the other one does.

In Section 4, we introduce the notion of a Galois covering for valued translation quivers, and show that a Galois covering between Hom-finite Krull-Schmidt categories induces a Galois covering of their Auslander-Reiten quivers.

In Sections 5, we shall strengthen Milicic's result that an adjoint pair of exact functors between abelian categories induces an adjoint pair between their derived categories; see [14], by showing that a graded adjoint pair between abelian categories having essential direct sums induces a graded adjoint pair between their derived categories.

In Section 6, we apply our results to study the derived push-down functor associated to a Galois covering between locally bounded linear categories. By showing that the push-down functor and the pull-up functor between the module categories form a graded adjoint pair, we obtain a graded adjoint pair formed by the derived push-down functor and the derived pull-up functor. Restricting the derived push-down functor, we obtain a precovering between the bounded derived categories of finite dimensional modules, and in case the group is torsion-free, it is a Galois covering if and only if it is dense.

In Section 7, specializing to locally bounded linear categories with radical squared zero, we prove that such a linear category admits a gradable Galois covering, which induces a Galois covering between the bounded derived categories of finite dimensional modules, and a Galois covering between the Auslander-Reiten quivers of these derived categories.

1. Preliminaries

Throughout this paper, all categories are skeletally small, and morphisms are composed from the right to the left. Let R be a commutative ring. An *R-linear category* is a category in which the morphism sets are R-modules such that the composition of morphisms is R-bilinear. All functors between R-linear categories are assumed to be R-linear. An R-linear category is called *Hom-finite* if the morphism modules are of finite R-length, *additive* if it has finite direct sums, and *skeletal* if the endomorphisms algebras are local and the isomorphisms are automorphisms. In the sequel, a *linear category* refers to a $\mathbb{Z}$-linear category and an *additive category* refers to an additive $\mathbb{Z}$-linear category. Moreover, a *Krull-Schmidt category* is an additive category in which every non-zero object is a finite direct sum of objects with a local endomorphism algebra.

Throughout this section, $\mathcal{A}$ stands for an additive category, whose object class is written as $\mathcal{A}_0$. We shall say that $\mathcal{A}$ *has direct sums* provided that any set-indexed family of objects in $\mathcal{A}$ has a direct sum. For brevity, we assume that any family of objects in $\mathcal{A}$ is always set-indexed. Let X_i, $i \in I$, be objects in $\mathcal{A}$ such that $\oplus_{i\in I} X_i$ exists with canonical injections $q_j : X_j \to \oplus_{i\in I} X_i$, $j \in I$. By definition, $\mathcal{A}$ has unique morphisms $p_j : \oplus_{i\in I} X_i \to X_j$, $j \in I$, called *pseudo-projections*, such that

$$(*) \qquad p_i q_j = \begin{cases} 1_{X_i}, & \text{if } i = j; \\ 0, & \text{if } i \neq j, \end{cases}$$

for all $i, j \in I$. An object $M \in \mathcal{A}$ is called *essential in* $\oplus_{i\in I} X_i$ provided, for any morphism $f : M \to \oplus_{i\in I} X_i$, that $f = 0$ if and only if $p_j f = 0$ for all $j \in I$. If every object in $\mathcal{A}$ is essential in $\oplus_{i\in I} X_i$, then $\oplus_{i\in I} X_i$ is called an *essential direct sum.* By saying that $\mathcal{A}$ *has essential direct sums*, we mean that each family of objects in $\mathcal{A}$ has an essential direct sum. Suppose that the product $\Pi_{i\in I} X_i$ exists with canonical projections $\pi_j : \Pi_{i\in I} X_i \to X_j$, $j \in J$. Then $\mathcal{A}$ has a canonical morphism $\mu : \oplus_{i\in I} X_i \to \Pi_{i\in I} X_i$, which makes the diagram

$$\begin{tikzcd} \oplus_{i\in I} X_i \arrow[r, "\mu"] \arrow[dr, "p_j"'] & \Pi_{i\in I} X_i \arrow[d, "\pi_j"] \\ & X_j \end{tikzcd}$$

commute, for every $j \in I$. The following observation explains the essentialness of a direct sum.

1.1. Lemma. *Let $\mathcal{A}$ be a linear category with objects X_i, $i \in I$. If both $\oplus_{i\in I} X_i$ and $\Pi_{i\in I} X_i$ exist in $\mathcal{A}$, then $\oplus_{i\in I} X_i$ is essential in $\mathcal{A}$ if and only if the canonical morphism $\mu : \oplus_{i\in I} X_i \to \Pi_{i\in I} X_i$ is a monomorphism.*

Proof. Suppose that both $\oplus_{i\in I} X_i$ and $\Pi_{i\in I} X_i$ exist in $\mathcal{A}$, with canonical injections $q_j : X_j \to \oplus_{i\in I} X_i$, pseudo-projections $p_j : \oplus_{i\in I} X_i \to X_j$, and canonical projections $\pi_j : \Pi_{i\in I} X_i \to X_j$, $j \in I$. Assume first that $\mu : \oplus_{i\in I} X_i \to \Pi_{i\in I} X_i$ is a monomorphism. If $f : M \to \oplus_{i\in I} X_i$ is such that $p_j f = 0$ for all $j \in I$, then $\pi_j \mu f = 0$ for all $j \in I$. Thus $\mu f = 0$, and hence $f = 0$. That is, $\oplus_{i\in I} X_i$ is essential.

Assume conversely that $\oplus_{i\in I} X_i$ is essential. If $g : N \to \oplus_{i\in I} X_i$ is such that $\mu g = 0$, then $p_j g = \pi_j \mu g = 0$ for all $j \in I$, and hence $g = 0$. That is, μ is a monomorphism. The proof of the lemma is completed.

Remark. (1) If I is finite, then $\mu : \oplus_{i\in I} X_i \to \Pi_{i\in I} X_i$ is an isomorphism. Therefore, finite direct sums are always essential.

(2) Let S be a ring. The category Mod$S$ of all left S-modules has direct sums and products, and each direct sum embeds canonically in the corresponding product. By Lemma 1.1, Mod$S$ has essential direct sums.

1.2. Lemma. *Let $\mathcal{A}, \mathcal{B}$ be linear categories, and let $\mathscr{F}(\mathcal{A}, \mathcal{B})$ be the category of linear functors $F : \mathcal{A} \to \mathcal{B}$. If $\mathcal{B}$ has* (*essential*) *direct sums, then so does $\mathscr{F}(\mathcal{A}, \mathcal{B})$.*

Proof. It is evident that $\mathscr{F}(\mathcal{A}, \mathcal{B})$ is a linear category. Suppose that $\mathcal{B}$ has direct sums. Consider a family of linear functors $F_i : \mathcal{A} \to \mathcal{B}$, $i \in I$. For each $a \in \mathcal{A}_0$, let $\oplus_{i\in I} F_i(a)$ be the direct sum of $F_i(a)$, $i \in I$, in $\mathcal{B}$ with canonical injections $q_j(a) : F_j(a) \to \oplus_{i\in I} F_i(a)$ and pseudo-projections $p_j : \oplus_{i\in I} F_i(a) \to F_j(a)$, $j \in I$. There exists a unique linear functor $F : \mathcal{A} \to \mathcal{B}$ such that $F(a) = \oplus_{i\in I} F_i(a)$ for $a \in \mathcal{A}_0$. It is easy to see that F is the direct sum of F_i, $i \in I$, with canonical injections $q_j = (q_j(a))_{a\in\mathcal{A}_0}$ and pseudo-projections $p_j = (p_j(a))_{a\in\mathcal{A}_0}$, $j \in I$. Moreover, if $\oplus_{i\in I} F_i(a)$ is an essential direct sum for every $a \in \mathcal{A}_0$, then F is an essential direct sum of the F_i. The proof of the lemma is completed.

Let X_i, $i \in I$, be objects in $\mathcal{A}$ such that $\oplus_{i\in I} X_i$ exists with canonical injections $q_j : X_j \to \oplus_{i\in I} X_i$, $j \in I$. For each $M \in \mathcal{A}_0$, the category of abelian groups has a direct sum $\oplus_{i\in I} \mathcal{A}(M, X_i)$ with canonical injections $u_j : \mathcal{A}(M, X_j) \to \oplus_{i\in I} \mathcal{A}(M, X_i)$. Considering the maps $\mathcal{A}(M, q_j) : \mathcal{A}(M, X_j) \to \mathcal{A}(M, \oplus_{i\in I} X_i)$, $j \in I$, we get a canonical morphism $\nu_M : \oplus_{i\in I} \mathcal{A}(M, X_i) \to \mathcal{A}(M, \oplus_{i\in I} X_i)$ such, for every $j \in I$, that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{A}(M, X_j) \arrow[r, "u_j"] \arrow[dr, "\mathcal{A}(M, q_j)"'] & \oplus_{i\in I} \mathcal{A}(M, X_i) \arrow[d, "\nu_M"] \\
& \mathcal{A}(M, \oplus_{i\in I} X_i).
\end{tikzcd}$$

In view of the equations stated in $(*)$, we see that ν_M is a monomorphism. It is important to find conditions for ν_M to be an isomorphism.

1.3. Proposition. *Let $\mathcal{A}$ be a linear category with a direct sum $\oplus_{i\in I} X_i$. If M is essential in $\oplus_{i\in I} X_i$ with $\mathcal{A}(M, X_i) = 0$ for all but finitely many $i \in I$, then the canonical morphism $\nu_M : \oplus_{i\in I} \mathcal{A}(M, X_i) \longrightarrow \mathcal{A}(M, \oplus_{i\in I} X_i)$ is an isomorphism.*

Proof. Assume that M is essential in $\oplus_{i\in J} X_i$ and $J \subseteq I$ is finite such that $\mathcal{A}(M, X_i) = 0$ for $i \in I \backslash J$. Let $q_j : X_j \to \oplus_{i\in I} X_i$ be the canonical injections and $p_j : \oplus_{i\in I} X_i \to X_j$ the pseudo-projections. Given any morphism $f : M \to \oplus_{i\in I} X_i$ in $\mathcal{A}$, we see that $g = \sum_{j\in J} p_j f$ is in $\oplus_{i\in I} \mathcal{A}(M, X_i)$. Consider the morphism $h = \nu_M(g) = \sum_{j\in J} q_j p_j f$. If $i \in J$, then $p_i h = \sum_{j\in J} p_i q_j p_j f = p_i f$. Otherwise, $p_i h = \sum_{j\in J} p_i q_j p_j f = 0 = p_i f$. Thus, $f = h = \nu_M(g)$. This shows that ν_M is an epimorphism. The proof of the proposition is completed.

Assume now that $\mathcal{A}$ is a full additive subcategory of an abelian category $\mathfrak{A}$. A *complex* $(X^\bullet, d_X^\bullet)$, or simply $X^\bullet$, over $\mathcal{A}$ is a double infinite chain

$$\cdots \longrightarrow X^n \xrightarrow{d_X^n} X^{n+1} \xrightarrow{d_X^{n+1}} X^{n+2} \longrightarrow \cdots, \quad n \in \mathbb{Z}$$

of morphisms in $\mathcal{A}$ such that $d_X^{n+1} d_X^n = 0$ for all n, where X^n is the *component of degree* n of $X^\bullet$, and d_X^n is the *differential of degree* n. Such a complex $X^\bullet$ is called *bounded-above* if $X^n = 0$ for all but finitely many positive integers n, *bounded* if $X^n = 0$ for all but finitely many integers n, and a *stalk complex concentrated in degree* s if $X^n = 0$ for all integers $n \neq s$. The *n-th cohomology* of a complex $X^\bullet$ is $\mathrm{H}^n(X^\bullet) = \mathrm{Ker}(d_X^n)/\mathrm{Im}(d_X^{n-1}) \in \mathfrak{A}$. One says that $X^\bullet$ has *bounded cohomology* if $\mathrm{H}^n(X^\bullet) = 0$ for all but finitely many integers n and that $X^\bullet$ is *acyclic* if $\mathrm{H}^n(X^\bullet) = 0$

for all integers n. A *morphism of complexes* $f^\bullet : X^\bullet \to Y^\bullet$ consists of morphisms $f^n : X^n \to Y^n$, $n \in \mathbb{Z}$, such that $f^n d_Y^n = d_X^n f^{n+1}$ for all $n \in \mathbb{Z}$. Such a morphism $f^\bullet$ is a *quasi-isomorphism* if f^n induces an isomorphism $H^n(f^\bullet) : H^n(X^\bullet) \to H^n(Y^\bullet)$ for every $n \in \mathbb{Z}$; and *null-homotopic* if there exist $h^n : X^n \to Y^{n-1}$, $n \in \mathbb{Z}$, such that $f^n = d_X^n h^{n+1} + h^n d_Y^n$, for all $n \in \mathbb{Z}$.

The complexes over $\mathcal{A}$ form an additive category $C(\mathcal{A})$. For $X^\bullet \in C(\mathcal{A})$ and $s \in \mathbb{Z}$, the *shift* of $X^\bullet$ by s is the complex $X^\bullet[s]$ of which the component of degree n is X^{n+s} and the differential of degree n is $(-1)^s d_X^{n+s}$. The automorphism of $C(\mathcal{A})$ sending $X^\bullet$ to $X^\bullet[1]$ is called the *shift functor* of $C(\mathcal{A})$. The full subcategories of $C(\mathcal{A})$ generated by the bounded-above complexes and by the bounded complexes will be denoted by $C^-(\mathcal{A})$ and $C^b(\mathcal{A})$, respectively. Moreover, $C^{-,\,b}(\mathcal{A})$ denotes the full subcategory of $C^-(\mathcal{A})$ generated by the complexes of bounded cohomologies.

Fix $* \in \{\emptyset, -, b, \{-, b\}\}$. The *homotopy category* $K^*(\mathcal{A})$ is the quotient category of $C^*(\mathcal{A})$ modulo the ideal of null-homotopic morphisms. This is a triangulated category whose translation functor is the shift by one and whose exact triangles are induced from the mapping cones. Let $P_{\mathcal{A}}^* : C^*(\mathcal{A}) \to K^*(\mathcal{A})$ be the canonical projection functor. For a morphism $f^\bullet \in C(\mathcal{A})$, we shall write $\bar{f}^\bullet = P_{\mathcal{A}}^*(f^\bullet) \in K^*(\mathcal{A})$. The *quasi-isomorphisms* in $K^*(\mathcal{A})$ are the images of the quasi-isomorphisms in $C^*(\mathcal{A})$ under $P_{\mathcal{A}}^*$, which form a multiplicative system. The *derived category* $D^*(\mathcal{A})$ of $\mathcal{A}$ is the localization of $K^*(\mathcal{A})$ with respect to the quasi-isomorphisms, which is also a triangulated category with the exact triangles induced from those of $K^*(\mathcal{A})$. The morphisms in $D^*(\mathcal{A})$ are the equivalence classes $\bar{f}^\bullet/\bar{s}^\bullet$ of the diagrams

$$X^\bullet \xleftarrow{\bar{s}^\bullet} Y^\bullet \xrightarrow{\bar{f}^\bullet} Z^\bullet$$

in $K^*(\mathcal{A})$ with $\bar{s}^\bullet$ a quasi-isomorphism. We have an exact functor of triangulated categories, called the *localization functor*, $L_{\mathcal{A}}^* : K^*(\mathcal{A}) \to D^*(\mathcal{A})$, sending $\bar{f}^\bullet$ to $\bar{f}^\bullet/1$. For a morphisms $f^\bullet$ in $C^*(\mathcal{A})$, we shall write $\widetilde{f}^\bullet = L_{\mathcal{A}}^*(P_{\mathcal{A}}^*(f^\bullet)) \in D^*(\mathcal{A})$.

Next, we shall study the existence of direct sums in complex categories, homotopy categories and derived categories.

1.4. Lemma. *Let $\mathcal{A}$ be a full additive subcategory of an abelian category. If $\mathcal{A}$ has direct sums, then so do $C(\mathcal{A})$ and $K(\mathcal{A})$. Moreover, if $\mathcal{A}$ has essential direct sums, then so does $C(\mathcal{A})$.*

Proof. Suppose that $\mathcal{A}$ has direct sums. Let $X_i^\bullet$, $i \in I$, be complexes over $\mathcal{A}$. For $n \in \mathbb{Z}$, let X^n be the direct sum in $\mathcal{A}$ of the X_i^n, $i \in I$, with canonical injections $q_i^n : X_i^n \to X^n$ and pseudo-projections $p_i^n : X^n \to X_i^n$, and set $d_X^n = \oplus_{i \in I}\, d_{X_i}^n$. This yields a complex $(X^\bullet, d_X^\bullet)$ over $\mathcal{A}$, which is clearly the direct sum in $C(\mathcal{A})$ of the $X_i^\bullet$ with canonical injections $q_i^\bullet = (q_i^n)_{n \in \mathbb{Z}} : X_i^\bullet \to X^\bullet$ and pseudo-projections $p_i^\bullet = (p_i^n)_{n \in \mathbb{Z}} : X^\bullet \to X_i^\bullet$, $i \in I$. Moreover, one sees easily that this direct sum is essential in $C(\mathcal{A})$ if all the direct sums $\oplus_{i \in I} X_i^n$, $n \in \mathbb{Z}$, are essential in $\mathcal{A}$.

Next, we claim that $X^\bullet$ is the direct sum of the $X_i^\bullet$, $i \in I$, in $K(\mathcal{A})$ with canonical injections $\bar{q}_i^\bullet$ and pseudo-projections $\bar{p}_i^\bullet$. It suffices to show the following fact: if $f^\bullet : X^\bullet \to Y^\bullet$ is a morphism in $C(\mathcal{A})$ such that $f^\bullet q_i^\bullet$ is null-homotopic for every $i \in I$, then $f^\bullet$ is null-homotopic. Indeed, let $h_i^n : X_i^n \to Y^{n-1}$ be such that $f^n q_i^n = d_Y^{n-1} h_i^n + h_i^{n+1} d_{X_i}^n$, for $i \in I$ and $n \in \mathbb{Z}$. Then, for each $n \in \mathbb{Z}$, there exists some $u^n : X^n \to Y^{n-1}$ such that $h_i^n = u^n q_i^n$, for every $i \in I$. This yields

$f^n q_i^n = d_Y^{n-1}u^n q_i^n + u^{n+1}q_i^{n+1}d_{X_i}^n = (d_Y^{n-1}u^n + u^{n+1}d_X^n)q_i^n$, for $i \in I$ and $n \in \mathbb{Z}$. As a consequence, $f^n = d_Y^{n-1}u^n + u^{n+1}d_X^n$, for every $n \in \mathbb{Z}$, that is, $f^\bullet$ is null-homotopic. The proof of the lemma is completed.

In general, the direct sums in $K(\mathcal{A})$ are not necessarily essential even if those in $\mathcal{A}$ are essential. Nevertheless, we have the following partial result.

1.5. Lemma. *Let $\mathcal{A}$ be a full additive subcategory having essential direct sums of an abelian category, and let $M^\bullet, X_i^\bullet, i \in I$, be complexes over $\mathcal{A}$. If $C(\mathcal{A})(M^\bullet, X_i^\bullet) = 0$ for all but finitely many $i \in I$, then $M^\bullet$ is essential in the direct sum of the $X_i^\bullet$, $i \in I$, in $K(\mathcal{A})$.*

Proof. Assume that J is a finite subset of I such that $C(\mathcal{A})(M^\bullet, X_i^\bullet) = 0$ for $i \in I \backslash J$. Let $X^\bullet$ be the direct sum of the $X_i^\bullet$, $i \in I$, in $C(\mathcal{A})$ with pseudo-projections $p_i^\bullet : X^\bullet \to X_i^\bullet$. Then X^n is the direct sum of the X_i^n, $i \in I$, in $\mathcal{A}$ with pseudo-projections $p_i^n : X^n \to X_i^n$, for every $n \in \mathbb{Z}$. By Lemma 1.4, $X^\bullet$ is the direct sum of the $X_i^\bullet$, $i \in I$, in $K(\mathcal{A})$ with pseudo-projections $\bar{p}_i^\bullet : X^\bullet \to X_i^\bullet$.

Let $\bar{f}^\bullet : M^\bullet \to X^\bullet$ be a morphism in $K(\mathcal{A})$ such that $\bar{p}_i^\bullet \bar{f}^\bullet = \bar{0}$, that is, $p_i^\bullet f^\bullet$ is null-homotopic, for all $i \in I$. Let $h_i^n : M^n \to X_i^{n-1}$ be morphisms such that $p_i^n f^n = d_{X_i}^{n-1}h_i^n + h_i^{n+1}d_M^n$, for $i \in I$ and $n \in \mathbb{Z}$. Setting $h^n = \sum_{j \in J} q_j^{n-1}h_j^n$, we obtain $p_i^n f^n = p_i^n(d_X^{n-1}h^n + h^{n+1}d_M^n)$, for $i \in I$ and $n \in \mathbb{Z}$. Since the direct sums in $\mathcal{A}$ are essential, $f^n = d_X^{n-1}h^n + h^{n+1}d_M^n$, for $n \in \mathbb{Z}$. This shows that $f^\bullet$ is null-homotopic, that is, $\bar{f}^\bullet = \bar{0}$. The proof of the lemma is completed.

For the existence of direct sums in derived categories, we shall deal only with derived categories of abelian categories. Let us start with an easy observation.

1.6. Lemma. *Let $\mathfrak{A}$ be an abelian category with a family of short exact sequences $0 \longrightarrow L_i \xrightarrow{f_i} M_i \xrightarrow{g_i} N_i \longrightarrow 0$, $i \in I$. If $\mathfrak{A}$ has essential direct sums, then it has a short exact sequence as follows:*

$$0 \longrightarrow \oplus_{i \in I} L_i \xrightarrow{\oplus f_i} \oplus_{i \in I} M_i \xrightarrow{\oplus g_i} \oplus_{i \in I} N_i \longrightarrow 0.$$

Proof. Let $0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$ denote the sequence stated in the lemma. Using the universal properties of direct sums, one sees that g is the cokernel of f. Let $q_i : L_i \to L$ and $u_i : M_i \to M$ be the canonical injections, and $p_i : L \to L_i$ and $v_i : M \to M_i$ the pseudo-projections. Fix $i \in I$. Then $v_i f q_i = v_i u_i f_i = f_i$. For any $j \in I$, if $j = i$, then $(v_i f - f_i p_i)q_j = v_i f q_i - f_i p_i q_i = 0$; and otherwise, $(v_i f - f_i p_i)q_j = v_i f q_j = v_i u_j f_j = 0$. Thus, $v_i f - f_i p_i = 0$.

Suppose that L is an essential direct sum of the L_i. Let $h : X \to M$ be such that $fh = 0$. Then $f_i p_i h = v_i f h = 0$, and since f_i is a monomorphism, $p_i h = 0$ for all $i \in I$. Since X is essential in L, we obtain $h = 0$. This proves that f is a monomorphism. Since $\mathfrak{A}$ is abelian, f is the kernel of its cokernel, that is g. The proof of the lemma is completed.

1.7. Corollary. *Let $\mathfrak{A}$ be an abelian category, having essential direct sums. If $s_i^\bullet : X_i^\bullet \to Y_i^\bullet$, $i \in I$, are quasi-isomorphisms in $C(\mathfrak{A})$, then the canonical morphism $\oplus_{i \in I} s_i^\bullet : \oplus_{i \in I} X_i^\bullet \to \oplus_{i \in I} Y_i^\bullet$ is a quasi-isomorphism.*

Proof. Let $s_i^\cdot : X_i^\cdot \to Y_i^\cdot$, $i \in I$, be quasi-isomorphisms in $C(\mathfrak{A})$. We write $X^\cdot = \oplus_{i\in I} X_i^\cdot$, $Y^\cdot = \oplus_{i\in I} Y_i^\cdot$ and $s^\cdot = \oplus_{i\in I} s_i^\cdot$. In view of Lemma 1.6, we see that $\mathrm{Im}(d_X^n) = \oplus_{i\in I} \mathrm{Im}(d_{X_i}^n)$ and $\mathrm{Ker}(d_X^{n-1}) = \oplus_{i\in I} \mathrm{Ker}(d_{X_i}^n)$, for all $n \in \mathbb{Z}$. This in turn implies that $\mathrm{H}^n(s^\cdot) = \oplus_{i\in I} \mathrm{H}^n(s_i^\cdot)$, and consequently, $\mathrm{H}^n(s^\cdot)$ is an isomorphism. The proof of the corollary is completed.

We are ready to state the main result of this section; compare [13, (3.5.1)].

1.8. Theorem. *Let $\mathfrak{A}$ be an abelian category. If $\mathfrak{A}$ has essential direct sums, then $D(\mathfrak{A})$ has direct sums.*

Proof. Suppose that $\mathfrak{A}$ has essential direct sums. Let $X_i^\cdot$, $i \in I$, be complexes over $\mathfrak{A}$. By Lemma 1.4, $C(\mathfrak{A})$ has an essential direct sum $X^\cdot$ of the $X_i^\cdot$ with canonical injections $q_i^\cdot : X_i^\cdot \to X^\cdot$. We shall show that $X^\cdot$ is the direct sum of the $X_i^\cdot$ in $D(\mathfrak{A})$ with canonical injections $\tilde{q}_i^\cdot : X_i^\cdot \to X^\cdot$.

Let $\theta_i^\cdot : X_i^\cdot \to Y^\cdot$, $i \in I$, be morphisms in $D(\mathfrak{A})$. Write $\theta_i^\cdot = \bar{f}_i^\cdot/\bar{s}_i^\cdot$, where $s_i^\cdot : Z_i^\cdot \to X_i^\cdot$ is a quasi-isomorphism and $f_i^\cdot : Z_i^\cdot \to Y^\cdot$ is a morphism in $C(\mathfrak{A})$. Let $Z^\cdot$ be the direct sum in $C(\mathfrak{A})$ of the $Z_i^\cdot$ with canonical injections $u_i^\cdot : Z_i^\cdot \to Z^\cdot$. By Corollary 1.7, $s^\cdot = \oplus_{i\in I} s_i^\cdot : Z^\cdot \to X^\cdot$ is a quasi-isomorphism. Moreover, there exists a morphism $f^\cdot : Z^\cdot \to Y^\cdot$ in $C(\mathfrak{A})$ such that $f_i^\cdot = f^\cdot u_i^\cdot$, for all $i \in I$. Set $\eta^\cdot = \bar{f}^\cdot/\bar{s}^\cdot : X^\cdot \to Y^\cdot$. For any $i \in I$, since $s^\cdot u_i^\cdot = q_i^\cdot s_i^\cdot$, we obtain

$$\eta^\cdot \tilde{q}_i^\cdot = \bar{f}^\cdot \bar{u}_i^\cdot/\bar{s}_i^\cdot = \bar{f}_i^\cdot/\bar{s}_i^\cdot = \theta_i^\cdot.$$

For proving the uniqueness of $\eta^\cdot$, it suffices to show that $\eta^\cdot = \tilde{0}$ in case $\theta_i^\cdot = \tilde{0}$, for all $i \in I$. Indeed, in this case, $C(\mathfrak{A})$ has quasi-isomorphisms $r_i^\cdot : L_i \to Z_i$, $i \in I$, such that $\bar{f}_i^\cdot \bar{r}_i^\cdot = \bar{0}$, for every $i \in I$. Let $L^\cdot = \oplus_{i\in I} L_i^\cdot \in C(\mathfrak{A})$ with canonical injections $v_i^\cdot : L_i^\cdot \to L^\cdot$. By Corollary 1.7, $r^\cdot = \oplus_{i\in I} r_i^\cdot$ is a quasi-isomorphism, such that $r^\cdot v_i^\cdot = u_i^\cdot r_i^\cdot$. This yields $\bar{f}^\cdot \bar{r}^\cdot\, \bar{v}_i^\cdot = \bar{f}^\cdot \bar{u}_i^\cdot \bar{r}_i^\cdot = \bar{f}_i^\cdot \bar{r}_i^\cdot = \bar{0}$, for all $i \in I$. By Lemma 1.4, $L^\cdot$ is a direct sum in $K(\mathfrak{A})$ of the $L_i^\cdot$ with canonical injections $\bar{v}_i^\cdot$. Hence, $\bar{f}^\cdot \bar{r}^\cdot = \bar{0}$, and thus, $\eta^\cdot = \bar{f}^\cdot/\bar{s}^\cdot = \tilde{0}$. The proof of the theorem is completed.

For later application, we study some useful properties of bounded-above complexes of projective objects.

1.9. Lemma. *Let $\mathfrak{A}$ be an abelian category, and let $X^\cdot, Y^\cdot, P^\cdot$ be complexes over $\mathfrak{A}$, where $P^\cdot$ is bounded-above of projective objects.*

(1) *The localization functor $L_{\mathfrak{A}} : K(\mathfrak{A}) \to D(\mathfrak{A})$ induces an isomorphism*
$$L_{P^\cdot,X^\cdot} : K(\mathfrak{A})(P^\cdot, X^\cdot) \to D(\mathfrak{A})(P^\cdot, X^\cdot) : \bar{f} \mapsto \tilde{f}.$$

(2) *If $\bar{s}^\cdot : X^\cdot \to Y^\cdot$ is a quasi-isomorphism and $\bar{f}^\cdot : P^\cdot \to Y^\cdot$ is a morphism in $K(\mathfrak{A})$, then $\bar{f}^\cdot = \bar{s}^\cdot \bar{g}^\cdot$, for some morphism $\bar{g}^\cdot : P^\cdot \to X^\cdot$ in $K(\mathfrak{A})$.*

Proof. Statement (1) is well known; see, for example, [17, (10.4.7)]. Let $\bar{s}^\cdot : X^\cdot \to Y^\cdot$ be a quasi-isomorphism and $\bar{f}^\cdot : P^\cdot \to Y^\cdot$ a morphism in $K(\mathfrak{A})$. Observing that $(\tilde{s}^\cdot)^{-1}\tilde{f}^\cdot \in D(\mathfrak{A})(P^\cdot, X^\cdot)$, we get some morphism $g^\cdot : P^\cdot \to X^\cdot$ in $C(\mathfrak{A})$ such that $(\tilde{s}^\cdot)^{-1}\tilde{f}^\cdot = \tilde{g}^\cdot$, that is, $\tilde{f}^\cdot = \tilde{s}^\cdot \tilde{g}^\cdot$. Since $L_{P^\cdot,X^\cdot}$ is an injective map, $\bar{f}^\cdot = \bar{s}^\cdot \bar{g}^\cdot$. The proof of the lemma is completed.

1.10. Proposition. *Let $\mathfrak{A}$ be an abelian category having essential direct sums, and let $P^\cdot, X_i^\cdot$, $i \in I$, be complexes over $\mathfrak{A}$, where $P^\cdot$ is bounded-above of projective objects. If $C(\mathfrak{A})(P^\cdot, X_i^\cdot) = 0$ for all but finitely many $i \in I$, then $P^\cdot$ is essential in the direct sum of the $X_i^\cdot$, $i \in I$, in $D(\mathfrak{A})$.*

Proof. Let $X^\bullet$ be the direct sum of the $X_i^\bullet$, $i \in I$, in $C(\mathfrak{A})$ with pseudo-projections $p_i^\bullet : X^\bullet \to X_i^\bullet$. By Lemma 1.4 and Theorem 1.8, $X^\bullet$ is the direct sum of the $X_i^\bullet$, $i \in I$, in $K(\mathfrak{A})$ and in $D(\mathfrak{A})$ with pseudo-projections $\bar{p}_i^\bullet : X^\bullet \to X_i^\bullet$, $i \in I$ and $\tilde{p}_i^\bullet : X^\bullet \to X_i^\bullet$, respectively.

Assume that J is a finite subset of I such that $C(\mathfrak{A})(P^\bullet, X_i^\bullet) = 0$ for $i \in I \backslash J$. In particular, $K(\mathfrak{A})(P^\bullet, X_i^\bullet) = 0$ for $i \in I \backslash J$. By Lemma 1.9(1), $D(\mathfrak{A})(P^\bullet, X_i^\bullet) = 0$, for $i \in I \backslash J$. Let $\theta^\bullet : P^\bullet \to X^\bullet$ be a morphism in $D(\mathfrak{A})$ such that $\tilde{p}_i^\bullet \theta^\bullet = 0$, for all $i \in I$. By Lemma 1.9(1), $\theta^\bullet = \tilde{g}^\bullet$ for some morphism $g^\bullet : M^\bullet \to X^\bullet$ in $C(\mathfrak{A})$. This yields $\tilde{p}_i^\bullet \tilde{g}^\bullet = 0$, and by Lemma 1.9(1), $\bar{p}_i^\bullet \bar{g}^\bullet = 0$ for all $i \in I$. Since $P^\bullet$ is essential in the direct sum $\oplus_{i \in I} X_i^\bullet$ in $K(\mathfrak{A})$; see (1.5), we obtain $\bar{g}^\bullet = 0$. Hence, $\theta^\bullet = 0$. The proof of the proposition is completed.

Let $\mathfrak{C}$ be a full abelian subcategory of $\mathfrak{A}$. We shall say that $\mathfrak{C}$ has *enough $\mathfrak{A}$-projective objects* provided that, for any $X \in \mathfrak{C}_0$, there exists an epimorphism $\varepsilon : P \to X$ in $\mathfrak{C}$ with P being projective in $\mathfrak{A}$. The following result is useful.

1.11. Lemma. *Let $\mathfrak{A}$ be an abelian category, and let $\mathfrak{C}$ be a full abelian subcategory of $\mathfrak{A}$. If $\mathfrak{C}$ has enough $\mathfrak{A}$-projective objects, then $D^b(\mathfrak{C})$ can be regarded as a full triangulated subcategory of $D(\mathfrak{A})$.*

Proof. Assume that $\mathfrak{C}$ has enough $\mathfrak{A}$-projective objects. It is well known that $D^b(\mathfrak{C})$ can be regarded as a full triangulated subcategory of $D^-(\mathfrak{C})$; see [10, (6.15)]. Hence, it suffices to show that $D^-(\mathfrak{C})$ can be regarded as a full triangulated subcategory of $D(\mathfrak{A})$. Indeed, $K^-(\mathfrak{C})$ is a full triangulated subcategory of $K(\mathfrak{A})$. The inclusion functor $j : K^-(\mathfrak{C}) \to K(\mathfrak{A})$ induces an exact functor of triangulated categories $j^D : D^-(\mathfrak{C}) \to D(\mathfrak{A})$ such that $L_{\mathfrak{A}} \circ j = j^D \circ L$, where $L : K^-(\mathfrak{C}) \to D^-(\mathfrak{C})$ is the localization functor. We need only to show that j^D is fully faithful.

Fix $X^\bullet, Y^\bullet \in C^-(\mathfrak{C})$. Since $\mathfrak{C}$ has enough $\mathfrak{A}$-projective objects, $K^-(\mathfrak{C})$ has a quasi-isomorphism $\bar{s}^\bullet : P^\bullet \to X^\bullet$, where $P^\bullet$ is bounded-above of projective objects in $\mathfrak{A}$; see [10, (7.5)]. Let $\theta^\bullet : X^\bullet \to Y^\bullet$ be a morphism in $D(\mathfrak{A})$. Write $\theta^\bullet = \bar{f}^\bullet / \bar{r}^\bullet$, where $\bar{r}^\bullet : M^\bullet \to X^\bullet$ is a quasi-isomorphism in $K(\mathfrak{A})$. By Lemma 1.9(2), $\bar{s}^\bullet = \bar{g}^\bullet \bar{r}^\bullet$ for some $\bar{g}^\bullet : P^\bullet \to M^\bullet$ in $K(\mathfrak{A})$, and hence, $\theta^\bullet = (\bar{f}^\bullet \bar{g}^\bullet)/\bar{s}^\bullet = j^D(L(\bar{f}^\bullet \bar{g}^\bullet) L(\bar{s}^\bullet)^{-1})$.

Now, let $\eta^\bullet : X^\bullet \to Y^\bullet$ be in $D^-(\mathfrak{C})$ with $j^D(\eta^\bullet) = 0$. Write $\eta^\bullet = L(\bar{h}^\bullet) L(\bar{t}^\bullet)^{-1}$, where $\bar{t}^\bullet : N^\bullet \to X^\bullet$ is a quasi-isomorphism in $K^-(\mathfrak{C})$. Then $\bar{h}^\bullet / \bar{t}^\bullet$ is null in $D(\mathfrak{A})$. Hence, $K(\mathfrak{A})$ has a quasi-morphism $\bar{u}^\bullet : U^\bullet \to N^\bullet$ such that $\bar{h}^\bullet \bar{u}^\bullet = \bar{0}$. By Lemma 1.9(2), $K^-(\mathfrak{C})$ has a morphism $\bar{v}^\bullet : P^\bullet \to N^\bullet$ such that $\bar{s}^\bullet = \bar{t}^\bullet \bar{v}^\bullet$. Since $\bar{s}^\bullet, \bar{t}^\bullet$ are quasi-isomorphisms in $K^-(\mathfrak{C})$, so is $\bar{v}^\bullet$. Using Lemma 1.9(2) again, we obtain a morphism $\bar{w}^\bullet : P^\bullet \to U^\bullet$ in $K(\mathfrak{A})$ such that $\bar{v}^\bullet = \bar{u}^\bullet \bar{w}^\bullet$. This yields $\bar{h}^\bullet \bar{v}^\bullet = \bar{h}^\bullet \bar{u}^\bullet \bar{w}^\bullet = 0$. Therefore, $\eta^\bullet = 0$. The proof of the lemma is completed.

2. Galois covering of linear categories

The main objective of this section is to extend Gabriel's notion of a Galois covering for skeletal linear categories to general linear categories. Throughout this section, let $\mathcal{A}$ be a linear category equipped with an action of a group G, that is, there exists a group homomorphism ρ from G into $\mathrm{Aut}(\mathcal{A})$, the group of automorphisms of $\mathcal{A}$.

2.1. Definition. Let $\mathcal{A}$ be a linear category with G a group acting on $\mathcal{A}$. The G-action on $\mathcal{A}$ is called

(1) *free* provided that $g\cdot X \ncong X$, for any indecomposable $X \in \mathcal{A}$ and any non-identity $g \in G$;
(2) *locally bounded* provided, for any indecomposable $X, Y \in \mathcal{A}$, that $\mathcal{A}(X, g\cdot Y) = 0$ for all but finitely many $g \in G$;
(3) *directed* provided, for any indecomposable $X, Y \in \mathcal{A}$, that $\mathcal{A}(X,\ g\cdot Y) = 0$ or $\mathcal{A}(g\cdot Y,\ X) = 0$ for all but at most one $g \in G$;
(4) *admissible* provided that it is both free and locally bounded.

A group is called *torsion-free* if every non-identity element is of infinite order.

2.2. Lemma. *Let $\mathcal{A}$ be a linear category with G a torsion-free group acting on $\mathcal{A}$. If the G-action on $\mathcal{A}$ is locally bounded, then it is free.*

Proof. Let $X \in \mathcal{A}_0$ be indecomposable such that there exists an isomorphism $u : X \to g\cdot X$, for some $g \in G$. Then

$$(g^i\cdot u) \circ \cdots \circ (g\cdot u) \circ u : X \to g^{i+1}\cdot X$$

is an isomorphism, for every $i \geq 1$. If the G-action is locally bounded, then g is of finite order, and hence, g is the identity of G. The proof of the lemma is completed.

Let $F : \mathcal{A} \to \mathcal{B}$ be a functor between linear categories. By abuse of notation, we identify $g \in G$ with $\rho(g) \in \mathrm{Aut}(\mathcal{A})$, where ρ is the homomorphism from G into $\mathrm{Aut}(\mathcal{A})$. In this way, $F \circ g : \mathcal{A} \to \mathcal{B}$ is an additive functor. Recall that a *functorial (iso)morphism* $\delta_g : F\circ g \to F$ consists of (iso)morphisms $\delta_{g,X} : (F \circ g)(X) \to F(X)$ with $X \in \mathcal{B}_0$, which are natural in X.

The following definition is due to Asashiba originally under the name of *invariance adjuster*; see [2, (1.1)].

2.3. Definition. Let $\mathcal{A}, \mathcal{B}$ be linear categories with G a group acting on $\mathcal{A}$. A functor $F : \mathcal{A} \to \mathcal{B}$ is called *G-stable* provided there exist functorial isomorphisms $\delta_g : F \circ g \to F$, $g \in G$, such that

$$\delta_{h,X} \circ \delta_{g,\,h\cdot X} = \delta_{gh,X},$$

for any $g, h \in G$ and $X \in \mathcal{A}_0$. In this case, we call $\delta = (\delta_g)_{g\in G}$ a *G-stabilizer* for F.

Remark. (1) By definition, $\delta_{g,X}^{-1} = \delta_{g^{-1},g\cdot X}$ for $g \in G$ and $X \in \mathcal{A}_0$; and $\delta_e = 1_F$, where e is the identity of G.

(2) If $X = Y \oplus Z$, then

$$\delta_{g,X} = \begin{pmatrix} \delta_{g,Y} & 0 \\ 0 & \delta_{g,Z} \end{pmatrix}.$$

(3) The G-stabilizer δ for F is called *trivial* if $\delta_g = 1_F$, for every $g \in G$. In this case, we shall say that F is *G-invariant*.

2.4. Lemma. *Let $\mathcal{A}, \mathcal{B}$ be linear categories with G a group acting on $\mathcal{A}$, and let $F : \mathcal{A} \to \mathcal{B}$ be a functor with a G-stabilizer δ. If $u : X \to g\cdot Y$ and $v : Y \to h\cdot Z$, where $g, h \in G$, are morphisms in $\mathcal{A}$, then*

$$(\delta_{h,Z} \circ F(v)) \circ (\delta_{g,Y} \circ F(u)) = \delta_{gh,Z} \circ F((g\cdot v) \circ u).$$

Proof. Let $u : X \to g \cdot Y$ and $v : Y \to h \cdot Z$, with $g, h \in G$, be morphisms in $\mathcal{A}$. Applying F yields a diagram

$$\begin{tikzcd} F(X) \arrow[r, "F(u)"] & F(g{\cdot}Y) \arrow[r, "F(g\cdot v)"] \arrow[d, "\delta_{g,Y}"'] & F(g{\cdot}(h{\cdot}Z)) \arrow[r, equal] \arrow[d, "\delta_{g,h\cdot Z}"'] & F((gh){\cdot}Z) \arrow[d, "\delta_{gh,Z}"] \\ & F(Y) \arrow[r, "F(v)"] & F(h{\cdot}Z) \arrow[r, "\delta_{h,Z}"] & F(Z), \end{tikzcd}$$

where the left square is commutative since $\delta_{g,Y}$ is natural in Y, and the right square is commutative by Definition 2.3. The proof of the lemma is completed.

The following definition is also due to Asashiba; see [2, (1.7)].

2.5. Definition. Let $\mathcal{A}, \mathcal{B}$ be linear categories with G a group acting on $\mathcal{A}$. A functor $F : \mathcal{A} \to \mathcal{B}$ is called a *G-precovering* provided that F has a G-stabilizer δ such that, for any $X, Y \in \mathcal{A}_0$, the following two maps are isomorphisms:

$$F_{X,Y} : \oplus_{g\in G}\, \mathcal{A}(X, g{\cdot}Y) \to \mathcal{B}(F(X), F(Y)) : \ (u_g)_{g\in G} \mapsto \textstyle\sum_{g\in G} \delta_{g,Y} \circ F(u_g).$$

$$F^{X,Y} : \oplus_{g\in G}\, \mathcal{A}(g{\cdot}X, Y) \to \mathcal{B}(F(X), F(Y)) : \ (v_g)_{g\in G} \mapsto \textstyle\sum_{g\in G} F(v_g) \circ \delta^{-1}_{g,X}.$$

Remark. In the above definition, as observed by Asashiba, it is sufficient to require all $F_{X,Y}$ be isomorphisms, or all $F^{X,Y}$ be isomorphisms; see [2, (1.6)].

In the following two results, we collect some properties of a precovering functor.

2.6. Lemma. *Let $\mathcal{A}, \mathcal{B}$ be linear categories with G a group acting on $\mathcal{A}$, and let $F : \mathcal{A} \to \mathcal{B}$ be a G-precovering with a G-stabilizer δ.*

(1) *For any $X, Y \in \mathcal{A}_0$, we have the following decompositions*

$$\mathcal{B}(F(X), F(Y)) = \oplus_{g\in G}\, \delta_{g,Y} \circ F(\mathcal{A}(X, g{\cdot}Y)) = \oplus_{g\in G}\, F(\mathcal{A}(g{\cdot}X, Y)) \circ \delta^{-1}_{g,X}.$$

(2) *The functor F is faithful, and in particular, it sends decomposable objects to decomposable ones.*

Proof. Fix $X, Y \in \mathcal{A}_0$. By definition, we have an isomorphism

$$F_{X,Y} : \oplus_{g\in G}\, \mathcal{A}(X, g{\cdot}Y) \to \mathcal{B}(F(X), F(Y)) : (u_g)_{g\in G} \mapsto \textstyle\sum_{g\in G} \delta_{g,Y} \circ F(u_g).$$

This yields immediately the first decomposition stated in Statement (1). Similarly, the second decomposition follows from the defining isomorphisms $F^{X,Y}$ stated in Definition 2.5. Furthermore, since $\delta_{e,Y} = 1_{F(Y)}$, where e is the identity of G, the isomorphism $F_{X,Y}$ restricts to a monomorphism

$$F_e : \mathcal{A}(X, Y) \to \mathcal{B}(F(X), F(Y)) : \ u \mapsto F(u).$$

The proof of the lemma is completed.

2.7. Lemma. *Let $\mathcal{A}, \mathcal{B}$ be linear categories with G a group acting on $\mathcal{A}$, and let $F : \mathcal{A} \to \mathcal{B}$ be a G-precovering. Consider a morphism $u : X \to Y$ in $\mathcal{A}$.*

(1) *If $v : X \to Z$ or $v : Z \to Y$ is a morphism in $\mathcal{A}$, then v factorizes through u if and only if $F(v)$ factorizes through $F(u)$.*

(2) *The morphism u is a section, retraction, or isomorphism if and only if $F(u)$ is a section, retraction, or isomorphism, respectively.*

Proof. (1) Let δ be the G-stabilizer for F. Assume first that $v \in \mathcal{A}(X, Z)$. If v factorizes through u, then $F(v)$ evidently factorizes through $F(u)$. Suppose conversely that $F(v) = w \circ F(u)$ for some $w : F(Y) \to F(Z)$ in $\mathcal{B}$. By Lemma 2.6(1), we may write $w = \sum_{i=1}^{n} \delta_{g_i,Z} \circ F(w_i)$, where $g_1, \ldots, g_n \in G$ are distinct, and $w_i \in \mathcal{A}(Y, g_i{\cdot}Z)$. This gives rise to

$$F(v) = \textstyle\sum_{i=1}^{n} \delta_{g_i,Z} \circ F(w_i) \circ F(u) = \sum_{i=1}^{n} \delta_{g_i,Z} \circ F(w_i \circ u).$$

In view of Lemma 2.6(1), there exists some $1 \le s \le n$ such that $g_s = e$, the identity of G, and $F(v) = F(w_s \circ u)$. Since F is faithful by Lemma 2.6(2), $v = w_s u$. In case $v \in \mathcal{A}(Z, Y)$, we can establish Statement (1) in a dual manner.

(2) Specializing Statement (1) to the case where $v = 1_X$ or $v = 1_Y$, we obtain the first two parts of Statement(2), and which in turn imply the third part. The proof of the lemma is completed.

A functor $F : \mathcal{A} \to \mathcal{B}$ between linear categories is called *almost dense* if each indecomposable object in $\mathcal{B}$ is isomorphic to an object lying in the image of F.

2.8. Definition. Let $\mathcal{A}, \mathcal{B}$ be linear categories with G a group acting admissibly on $\mathcal{A}$. A G-precovering $F : \mathcal{A} \to \mathcal{B}$ is called a *Galois G-covering* provided that the following conditions are verified.
(1) The functor F is almost dense.
(2) If $X \in \mathcal{A}$ is indecomposable, then $F(X)$ is indecomposable.
(3) If $X, Y \in \mathcal{A}$ are indecomposable with $F(X) \cong F(Y)$, then there exists some $g \in G$ such that $Y = g{\cdot}X$.

Remark. (1) In case $\mathcal{A}, \mathcal{B}$ are Krull-Schmidt, a Galois G-covering $F : \mathcal{A} \to \mathcal{B}$ is a dense functor, and consequently, F is an equivalence if and only if G is trivial.

(2) If $\mathcal{A}, \mathcal{B}$ are skeletal linear categories over a field, then a Galois covering $F : \mathcal{A} \to \mathcal{B}$ in Gabriel's sense; see [8, (3.1)] is simply a G-invariant Galois G-covering.

The next two results will be useful for determining when a precovering is a Galois covering.

2.9. Lemma. *Let $\mathcal{A}, \mathcal{B}$ be linear categories with G a group acting admissibly on $\mathcal{A}$, and let $F : \mathcal{A} \to \mathcal{B}$ be a G-precovering. Consider an object X in $\mathcal{A}$ such that* $\mathrm{End}_{\mathcal{A}}(X)$ *is local with a nilpotent radical.*
(1) $\mathrm{End}_{\mathcal{B}}(F(X))$ *is local with a nilpotent radical.*
(2) *If $Y \in \mathcal{A}$ with $F(Y) \cong F(X)$, then $Y \cong g{\cdot}X$ for some $g \in G$.*

Proof. Let δ be the G-stabilizer for F. By Definition 2.5, we have an isomorphism

$$F_X : \oplus_{g\in G}\, \mathcal{A}(X, g{\cdot}X) \to \mathrm{End}_{\mathcal{B}}(F(X)) : (u_g)_{g\in G} \mapsto \textstyle\sum_{g\in G} \delta_{g,Y} \circ F(u_g).$$

(1) Let e denote the identity of G. Consider the additive subgroup

$$J = F(\mathrm{rad}\,(\mathrm{End}_{\mathcal{A}}(X))) + \textstyle\sum_{e\ne g\in G} \delta_{g,X} \circ F(\mathcal{A}(X,\, g{\cdot}X))$$

of $\mathrm{End}_{\mathcal{B}}(F(X))$. By hypothesis, $\mathrm{End}_{\mathcal{A}}(g\cdot X)$ is local, for every $g \in G$. Since the G-action on $\mathcal{A}$ is free, any morphism $u : X \to g \cdot X$ with $g \ne e$ is not invertible. Let $u : X \to g \cdot X$ and $v : X \to h \cdot X$, with $g, h \in G$, be morphisms in $\mathcal{A}$. If u or v is not invertible, then $(g \cdot v) \circ u : X \to (gh) \cdot X$ is not invertible. In view of Lemma 2.4, we see that

$$\delta_{h,X} \circ F(v) \circ \delta_{g,X} \circ F(u) = \delta_{gh,X} \circ F((g{\cdot}v) \circ u) \in J.$$

This implies that J is a two-sided ideal in $\mathrm{End}_{\mathcal{B}}(F(X))$. As a consequence, the isomorphism F_X induces a surjective algebra homomorphism

$$\bar{F}_X : \mathrm{End}_{\mathcal{A}}(X)/\mathrm{rad}(\mathrm{End}_{\mathcal{A}}(X)) \to \mathrm{End}_{\mathcal{B}}(F(X))/J.$$

Since $\mathrm{End}_{\mathcal{A}}(X)/\mathrm{rad}(\mathrm{End}_{\mathcal{A}}(X))$ is a division algebra, $\bar{F}_X$ is an isomorphism. In particular, $\mathrm{End}_{\mathcal{B}}(F(X))/J$ is a division algebra.

Since the G-action on $\mathcal{A}$ is locally bounded, there exists a finite subset G_0 of G, say of n elements, such that $\mathcal{A}(X, g\cdot X) = 0$ for all $g \in G\backslash G_0$. Moreover, by hypothesis, $\mathrm{rad}^d(\mathrm{End}_{\mathcal{A}}(X)) = 0$, for some $d > 0$. Then, $\mathrm{rad}^d(\mathrm{End}_{\mathcal{A}}(g\cdot X)) = 0$, for every $g \in G$. Set $m = nd+1$. We claim that $u_m \cdots u_2u_1 = 0$, for $u_1, u_2, \ldots, u_m \in J$. Indeed, with no loss of generality, we may assume that $u_i = \delta_{g_i,X} \circ F(v_i)$, where $g_i \in G$ and $v_i : X \to g_i\cdot X$ is a non-invertible morphism, $i = 1, \ldots, m$. Put $h_0 = e$, and $h_i = g_1 \cdots g_i$ and $w_i = h_{i-1} \cdot v_i$, for $i = 1, \ldots, m$. Consider the sequence

$$X \xrightarrow{w_1} h_1\cdot X \xrightarrow{w_2} h_2\cdot X \longrightarrow \cdots \longrightarrow h_{m-1}\cdot X \xrightarrow{w_m} h_m\cdot X$$

of non-invertible morphisms. If $h_i \notin G_0$ for some $1 \le i \le m$, then $w_m \cdots w_2w_1 = 0$. Otherwise, since $m > nd$, there exists some t with $1 \le t \le n$ such that the number of indices j with $1 \le j \le m$ for which $h_j = h_t$ is greater than d. Since $\mathrm{rad}^d(\mathrm{End}_{\mathcal{A}}(h_t\cdot X)) = 0$, we have $w_m \cdots w_2w_1 = 0$. By Lemma 2.4, we have

$$u_m \cdots u_2u_1 = \delta_{h_m,X} \circ F(w_m \cdots w_2w_1) = 0.$$

This proves our claim, and hence, $J^m = 0$. In particular, $J \subseteq \mathrm{rad}(\mathrm{End}_{\mathcal{B}}(F(X)))$. Since $\mathrm{End}_{\mathcal{B}}(F(X))/J$ is a division algebra, we obtain $J = \mathrm{rad}(\mathrm{End}_{\mathcal{B}}(F(X)))$. That is, $\mathrm{End}_{\mathcal{B}}(F(X))$ is local with a nilpotent radical.

(2) Let $Y \in \mathcal{A}$ be such that $F(X) \cong F(Y)$. By Statement (1), $\mathrm{End}_{\mathcal{B}}(F(Y))$ is local. Since F is faithful by Lemma 2.6(2), $\mathrm{End}_{\mathcal{A}}(Y)$ has no proper idempotent. Let now $u : F(X) \to F(Y)$ and $v : F(Y) \to F(X)$ be morphisms in $\mathcal{B}$ such that $vu = 1_{F(X)}$. By Lemma 2.6(1), we may write $u = \sum_{i=1}^{r} \delta_{g_i,Y} \circ F(u_i)$, where $g_1, \ldots, g_r \in G$ are distinct and $u_i \in \mathcal{A}(X, g_i\cdot Y)$; and $v = \sum_{j=1}^{s} \delta_{h_j,X} \circ F(v_i)$, where $h_1, \ldots, h_s \in G$ are distinct and $v_j \in \mathcal{A}(Y, h_j\cdot X)$. Applying Lemma 2.4, we obtain

$$1_{F(X)} = \sum\nolimits_{1\le i\le r; 1\le j\le s} \delta_{g_ih_j} \circ F((g_i\cdot v_j) \circ u_i).$$

By Lemma 2.6(1), $1_{F(X)} = \sum_{i,j;\, g_ih_j=e} F((g_i\cdot v_j) \circ u_i)$. Since F is faithful by Lemma 2.6(2), $1_X = \sum_{i,j;\, g_ih_j=e} (g_i \cdot v_j) \circ u_i$. Since $\mathrm{End}_{\mathcal{A}}(X)$ is local, there exist i, j such that $h_j = g_i^{-1}$ and $(g_i \cdot v_j) \circ u_i$ is invertible. Since $\mathrm{End}_{\mathcal{A}}(Y)$ has no proper idempotent, $u_i : X \to g_i \cdot Y$ is an isomorphism. The proof of the lemma is completed.

An object $X \in \mathcal{A}$ is called *properly indecomposable* if $\mathrm{End}_{\mathcal{A}}(X)$ has no proper idempotent. Observe that if all idempotents in $\mathcal{A}$ split, then every indecomposable object in $\mathcal{A}$ is properly indecomposable.

2.10. Lemma. *Let $F : \mathcal{A} \to \mathcal{B}$ be a G-precovering between linear categories, where G is a group acting on $\mathcal{A}$ with a directed and locally bounded action. Let X be a properly indecomposable object in $\mathcal{A}$.*

(1) *The image $F(X)$ is properly indecomposable.*

(2) *If $Y \in \mathcal{A}$ with $F(Y) \cong F(X)$, then $Y \cong g\cdot X$ for some $g \in G$.*

Proof. Let δ be the G-stabilizer for F. By Lemma 2.6(1), we have

$$(*) \qquad \mathrm{End}_{\mathcal{B}}(F(X)) = \oplus_{g\in G}\, \delta_{g,X} \circ F(\mathcal{A}(X,\, g\cdot X)).$$

Denote by e the identity of G. We claim that

$$I = \textstyle\sum_{e\neq g\in G}\, \delta_{g,X} \circ F(\mathcal{A}(X,\, g\cdot X))$$

is a two-sided ideal in $\mathrm{End}_{\mathcal{B}}(F(X))$. That is, if $u \in I$ and $v \in \mathrm{End}_{\mathcal{B}}(F(X))$, then $uv,\, vu \in I$. Indeed, with no loss of generality, we may assume that $u = \delta_{g,X} \circ F(u_g)$ with $e \neq g \in G$ and $u_g \in \mathcal{A}(X, g\cdot X)$, and $v = \delta_{h,X} \circ F(v_h)$ with $h \in G$ and $v_h \in \mathcal{A}(X, h\cdot X)$. By Lemma 2.4, $uv = \delta_{hg,X} \circ F((h\cdot u_g)\circ v_h)$. If $hg \neq e$, then $uv \in I$ by definition. Otherwise, $h \neq e$. Since the G-action on $\mathcal{A}$ is directed, using the fact that $\mathcal{A}(X, e\cdot X) \neq 0$ and $\mathcal{A}(e\cdot X, X) \neq 0$, we see that the composition map

$$(**) \qquad \mathcal{A}(h\cdot X, X) \otimes_{\mathbb{Z}} \mathcal{A}(X, h\cdot X) \longrightarrow \mathrm{End}_{\mathcal{A}}(X)$$

vanishes. In particular, $(h\cdot u_g)\circ v_h = 0$, and consequently, $uv \in I$. Similarly, we can show that $vu \in I$. This proves our claim.

Next, since the G-action on $\mathcal{A}$ is locally bounded, there exists a finite subset G_0 of G, say of n elements, such that $\mathcal{A}(X, g{\cdot}X) = 0$, for any $g \in G\backslash G_0$. Set $m = n+1$. We shall show that $u_m \cdots u_2 u_1 = 0$, for any $u_1, u_2, \ldots, u_m \in I$. Indeed, with no loss of generality, we may assume that $u_i = \delta_{g_i,X} \circ F(v_i)$ with $e \neq g_i \in G$ and $v_i \in \mathcal{A}(X, g_i\cdot X)$, for $i = 1, \ldots, m$. Write $h_0 = e$, $h_i = g_1 \cdots g_i$ and $w_i = h_{i-1}\cdot v_i$, for $i = 1, \ldots, m$. Consider the sequence

$$X \xrightarrow{w_1} h_1\cdot X \xrightarrow{w_2} h_2\cdot X \longrightarrow \cdots \longrightarrow h_{m-1}\cdot X \xrightarrow{w_m} h_m\cdot X.$$

If $h_i \notin G_0$ for some $1 \le i \le m$, then $w_m \cdots w_2 w_1 = 0$. By Lemma 2.4,

$$u_m \cdots u_2 u_1 = \delta_{h_m,X} \circ F(w_m \cdots w_2 w_1) = 0.$$

Otherwise, since $m > n$, there exist r, s with $1 \le r < s \le m$ such that $h_r = h_s$. As a consequence, $g_{r+1} \cdots g_s = e$, and in particular, $r+1 < s$. Since $g_{r+1} \neq e$, in view of the vanishing map $(**)$ for $h = g_{r+1}$, we see that

$$((g_{r+1} \cdots g_{s-1})\cdot v_s) \circ \cdots \circ (g_{r+1}\cdot v_{r+2}) \circ v_{r+1} = 0.$$

By Lemma 2.4, $u_s \cdots u_{r+1} = 0$, and consequently, $u_m \cdots u_2 u_1 = 0$. This shows that I is a nilpotent.

(1) Let $f \in \mathrm{End}_{\mathcal{B}}(F(X))$ be an idempotent. In view of the direct decomposition $(*)$, we may uniquely write $f = u - v$ with $u \in F(\mathcal{A}(X, X))$ and $v \in I$. Since $f = f^2 = u^2 - (uv + vu - v^2)$ with $u^2 \in F(\mathcal{A}(X, X))$ and $uv + vu - v^2 \in I$, we infer that $u = u^2$ and $v = uv + vu - v^2$. Write $u = F(u_0)$ for some $u_0 \in \mathrm{End}_{\mathcal{A}}(X)$. Since F is faithful by Lemma 2.6(2), $u_0^2 = u_0$, and since X is properly indecomposable, $u_0 = 0$ or $u_0 = 1_X$. As a consequence, $u = 0$ or $u = 1_{F(X)}$. In the first case, since $-v$ is nilpotent, $f = 0$. In the second case, $v = v + v - v^2$, and hence $v = v^2$. Since v is nilpotent, $v = 0$, and hence, $f = 1_{F(X)}$. This establishes Statement (1).

(2) Let $Y \in \mathcal{A}$ be such that $F(X) \cong F(Y)$. By Statement (1), $F(Y)$ is properly indecomposable, and by Lemma 2.6(2), so is Y. Let $u : F(X) \to F(Y)$ and $v : F(Y) \to F(X)$ be morphisms such that $vu = 1_{F(X)}$. In view of the direct decomposition $(*)$, we write uniquely $u = \sum_{i=1}^{r} \delta_{g_i,Y} \circ F(u_i)$ with $g_1, \ldots, g_r \in G$ distinct and $u_i \in \mathcal{A}(X, g_i\cdot Y)$, and $v = \sum_{j=1}^{s} \delta_{h_j,X} \circ F(v_i)$ with $h_1, \ldots, h_s \in G$ distinct and $v_j \in \mathcal{A}(Y, h_j\cdot X)$. Then, $1_{F(X)} = vu = \sum_{i,j} \delta_{g_i h_j} \circ F((g_i\cdot v_j)\circ u_i)$; see (2.4), and hence, $1_{F(X)} = \sum_{i,j;\, g_i h_j = e} F((g_i\cdot v_j)\circ u_i)$. Since F is faithful; see (2.6),

$1_X = \sum_{i,j;\, g_i h_j = e} (g_i \cdot v_j) \circ u_i$. Since the G-action on $\mathcal{A}$ is directed and the g_i are pairwise distinct, we may assume that $(g_i \cdot v_j) \circ u_i = 0$ for all i, j with $1 < i \le r$ and $h_j g_i = e$. Since the h_j are pairwise distinct, we may assume that $g_1 h_1 = e$ and $g_1 h_j \ne e$ for all $1 < j \le s$. This implies that $1_X = (g_1 \cdot v_1) \circ u_1$. Since $\mathrm{End}_{\mathcal{A}}(Y)$ has no proper idempotent, $u_1 \circ (g_1 \cdot v_1) = 1_Y$, and hence, $X \cong g_1 \cdot Y$. The proof of the lemma is completed.

For the rest of this section, we assume that $\mathcal{A}$ has direct sums. By Lemma 1.2, the category of linear endofunctors of $\mathcal{A}$ also has direct sums. Regarding each $g \in G$ as an automorphism of $\mathcal{A}$, we obtain a direct sum $\mathcal{G} = \oplus_{g \in G}\, g$ with canonical injections $j_g : g \to \mathcal{G}$, $g \in G$. More explicitly, for each $X \in \mathcal{A}$, we have $\mathcal{G}(X) = \oplus_{g \in G}\, g \cdot X$ with canonical injections $j_{g,X} : g \cdot X \to \mathcal{G}(X)$, with $g \in G$.

2.11. Definition. Let $\mathcal{A}, \mathcal{B}$ be linear categories such that $\mathcal{A}$ has direct sums and admits an action of a group G. Let $F : \mathcal{A} \to \mathcal{B}$ and $E : \mathcal{B} \to \mathcal{A}$ be functors such that (F, E) is an adjoint pair with adjoint isomorphism ϕ. We say that (E, F) is *G-graded* provided that the following conditions are verified.

(1) There exists a functorial isomorphism $\gamma : \oplus_{g \in G}\, g \to E \circ F$.
(2) There exists a G-stabilizer δ for F such that

$$\phi_{X,F(Y)}(\gamma_Y \circ j_{g,Y} \circ u) = \delta_{g,Y} \circ F(u),$$

for any $X, Y \in \mathcal{A}_0$; $g \in G$ and $u \in \mathcal{A}(X, g \cdot Y)$.

A full subcategory $\mathcal{C}$ of $\mathcal{A}$ is called *stable* under the G-action on $\mathcal{A}$, or simply, a *G-subcategory* of $\mathcal{A}$, provided, for any $X \in \mathcal{C}_0$ and $g \in G$, that $g \cdot X \in \mathcal{C}$. In this case, the G-action on $\mathcal{A}$ restricts to a G-action on $\mathcal{C}$. This restricted G-action is called *$\mathcal{A}$-essential* provided, for any $X, Y \in \mathcal{C}_0$, that X is essential in the direct sum $\mathcal{G}(Y) = \oplus_{g \in G}\, g \cdot Y \in \mathcal{A}$.

2.12. Theorem. *Let $\mathcal{A}, \mathcal{B}$ be linear categories such that $\mathcal{A}$ has direct sums and admits an action of a group G. Let $F : \mathcal{A} \to \mathcal{B}$ and $E : \mathcal{B} \to \mathcal{A}$ be functors forming a G-graded adjoint pair (F, E). Let $\mathcal{C}$ be a G-subcategory of $\mathcal{A}$ with a locally bounded and $\mathcal{A}$-essential G-action, and $\mathcal{D}$ a full subcategory of $\mathcal{B}$. If F sends $\mathcal{C}$ into $\mathcal{D}$, then it restricts to a G-precovering $F' : \mathcal{C} \to \mathcal{D}$.*

Proof. By Definition 2.11, F has a G-stabilizer δ, which restricts to a G-stabilizer δ' for the restriction $F' : \mathcal{C} \to \mathcal{D}$. Consider the direct sum $\mathcal{G} = \oplus_{g \in G}\, g$, where g is regarded as an automorphism of $\mathcal{A}$, with canonical injections $j_g : g \to \mathcal{G}$, $g \in G$.

Let $X, Y \in \mathcal{C}_0$. By the assumption, $\mathcal{A}(X, g \cdot Y) = 0$ for all but finitely many $g \in G$ and X is essential in $\oplus_{g \in G}\, g \cdot Y$. By Proposition 1.3, there exists an isomorphism

$$\nu_X : \oplus_{g \in G}\, \mathcal{A}(X, g \cdot Y) \longrightarrow \mathcal{A}(X, \mathcal{G}(Y))$$

such that $\nu_X(u_g) = j_{g,Y} \circ u_g$, for any $u_g \in \mathcal{A}(X, g \cdot Y)$. By Definition 2.11(1), there exists a functorial isomorphism $\gamma : \mathcal{G} \to E \circ F$, which yields an isomorphism

$$\mathcal{A}(X, \gamma_Y) : \mathcal{A}(X, \mathcal{G}(Y)) \longrightarrow \mathcal{A}(X, E(F(Y))).$$

Moreover, the adjoint isomorphism ϕ yields an isomorphism:

$$\phi_{X,F(Y)} : \mathcal{A}(X, E(F(Y))) \longrightarrow \mathcal{B}(F(X), F(Y)).$$

Composing the above three isomorphisms yields an isomorphism

$$F_{X,Y} : \oplus_{g\in G}\,\mathcal{A}(X, g\cdot Y) \longrightarrow \mathcal{B}(F(X), F(Y)).$$

By Definition 2.11(2), for any $g \in G$ and $u_g \in \mathcal{A}(X, g \cdot Y)$, we have

$$F_{X,Y}(u_g) = \phi_{X,F(Y)}(\gamma_Y \circ j_{g,Y} \circ u_g) = \delta_{g,X} \circ F(u_g) = \delta_{g,X} \circ F'(u_g).$$

That is, F' is a G-precovering. The proof of the theorem is completed.

3. Auslander-Reiten theory under a Galois covering

The main objective of this section is to show that a Galois covering between Krull-Schmidt categories preserves irreducible morphisms and almost split sequences. Throughout this section, let $\mathcal{A}$ be a Krull-Schmidt category equipped with an admissible action of a group G. Recall that the *radical* $\mathrm{rad}(\mathcal{A})$ of $\mathcal{A}$ is the ideal generated by the non-invertible morphisms between indecomposable objects. A morphism in $\mathcal{A}$ is called *radical* if it lies in $\mathrm{rad}(\mathcal{A})$.

3.1. Lemma. *Let $\mathcal{A}, \mathcal{B}$ be Krull-Schmidt categories with G a group acting admissibly on $\mathcal{A}$, and let $F : \mathcal{A} \to \mathcal{B}$ be a Galois G-covering. If u is a morphism in $\mathcal{A}$, then it is radical if and only if $F(u)$ is radical.*

Proof. Let $u : X \to Y$ be a morphism in $\mathcal{A}$. Write $X = X_1 \oplus \cdots \oplus X_n$ and $Y = Y_1 \oplus \cdots \oplus Y_m$, with X_i, Y_j being indecomposable. Then, $u = (u_{ij})_{n\times m}$ with $u_{ij} \in \mathcal{A}(X_j, Y_i)$. Since F is linear, we have $F(X) = F(X_1) \oplus \cdots \oplus F(X_n)$ and $F(Y) = F(Y_1)\oplus\cdots\oplus F(Y_m)$, where $F(X_i)$, $F(Y_j)$ are indecomposable by Definition 2.8(2), and $F(u) = (F(u_{ij}))_{m\times n}$ with $F(u_{ij}) \in \mathcal{B}(F(X_j), F(Y_i))$.

Now, u is radical if and only if u_{ij} is not invertible for all $1 \le i \le m$ and $1 \le j \le n$. By Lemma 2.7(2), this is equivalent to $F(u_{rs})$ being not invertible, for all $1 \le i \le m$ and $1 \le j \le n$, that is, $F(u)$ is radical. The proof of the lemma is completed.

For each integer $n > 1$, the *n-th radical* $\mathrm{rad}^n(\mathcal{A})$ of $\mathcal{A}$ is the ideal generated by the composites of n radical morphisms between indecomposable objects.

3.2. Lemma. *Let $\mathcal{A}, \mathcal{B}$ be Krull-Schmidt categories with G a group acting admissibly on $\mathcal{A}$, and let $F : \mathcal{A} \to \mathcal{B}$ be a Galois G-covering with a G-stabilizer δ. For some $X, Y \in \mathcal{A}$, consider*

$$u = \textstyle\sum_{g\in G} \delta_{g,Y} \circ F(u_g) \in \mathcal{B}(F(X), F(Y)),$$

where $u_g \in \mathcal{A}(X, g \cdot Y)$ such that $u_g = 0$ for all but finitely many $g \in G$. If m is a positive integer, then $u \in \mathrm{rad}^m(\mathcal{B})$ if and only if $u_g \in \mathrm{rad}^m(\mathcal{A})$, for all $g \in G$.

Proof. We shall need only to prove the necessity, since the sufficiency follows easily from Lemma 3.1. Assume that $u \in \mathrm{rad}^m(\mathcal{B})(F(X), F(Y))$ for some integer $m > 0$. Let $G_0 = \{g_1, \ldots, g_n\}$ be a subset of G such that $u_g = 0$, for all $g \in G\backslash G_0$. Writing $u_{g_i} = u_i$, we obtain $u = \sum_{i=1}^n \delta_{g_i,Y} \circ F(u_i)$. In order to prove that $u_i \in \mathrm{rad}^m(\mathcal{A})(X, g_i \cdot Y)$, for $i = 1, \ldots, n$, there exists no loss of generality in assuming that X is indecomposable.

Let $m = 1$. Suppose on the contrary that some of the u_i, say u_1, is not radical. Since X is indecomposable, $p_1 \circ u_1 = 1_X$ for some $p_1 \in \mathcal{A}(g_1{\cdot}Y, X)$. Observing that

$$\delta^{-1}_{g_1,Y} \circ \delta_{g_j,Y} = \delta_{g_1^{-1},\, g_1\cdot Y} \circ \delta_{g_j,\, g_1^{-1}(g_1\cdot Y)} = \delta_{g_j g_1^{-1},\, g_1\cdot Y}, \text{ for } j = 2, \ldots, n,$$

we obtain

$$\begin{array}{rcl} F(p_1)\circ\delta_{g_1,Y}^{-1}\circ u & = & F(p_1)\circ F(u_1)+\sum_{j=2}^r F(p_1)\circ\delta_{g_1,Y}^{-1}\circ\delta_{g_j,Y}\circ F(u_j) \\ & = & F(1_X)+\sum_{j=2}^r F(p_1)\circ\delta_{g_jg_1^{-1},\,g_1\cdot Y}\circ F(u_j) \\ & = & 1_{F(X)}+\sum_{j=2}^r \delta_{g_jg_1^{-1},X}\circ F((g_jg_1^{-1}\cdot p_1)\circ u_j) \end{array}$$

For any $j>1$, since the G-action on $\mathcal{A}$ is free, $(g_jg_1^{-1}\cdot p_1)\circ u_j : X\to g_jg_1^{-1}\cdot X$ is radical, and so is $\delta_{g_jg_1^{-1},X}\circ F((g_jg_1^{-1}\cdot p_1)\circ u_j)$ by Lemma 3.1. As a consequence, $F(p_1)\circ\delta_{g_1,Y}^{-1}\circ u$ is an automorphism of $F(X)$, which is absurd.

Assume that $m>1$ and the necessity holds for $m-1$. Write $u=vw$, where $v\in \mathrm{rad}(\mathcal{B})(M,F(Y))$ and $w\in\mathrm{rad}^{m-1}(\mathcal{B})(F(X),M)$. Since F is dense, we may assume that $M=F(N)$ for some $N\in\mathcal{A}_0$. By Lemma 2.6(1), we have

$$\mathcal{B}(F(X),F(N))=\oplus_{g\in G}\,\delta_{g,N}\circ F(\mathcal{A}(X,g\cdot N))$$

and

$$\mathcal{B}(F(N),F(Y))=\oplus_{g\in G}\,F(\mathcal{A}(g\cdot N,Y))\circ\delta_{g,N}^{-1}.$$

Adding some zero summands for each of u,v,w, if necessary, we may assume that $w=\sum_{i=1}^n\delta_{g_i,N}\circ F(w_i)$ for some $w_i: X\to g_i\cdot N$, and $v=\sum_{i=1}^n F(v_i)\circ\delta_{g_i,N}^{-1}$ for some $v_i: g_i\cdot N\to Y$. By the induction hypothesis and its dual, $w_i\in\mathrm{rad}^{m-1}(\mathcal{A})(X,g_i\cdot N)$ and $v_i\in\mathrm{rad}(\mathcal{A})(g_i\cdot N,Y)$, for $i=1,\ldots,n$. This yields

$$\begin{array}{rcl} \sum_{i=1}^n\,\delta_{g_i,Y}\circ F(u_i) & = & \sum_{1\le r,s\le n}F(v_r)\circ\delta_{g_r,N}^{-1}\circ\delta_{g_s,N}\circ F(w_s) \\ & = & \sum_{1\le r,s\le n}\,F(v_r)\circ\delta_{g_sg_r^{-1},\,g_r\cdot N}\circ F(w_s) \\ & = & \sum_{1\le r,s\le n}\,\delta_{g_sg_r^{-1},Y}\circ F((g_sg_r^{-1}\cdot v_r)\circ w_s). \end{array}$$

For each integer i with $1\le i\le n$, set $\Omega_i=\{(r,s)\mid 1\le r,s\le n;\ g_rg_s^{-1}=g_i\}$. Since $\mathcal{B}(F(X),F(Y))=\oplus_{g\in G}\,\delta_{g,Y}\circ F(\mathcal{A}(X,g\cdot Y))$, we deduce that

$$\delta_{g_i,Y}\circ F(u_i)=\sum_{(r,s)\in\Omega_i}\delta_{g_i,Y}\circ F((g_i\cdot v_r)\circ w_s).$$

Since $\delta_{g_i,Y}$ is an isomorphism and F is faithful, we conclude that

$$u_i=\sum_{(r,s)\in\Omega_i}(g_i\cdot v_r)\circ w_s\in\mathrm{rad}^m(\mathcal{A})(X,g_i\cdot Y),\ i=1,\ldots,n.$$

The proof of the lemma is completed.

If $M\in\mathcal{A}$ is indecomposable, then we call $D_{M}=\mathrm{End}_{\mathcal{A}}(M)/\mathrm{rad}(\mathrm{End}_{\mathcal{A}}(M))$ the *automorphism field* of M. Let $f: X\to Y$ be a morphism in $\mathcal{A}$. Recall that f is *irreducible* if f is neither a section nor a retraction, and every factorization $f=gh$ implies that g is a section or h is a retraction. If X,Y are indecomposable, then

$$\mathrm{irr}(X,Y)=\mathrm{rad}(\mathcal{A})(X,Y)/\mathrm{rad}^2(\mathcal{A})(X,Y)$$

is a D_X-D_Y-bimodule such that f is irreducible if and only if f is radical with a non-zero image in $\mathrm{irr}(X,Y)$; see [5].

3.3. Proposition. *Let $\mathcal{A},\mathcal{B}$ be Krull-Schmidt categories with G a group acting admissibly on $\mathcal{A}$, and let $F:\mathcal{A}\to\mathcal{B}$ be a Galois G-covering. If $u: X\to Y$ is a morphism in $\mathcal{A}$ with X or Y indecomposable, then u is irreducible in $\mathcal{A}$ if and only if $F(u)$ is irreducible in $\mathcal{B}$.*

Proof. Let $u : X \to Y$ be a morphism in $\mathcal{A}$. We shall consider only the case where X is indecomposable. Suppose that $F(u)$ is irreducible. If $v : X \to M$ and $w : M \to Y$ are morphisms in $\mathcal{A}$ such that $u = vw$, then $F(v)$ is a section or $F(w)$ is a retraction. By Lemma 2.7(2), v is a section or w is a retraction. That is, u is irreducible in $\mathcal{A}$.

For proving the necessity, suppose that u is irreducible in $\mathcal{A}$. Since F is dense, we may assume that $F(Y) = F(Y_1)^{n_1} \oplus \cdots \oplus F(Y_r)^{n_r}$, where $n_i > 0$ and the Y_i are indecomposable objects in $\mathcal{A}$ such that the $F(Y_i)$ are indecomposable and pairwise non-isomorphic. Consider first the case where $r = 1$. By Definition 2.8(3), we may assume that $Y = (g_1 \cdot Y_1)^{m_1} \oplus \cdots \oplus (g_s \cdot Y_1)^{m_s}$, where $g_1, \ldots, g_s \in G$ are distinct and $m_1, \ldots, m_s > 0$ with $m_1 + \cdots + m_s = n_1$. For each $1 \le i \le s$, denote by D_i the automorphism field of $g_i{\cdot}Y_1$. Since X is indecomposable, u is radical. Thus,

$$u = (u_{11}, \cdots, u_{1,m_1}, \cdots, u_{s1}, \cdots, u_{s,m_s})^T,$$

where $u_{i1}, \ldots, u_{i,m_i} \in \mathrm{rad}(\mathcal{A})(X, g_i{\cdot}Y_1)$, with $1 \le i \le s$, are D_i-linearly independent modulo $\mathrm{rad}^2(\mathcal{A})(X, g_i{\cdot}Y_1)$; see [5, (3.5)]. Consider the isomorphism

$$w = \mathrm{diag}\{ \overbrace{\delta_{g_1,Y_1}, \cdots, \delta_{g_1,Y_1}}^{m_1}, \cdots, \overbrace{\delta_{g_s,Y_1}, \cdots, \delta_{g_s,Y_1}}^{m_s}\},$$

and set

$$v = w \circ F(u) = (v_{11}, \cdots, v_{1,m_1}, \cdots, v_{s1}, \cdots, v_{s,m_s})^T : \; F(X) \longrightarrow F(Y_1)^{n_1} \;,$$

where $v_{ij} = \delta_{g_i,Y} \circ F(u_{ij})$, for $j = 1, \ldots, m_i$; $i = 1, \ldots, s$.

We claim that if $a_{ij} \in \mathrm{End}_{\mathcal{B}}(F(Y_1))$ are such that

$$\textstyle\sum_{i=1}^{s} \sum_{j=1}^{m_i} a_{ij} \circ v_{ij} \in \mathrm{rad}^2(\mathcal{B})(F(X), F(Y_1)),$$

then all the a_{ij} are radical. Indeed, $\mathrm{End}_{\mathcal{B}}(F(Y_1)) = \oplus_{g \in G}\, \delta_{g,Y_1} \circ F(\mathcal{A}(Y_1, g{\cdot}Y_1))$ by Lemma 2.6(1). If $g \in G$ is non-identity, then $\mathcal{A}(Y_1, g{\cdot}Y_1) = \mathrm{rad}(\mathcal{A})(Y_1, g{\cdot}Y_1)$, and by Lemma 3.1, $\delta_{g,Y_1} \circ F(\mathcal{A}(Y_1, g{\cdot}Y_1)) \subseteq \mathrm{rad}(\mathcal{B})(F(Y_1), F(Y_1))$. Therefore, we may assume that $a_{ij} = F(b_{ij})$ with $b_{ij} \in \mathrm{End}_{\mathcal{A}}(Y_1)$, for all i, j. This yields

$$\begin{aligned}
\textstyle\sum_{i=1}^{s} \sum_{j=1}^{m_i} a_{ij} \circ v_{ij} &= \textstyle\sum_{i=1}^{s} \sum_{j=1}^{m_i} F(b_{ij}) \circ \delta_{g_i,Y} \circ F(u_{ij}) \\
&= \textstyle\sum_{i=1}^{s} \sum_{j=1}^{m_i} \delta_{g_i,Y} \circ F((g_i \cdot b_{ij}) \circ u_{ij}) \\
&= \textstyle\sum_{i=1}^{s} \delta_{g_i,Y_1} \circ F\left(\sum_{j=1}^{m_i} (g_i \cdot b_{ij}) \circ u_{ij}\right).
\end{aligned}$$

By Lemma 3.2, $\sum_{j=1}^{m_i} (g_i \cdot b_{ij}) \circ u_{ij} \in \mathrm{rad}^2(\mathcal{A})(X, g_i \cdot Y_1)$, for $i = 1, \ldots, s$. Since $u_{i1}, \ldots, u_{i,m_i}$ are D_i-linearly independent modulo $\mathrm{rad}^2(\mathcal{A})(X, g_i{\cdot}Y_1)$, we deduce that $g_i \cdot b_{ij} \in \mathrm{rad}(\mathcal{A})(g_i{\cdot}Y_1,\, g_i{\cdot}Y_1)$, and thus, $b_{ij} \in \mathrm{rad}(\mathcal{A})(Y_1, Y_1)$. By Lemma 3.1, $a_{ij} = F(b_{ij}) \in \mathrm{rad}(\mathcal{B})(F(Y_1), F(Y_1))$, for $j = 1, \ldots, m_i$; $i = 1, \ldots, s$. This establishes our claim. As a consequence, v is irreducible in $\mathcal{B}$; see [5, (3.5)], and so is $F(u)$ since w is an isomorphism.

Suppose now that $r > 1$ and the necessity holds for $r-1$. Write $Y = M \oplus N$ such that $F(M) = F(Y_1)^{n_1} \oplus \cdots \oplus F(Y_{r-1})^{n_{r-1}}$ and $F(N) = F(Y_r)^{n_r}$. Then, we have $u = (v, w)^T : X \to M \oplus N$ and $F(u) = (F(v), F(w))^T : F(X) \to F(M) \oplus F(N)$. Since v, w are irreducible in $\mathcal{A}$; see [5, (3.2)], $F(v), F(w)$ are irreducible in $\mathcal{B}$ by the induction hypothesis. Since $F(M)$, $F(N)$ have no common indecomposable direct summand, $F(u)$ is irreducible; see [5]. The proof of the proposition is completed.

As the converse of the previous result, we have the following statement.

3.4. Proposition. *Let $\mathcal{A}, \mathcal{B}$ be Krull-Schmidt categories with G a group acting admissibly on $\mathcal{A}$, and let $F : \mathcal{A} \to \mathcal{B}$ be a Galois G-covering. Consider an indecomposable object X in $\mathcal{A}$. If $\mathcal{B}$ has an irreducible morphism $v : F(X) \to Z$ or $v : Z \to F(X)$, then $\mathcal{A}$ has an irreducible morphism $u : X \to Y$ or $u : Y \to X$, respectively, such that $F(Y) \cong Z$.*

Proof. We shall consider only the case where there exists an irreducible morphism $v : F(X) \to Z$ in $\mathcal{B}$. Write $Z = Z_1^{n_1} \oplus \cdots \oplus Z_r^{n_r}$, where the n_i are positive integers and the Z_i are pairwise non-isomorphic indecomposable objects in $\mathcal{B}$. Since F is dense, we may assume that $Z_i = F(Y_i)$ with Y_i some indecomposable object in $\mathcal{A}$, for $i = 1, \ldots, r$.

Suppose that $r = 1$. Let D be the automorphism field of $F(Y_1)$. Since $F(X)$ is indecomposable, $v = (v_1, \ldots, v_{n_1})^T$, where $v_1, \ldots, v_{n_1} \in \operatorname{rad}(\mathcal{B})(F(X), F(Y_1))$ are D-linearly independent modulo $\operatorname{rad}^2(\mathcal{B})(F(X), F(Y_1))$; see [5, (3.4)]. Applying Lemma 3.2, we obtain

$$\operatorname{rad}(\mathcal{B})(F(X), F(Y_1)) = \oplus_{g \in G}\, \delta_{g,Y_1} \circ F(\operatorname{rad}(\mathcal{A})(X, g \cdot Y_1)).$$

Therefore, $v_1 = \sum_{j=1}^{m} \delta_{h_{1j},Y_1} \circ F(v_{1j})$, with $h_{1j} \in G$ and $v_{1j} \in \operatorname{rad}(\mathcal{A})(X, h_{1j}{\cdot}Y_1)$. Observe that one of the $\delta_{h_{1j},Y_1} \circ F(v_{1j})$, say $\delta_{h_{11},Y_1} \circ F(v_{11})$, is not a D-linear combination of $v_2, \ldots, v_n$ modulo $\operatorname{rad}^2(\mathcal{B})(F(X), F(Y_1))$. Setting $v_1' = \delta_{h_{11},Y_1} \circ F(v_{11})$, we see that $v_1', v_2, \ldots, v_n$ are D-linearly independent modulo $\operatorname{rad}^2(\mathcal{B})(F(X), F(Y_1))$. Repeating this process, we may assume that $v_i = \delta_{h_i,Y_1} \circ F(u_i)$ with $h_i \in G$ and $u_i \in \operatorname{rad}(\mathcal{A})(X, h_i{\cdot}Y_1)$, for $i = 1, \ldots, n_1$. Up to permutation, we may assume that

$$\{h_1, \ldots, h_{n_1}\} = \{\overbrace{g_1, \cdots, g_1}^{m_1}, \cdots, \overbrace{g_s, \cdots, g_s}^{m_s}\},$$

where $g_1, \ldots, g_s \in G$ are distinct and $m_1, \ldots, m_s > 0$.

We claim that if $\sum_{i=1}^{m_1} w_i \circ u_i \in \operatorname{rad}^2(\mathcal{A})(X, g_1{\cdot}Y_1)$ with $w_i \in \operatorname{End}_{\mathcal{A}}(g_1{\cdot}Y_1)$, then all the w_i are radical. Indeed, observing $v_i = \delta_{h_i,Y_1} \circ F(u_i) = \delta_{g_1,Y_1} \circ F(u_i)$ and applying Lemma 2.4, we obtain

$$\begin{aligned}
\textstyle\sum_{i=1}^{m_1} F(g_1^{-1}{\cdot}w_i) \circ v_i &= \textstyle\sum_{i=1}^{m_1} F(g_1^{-1}{\cdot}w_i) \circ \delta_{g_1,Y_1} \circ F(u_i) \\
&= \textstyle\sum_{i=1}^{m_1} \delta_{g_1,Y_1} \circ F(w_i \circ u_i) \\
&= \delta_{g_1,Y_1} \circ F(\textstyle\sum_{i=1}^{m_1} w_i \circ u_i)
\end{aligned}$$

which, by Lemma 3.1, lies in $\operatorname{rad}^2(\mathcal{B})(F(X), F(Y_1))$. Since the v_i are D-linearly independent modulo $\operatorname{rad}^2(\mathcal{B})(F(X), F(Y_1))$, we have $F(g_1^{-1} \cdot w_i) \in \operatorname{rad}(\operatorname{End}_{\mathcal{B}}(F(Y_1))$, and $g_1^{-1}{\cdot}w_i \in \operatorname{rad}(\operatorname{End}_{\mathcal{B}}(Y_1))$ by Lemma 3.2, that is, $w_i \in \operatorname{rad}(\operatorname{End}_{\mathcal{B}}(g_1{\cdot}Y_1))$, for $i = 1, \ldots, m_1$. This proves our claim, and consequently, $(u_1, \cdots, u_{m_1})^T$ is irreducible in $\mathcal{A}$; see [5, (3.4)]. Similarly, $(u_{m_{j-1}+1}, \cdots, u_{m_{j-1}+m_j})^T$ is irreducible, for $j = 2, \ldots, s$. Since $g_1{\cdot}Y_1, \ldots, g_s{\cdot}Y_1$ are pairwise non-isomorphic,

$$u = (u_1, \ldots, u_{n_1})^T : X \to \oplus_{i=1}^{n_1} h_i{\cdot}Y_1$$

is irreducible in $\mathcal{A}$ such that $F(\oplus_{i=1}^{n_1} h_i{\cdot}Y_1) \cong F(Y_1)^{n_1}$.

Suppose now that $r > 1$ and the proposition holds for $r-1$. Set $Z_1 = F(Y_1)^{n_1}$ and $Z_2 = F(Y_2)^{n_2} \oplus \cdots \oplus F(Y_r)^{n_r}$. Then $Z = Z_1 \oplus Z_2$ and $v = (w_1, w_2)^T : X \to Z_1 \oplus Z_2$, where $w_i : Z \to Z_i$ is irreducible, for $i = 1, 2$. By the induction hypothesis, $\mathcal{A}$ has irreducible morphisms $f_i : X \to M_i$ with $F(M_i) \cong Z_i$, $i = 1, 2$. Since Z_1 and Z_2 have no common indecomposable direct summand, neither do M_1 and M_2. As a consequence, $f = (f_1, f_2) : X \to M_1 \oplus M_2$ is irreducible in $\mathcal{A}$; see [5, (3.2)], such that $F(M_1 \oplus M_2) \cong Z$. The proof of the proposition is completed.

Let $f : X \to Y$ be a morphism in $\mathcal{A}$. Recall that f is *left minimal* if every factorization $f = hf$ implies that h is an automorphism of X; and *left almost split* if f is not a section and every non-section morphism $g : X \to Y$ factors through f; and a *source morphism* if f is left minimal and left almost split. In dual situations, one says that f is *right minimal*, *right almost split*, and a *sink morphism*, respectively. Observe that X or Y is indecomposable in case f is left almost or right almost, respectively; see [3], and also [15].

3.5. Proposition. *Let $\mathcal{A}, \mathcal{B}$ be Krull-Schmidt categories with G a group acting admissibly on $\mathcal{A}$, and let $F : \mathcal{A} \to \mathcal{B}$ be a Galois G-covering. If $u : X \to Y$ is a morphism in $\mathcal{A}$, then u is a source morphism or sink morphism if and only if $F(u)$ is a source morphism or sink morphism in $\mathcal{B}$, respectively.*

Proof. Let $u : X \to Y$ be a morphism in $\mathcal{A}$. We shall prove only the first part of the proposition. Assume that $F(u)$ is a source morphism in $\mathcal{B}$. By Lemma 2.7(2), u is not a section. Let $v : X \to L$ be a non-section morphism in $\mathcal{A}$. By Lemma 2.7(2), $F(v)$ is not a section morphism in $\mathcal{B}$, and hence, $F(v)$ factorizes through $F(u)$. By Lemma 2.7(1), v factorizes through u. If $w : Y \to Y$ is such that $u = wu$, then $F(u) = F(w)F(u)$. Therefore, $F(w)$ is an automorphism of $F(Y)$, and by Lemma 2.7(2), w is an automorphism of Y. That is, u is a source morphism in $\mathcal{A}$.

Conversely, suppose that u is a source morphism in $\mathcal{A}$. In particular, X is indecomposable. By Lemma 2.7(2), $F(u)$ is not a section. Let $v : F(X) \to M$ be a non-section morphism in $\mathcal{B}$. Since F is dense, we may assume that $M = F(N)$ with $N \in \mathcal{A}$. Let δ be the G-stabilizer for F. Then, $v = \sum_{i=1}^{n} \delta_{g_i,N} \circ F(v_i)$, where $g_1, \ldots, g_n \in G$ are distinct and $v_i \in \mathcal{A}(X, g_i \cdot N)$. Since $F(X)$ is indecomposable by Definition 2.8(2), v is radical. Then, v_i is radical by Lemma 3.2, and hence, $v_i = uu_i$ for some $u_i \in \mathcal{A}(X, g_i{\cdot}N)$, $i = 1, \ldots, n$. This yields $v = (\sum_{i=1}^{n} \delta_{g_i,N} \circ F(v_i)) \circ F(u)$. Furthermore, if $u \neq 0$, then u is irreducible, and so is $F(u)$ by Proposition 3.3. Since $\mathcal{B}$ is Krull-Schmidt, $F(u)$ is left minimal. If $u = 0$, then $Y = 0$; see [12, (1.1)], and consequently, $F(u)$ is left minimal. This shows that $F(u)$ is a source morphism in $\mathcal{B}$. The proof of the proposition is completed.

Finally, we shall study the behavior of almost split sequences under a Galois covering. A *short sequence* in $\mathcal{A}$ is a sequence of two morphisms

$$\eta : X \xrightarrow{u} Y \xrightarrow{v} Z,$$

which is called *pseudo-exact* if u is a *pseudo-kernel* of v, while v is a pseudo-cokernel of u. Moreover, η is called an *almost split sequence* if it is pseudo-exact such that Y is non-zero, u is a source morphism, and v is a sink morphism; see [12].

3.6. Lemma. *Let $\mathcal{A}, \mathcal{B}$ be Krull-Schmidt categories with G a group acting admissibly on $\mathcal{A}$, and let $F : \mathcal{A} \to \mathcal{B}$ be a Galois G-covering. If η is a short sequence in $\mathcal{A}$, then it is pseudo-exact if and only if $F(\eta)$ is pseudo-exact.*

Proof. Consider a short sequence $\eta : X \xrightarrow{u} Y \xrightarrow{v} Z$ in $\mathcal{A}$. Suppose first that

$$F(\eta) : F(X) \xrightarrow{F(u)} F(Y) \xrightarrow{F(v)} F(Z)$$

is pseudo-exact. Since F is faithful by Lemma 2.6(2), $vu = 0$. Let $w : Y \to W$ be a morphism in $\mathcal{A}$ such that $wu = 0$. Since $F(w) \circ F(u) = 0$, we see that $F(w)$ factorizes through $F(v)$. By Lemma 2.7(1), w factorizes through v. That is, v is a pseudo-cokernel of u. Dually, one can show that u is a pseudo-kernel of v.

Suppose conversely that η is pseudo-exact. In particular, $F(v)F(u) = 0$. Let $w : F(Y) \to M$ be a morphism in $\mathcal{B}$ such that $wF(u) = 0$. Since F is dense, we may assume that $M = F(L)$ for some $L \in \mathcal{A}$. By Lemma 2.6(1), $w = \sum_{i=1}^n \delta_{g_i,L} \circ F(w_i)$, where $g_1, \ldots, g_n \in G$ are distinct and $w_i \in \mathcal{A}(Y, g_i{\cdot}L)$. This gives rise to

$$0 = wF(u) = \textstyle\sum_{i=1}^n \delta_{g_i,L} \circ F(w_i)F(u) = \sum_{i=1}^n \delta_{g_i,L} \circ F(w_i u).$$

Then, for any $1 \le i \le n$, we have $\delta_{g_i,L} \circ F(w_i u) = 0$, and hence, $w_i u = 0$ since F is faithful. Therefore, $w_i = v_i v$ for some $v_i \in \mathcal{A}(Z, g_i{\cdot}L)$, $i = 1, \ldots, n$. This yields $w = (\sum_{i=1}^n \delta_{g_i,L} \circ F(v_i))\, F(v)$. That is, $F(v)$ is a pseudo-cokernel of $F(u)$. Dually, one can show that $F(u)$ is a pseudo-kernel of $F(v)$. The proof of the lemma is completed.

Remark. The above result says particularly that a Galois covering between Krull-Schmidt abelian categories is an exact functor.

We are now ready to obtain the main result of this section.

3.7. Theorem. *Let $\mathcal{A}, \mathcal{B}$ be Krull-Schmidt categories with G a group acting admissibly on $\mathcal{A}$, and let $F : \mathcal{A} \to \mathcal{B}$ be a Galois G-covering.*

(1) *A short sequence η in $\mathcal{A}$ is almost split if and only if $F(\eta)$ is almost split.*

(2) *An object X in $\mathcal{A}$ is the starting term or the ending term of an almost split sequence if and only if $F(X)$ is the starting term or the ending term of an almost split sequence, respectively.*

Proof. Consider a short sequence $\eta : X \longrightarrow Y \longrightarrow Z$ in $\mathcal{A}$. Since F is faithful by Lemma 2.6(2), $Y \neq 0$ if and only if, $F(Y) \neq 0$. Now, it follows from Proposition 3.5 and Lemma 3.6 that η is almost split if and only if $F(\eta)$ is almost split. This establishes Statement (1).

Next, the necessity of Statement (2) follows immediately from Statement (1). Let $X \in \mathcal{A}_0$ be such that $\mathcal{B}$ has an almost split sequence

$$F(X) \xrightarrow{f} M \xrightarrow{g} N.$$

Note that f is irreducible in $\mathcal{B}$ because $M \neq 0$. By Proposition 3.4, $\mathcal{A}$ has an irreducible morphism $u : X \to Y$ with $F(Y) \cong M$. By Proposition 3.3, $F(u)$ is irreducible in $\mathcal{B}$. Therefore, $F(u) = fw$ for some retraction $w : M \to F(Y)$. Since $\mathcal{B}$ is Krull-Schmidt, w is an isomorphism. Setting $v = gw^{-1}$, we see that

$$F(X) \xrightarrow{F(u)} F(Y) \xrightarrow{v} N$$

is an almost split sequence in $\mathcal{B}$. Since F is dense, we may assume that $N = F(Z)$ for some indecomposable $Z \in \mathcal{A}_0$. By Lemma 2.6(1), $v = \sum_{i=1}^n \delta_{g_i,Z} \circ F(w_i)$, where $g_1, \ldots, g_n \in G$ are distinct and $w_i \in \mathcal{A}(Y, g_i{\cdot}Z)$. This yields

$$\textstyle\sum_{i=1}^n \delta_{g_i,Z} \circ F(w_i u) = vF(u) = 0.$$

Then, $(\delta_{g_i,Z} \circ F(w_i))F(u) = 0$, and thus, $\delta_{g_i,Z} \circ F(w_i) = a_i v$ with $a_i \in \mathrm{End}_{\mathcal{B}}(F(Z))$, for $i = 1, \ldots, n$. This gives rise to $v = v(\sum_{i=1}^n a_i)$. Since $\mathrm{End}_{\mathcal{B}}(F(Z))$ is local, we may assume that a_1 is an automorphism. Thus, $\mathcal{B}$ has an almost split sequence

$$F(X) \xrightarrow{F(u)} F(Y) \xrightarrow{\delta_{g_1,Z}\circ F(w_1)} F(Z),$$

and consequently, $\mathcal{B}$ has an almost split sequence

$$F(X) \xrightarrow{F(u)} F(Y) \xrightarrow{F(w_1)} F(g_1{\cdot}Z).$$

By Statement (1), $X \xrightarrow{u} Y \xrightarrow{w_1} g_1{\cdot}Z$ is an almost split sequence in $\mathcal{A}$. This proves the first part of the sufficiency, and the second part follows dually. The proof of the theorem is completed.

Remark. The above theorem says in particular that if $F : \mathcal{A} \to \mathcal{B}$ is a Galois covering, then $\mathcal{A}$ has (left, right) almost split sequences if and only if $\mathcal{B}$ has (left, right) almost split sequences.

4. Galois coverings for Auslander-Reiten quivers

The classical notion of a Galois covering for translation quivers works only in the unvalued context; see [6, 16]. In this section, we extend this to the valued context, and show that a Galois covering between Hom-finite Krull-Schmidt categories induces a Galois covering between their Auslander-Reiten quivers.

We start with a brief recall on some combinatorial background. Let $Q = (Q_0, Q_1)$ be a quiver, where Q_0 is the set of vertices and Q_1 is the set of arrows. If $\alpha : x \to y$ is in Q_1, then we shall write $x = s(\alpha)$ and $y = e(\alpha)$. For $a \in Q_0$, denote by x^+ the set of arrows α with $s(\alpha) = x$, and by x^- the set of arrows β with $e(\beta) = x$. One says that Q is *locally finite* if x^+ and x^- are both finite for every $x \in Q_0$. A sequence $\rho = \alpha_n \cdots \alpha_1$ with $n > 0$ and $\alpha_i \in Q_1$ such that $s(\alpha_{i+1}) = e(\alpha_i)$ for all $1 \le i < n$ is called a *path* of length n. To each $x \in Q_0$, one associates a *trivial path* ε_x with $s(\varepsilon_x) = e(\varepsilon_x) = x$ which, by convention, is of length 0. For $x, y \in Q_0$, denote by $Q_1(x, y)$ the set of arrows from x to y; by $Q_{\le 1}(x, y)$ the set of paths of length ≤ 1 from x to y, and by $Q(x, y)$ the set of all paths from x to y. For each $\alpha : x \to y$ in Q_1, one introduces a formal inverse α^{-1} with $s(\alpha^{-1}) = y$ and $e(\alpha^{-1}) = x$. A sequence $w = c_n \cdots c_2 c_1$, where the c_i are trivial paths, arrows or the inverses of arrows in Q such that $s(c_{i+1}) = e(c_i)$ for $1 \le i < n$, is called a *walk*. In this case, we write $s(w) = s(c_1)$ and $e(w) = e(c_n)$. The set of walks in Q will be denoted by $W(Q)$. A walk w in Q is called *closed* if $s(w) = e(w)$; *reduced* if w is either a trivial path, or $w = c_n \cdots c_1$ with $c_i \in Q_1$ or $c_i^{-1} \in Q_1$ such that $c_{i+1} \ne c_i^{-1}$ for all $1 \le i < n$; and a *cycle* if w is non-trivial, reduced and closed. A *quiver-morphism* $\varphi : Q \to Q'$ consists of two maps $\varphi_0 : Q_0 \to Q'_0$ and $\varphi_1 : Q_1 \to Q'_1$ such that $\varphi_1(Q_1(x, y)) \subseteq Q'_1(\varphi_0(x), \varphi_0(y))$ for all $x, y \in Q_0$. Such a morphism induces a map from $W(Q)$ to $W(Q')$, which will be denoted again by φ. Observe that a quiver-morphism $\varphi : Q \to Q'$ is an isomorphism if both φ_0 and φ_1 are bijective.

Suppose that G is a group acting on Q, that is, there exists a homomorphism from G into the group of automorphisms of Q. One says that the G-action on Q is *free* provided that $g{\cdot}x \ne x$, for any $x \in Q_0$ and any non-identity $g \in G$. As usual, we shall regard $g \in G$ as an automorphism of Q. The following definition is well known; see, for example, [8, 9].

4.1. Definition. Let Q be a quiver with a free action of a group G. A quiver-morphism $\varphi : Q \to Q'$ is called a *Galois G-covering* provided that the following conditions are satisfied.

(1) The map φ_0 is surjective.
(2) If $g \in G$, then $\varphi \circ g = g$.
(3) If $x, y \in Q_0$ with $\varphi_0(x) = \varphi_0(y)$, then $y = g \cdot x$ for some $g \in G$.
(4) If $x \in Q_0$, then φ_1 induces two bijections $x^+ \to \varphi_0(x)^+$ and $x^- \to \varphi_0(x)^-$.

Remark. (1) A quiver-morphism $\varphi : Q \to Q'$ satisfying the above conditions (1) and (4) is called a *quiver-covering*.

(2) It is easy to see that a Galois G-covering $\varphi : Q \to Q'$ of quivers is an isomorphism, if and only if, $\varphi_0 : Q_0 \to Q'_0$ is an injection, if and only if G is trivial.

Next, a *valued quiver* is a pair (Δ, v), where Δ is a quiver without multiple arrows and v is a valuation on the arrows, that is, each arrow $x \to y$ is endowed with a pair (v_{xy}, v'_{xy}) of positive integers. The valuation (v_{xy}, v'_{xy}) of an arrow $x \to y$ is called *symmetric* if $v_{xy} = v'_{xy}$. More generally, we say that Δ is *symmetrically valued* if all the arrows have a symmetric valuation.

4.2. Definition. A *valued-quiver-morphism* $\varphi : (\Delta, v) \to (\Omega, u)$ is a quiver-morphism $\varphi : \Delta \to \Omega$ such that $v_{xy} \le u_{\varphi(x),\varphi(y)}$ and $v'_{xy} \le u'_{\varphi(x),\varphi(y)}$, for any $x \to y$ in Δ_1.

Remark. A valued-quiver-morphism $\varphi : (\Delta, v) \to (\Omega, u)$ is an isomorphism if and only if $\varphi : \Delta \to \Omega$ is a quiver-isomorphism such that $v_{xy} = u_{\varphi(x),\varphi(y)}$ and $v'_{xy} = u'_{\varphi(x),\varphi(y)}$, for any arrow $x \to y$ in Δ.

Sometimes, it will be convenient to identify a non-valued quiver Q with a symmetrically valued quiver $\Delta(Q)$ defined as follows: the vertices are those of Q, and there exists an unique arrow $x \to y$ in $\Delta(Q)$ with valuation (d_{xy}, d_{xy}) if and only if there exist d_{xy} arrows from x to y in Q.

4.3. Definition. Let (Δ, v) be a valued quiver with a free action of a group G. A valued-quiver-morphism $\varphi : (\Delta, v) \to (\Omega, u)$ is called a *Galois G-covering* provided that the following conditions are verified.

(1) The map $\varphi_0 : \Delta_0 \to \Omega_0$ is surjective.
(2) If $g \in G$, then $\varphi \circ g = \varphi$.
(3) If $x, y \in \Delta_0$ with $\varphi(x) = \varphi(y)$, then $y = g \cdot x$ for some $g \in G$.
(4) If $x \in \Delta_0$ with $a \in \varphi(x)^+$ and $b \in \varphi(x)^-$, then

$$u_{\varphi(x),a} = \textstyle\sum_{y \in x^+ \cap \varphi^-(a)} v_{x,y} \quad \text{and} \quad u'_{b,\varphi(x)} = \textstyle\sum_{z \in x^- \cap \varphi^-(b)} v'_{z,x}.$$

Remark. (1) We deduce from Definition 4.3(4) that φ induces, for any $x \in \Delta_0$, two surjections $x^+ \to (\varphi(x))^+$ and $x^- \to (\varphi(x))^-$; compare Definition 4.1(4). In particular, the map $\varphi_1 : \Delta_1 \to \Omega_1$ is surjective.

(2) A Galois G-covering $\varphi : (\Delta, v) \to (\Omega, u)$ of valued quivers is an isomorphism if and only if $\varphi_0 : \Delta_0 \to \Omega_0$ is an injection, if and only if G is trivial.

(3) Given non-valued quivers Q and Q', each quiver-morphism $\varphi : Q \to Q'$ induces a valued-quiver-morphism $\Delta(\varphi) : \Delta(Q) \to \Delta(Q')$ in such a way that φ is a Galois covering of quivers if and only if $\Delta(\varphi)$ is a Galois covering of valued quivers.

Example. Let $\mathbb{A}_{2,3}$ be the valued quiver consisting of two vertices a, b, and one arrow $a \to b$ with valuation $(2, 3)$. It is easy to see that $\mathbb{A}_{2,3}$ admits a Galois covering

$\mathbb{A}_\infty^\infty$, consisting of the vertices a_i, b_i, $i \in \mathbb{Z}$, and arrows $a_i \to b_i$ with valuation $(1,1)$ and arrows $a_i \to b_{i+1}$ with valuation $(1,2)$, $i \in \mathbb{Z}$.

Let Δ be a valued quiver with an action of a group G, and let $\mathcal{C}$ be a connected component of Δ. If $g \in G$, then $g \cdot \mathcal{C}$ is a connected component of Δ, and consequently, either $g \cdot \mathcal{C} = \mathcal{C}$ or $\mathcal{C} \cap g \cdot \mathcal{C} = \emptyset$. The elements $g \in G$ such that $g \cdot \mathcal{C} = \mathcal{C}$ form a subgroup of G, which is written as $G_\mathcal{C}$. Clearly, the G-action on Δ restricts to a $G_\mathcal{C}$-action on $\mathcal{C}$.

4.4. Lemma. *Let $\varphi : (\Delta, v) \to (\Omega, u)$ be a Galois G-covering of valued quivers, where G is a group acting freely on (Δ, v). If $\mathcal{C}$ is a connected component of Δ, then $\varphi(\mathcal{C})$ is a connected component of Ω such that φ restricts to a Galois $G_\mathcal{C}$-covering $\varphi_c : (\mathcal{C}, v) \to (\varphi(\mathcal{C}), u)$.*

Proof. First of all, the $G_\mathcal{C}$-action on $\mathcal{C}$ is free such that $\varphi_c \circ g = \varphi_c$ for all $g \in G_\mathcal{C}$. Moreover, since φ satisfies Condition (4) stated in Definition 4.3, so does φ_c.

Now, since $\mathcal{C}$ is connected, $\varphi(\mathcal{C})$ is a connected full subquiver of Ω. Let x be a vertex in $\varphi(\Delta)$, and let $y \in \Omega_0$ for which there exists a walk w with $s(w) = x$ and $e(w) = y$. Write $x = \varphi(a)$ with $a \in \Delta_0$. Since $\varphi_1 : \Delta_1 \to \Omega_1$ is surjective, Δ has a walk w' from some vertex b to a such that $\varphi(w') = w$. In particular, $y = \varphi(b)$. Since $\mathcal{C}$ is a connected component of Δ, we have $y \in \varphi(\mathcal{C})$. This shows that $\varphi(\mathcal{C})$ is a connected component of Ω and the action of φ_c on the vertices is surjective.

Next, let $y, z \in \mathcal{C}_0$ with $\varphi_c(y) = \varphi_c(z)$, that is, $\varphi(y) = \varphi(z)$. By Definition 4.1(2), $z = g \cdot y$ for some $g \in G$. Since $z \in \mathcal{C} \cap g \cdot \mathcal{C}$, we have $\mathcal{C} = g \cdot \mathcal{C}$, that is, $g \in G_\mathcal{C}$. Thus, φ_c satisfies Condition (3) stated in Definition 4.1, and hence, is a Galois $G_\mathcal{C}$-covering. The proof of the lemma is completed.

Finally, a *valued translation quiver* is a triple (Γ, v, τ), where (Γ, v) is a valued quiver and τ is a *translation*, that is a bijection from one subset of Γ_0 to another one such that, for any $x \in \Gamma_0$ with τx defined, we have $x^+ = (\tau x)^- \neq \emptyset$ and $(v_{\tau x, y}, v'_{\tau x, y}) = (v'_{yx}, v_{yx})$ for every $y \in x^+$. In this case, $x \in \Gamma_0$ is called *projective* or *injective* if τx or $\tau^- x$ is not defined, respectively. Note that valued translation quivers considered here are not necessarily locally finite and may contain loops; compare [11, Section 2].

4.5. Definition. A *morphism* of valued translation quivers $\varphi : (\Delta, v, \tau) \to (\Omega, u, \rho)$ is a valued-quiver-morphism $\varphi : (\Delta, v) \to (\Omega, u)$ satisfying the condition: for any non-projective $x \in \Gamma_0$, the image $\varphi(x)$ is not projective with $\rho(\varphi(x)) = \varphi(\tau(x))$; or equivalently, for any non-injective $x \in \Gamma_0$, the image $\varphi(x)$ is not injective with $\rho^-(\varphi(x)) = \varphi(\tau^-(x))$.

Remark. A morphism of valued translation quivers $\varphi : (\Gamma, \tau, v) \to (\Delta, \rho, d)$ is an isomorphism if and only if $\varphi : (\Gamma, v) \to (\Delta, d)$ is a valued-quiver-isomorphism satisfying the condition: for any projective $x \in \Gamma_0$, the image $\varphi(x)$ is projective; or equivalently, for any injective $x \in \Gamma_0$, the image $\varphi(x)$ is injective.

The following definition extends the notion of a Galois covering for unvalued translation quivers introduced by Riedtmann; see [16], and also [6].

4.6. Definition. Let (Γ, v, τ) be a valued translation quiver with a free action of a group G. A morphism of valued translation quivers $\varphi : (\Gamma, v, \tau) \to (\Delta, d, \rho)$ is called a *Galois G-covering* provided that $\varphi : (\Gamma, v) \to (\Delta, d)$ is a Galois G-covering of valued quivers with the property: for any projective $x \in \Gamma_0$, the image $\varphi(x)$ is projective; or equivalently, for any injective $x \in \Gamma_0$, the image $\varphi(x)$ is injective.

Remark. (1) The equivalence of the two conditions stated in Definition 4.6 follows from the conditions (1) and (3) stated in Definition 4.3.

(2) A Galois G-covering of valued translation quivers $\varphi : (\Delta, v, \tau) \to (\Omega, u, \rho)$ is an isomorphism if and only if G is trivial, if and only if $\varphi_0 : \Delta_0 \to \Omega_0$ is an injection.

For the rest of this section, let $\mathcal{A}$ be a Hom-finite Krull-Schmidt R-linear category, where R is a commutative artinian ring. If $M \in \mathcal{A}_0$, then M^n with $n > 0$ will denote the direct sum of n copies of M. If $X, Y \in \mathcal{A}_0$ are indecomposable, then we shall denote by d'_{XY} and d_{XY} the dimensions of $\operatorname{irr}(X, Y)$ over D_X and D_Y, respectively. If $\operatorname{irr}(X, Y) \neq 0$, then d'_{XY} is the maximal integer such that $\mathcal{A}$ admits an irreducible morphism from $X^{d'_{XY}}$ to Y, while d_{XY} is the maximal integer such that $\mathcal{A}$ admits an irreducible morphism from X to $Y^{d_{XY}}$; see [5, (3.4)]. As a consequence, if $M \to Y$ is a sink morphism, then d'_{XY} is the multiplicity of X as a direct summand of M, and if $X \to N$ is a source morphism, then d_{XY} is the multiplicity of Y as a direct summand of N. Let $\operatorname{ind}\mathcal{A}$ denote a complete set of non-isomorphic indecomposable objects in $\mathcal{A}$. Then the *Auslander-Reiten quiver* $\Gamma_{\mathcal{A}}$ of $\mathcal{A}$ is a valued translation quiver defined as follows; see [12, (2.1)]: the vertex set is $\operatorname{ind}\mathcal{A}$; for any vertices X, Y, there exists a single arrow $X \to Y$ with valuation (d_{XY}, d'_{XY}) if and only if $d_{XY} > 0$; and the translation $\tau_{\mathcal{A}}$ is the *Auslander-Reiten translation*, that is, $X = \tau_{\mathcal{A}} Z$ if and only if $\mathcal{A}$ has an almost split sequence $X \to Y \to Z$.

Suppose now that G is a group acting freely on $\mathcal{A}$. Let Σ be a complete set of representatives of the G-orbits in $\mathcal{A}_0$. Then, we can choose $\operatorname{ind}\mathcal{A}$ to be the set of objects $g \cdot X$ with $X \in \Sigma$ and $g \in G$. In this way, $\operatorname{ind}\mathcal{A}$ becomes G-stable, that is, if $M \in \operatorname{ind}\mathcal{A}$, then $g \cdot M \in \operatorname{ind}\mathcal{A}$, for every $g \in G$. It is easy to see that the free G-action on $\mathcal{A}$ induces a free G-action on the valued translation quiver $\Gamma_{\mathcal{A}}$.

4.7. Theorem. *Let $\mathcal{A}, \mathcal{B}$ be Hom-finite Krull-Schmidt R-linear categories with G a group acting admissibly on $\mathcal{A}$, where R is a commutative artinian ring, and let $F : \mathcal{A} \to \mathcal{B}$ be a Galois G-covering.*

(1) *The functor F induces a Galois G-covering $\pi : \Gamma_{\mathcal{A}} \to \Gamma_{\mathcal{B}}$ of valued translation quivers.*

(2) *If Γ is a connected component of $\Gamma_{\mathcal{A}}$, then $\pi(\Gamma)$ is a connected component of $\Gamma_{\mathcal{B}}$ and π restricts to a Galois G_Γ-covering $\pi_\Gamma : \Gamma \to \pi(\Gamma)$.*

Proof. Let Σ be a complete set of representatives of the G-orbits in $\operatorname{ind}\mathcal{A}$. Then the vertices in $\Gamma_{\mathcal{A}}$ are the objects $g \cdot X$, with $g \in G$ and $X \in \Sigma$. By Definition 2.8, we may choose $F(\Sigma) = \{F(X) \mid X \in \Sigma\}$ to be the vertex set of $\Gamma_{\mathcal{B}}$.

Let $M = g_0 \cdot X$ be a vertex in $\Gamma_{\mathcal{A}}$, where $g_0 \in G$ and $X \in \Sigma$. Set $\pi_0(M) = F(X)$. This defines a surjection $\pi_0 : (\Gamma_{\mathcal{A}})_0 \to (\Gamma_{\mathcal{B}})_0$ such that $\pi_0(g \cdot M) = \pi_0(M)$, for any $g \in G$. Moreover, by Definition 2.8(3), π_0 satisfies Condition (3) stated in Definition 4.3. Furthermore, $F(M) \cong F(X) = \pi_0(M)$. By Theorem 3.7(3), M is not projective in $\Gamma_{\mathcal{A}}$ if and only if $\pi_0(M)$ is not projective in $\Gamma_{\mathcal{B}}$, and in this case, $\pi_0(\tau_{\mathcal{A}} X) = \tau_{\mathcal{B}}(\pi_0(X))$ by Theorem 3.7(2). This shows that π_0 commutes with the translations and has the property stated in Definition 4.6.

Next, let $\alpha : M \to N$ be an arrow in $\Gamma_{\mathcal{A}}$, where $N = h \cdot Y$ with $h \in G$ and $Y \in \Sigma$. By Proposition 3.3, $\operatorname{irr}(F(M), F(N)) \neq 0$ with $d_{M,N} \leq d_{F(M),F(N)}$ and $d'_{M,N} \leq d'_{F(M),F(N)}$. Assume that δ is the G-stabilizer for F. In view of the isomorphisms

$\delta_{g,X}$ and $\delta_{h,Y}$, we see that $(d_{F(X),F(Y)}, d'_{F(X),F(Y)}) = (d_{F(M),F(N)}, d'_{F(M),F(N)})$. Therefore, $\Gamma_{\mathcal{B}}$ has an arrow $\beta : \pi_0(M) \to \pi_0(N)$ with $d_{M,N} \le d_{\pi_0(M),\pi_0(N)}$ and $d'_{M,N} \le d'_{\pi_0(M),\pi_0(N)}$. Set $\pi_1(\alpha) = \beta$. Then, $\pi_1(g\cdot\alpha) = \pi_1(\alpha)$ for any $g \in G$. This yields a morphism of valued translation quivers $\pi = (\pi_0, \pi_1) : \Gamma_{\mathcal{A}} \to \Gamma_{\mathcal{B}}$, which verifies the property stated in Definition 4.6.

It remains to show that π verifies Condition (4) stated in Definition 4.3. Consider $U \in (\pi(M))^+$. Let $X^+ \cap \pi^-(U) = \{L_1, \dots, L_r\}$ and $m = m_1 + \cdots + m_r$, where $m_i = d_{X,L_i}$. Then $M^+ \cap \pi^-(U) = \{g_0 \cdot L_1, \dots, g_0 \cdot L_r\}$ and $d_{M,g_0\cdot L_i} = d_{X,L_i} = m_i$, for $i = 1, \dots, r$. This implies that $\sum_{V\in M^+\cap\pi^-(U)} d_{M,V} = \sum_{i=1}^r d_{M,g_0\cdot L_i} = m$.

Set $n = d_{\pi(M),U} = d_{F(X),U}$, which is maximal such that $\mathcal{B}$ has an irreducible morphism $f : F(X) \to U^n$. By Proposition 3.4, $\mathcal{A}$ has an irreducible morphism $u : X \to N$ with $F(N) \cong U^n$. We may assume that $N = N_1^{n_1} \oplus \cdots \oplus N_s^{n_s}$, where $n_i > 0$ and the N_i are distinct vertices in $\Gamma_{\mathcal{A}}$. Since u co-restricts to an irreducible morphism $u_i : X \to N_i^{n_i}$, we see that $N_i \in X^+ \cap \pi^-(U)$ and $d_{X,N_i} \ge n_i$, for $i = 1, \dots, s$. This yields

$$n = n_1 + \cdots + n_s \le \textstyle\sum_{i=1}^s d_{X,N_i} \le \sum_{L\in X^+\cap\pi^-(U)} d_{X,L} = \sum_{i=1}^r d_{X,L_i} = m.$$

On the other hand, since $m_i = d_{X,L_i}$, there exists an irreducible morphism $w_i : X \to L_i^{m_i}$ in $\mathcal{A}$, for $i = 1, \dots, r$. Hence, $w = (w_1, \cdots, w_r)^T : X \to \oplus_{i=1}^r L_i^{m_i}$ is irreducible in $\mathcal{A}$; see [5, (3.2)]. By Proposition 3.3, $F(w) : F(X) \to \oplus_{i=1}^r F(L_i)^{m_i}$ is irreducible in $\mathcal{B}$. Since $F(L_i) \cong \pi(L_i) = U$, we see that $\mathcal{B}$ has an irreducible morphism $g : F(X) \to U^m$, and hence, $m \le n$. As a consequence, $n = m$, that is, π verifies the first equation stated in Definition 4.3(4); and dually, it also verifies the second equation. This establishes Statement (1).

For proving Statement (2), let Γ be a connected component of $\Gamma_{\mathcal{A}}$. By Lemma 4.4, $\pi(\Gamma)$ is a connected component of $\Gamma_{\mathcal{B}}$ such that $\pi_{\Gamma} : \Gamma \to \pi(\Gamma)$ is a Galois G_Γ-covering of valued quivers. Since π is a morphism of valued translation quivers and verifies the property stated in Definition 4.6, the same holds for π_Γ. That is, $\pi_\Gamma : \Gamma \to \pi(\Gamma)$ is a Galois G_Γ-covering of valued translation quivers. The proof of the theorem is completed.

5. Deriving adjoint pairs

Throughout this section, let $\mathfrak{A}, \mathfrak{B}$ be abelian categories. The objective of this section is to show that a graded adjoint pair between $\mathfrak{A}$ and $\mathfrak{B}$ induces a graded adjoint pair between $D(\mathfrak{A})$ and $D(\mathfrak{B})$.

Let $F : \mathfrak{A} \to \mathfrak{B}$ be a linear functor. For $X^\bullet \in C(\mathfrak{A})$, define $F^C(X^\bullet)$ to be the complex of which the component and the differentiation of degree n are $F(X^n)$ and $F(d_X^n)$ respectively, for $n \in \mathbb{Z}$. This yields a linear functor $F^C : C(\mathfrak{A}) \to C(\mathfrak{B})$. The following result is well known; see, for example, [14, (V.1.2.2)].

5.1. Proposition. *Let $\mathfrak{A}, \mathfrak{B}$ be abelian categories. If $F : \mathfrak{A} \to \mathfrak{B}$ is an exact functor, then it induces a commutative diagram of functors*

$$\begin{tikzcd}
C(\mathfrak{A}) \arrow[r, "P_{\mathfrak{A}}"] \arrow[d, "F^C"'] & K(\mathfrak{A}) \arrow[r, "L_{\mathfrak{A}}"] \arrow[d, "F^K"'] & D(\mathfrak{A}) \arrow[d, "F^D"'] \\
C(\mathfrak{B}) \arrow[r, "P_{\mathfrak{B}}"] & K(\mathfrak{B}) \arrow[r, "L_{\mathfrak{B}}"] & D(\mathfrak{B}),
\end{tikzcd}$$

where F^C is an exact functor between abelian categories, while F^K, F^D are exact functors between triangulated categories.

In our late investigation, we shall need the following easy observation.

5.2. Lemma. *Let $\mathfrak{A}, \mathfrak{B}$ be abelian categories, and let $E, F : \mathfrak{A} \to \mathfrak{B}$ be exact functors. If $\eta : E \to F$ is a functorial (iso)morphism, then it induces functorial (iso)morphisms $\eta^C : E^C \to F^C$, and $\eta^K : E^K \to F^K$, and $\eta^D : E^D \to F^D$ such that, for any $X^\bullet \in C(\mathfrak{A})$, we have $\eta^K_{X^\bullet} = P_{\mathfrak{B}}(\eta^C_{X^\bullet})$ and $\eta^D_{X^\bullet} = L_{\mathfrak{B}}(\eta^C_{X^\bullet})$.*

Proof. Suppose that $\eta : E \to F$ is a functorial morphism. For each $X^\bullet \in C(\mathfrak{B})$, we define $\eta^C_{X^\bullet} : E^C(X^\bullet) \to F^C(X^\bullet)$ by setting its n-th component to be the morphism $\eta_{X^n} : E(X^n) \to F(X^n)$. Since η_X is natural in X, we see that $\eta^C_{X^\bullet}$ is a morphism in $C(\mathfrak{B})$ which is natural in $X^\bullet$. Now, $\eta^K_{X^\bullet} = P_{\mathfrak{B}}(\eta^C_{X^\bullet}) : E^K(X^\bullet) \to F^K(X^\bullet)$ is a morphism in $K(\mathfrak{B})$, and $\eta^D_{X^\bullet} = L_{\mathfrak{B}}(\eta^K_{X^\bullet}) : E^D(X^\bullet) \to F^D(X^\bullet)$ is a morphism in $D(\mathfrak{B})$, both are natural in $X^\bullet$. Finally, if the η_X with $X \in \mathfrak{B}$ are all isomorphisms, then the $\eta^C_{X^\bullet}$, $\eta^K_{X^\bullet}$ and $\eta^D_{X^\bullet}$ with $X^\bullet \in C(\mathfrak{B})$ are all isomorphisms. The proof of the lemma is completed.

The following result has been widely believed to be true. However, we find a rigorous proof only in Milicic's unpublished lectures notes; see [14, (V.1.7.1)].

5.3. Theorem (Milicic). *Let $\mathfrak{A}, \mathfrak{B}$ be abelian categories, and let $F : \mathfrak{A} \to \mathfrak{B}$ and $E : \mathfrak{B} \to \mathfrak{A}$ be exact functors. If (F, E) is an adjoint pair, then the induced pairs (F^C, E^C), (F^K, E^K), (F^D, E^D) are adjoint pairs.*

Proof. We recall the description of the induced adjoint isomorphisms for our later investigation, and refer the details to [14, (V.1.7.1)]. Assume that (F, E) is an adjoint pair. For each pair $(X, Y) \in \mathfrak{A}_0 \times \mathfrak{B}_0$, there exists an isomorphism

$$\phi_{X,Y} : \mathfrak{A}(X, E(Y)) \to \mathfrak{B}(F(X), Y),$$

which is natural in both variables X, Y.

Fix $X^\bullet \in C(\mathfrak{A})$ and $Y^\bullet \in C(\mathfrak{B})$. If $f^\bullet : X^\bullet \to E^C(Y^\bullet)$ is a morphism in $C(\mathfrak{A})$, then we set $\phi^C_{X^\bullet, Y^\bullet}(f^\bullet) = (\phi_{X^i, Y^i}(f^i))_{i \in \mathbb{Z}} : F^C(X^\bullet) \to Y^\bullet$, which is a morphism in $C(\mathfrak{B})$. This yields an isomorphism

$$\phi^C_{X^\bullet, Y^\bullet} : \; C(\mathfrak{A})(X^\bullet, \, E^C(Y^\bullet)) \to C(\mathfrak{B})(F^C(X^\bullet), \, Y^\bullet),$$

which is natural in both variables $X^\bullet$ and $Y^\bullet$. Thus, (E^C, F^C) is an adjoint pair.

Next, the naturality of $\phi_{X,Y}$ in X, Y ensures that $\phi^C_{X^\bullet, Y^\bullet}$ sends null-homotopic morphisms to null-homotopic ones. Therefore, $\phi^C_{X^\bullet, Y^\bullet}$ induces an isomorphism

$$\phi^K_{X^\bullet, Y^\bullet} : \; K(\mathfrak{A})(X^\bullet, \, E^K(Y^\bullet)) \to K(\mathfrak{B})(F^K(X^\bullet), \, Y^\bullet),$$

which is natural in both variables $X^\bullet, Y^\bullet$. That is, (E^K, F^K) is an adjoint pair.

Finally, observe that F^K sends quasi-isomorphisms to quasi-isomorphisms. For each morphism $\theta^\bullet = \bar{f}^\bullet / \bar{s}^\bullet \in D(\mathfrak{A})(X^\bullet, E^D(Y^\bullet))$, where $\bar{s}^\bullet : M^\bullet \to X^\bullet$ is a quasi-isomorphism and $\bar{f}^\bullet : M^\bullet \to Y^\bullet$ is a morphism in $K(\mathfrak{A})$, set

$$\phi^D_{X^\bullet, Y^\bullet}(\theta^\bullet) = \phi^K_{M^\bullet, Y^\bullet}(\bar{f}^\bullet) / F(\bar{s}^\bullet).$$

This yields an isomorphism

$$\phi^D_{X^\bullet, Y^\bullet} : \; D(\mathfrak{A})(X^\bullet, \, E^D(Y^\bullet)) \to D(\mathfrak{B})(F^D(X^\bullet), \, Y^\bullet),$$

which is natural in both variables $X^{\bullet}, Y^{\bullet}$. The proof of the theorem is completed.

For the rest of this section, assume that $\mathfrak{A}$ is equipped with an action of a group G. Regarding $g \in G$ as an automorphism of $\mathfrak{A}$, we deduce immediately the following statement from Proposition 5.1.

5.4. LEMMA. *Let $\mathfrak{A}$ be an abelian category. If G is a group acting on $\mathfrak{A}$, then it acts on each of $C(\mathfrak{A})$, $K(\mathfrak{A})$, $D(\mathfrak{A})$ in such a way that, for any $g \in G$, the following diagram commutes:*

$$\begin{tikzcd} C(\mathfrak{A}) \arrow[r, "P_{\mathfrak{A}}"] \arrow[d, "g"] & K(\mathfrak{A}) \arrow[r, "L_{\mathfrak{A}}"] \arrow[d, "g"] & D(\mathfrak{A}) \arrow[d, "g"] \\ C(\mathfrak{A}) \arrow[r, "P_{\mathfrak{A}}"] & K(\mathfrak{A}) \arrow[r, "L_{\mathfrak{A}}"] & D(\mathfrak{A}). \end{tikzcd}$$

We shall say that $\mathfrak{A}$ has *enough projective objects* provided that every object X in $\mathfrak{A}$ admits an epimorphism $\varepsilon : P \to X$ with P being projective. Recall that $D^b(\mathfrak{A})$ can be regarded as a full triangulated subcategory of $D(\mathfrak{A})$; see [10, (6.15)].

5.5. LEMMA. *Let $\mathfrak{A}$ be an abelian category having enough projective objects and equipped with a locally bounded action of a group G.*

(1) *If $X^{\bullet}, Y^{\bullet} \in C^b(\mathfrak{A})$, then $C(\mathfrak{A})$ has a quasi-isomorphism $s^{\bullet} : P^{\bullet} \to X^{\bullet}$ with $P^{\bullet}$ a bounded-above complex of projective objects such that $C(\mathfrak{A})(P^{\bullet}, g \cdot Y^{\bullet}) = 0$, for all but finitely many $g \in G$.*

(2) *The category $D^b(\mathfrak{A})$ is a G-subcategory of $D(\mathfrak{A})$ with a locally bounded G-action.*

Proof. Clearly, $D^b(\mathfrak{A})$ is stable under the G-action on $D(\mathfrak{A})$. Let $X^{\bullet}, Y^{\bullet} \in C^b(\mathfrak{A})$. Since $\mathfrak{A}$ has enough projective objects, $C(\mathfrak{A})$ has a quasi-isomorphism $s^{\bullet} : P^{\bullet} \to X^{\bullet}$ with $P^{\bullet}$ a bounded-above complex of projective objects; see [10, (7.5)]. Let $m \le n$ be integers such that $Y^i = 0$ for $i \notin [m, n]$. Since the G-action on $\mathfrak{A}$ is locally bounded, there exists a finite subset G_0 of G such that $\mathfrak{A}(P^i, gY^i) = 0$, for $g \in G \backslash G_0$ and $m \le i \le n$. Thus, $C(\mathfrak{A})(P^{\bullet}, g \cdot Y^{\bullet}) = 0$, and consequently, $K(\mathfrak{A})(P^{\bullet}, g \cdot Y^{\bullet}) = 0$, for any $g \in G \backslash G_0$. Since $\tilde{s}^{\bullet}$ is an isomorphism $D(\mathfrak{A})$, we have

$$D^b(\mathfrak{A})(X^{\bullet}, g \cdot Y^{\bullet}) \cong D(\mathfrak{A})(X^{\bullet}, g \cdot Y^{\bullet}) \cong D(\mathfrak{A})(P^{\bullet}, g \cdot Y^{\bullet}) \cong K(\mathfrak{A})(P^{\bullet}, g \cdot Y^{\bullet}) = 0,$$

where the last isomorphism follows from Lemma 1.9(1), for any $g \in G \backslash G_0$. The proof of the lemma is completed.

We shall need the following easy result.

5.6. LEMMA. *Let $F : \mathfrak{A} \to \mathfrak{B}$ be a functor between abelian categories, and let G be a group acting on $\mathfrak{A}$. If δ is a G-stabilizer for F, then it induces G-stabilizers δ^C, δ^K, and δ^D for the functors F^C, F^K, and F^D, respectively.*

Proof. Let δ be a G-stabilizer for F. Then, there exist functorial isomorphism $\delta_g : F \circ g \to F$, with $g \in G$, such that $\delta_{h,X}\, \delta_{g,\, h \cdot X} = \delta_{gh,X}$, for $X \in \mathcal{A}_0$ and $g, h \in G$. For each $g \in G$, observing that $(F \circ g)^C = F^C \circ g$, and $(F \circ g)^K = F^K \circ g$, and $(F \circ g)^D = F^D \circ g$, we deduce from Lemma 5.2 that δ_g induces functorial isomorphisms $\delta_g^C : F^C \circ g \to F^C$, and $\delta_g^K : F^K \circ g \to F^K$, and $\delta_g^D : F^D \circ g \to F^D$.

Fix $X^{\bullet} \in C(\mathfrak{A})$ and $g, h \in G$. For any $n \in \mathbb{Z}$, we have $\delta_{h,X^n} \circ \delta_{g,\, h \cdot X^n} = \delta_{gh,X^n}$, that is, $(\delta^C_{h,X^{\bullet}})^n \circ (\delta^C_{g,\, h \cdot X^{\bullet}})^n = (\delta^C_{gh,X^{\bullet}})^n$. This implies that $\delta^C_{h,X^{\bullet}} \circ \delta^C_{g,\, h \cdot X^{\bullet}} = \delta^C_{gh,X^{\bullet}}$. Applying first the functor $P_{\mathfrak{A}}$ and then $L_{\mathfrak{A}}$ yields $\delta^K_{h,X^{\bullet}} \circ \delta^K_{g,\, h \cdot X^{\bullet}} = \delta^K_{gh,X^{\bullet}}$ and

$\delta^D_{h,X^\bullet}\circ\delta^D_{g,\,h\cdot X^\bullet}=\delta^D_{gh,X^\bullet}$. The proof of the lemma is completed.

Furthermore, assume that $\mathfrak{A}$ has essential direct sums. By Lemma 1.2, the category of endofunctors of $\mathfrak{A}$ has a direct sum $\mathcal{G}$ of the $g\in G$, regarded as automorphisms of $\mathcal{A}$ with canonical injection $j_g: g\to\mathcal{G}$. Being exact by Lemma 1.6, $\mathcal{G}$ induces a commutative diagram

$$\begin{array}{ccccc} C(\mathfrak{A}) & \xrightarrow{P_{\mathfrak{A}}} & K(\mathfrak{A}) & \xrightarrow{L_{\mathfrak{A}}} & D(\mathfrak{A}) \\ \downarrow{\scriptstyle \mathcal{G}^C} & & \downarrow{\scriptstyle \mathcal{G}^K} & & \downarrow{\scriptstyle \mathcal{G}^D} \\ C(\mathfrak{A}) & \xrightarrow{P_{\mathfrak{A}}} & K(\mathfrak{A}) & \xrightarrow{L_{\mathfrak{A}}} & D(\mathfrak{A}). \end{array}$$

On the other hand, by Lemma 1.4 and Theorem 1.8, $C(\mathfrak{A})$, $K(\mathfrak{A})$, and $D(\mathfrak{A})$ all have direct sums, and so does the category of linear endofunctors of each of $C(\mathfrak{A})$, $K(\mathfrak{A})$ and $D(\mathfrak{A})$. It is easy to see that $\mathcal{G}^C$, $\mathcal{G}^K$, $\mathcal{G}^D$ are the direct sums of the $g\in G$, considered as automorphisms of $C(\mathfrak{A})$, $K(\mathfrak{A})$, $D(\mathfrak{A})$, respectively, with canonical injections $j^C_g: g\to\mathcal{G}^C$, and $j^K_g: g\to\mathcal{G}^K$, and $j^D_g: g\to\mathcal{G}^D$, which are induced as described in Lemma 5.2 from the canonical injections $j_g: g\to\mathcal{G}$.

5.7. Proposition. *Let $\mathfrak{A},\mathfrak{B}$ be abelian categories such that $\mathfrak{A}$ has essential direct sums and admits an action of a group G. Let $F:\mathfrak{A}\to\mathfrak{B}$ and $E:\mathfrak{B}\to\mathfrak{A}$ be exact functors. If (F,E) is a G-graded adjoint pair, then the induced pairs $(F^C,E^C),(F^K,E^K)$ and (F^D,E^D) are G-graded adjoint pairs.*

Proof. Assume that (F,E) is a G-graded adjoint pair. Then, we have a functorial isomorphism $\gamma:\mathcal{G}\to E\circ F$, a G-stabilizer δ for F, and an adjoint isomorphism ϕ for (E,F). It follows from Lemma 5.2 that γ induces functorial isomorphisms $\gamma^C:\mathcal{G}^C\to E^C\circ F^C$, and $\gamma^K:\mathcal{G}^K\to E^K\circ F^K$, and $\gamma^D:\mathcal{G}^D\to E^C\circ F^D$. Moreover, by Lemma 5.6, δ induces G-stabilizers δ^C, δ^K, δ^D for F^C, F^K, F^D, respectively. Finally, as described in the proof of Theorem 5.3, ϕ induces adjoint isomorphisms ϕ^C,ϕ^K,ϕ^D for $(F^C,E^C),(F^K,E^K),(F^D,E^D)$, respectively.

Consider a morphism $u^\cdot: X^\bullet\to g\cdot Y^\bullet$ in $C(\mathfrak{A})$, where $g\in G$. By Definition 2.11(2), we have

$$\phi_{X^n,F(Y^n)}(\gamma_{Y^n}\circ j_{g,Y^n}\circ u^n)=\delta_{g,Y^n}\circ F(u^n),$$

for all $n\in\mathbb{Z}$. This yields

$$\phi^C_{X^\bullet,F^C(Y^\bullet)}(\gamma^C_{Y^\bullet}\circ j^C_{g,Y^\bullet}\circ u^\cdot)=\delta^C_{g,Y^\bullet}\circ F^C(u^\cdot).$$

Applying the projection functor $P_{\mathfrak{B}}: C(\mathfrak{B})\to K(\mathfrak{B})$, we obtain

$$\phi^K_{X^\bullet,F^K(Y^\bullet)}(\gamma^K_{Y^\bullet}\circ j^K_{g,Y^\bullet}\circ \bar u^\cdot)=\delta^K_{g,Y^\bullet}\circ F^K(\bar u^\cdot).$$

Finally, let $\theta^\cdot=\bar u^\cdot/\bar s^\cdot: X^\bullet\to g\cdot Y^\bullet$ with $g\in G$ be a morphism in $D(\mathfrak{A})$, where $\bar s^\cdot: Z^\bullet\to X^\bullet$ is a quasi-morphism and $\bar u^\cdot: Z^\bullet\to g\cdot Y^\bullet$ is a morphism in $K(\mathfrak{A})$. Observing that $\gamma^D_{Y^\bullet}\circ j^D_{g,Y^\bullet}\circ\theta^\cdot=(\gamma^K_{Y^\bullet}\circ j^K_{g,Y^\bullet}\circ\bar u^\cdot)/\bar s^\cdot$, we obtain

$$\begin{aligned}\phi^D_{X^\bullet,\,F^D(Y^\bullet)}(\gamma^D_{Y^\bullet}\circ j^D_{g,Y^\bullet}\circ\theta^\cdot) &= \phi^K_{Z^\bullet,\,F^K(Y^\bullet)}(\gamma^K_{Y^\bullet}\circ j^K_{g,Y^\bullet}\circ\bar u^\cdot)/F^K(\bar s^\cdot)\\ &= \left(\delta^K_{g,Y^\bullet}\circ F^K(\bar u^\cdot)\right)/F^K(\bar s^\cdot)\\ &= \delta^D_{g,Y^\bullet}\circ\left(F^K(\bar u^\cdot)/F^K(\bar s^\cdot)\right)\\ &= \delta^D_{g,Y^\bullet}\circ F^D(\theta^\cdot).\end{aligned}$$

The proof of the proposition is completed.

6. Module categories and their derived categories

The objective of this section is to apply our previous results to study push-down functors between the module categories and between their derived categories induced from a Galois covering of locally bounded linear categories.

Throughout this section, k denotes a field, and all tensor products are over k. Let Λ stand for a locally bounded linear category over k, that is, Λ is skeletal such that $\oplus_{x\in\Lambda_0}(\Lambda(a,x)\oplus\Lambda(a,x))$ is finite dimensional, for any $a\in\Lambda_0$; see [6, (2.1)]. A *left Λ-module* is a k-linear functor $M:\Lambda\to \mathrm{Mod}k$, where $\mathrm{Mod}k$ is the category of all k-vector spaces. If M,N are left Λ-modules, then a Λ-linear morphism $f:M\to N$ consists of k-linear maps $f(x):M(x)\to N(x)$, $x\in\Lambda_0$, such that $f(y)M(\alpha)=N(\alpha)f(x)$, for every morphism $\alpha:x\to y$ in Λ. We denote by $\mathrm{Mod}\Lambda$ the category of all left Λ-modules, which has essential direct sums by Lemma 1.2. For $M\in\mathrm{Mod}\Lambda$, one defines its *support* $\mathrm{supp}M$ to be the set of $x\in\Lambda_0$ for which $M(x)\neq 0$. One says that M is *finitely supported* if $\mathrm{supp}M$ is finite, and *finite dimensional* if $\sum_{x\in\Lambda_0}\dim_k M(x)$ is finite. We shall denote by $\mathrm{Mod}^b\!\Lambda$ and $\mathrm{mod}^b\!\Lambda$ the full abelian subcategories of $\mathrm{Mod}\Lambda$ generated by the finitely supported modules and by the finite dimensional modules, respectively.

For each $x\in\Lambda_0$, it is well known that $P[x]=\Lambda(x,-)$ is an indecomposable projective object in $\mathrm{Mod}\Lambda$. Since Λ is locally bounded, $P[x]\in\mathrm{mod}^b\!\Lambda$. Hence, for any $V\in\mathrm{Mod}k$, the Λ-module $P[x]\otimes V$ lies in $\mathrm{Mod}^b\!\Lambda$ and is projective in $\mathrm{Mod}\Lambda$. We denote by $\mathrm{proj}\,\Lambda$ the full additive subcategory of $\mathrm{mod}^b\!\Lambda$ generated by the modules isomorphic to $P[x]$ for some $x\in\Lambda_0$, and by $\mathrm{Proj}\,\Lambda$ the full additive subcategory of $\mathrm{Mod}^b\!\Lambda$ generated by the modules isomorphic to some $P[x]\otimes V$ with $x\in\Lambda_0$ and $V\in\mathrm{Mod}k$.

6.1. Lemma. *Let Λ be a locally bounded k-linear category. If $M\in\mathrm{Mod}^b\!\Lambda$, then it admits a projective cover $\varepsilon:P\to X$ with $P\in\mathrm{Proj}\,\Lambda$ in such a way that $M\in\mathrm{mod}^b\!\Lambda$ if and only if $P\in\mathrm{proj}\,\Lambda$.*

Proof. Let $M\in\mathrm{Mod}^b\!\Lambda$ having a finite support Σ. Since Λ is locally bounded, $M/\mathrm{rad}\,M\cong\oplus_{x\in\Sigma}S[x]\otimes U_x$, where $S[x]$ is the simple Λ-module supported by x and $U_x\in\mathrm{Mod}k$. Hence, $P=\oplus_{x\in\Sigma}P[x]\otimes U_x$ is a projective cover of M, which lies in $\mathrm{Proj}\,\Lambda$. If M is finite dimensional, then U_x is finite dimensional for every $x\in\Sigma$, and hence, $P\in\mathrm{proj}\,\Lambda$. Conversely, if $P\in\mathrm{proj}\,\Lambda$, then M is clearly finite dimensional. The proof of the lemma is completed.

For the rest of this section, let G be a group acting on Λ. The G-action on Λ induces a G-action on $\mathrm{Mod}\Lambda$ as follows; see [8, (3.2)]. Fix $g\in G$. For a Λ-module $M:\Lambda\to\mathrm{Mod}k$, regarding g as an automorphism of Λ, one defines $g{\cdot}M=M\circ g^{-1}:\Lambda\to\mathrm{Mod}k$; and for a Λ-linear morphism $u:M\to N$, one defines $g{\cdot}u:g{\cdot}M\to g{\cdot}N$ by setting $(g\cdot u)(x)=u(g^{-1}{\cdot}x)$, for $x\in\Lambda_0$.

6.2. Lemma. *Let Λ be a locally bounded k-linear category with an action of a group G. If the G-action on Λ is free, then $\mathrm{Mod}^b\!\Lambda$ and $\mathrm{mod}^b\!\Lambda$ are G-subcategories of $\mathrm{Mod}\Lambda$ with a locally bounded G-action.*

Proof. Assume that the G-action on Λ is free. Let $M,N\in\mathrm{Mod}^b\!\Lambda$. For each $g\in G$, by the definition of the G-action on $\mathrm{Mod}\Lambda$, we have $\mathrm{supp}(g{\cdot}N)=g{\cdot}\mathrm{supp}N$. As an immediate consequence, $\mathrm{Mod}^b\!\Lambda$ and $\mathrm{mod}^b\!\Lambda$ are both G-subcategories of $\mathrm{Mod}\Lambda$.

Suppose that supp$M \cap g \cdot$ supp$N \neq \emptyset$ for infinitely many $g \in G$. Being finite, supp$M$ has an element x with $x \in g \cdot$ supp$N$, for infinitely many $g \in G$. Since supp$N$ is finite, there exists $y \in$ supp$N$ such that $x = g \cdot y$ for infinitely many $g \in G$. In particular, there exist two distinct elements $g, h \in G$ such that $g{\cdot}y = h{\cdot}y$, which is absurd. Thus, supp$M \cap g \cdot$ supp$N = \emptyset$ for all but finitely many $g \in G$. In particular, $\mathrm{Hom}_\Lambda(M,\, g \cdot N) = 0$ for all but finitely many $g \in G$. That is, the G-action on $\mathrm{Mod}^b\!\Lambda$ is locally bounded, and consequently, so is the G-action on $\mathrm{mod}^b\!\Lambda$. The proof of the lemma is completed.

Assume that the G-action on Λ is admissible. Let $\pi : \Lambda \to A$ be a G-invariant Galois G-covering between locally bounded k-linear categories. In [6, (3.2)], Bongartz and Gabriel constructed an exact functor

$$\pi_\lambda : \mathrm{Mod}\,\Lambda \to \mathrm{Mod}\,A,$$

called *push-down functor*. Indeed, for a module $M \in \mathrm{Mod}\,\Lambda$, one defines an A-module $\pi_\lambda(M)$ as follows. For $a \in A_0$, one sets $\pi_\lambda(M)(a) = \oplus_{x \in \pi^-(a)} M(x)$, where $\pi^-(a) = \{x \in \Lambda_0 \mid \pi(x) = a\}$. Let $\alpha : a \to b$ a morphism in A. Since π is G-invariant, for each pair $(x, y) \in \pi^-(a) \times \pi^-(b)$, there exists a unique $\alpha_{y,x} \in \Lambda(x, y)$ such that $\sum_{y \in \pi^-(b)} \pi(\alpha_{y,x}) = \alpha$, for every $x \in \pi^-(a)$. Observing that $M(\alpha_{y,x})$ is a k-linear map from $M(x)$ to $M(y)$, one sets

$$\pi_\lambda(M)(\alpha) = (M(\alpha_{y,x}))_{(y,x)\in\pi^-(b)\times\pi^-(a)} : \oplus_{x\in\pi^-(a)} M(x) \to \oplus_{y\in\pi^-(b)} M(y).$$

Next, let $f : M \to N$ be a morphism in $\mathrm{Mod}\,\Lambda$. Setting

$$\pi_\lambda(f)(a) = \mathrm{diag}\{f(x) \mid x \in \pi^-(a)\} : \ \oplus_{x\in\pi^-(a)} M(x) \to \oplus_{x\in\pi^-(a)} N(x),$$

for each $a \in A_0$, one obtains a morphism $\pi_\lambda(f) : \pi_\lambda(M) \to \pi_\lambda(N)$ in $\mathrm{Mod}\,A$.

The following result collects some properties of the push-down functor, which is partially due to Bongartz-Gabriel; see [6, 8].

6.3. Lemma. *Let Λ, A be locally bounded k-linear categories with G a group acting admissibly on Λ, and let $\pi : \Lambda \to A$ be a G-invariant Galois G-covering.*

(1) *The push-down functor $\pi_\lambda : \mathrm{Mod}\Lambda \to \mathrm{Mod}A$ admits a G-stabilizer δ.*

(2) *If $x \in \Lambda_0$, then $\pi_\lambda(P[x]) \cong P[\pi(x)]$.*

Proof. (1) Fix $g \in G$. For $M \in \mathrm{Mod}\Lambda$, we define $\delta_{g,M} : \pi_\lambda(g \cdot M) \to \pi_\lambda(M)$ by setting, for $a \in A_0$, that

$$\delta_{g,M}(a) = (\varepsilon_{y,x})_{(y,x)\in\pi^-(a)\times\pi^-(a)} : \oplus_{x\in\pi^-(a)} M(g^{-1}{\cdot}x) \to \oplus_{y\in\pi^-(a)} M(y),$$

where $\varepsilon_{y,x} : M(g^{-1} \cdot x) \to M(y)$ is a k-linear map so that $\varepsilon_{y,x} = \mathbb{1}$ if $g^{-1} \cdot x = y$; and $\varepsilon_{y,x} = 0$ otherwise. One verifies easily that $\delta_{g,M}$ is an A-linear isomorphism, which is natural in M. This yields a natural isomorphism $\delta_g : \pi_\lambda \circ g \to \pi_\lambda$.

Next, let $g, h \in G$. For each $a \in A_0$, we write

$$\delta_{gh,M}(a) = (\varepsilon_{z,x})_{(z,x)\in\pi^-(a)\times\pi^-(a)} : \oplus_{x\in\pi^-(a)} M((gh)^{-1} \cdot x) \to \oplus_{z\in\pi^-(a)} M(z),$$

where $\varepsilon_{z,x} : M((gh)^{-1} \cdot x) \to M(z)$ is defined such that $\varepsilon_{z,x} = \mathbb{1}$ if $(gh)^{-1} \cdot x = z$; and $\varepsilon_{z,x} = 0$ otherwise. Moreover, write

$$\delta_{g,h\cdot M}(a) = (\eta_{y,x})_{(y,x)\in\pi^-(a)\times\pi^-(a)} : \oplus_{x\in\pi^-(a)} M(h^{-1}(g^{-1}{\cdot}x)) \to \oplus_{y\in\pi^-(a)} M(h^{-1}{\cdot}y),$$

where $\eta_{y,x} : M(h^{-1}(g^{-1} \cdot x)) \to M(h^{-1} \cdot y)$ is such that $\eta_{y,x} = \mathbb{1}$ provided that $g^{-1} \cdot x = y$; and $\eta_{y,x} = 0$ otherwise, and write

$$\delta_{h,M}(a) = (\zeta_{z,y})_{(z,y)\in\pi^-(a)\times\pi^-(a)} : \oplus_{y\in\pi^-(a)} M(h^{-1} \cdot y) \to \oplus_{z\in\pi^-(a)} M(z),$$

where $\zeta_{z,y} : M(h^{-1} \cdot y) \to M(z)$ is such that $\zeta_{z,y} = \mathbb{1}$ if $h^{-1} \cdot y = z$; and $\zeta_{z,y} = 0$ otherwise. Therefore,

$$\delta_{h,M}(a) \circ \delta_{g,h\cdot M}(a) = (\xi_{z,x})_{(z,x)\in\pi^-(a)\times\pi^-(a)}, \text{ where } \xi_{z,x} = \textstyle\sum_{y\in\pi^-(a)} \zeta_{z,y} \circ \eta_{y,x}.$$

Assume that $z = (gh)^{-1} \cdot x = h^{-1}(g^{-1} \cdot x)$. For $y \in \pi^-(a)$, we have $\zeta_{z,y} \circ \eta_{y,x} = 0$ if $y \neq g^{-1} \cdot x$, and otherwise, $\zeta_{z,y} \circ \eta_{y,x} = 1$. As a consequence, $\xi_{z,x} = 1$ in case $z = (gh)^{-1} \cdot x$. If $z \neq (gh)^{-1} \cdot x$, then $\eta_{y,x} = 0$ in case $y \neq g^{-1} \cdot x$ and $\zeta_{z,y} = 0$ if $y = g^{-1} \cdot x$. Therefore, $\xi_{z,x} = 0$ in case $z \neq (gh)^{-1} \cdot x$. This implies that $\delta_{gh,M} = \delta_{h,M} \circ \delta_{g,h\cdot M}$, that is, δ is a G-stabilizer for π_λ.

(2) Fix $x \in \Lambda_0$. For each $a \in A_0$, since π is G-invariant, we have a k-linear isomorphism

$$\pi_{x,a} : \pi_\lambda(P[x])(a) = \oplus_{y\in\pi^-(a)} \Lambda(x, y) \to A(x, a) = P[\pi(x)](a),$$

sending $(f_{y,x})_{y\in\pi^-(a)}$ to $\sum_{y\in\pi^-(a)} \pi(f_{y,x})$, where $f_{y,x} \in \Lambda(x, y)$. Moreover, let $\alpha : a \to b$ be a morphism in A. For each pair $(z, y) \in \pi^-(b) \times \pi^-(a)$, there exists $\alpha_{z,y} \in \Lambda(y, z)$ such that $\sum_{z\in\pi^-(b)} \pi(\alpha_{z,y}) = \alpha$, for any $y \in \pi^-(a)$. By definition, for each $f = (f_{y,x})_{y\in\pi^-(a)} \in \pi_\lambda(P[x])(a)$, we have

$$\begin{aligned} (\pi_{x,b} \circ \pi_\lambda(P[x])(\alpha))\,(f) &= \textstyle\sum_{z\in\pi^-(b)} \sum_{y\in\pi^-(a)} \pi(\alpha_{z,y})\pi(f_{y,x}) \\ &= \alpha \textstyle\sum_{y\in\pi^-(a)} \pi(f_{y,x}) \\ &= (P[\pi(x)](\alpha) \circ \pi_{x,a})\,(f). \end{aligned}$$

This shows that $\pi_\lambda(P[x]) \cong P[\pi(x)]$. The proof of the lemma is completed.

Moreover, Bongartz and Gabriel observed that the push-down functor π_λ admits an exact right adjoint

$$\pi_\mu : \mathrm{Mod} A \to \mathrm{Mod} \Lambda,$$

called *pull-up functor*, such that if $N \in \mathrm{Mod} A$, then $\pi_\mu(N)$ is the composite of $\pi : \Lambda \to A$ and $N : A \to \mathrm{Mod} k$; and if $f : M \to N$ is a morphism in $\mathrm{Mod} A$, then $\pi_\mu(f) : \pi_\mu(M) \to \pi_\mu(N)$ is defined by $\pi_\mu(f)(x) = f(\pi(x))$ for all $x \in \Lambda_0$.

The following result is essentially due to Gabriel; see [8].

6.4. Proposition. *Let Λ, A be locally bounded k-linear categories with G a group acting admissibly on Λ. If $\pi : \Lambda \to A$ is a G-invariant Galois G-covering, then the functors $\pi_\lambda : \mathrm{Mod}\Lambda \to \mathrm{Mod}A$ and $\pi_\mu : \mathrm{Mod}A \to \mathrm{Mod}\Lambda$ form a G-graded adjoint pair (π_λ, π_μ).*

Proof. Let $\pi : \Lambda \to A$ be a G-invariant Galois G-covering. It is well known that (π_λ, π_μ) is an adjoint pair. For our purpose, we recall the definition of the adjoint isomorphism ϕ. Fix $M \in \mathrm{Mod}\,\Lambda$ and $N \in \mathrm{Mod}\,A$. Let $u \in \mathrm{Hom}_\Lambda(M, \pi_\mu(N))$, consisting of a family of k-linear maps $u(x) : M(x) \to \pi_\mu(N)(x) = N(\pi(x))$ with $x \in \Lambda_0$. For $a \in A_0$, define

$$\phi_{M,N}(u)(a) = (u(x))_{x\in\pi^-(a)} : \oplus_{x\in\pi^-(a)} M(x) \to N(a).$$

Then $\phi_{M,N}(u) = (\phi_{M,N}(u)(a))_{a\in A_0}$ is an A-linear morphism from $\pi_\lambda(M)$ to N. This yields a natural k-linear map

$$\phi_{M,N} : \mathrm{Hom}_\Lambda(M, \pi_\mu(N)) \to \mathrm{Hom}_A(\pi_\lambda(M), N).$$

Conversely, let $v \in \mathrm{Hom}_A(\pi_\lambda(M), N)$. For each $a \in A_0$, write

$$v(a) = (v_{a,x})_{x\in\pi^-(a)} : \oplus_{x\in\pi^-(a)} M(x) \to N(a),$$

where $v_{a,\,x} \in \mathrm{Hom}_k(M(x), N(a))$. For each $x \in \Lambda_0$, define

$$\psi_{M,\,N}(v)(x) = v_{\pi(x),\,x} : M(x) \to N(\pi(x)) = \pi_\mu(N)(x).$$

Then $\psi_{M,N}(v) = (\psi_{M,N}(v)(x))_{x\in\Lambda_0}$ is a Λ-linear morphism from M to $\pi_\mu(N)$. This yields a k-linear map

$$\psi_{M,\,N} : \mathrm{Hom}_\Lambda(\pi_\lambda(M), N) \to \mathrm{Hom}_A(M, \pi_\mu(N)),$$

which is the inverse of $\phi_{M,N}$.

Consider the direct sum $\mathcal{G} = \oplus_{g\in G}\, g$ of the $g \in G$, regarded as automorphisms of ModΛ, with canonical injections $j_g : g \to \mathcal{G}$. For each $M \in$ ModΛ, one defines a Λ-linear morphism $\gamma_{_M} : \mathcal{G}(M) \to \pi_\mu(\pi_\lambda(M))$ in such a way that, for any $x \in \Lambda_0$,

$$\gamma_{_M}(x) = (\varepsilon_{y,g})_{(y,g)\in\pi^-(\pi(x))\times G} : \oplus_{g\in G}\, M(g^{-1}{\cdot}x) \to \oplus_{y\in\pi^-(\pi(x))}\, M(y),$$

where $\varepsilon_{y,g} = \mathbb{1}$ if $g^{-1}(x) = y$; and $\varepsilon_{y,g} = 0$ otherwise. It is easy to see that $\gamma_{_M}$ is an isomorphism which is natural in M. This yields a natural isomorphism $\gamma : \mathcal{G} \to \pi_\mu \circ \pi_\lambda$.

Let $u : M \to g \cdot N$, where $g \in G$, be a morphism in ModΛ. Write $\rho_{g,N}$ for the composite of $j_g : g{\cdot}N \to \mathcal{G}(N)$ and $\gamma_N : \mathcal{G}(N) \to \pi_\mu(\pi_\lambda(N))$. Fix $a \in A_0$. For each $x \in \pi^-(a)$, we have a k-linear map $u(x) : M(x) \to (g \cdot N)(x) = N(g^{-1} \cdot x)$, and by definition, the k-linear map $\rho_{g,N}(x) : (g \cdot N)(x) \to \pi_\mu\pi_\lambda(N)(x) = \pi_\lambda(N)(a)$ is a column-matrix

$$\rho_{g,N}(x) = (\varepsilon_{y,x})_{y\in\pi^-(a)} : N(g^{-1}{\cdot}x) \to \oplus_{y\in\pi^-(a)} N(y),$$

where $\varepsilon_{y,x} = \mathbb{1}$ in case $g^{-1} \cdot x = y$; and $\varepsilon_{y,x} = 0$ otherwise. As a consequence, $(\rho_{g,N} \circ u)(x) : M(x) \to \pi_\lambda(N)(a)$ is the following column-matrix

$$(\rho_{g,N} \circ u)(x) = (\varepsilon_{y,x} \circ u(x))_{y\in\pi^-(a)} : M(x) \to \oplus_{y\in\pi^-(a)} N(y) = \pi_\lambda(N)(a).$$

In view of the above definition of ϕ, we see that the k-linear map

$$\phi_{M,\pi_\lambda(N)}(\rho_{g,N} \circ u)(a) : \pi_\lambda(M)(a) \to \pi_\lambda(N)(a)$$

is given by the following row-matrix

$$\phi_{M,\pi_\lambda(N)}(\rho_{g,N} \circ u)(a) = ((\rho_{g,N} \circ u)(x))_{x\in\pi^-(a)} : \oplus_{x\in\pi^-(a)}\, M(x) \to \pi_\lambda(N)(a),$$

which is indeed the following square matrix

$$(\varepsilon_{y,x} \circ u(x))_{(y,x)\in\pi^-(a)\times\pi^-(a)} : \oplus_{x\in\pi^-(a)}\, M(x) \to \oplus_{y\in\pi^-(a)} N(y).$$

On the other hand, $\pi_\lambda(u)(a) : \pi_\lambda(M)(a) \to \pi_\lambda(g{\cdot}N)(a)$ is a diagonal matrix

$$\pi_\lambda(u)(a) = \mathrm{diag}\,\{u(x)\}_{x\in\pi^-(a)} : \oplus_{x\in\pi^-(a)} M(x) \to \oplus_{x\in\pi^-(a)} N(g^{-1} \cdot x).$$

Let δ be the G-stabilizer for π_λ as described in Lemma 6.3. By definition, $\delta_{g,N}(a) : \pi_\lambda(g{\cdot}N) \to \pi_\lambda(N)$ is given by the following square matrix:

$$\delta_{g,N}(a) = (\varepsilon_{y,x})_{(y,x)\in\pi^-(a)\times\pi^-(a)} : \oplus_{x\in\pi^-(a)}\, N(g^{-1} \cdot x) \to \oplus_{y\in\pi^-(a)} N(y).$$

Therefore, $(\delta_{g,N} \circ \pi_\lambda(u))(a) : \pi_\lambda(M)(a) \to \pi_\lambda(N)(a)$ is given by the following square matrix

$$(\varepsilon_{y,x} \circ u(x))_{(y,x)\in\pi^-(a)\times\pi^-(a)} : \oplus_{x\in\pi^-(a)} M(x) \to \oplus_{y\in\pi^-(a)} N(y).$$

That is, $\phi_{M,\pi_\lambda(N)}(\rho_{g,N} \circ u)(a) = (\delta_{g,N} \circ \pi_\lambda(u))(a)$. This gives rise to

$$\phi_{M,\pi_\lambda(N)}(\gamma_N \circ j_{g,N} \circ u) = \delta_{g,N} \circ \pi_\lambda(u).$$

The proof of the proposition is completed.

Observe that the push-down functor $\pi_\lambda : \mathrm{Mod}\mathit{\Lambda} \to \mathrm{Mod}A$ sends $\mathrm{Mod}^b\mathit{\Lambda}$ and $\mathrm{mod}^b\mathit{\Lambda}$ into Mod^bA and mod^bA, respectively. Restricting this functor yields two functors $\mathrm{Mod}^b\mathit{\Lambda} \to \mathrm{Mod}^bA$ and $\mathrm{mod}^b\mathit{\Lambda} \to \mathrm{mod}^bA$ which, by abuse of notation, are both denoted by π_λ again.

6.5. Theorem. *Let $\mathit{\Lambda}, A$ be locally bounded k-linear categories with G a group acting admissibly on $\mathit{\Lambda}$. Let $\pi : \mathit{\Lambda} \to A$ be a G-invariant Galois G-covering.*

(1) *The push-down functor $\pi_\lambda : \mathrm{Mod}^b\mathit{\Lambda} \to \mathrm{Mod}^bA$ is G-precovering.*
(2) *The push-down functor $\pi_\lambda : \mathrm{mod}^b\mathit{\Lambda} \to \mathrm{mod}^bA$ is a G-precovering, and in case G is torsion-free, it has the following properties.*
   (a) *If $M \in \mathrm{mod}^b\mathit{\Lambda}$ is indecomposable, then $\pi_\lambda(M)$ is indecomposable.*
   (b) *If $M, N \in \mathrm{mod}^b\mathit{\Lambda}$ are indecomposable with $\pi_\lambda(M) \cong \pi_\lambda(N)$, then $N \cong g{\cdot}M$ for some $g \in G$.*

Proof. By Proposition 6.4, the push-down functor $\pi_\lambda : \mathrm{Mod}\mathit{\Lambda} \to \mathrm{Mod}A$ and the pull-up functor $\pi_\mu : \mathrm{Mod}A \to \mathrm{Mod}\mathit{\Lambda}$ form a G-graded adjoint pair (π_λ, π_μ). By Lemma 6.2, $\mathrm{Mod}^b\mathit{\Lambda}$ and $\mathrm{mod}^b\mathit{\Lambda}$ are abelian G-subcategories of $\mathrm{Mod}\mathit{\Lambda}$ with a locally bounded G-action. Since the direct sums in $\mathrm{Mod}\mathit{\Lambda}$ are essential, the G-actions on $\mathrm{Mod}^b\mathit{\Lambda}$ and $\mathrm{mod}^b\mathit{\Lambda}$ are $\mathrm{Mod}\mathit{\Lambda}$-essential. It follows from Theorem 2.12 that both $\pi_\lambda : \mathrm{Mod}^b\mathit{\Lambda} \to \mathrm{Mod}^bA$ and $\pi_\lambda : \mathrm{mod}^b\mathit{\Lambda} \to \mathrm{mod}^bA$ are G-precoverings.

Suppose that G is torsion-free. By Lemma 2.2, the G-action on $\mathrm{mod}^b\mathit{\Lambda}$ is free. Since $\mathrm{mod}^b\mathit{\Lambda}$ is Hom-finite and abelian, the endomorphism algebra of any indecomposable module is local with a nilpotent radical. Therefore, the Statements (a) and (b) follow immediately from Lemma 2.9. The proof of the theorem is completed.

Remark. (1) Theorem 6.5(1) generalizes slightly a result of Asashiba; see [2, (4.3)].

(2) Theorem 6.5(2) is essentially due to Gabriel; see [8]. It shows in particular that if G is torsion-free, then $\pi_\lambda : \mathrm{mod}^b\mathit{\Lambda} \to \mathrm{mod}^bA$ is a Galois covering if and only if it is dense.

Next, we shall study the functors between the derived categories of the module categories induced from the push-down functor. First of all, by Lemma 5.4, the G-action on $\mathrm{Mod}\mathit{\Lambda}$ induces a G-action on $D(\mathrm{Mod}\mathit{\Lambda})$.

6.6. Lemma. *Let $\mathit{\Lambda}$ be a locally bounded k-linear category with an action of a group G. If the G-action on $\mathit{\Lambda}$ is free, then $D^b(\mathrm{Mod}^b\mathit{\Lambda})$ and $D^b(\mathrm{mod}^b\mathit{\Lambda})$ are G-subcategories of $D(\mathrm{Mod}\mathit{\Lambda})$ with a locally bounded G-action.*

Proof. By Lemma 6.1, $\mathrm{Mod}^b\mathit{\Lambda}$ and $D^b(\mathrm{mod}^b\mathit{\Lambda})$ have enough $\mathrm{Mod}\mathit{\Lambda}$-projective objects. Thus, by Lemma 1.11, $D^b(\mathrm{Mod}^b\mathit{\Lambda})$ and $D^b(\mathrm{mod}^b\mathit{\Lambda})$ are full triangulated subcategories of $D(\mathrm{Mod}\mathit{\Lambda})$. Assume that the G-action on $\mathit{\Lambda}$ is free. By Lemma 6.2, $\mathrm{Mod}^b\mathit{\Lambda}$ and $\mathrm{mod}^b\mathit{\Lambda}$ are G-subcategories of $\mathrm{Mod}^b\mathit{\Lambda}$ with a locally bounded G-action. Hence, $D^b(\mathrm{Mod}^b\mathit{\Lambda})$ and $D^b(\mathrm{mod}^b\mathit{\Lambda})$ are G-subcategories of $D(\mathrm{Mod}\mathit{\Lambda})$. Since the

G-actions on $\mathrm{Mod}^b\Lambda$ and $\mathrm{mod}^b\Lambda$ are locally bounded, by Lemma 5.5, so are the G-actions on $D^b(\mathrm{Mod}^b\Lambda)$ and $D^b(\mathrm{mod}^b\Lambda)$. The proof of the lemma is completed.

Now, by Proposition 5.1, the push-down functor $\pi_\lambda : \mathrm{Mod}\Lambda \to \mathrm{Mod}A$ induces a commutative diagram of functors

$$\begin{tikzcd} C(\mathrm{Mod}\Lambda) \arrow[r, "P_\Lambda"] \arrow[d, "\pi_\lambda^C"'] & K(\mathrm{Mod}\Lambda) \arrow[r, "L_\Lambda"] \arrow[d, "\pi_\lambda^K"'] & D(\mathrm{Mod}\Lambda) \arrow[d, "\pi_\lambda^D"] \\ C(\mathrm{Mod}A) \arrow[r, "P_A"] & K(\mathrm{Mod}A) \arrow[r, "L_A"] & D(\mathrm{Mod}A), \end{tikzcd}$$

where the vertical functors are also called *push-down functors*. Moreover, the pull-up functor $\pi_\mu : \mathrm{Mod}A \to \mathrm{Mod}\Lambda$ induces a commutative diagram of functors:

$$\begin{tikzcd} C(\mathrm{Mod}A) \arrow[r, "P_A"] \arrow[d, "\pi_\mu^C"'] & K(\mathrm{Mod}A) \arrow[r, "L_A"] \arrow[d, "\pi_\mu^K"'] & D(\mathrm{Mod}A) \arrow[d, "\pi_\mu^D"] \\ C(\mathrm{Mod}\Lambda) \arrow[r, "P_\Lambda"] & K(\mathrm{Mod}\Lambda) \arrow[r, "L_\Lambda"] & D(\mathrm{Mod}\Lambda) \end{tikzcd}$$

where the vertical functors are also called *pull-up functors*.

Note that the functor $\pi_\lambda^D : D(\mathrm{Mod}\Lambda) \to D(\mathrm{Mod}A)$ sends $D^b(\mathrm{Mod}^b\Lambda)$ and $D^b(\mathrm{mod}^b\Lambda)$ into $D^b(\mathrm{Mod}^bA)$ and $D^b(\mathrm{mod}^bA)$, respectively. Restricting this functor, we obtain two functors $D^b(\mathrm{Mod}^b\Lambda) \to D^b(\mathrm{Mod}^bA)$ and $D^b(\mathrm{mod}^b\Lambda) \to D^b(\mathrm{mod}^bA)$ which, by abuse of notation, both are denoted by π_λ^D again.

6.7. Theorem. *Let Λ, A be locally bounded k-linear categories with G a group acting admissibly on Λ. Let $\pi : \Lambda \to A$ be a G-invariant Galois G-covering.*

(1) *The push-down functor $\pi_\lambda^D : D^b(\mathrm{Mod}^b\Lambda) \to D^b(\mathrm{Mod}^bA)$ is a G-precovering.*

(2) *The push-down functor $\pi_\lambda^D : D^b(\mathrm{mod}^b\Lambda) \to D^b(\mathrm{mod}^bA)$ is a G-precovering, and in case G is torsion-free, it has the following properties.*
   - (a) *If $M^\bullet \in D^b(\mathrm{mod}^b\Lambda)$ is indecomposable, then $\pi_\lambda^D(M^\bullet)$ is indecomposable.*
   - (b) *If $M^\bullet, N^\bullet \in D^b(\mathrm{mod}^b\Lambda)$ are indecomposable with $\pi_\lambda^D(M^\bullet) \cong \pi_\lambda^D(N^\bullet)$, then $N^\bullet \cong g{\cdot}M^\bullet$ for some $g \in G$.*

Proof. First of all, by Theorem 1.8, $D(\mathrm{Mod}\Lambda)$ has direct sums and is equipped with a G-action induced from the G-action on $\mathrm{Mod}\Lambda$. By Proposition 6.4, the two functors $\pi_\lambda : \mathrm{Mod}\Lambda \to \mathrm{Mod}A$ and $\pi_\mu : \mathrm{Mod}A \to \mathrm{Mod}\Lambda$ form a G-graded adjoint pair (π_λ, π_μ). By Proposition 5.7, the induced functors $\pi_\lambda^D : D(\mathrm{Mod}\Lambda) \to D(\mathrm{Mod}A)$ and $\pi_\mu^D : D(\mathrm{Mod}A) \to D(\mathrm{Mod}\Lambda)$ form a G-graded adjoint pair $(\pi_\lambda^D, \pi_\mu^D)$. On the other hand, by Lemma 6.6, $D^b(\mathrm{Mod}^b\Lambda)$ and $D^b(\mathrm{mod}^b\Lambda)$ are G-subcategories of $D(\mathrm{Mod}\Lambda)$ with a locally bounded G-action.

(1) Let $X^\bullet$, $Y^\bullet \in D^b(\mathrm{Mod}^b\Lambda)$. By Lemma 5.5(1), there exists a quasi-isomorphism $s^\bullet : P^\bullet \to X^\bullet$ in $C(\mathrm{Mod}^b\Lambda)$, where $P^\bullet$ is a bounded above-complex of projective Λ-modules such that $C(\mathrm{Mod}^b\Lambda)(P^\bullet, g{\cdot}Y^\bullet) = 0$, that is, $C(\mathrm{Mod}\Lambda)(P^\bullet, g{\cdot}Y^\bullet) = 0$, for all but finitely many $g \in G$. By Proposition 1.10, $P^\bullet$ is essential in the direct sum $\oplus_{g\in G}\, g{\cdot}Y^\bullet$ in $D(\mathrm{Mod}\Lambda)$, and so is $X^\bullet$ because $\tilde{s}^\bullet$ is an isomorphism in $D(\mathrm{Mod}\Lambda)$. That is, the G-action on $D^b(\mathrm{Mod}^b\Lambda)$ is $D(\mathrm{Mod}\Lambda)$-essential. By Theorem 2.12, we see that $\pi_\lambda^D : D^b(\mathrm{Mod}^b\Lambda) \to D^b(\mathrm{Mod}^bA)$ is a G-precovering.

(2) As argued above, we see that $\pi_\lambda^D : D^b(\mathrm{mod}^b\Lambda) \to D^b(\mathrm{mod}^bA)$ is a G-precovering. Suppose that G is torsion-free. By Lemma 2.2, the G-action on

$D^b(\mathrm{mod}^b\!\Lambda)$ is free. Since $\mathrm{mod}^b\!\Lambda$ is Hom-finite and has enough projective objects, it is well known that $D^b(\mathrm{mod}^b\!\Lambda)$ is Hom-finite. Moreover, since the idempotents in $D^b(\mathrm{mod}^b\!\Lambda)$ split; see [4, (2.10)], $D^b(\mathrm{mod}^b\!\Lambda)$ is Krull-Schmidt. Therefore, the endomorphism algebra of any indecomposable object in $D^b(\mathrm{mod}^b\!\Lambda)$ is local with a nilpotent radical. By Lemma 2.9, $\pi_\lambda^D : D^b(\mathrm{mod}^b\!\Lambda) \to D^b(\mathrm{mod}^b\!A)$ has the properties stated in Statements (a) and (b). The proof of the theorem is completed.

Remark. (1) Theorem 6.7 says in particular that if G is a torsion-free group, then $\pi_\lambda^D : D^b(\mathrm{mod}^b\!\Lambda) \to D^b(\mathrm{mod}^b\!A)$ is a Galois G-covering if and only if it is dense.

(2) The same result hold for the push-down functors between the complex categories and between the homotopy categories.

We conclude this section with a result which will be used in the next section. A morphism $f : M \to N$ in $\mathrm{Mod}\,\Lambda$ is called *radical* if the image of f is contained in the radical of N. More generally, a complex over $\mathrm{Mod}\,\Lambda$ is called *radical* if all its differentials are radical morphisms. For $* \in \{-, \{-, b\}\}$, we shall denote by $RC^*(\mathrm{Proj}\Lambda)$ the full subcategory of $C^*(\mathrm{Proj}\Lambda)$ generated by the radical complexes.

6.8. Lemma. *Let Λ be a locally bounded k-linear category. If $X^\bullet \in D^b(\mathrm{Mod}^b\!\Lambda)$ is indecomposable, then there exists some indecomposable complex $P^\bullet \in RC^-(\mathrm{Proj}\Lambda)$ such that $X^\bullet \cong P^\bullet$ in $D(\mathrm{Mod}^b\!\Lambda)$ in such a way that $P^\bullet \in RC^-(\mathrm{proj}\Lambda)$ whenever $X^\bullet \in D^b(\mathrm{mod}^b\!\Lambda)$.*

Proof. Let $X^\bullet \in D^b(\mathrm{Mod}^b\!\Lambda)$ be indecomposable. Making use of Lemma 6.1, we can find $P^\bullet \in RC^-(\mathrm{Proj}\,\Lambda)$ such that $X^\bullet \cong P^\bullet$ in $D(\mathrm{Mod}^b\!\Lambda)$; see [10, (7.5)], where $P^\bullet \in RC^-(\mathrm{proj}\Lambda)$ whenever $X^\bullet \in D^b(\mathrm{mod}^b\!\Lambda)$. By Lemmas 1.11 and 1.9, we obtain

$$\mathrm{End}_{D^b(\mathrm{Mod}^b\Lambda)}(X^\bullet) \cong \mathrm{End}_{D(\mathrm{Mod}^b\Lambda)}(X^\bullet) \cong \mathrm{End}_{D(\mathrm{Mod}^b\Lambda)}(P^\bullet) = \mathrm{End}_{K(\mathrm{Mod}^b\Lambda)}(P^\bullet).$$

Since idempotents in $D^b(\mathrm{Mod}^b\!\Lambda)$ split, $\mathrm{End}_{D^b(\mathrm{Mod}^b\Lambda)}(X^\bullet)$ has no proper idempotent, and neither does $\mathrm{End}_{K(\mathrm{Mod}^b\Lambda)}(P^\bullet)$. This implies that $P^\bullet$ is indecomposable in $K(\mathrm{Mod}^b\!\Lambda)$. Hence, $P^\bullet$ is indecomposable in $RC^-(\mathrm{Proj}\,\Lambda)$ because no non-zero complex in $RC^-(\mathrm{Proj}\,\Lambda)$ vanishes in $K(\mathrm{Mod}^b\!\Lambda)$. The proof of the lemma is completed.

7. The radical squared zero case

The objective of this section is to show how the covering technique can be applied to study derived categories of modules over a locally bounded linear category with radical squared zero.

We start with some combinatorial consideration. Throughout this section, let $Q = (Q_0, Q_1)$ be a connected locally finite quiver. Given a walk w in Q, we define its *degree* $\partial(w)$ in the following manner: if w is a trivial path, an arrow, or the inverse of an arrow, then $\partial(w) = 0, 1$, or -1, respectively, and this definition is extended to all walks in Q by $\partial(uv) = \partial(u) + \partial(v)$. In particular, the degree of a path is equal to its length.

7.1. Definition. A quiver Q is called *gradable* if all the closed walks in Q are of degree zero.

Remark. (1) A quiver without (oriented or non-oriented) cycles is evidently gradable. On the other hand, a gradable quiver contains no oriented cycle.

(2) Suppose that Q is gradable. Then, for any $x, y \in Q_0$, all the walks from x to y have the same degree, written as $d(x, y)$. Thus, each vertex a in Q determines a graduation on Q as follows. For each $i \in \mathbb{Z}$, denote by $Q^{(a,i)}$ the set of vertices x for which $d(a, x) = i$. In this way, Q_0 is the disjoint union of the $Q^{(a,i)}$, $i \in \mathbb{Z}$; and each arrow in Q is of the form $x \to y$ with $x \in Q^{(a,i)}$ and $y \in Q^{(a,i+1)}$ for some i. Moreover, if b is antoher vertex, then $Q^{(b,i)} = Q^{(a,i+d(a,b))}$ for all $i \in \mathbb{Z}$, where $d(a, b)$ is a constant.

Associated with a quiver Q, we define a new quiver $Q^{\mathbb{Z}}$ as follows: the vertices are the pairs (a, i) with $a \in Q_0$ and $i \in \mathbb{Z}$, and the arrows are $(\alpha, i) : (a, i) \to (b, i+1)$, where $i \in \mathbb{Z}$ and $\alpha : a \to b$ is an arrow in Q. As shown below, $Q^{\mathbb{Z}}$ is gradable.

7.2. Lemma. *Let Q be a quiver. If (a, m), $(b, n) \in Q^{\mathbb{Z}}$, then $Q^{\mathbb{Z}}$ has a walk from (a, m) to (b, n) if and only if Q has a walk of degree $n - m$ from a to b, and in this case, all the walks in $Q^{\mathbb{Z}}$ from (a, m) to (b, n) are of degree $n - m$.*

Proof. Let (a, m), (b, n) be vertices in $Q^{\mathbb{Z}}$. First, let w be a non-trivial walk in $Q^{\mathbb{Z}}$ from (a, m) to (b, n). We may assume that $w = (\alpha_r, m_r)^{d_r} \cdots (\alpha_1, m_1)^{d_1}$, where $r \geq 1$, $\alpha_i \in Q_1$, $m_i \in \mathbb{Z}$, and $d_i = \pm 1$. For $1 \leq i \leq r$, write $b_i = e(\alpha_i^{d_i})$, then $e((\alpha_i, m_i)^{d_i}) = (b_i, n_i)$ with $n_i \in \mathbb{Z}$. By definition, $n_i = m + d_1 + \cdots + d_i$, for $i = 1, \ldots, r$. In particular, $n = n_r = m + d_1 + \cdots + d_r = m + \partial(w)$. Thus $\partial(w) = n - m$, and $\alpha_r^{d_r} \cdots \alpha_1^{d_1}$ is a walk in Q from a to b of degree $n - m$. From this, we deduce the necessity of the first part and the second part of the lemma.

It remains to prove the sufficiency of the first part of the lemma. Indeed, let v be a walk of degree $n - m$ in Q from a to b. If v is trivial, then $(a, m) = (b, n)$, and hence, the trivial path in $Q^{\mathbb{Z}}$ at (a, m) is of degree $n - m$. Otherwise, we can write $v = \beta_s^{d_s} \cdots \beta_1^{d_1}$, where $s \geq 1$, $\beta_i \in Q_1$ and $d_i = \pm 1$. Write $a_0 = a$ and $m_0 = m$, and write $a_i = e(\beta_i^{d_i})$ and $m_i = m_0 + d_1 + \cdots + d_i$, for $i = 1, \ldots, s$. Moreover, define $n_i = m_{i-1}$ if $d_i = 1$ and $n_i = m_i$ if $d_i = -1$, for $i = 1, \ldots, s$. Then, $(\beta_i, n_i)^{d_i} \cdots (\beta_1, n_1)^{d_1}$ is a walk in $Q^{\mathbb{Z}}$ from (a_0, m_0) to (a_i, m_i), for $i = 1, \ldots, s$. In particular, since $n = m_s$, we obtain a walk $(\beta_s, n_s)^{d_s} \cdots (\beta_1, n_1)^{d_1}$ in $Q^{\mathbb{Z}}$ from (a, m) to (b, n) of degree $n - m$. The proof of the lemma is completed.

We shall need the following notion in order to describe the connected components of $Q^{\mathbb{Z}}$.

7.3. Definition. Let Q be a connected quiver. The *grading period* of Q is defined to be 0 in case Q is gradable, and otherwise, to be the minimum of the positive degrees of closed walks in Q.

Observe that $Q^{\mathbb{Z}}$ has an automorphism σ, called the *translation*, sending (a, i) to $(a, i+1)$, and (α, i) to $(\alpha, i+1)$, where $a \in Q_0$, $\alpha \in Q_1$ and $i \in \mathbb{Z}$. The group generated σ will be called the *translation group* of $Q^{\mathbb{Z}}$. Moreover, for an integer $s \geq 0$, we write $\mathbb{Z}_s = \mathbb{Z}$ if $s = 0$; and $\mathbb{Z}_s = \{0, 1, \cdots, s-1\}$ if $s > 0$.

7.4. Lemma. *Let Q be a connected quiver of grading period r. Let $\mathcal{C}$ be a connected component, and σ the translation, of $Q^{\mathbb{Z}}$.*

(1) *If $m, n \in \mathbb{Z}$, then $\sigma^m(\mathcal{C}) = \sigma^n(\mathcal{C})$ if and only if $m \equiv n \,(\mathrm{mod}\, r)$.*
(2) *The distinct connected components of $Q^{\mathbb{Z}}$ are $\sigma^n(\mathcal{C})$ with $n \in \mathbb{Z}_r$.*
(3) *If w is a closed walk in Q, then $\partial(w)$ is a multiple of r.*

Proof. Observe first that if η is an automorphism of $Q^{\mathbb{Z}}$, then $\eta(\mathcal{C})$ is a connected component of $Q^{\mathbb{Z}}$, and hence, $\mathcal{C} = \eta(\mathcal{C})$ or $\mathcal{C} \cap \eta(\mathcal{C}) = \emptyset$. Fix a vertex (a, n_0) in $\mathcal{C}$, where $a \in Q_0$ and $n_0 \in \mathbb{Z}$.

(1) By definition, Q contains a closed walk of degree r which, we may assume, is from a to a. By Lemma 7.2, $Q^{\mathbb{Z}}$ has a walk from (a, n_0) to $(a, n_0 + r)$. Therefore, $\sigma^r(\mathcal{C}) = \mathcal{C}$. As a consequence, $\sigma^m(\mathcal{C}) = \sigma^n(\mathcal{C})$ whenever $m \cong n \,(\mathrm{mod}\, r)$. Conversely, assume that $\sigma^m(\mathcal{C}) = \sigma^n(\mathcal{C})$ with $m \not\equiv n (\mathrm{mod}\, r)$. In particular, $Q^{\mathbb{Z}}$ has walk from $(a, n_0 + m)$ to $(a, n_0 + n)$. By Lemma 7.2, Q contains a walk from a to a of degree $m - n$. In particular, Q is not gradable, and hence $r > 0$. If m_1, n_1 are the remainders of m, n divided by r respectively, then $\sigma^{m_1}(\mathcal{C}) = \sigma^{n_1}(\mathcal{C})$. Then, Q contains a walk from a to a of degree $n_1 - m_1$, which is a contradiction since $0 < |m_1 - n_1| < r$. This establishes Statement (1).

(2) By Statement (1), the $\sigma^n(\mathcal{C})$ with $n \in \mathbb{Z}_r$ are distinct connected components of $Q^{\mathbb{Z}}$. Let (x, i) be an arbitrary vertex in $Q^{\mathbb{Z}}$. Being connected, Q has a walk from a to x, say, of degree d. By Lemma 7.2, $Q^{\mathbb{Z}}$ has a walk from $(a, i - d)$ to (x, i). Hence $(x, i) \in \sigma^{i-d-n_0}(\mathcal{C}) = \sigma^t(\mathcal{C})$, where $t \in \mathbb{Z}_r$ with $i - d - n_0 \equiv t \,(\bmod\, r)$. This establishes Statement (2).

(3) Let w be a closed walk of degree s in Q, which we may assume is from a to a. By Lemma 7.2, Q has a walk from $(a, 0)$ to (a, s). Suppose that $(a, 0)$ lies in a connected component $\mathcal{D}$ of $Q^{\mathbb{Z}}$. Then (a, s) lies in the connected component $\sigma^s(\mathcal{D})$, and hence, $\mathcal{D} = \sigma^s(\mathcal{D})$. In view of Statement (1), s is a multiple of r. The proof of the lemma is completed.

Remark. By Lemma 7.4, restricting σ^r to $\mathcal{C}$ yields an automorphism σ_c, called the *translation*, of $\mathcal{C}$. If Σ is the translation group of $Q^{\mathbb{Z}}$, then $\Sigma_c = \{g \in \Sigma \mid g(\mathcal{C}) = \mathcal{C}\}$ is generated by σ_c, which we shall call the *translation group* of $\mathcal{C}$.

It is evident that we have a quiver-morphism $q : Q^{\mathbb{Z}} \to Q$, sending (a, i) and (α, i) to a and α respectively. We shall call it the *canonical quiver-morphism.*

7.5. Theorem. *Let Q be a connected locally finite quiver. Let Σ be the automorphism group and $\mathcal{C}$ a connected component of $Q^{\mathbb{Z}}$.*

(1) *The canonical morphism $q : Q^{\mathbb{Z}} \to Q$ is a Galois Σ-covering.*
(2) *Restricting q yields a Galois Σ_c-covering $q_c : \mathcal{C} \to Q$.*
(3) *The quiver Q is gradable if and only if q_c is an isomorphism.*
(4) *If Q is finite, then Q is gradable if and only if $\mathcal{C}$ is finite.*
(5) *If $\phi : \Gamma \to Q$ is a quiver-covering with Γ gradable, then there exists a quiver-covering $\psi : \Gamma \to \mathcal{C}$ such that $\phi = \psi \circ q_c$.*

Proof. By Lemma 7.4, $\Sigma_c = \{\sigma^{ri} \mid i \in \mathbb{Z}\}$, where r is the grading period of Q.

(1) It is evident that Σ acts freely on $Q^{\mathbb{Z}}$, the action of q on the vertices is surjective, and $q \circ \sigma^i = q$ for any $i \in \mathbb{Z}$. Moreover, if $q(a, i) = (b, j)$, then $b = a$ and $(b, j) = \sigma^{j-i}(a, i)$. For any vertex (a, i) in $Q^{\mathbb{Z}}$, the arrows in $Q^{\mathbb{Z}}$ starting in (a, i) are the arrows (α, i), where α ranges over the arrows in Q starting in a, and the the arrows in $Q^{\mathbb{Z}}$ ending in (a, i) are the arrows $(\beta, i - 1)$, where β ranges over the arrows in Q ending in a. This shows that q is a Galois Σ-covering.

(2) Since Q is connected, Statement (2) follows immediately from Lemma 4.4.

(3) If q_c is an isomorphism, then Q is gradable because $\mathcal{C}$ is gradable. Conversely, if Q is gradable, then $r = 0$, and hence Σ_c is trivial. Being a Galois Σ_c-covering, q_c is an isomorphism.

(4) Suppose that Q is finite. By Lemma 7.4(1), $\sigma^{ir}(\mathcal{C}) = \mathcal{C}$ for $i \in \mathbb{Z}$. Let (a, m) be a vertex in $\mathcal{C}$. Then, $(a, m + ri) \in \mathcal{C}$ for all $i \in \mathbb{Z}$. If $r > 0$, then $\mathcal{C}$ is infinite. Otherwise, $\mathcal{C} \cong Q$ by Statement (3), and hence, $\mathcal{C}$ is finite.

(5) Let $\phi : \Gamma \to Q$ be a quiver-covering with Γ being gradable. Since Q is connected, by Lemma 4.4(1), the restriction of ϕ to any connected component of Γ is a quiver-covering. Thus, we may assume that Γ is connected. Choose $x^* \in \Gamma_0$. Then $y = \phi(x^*) \in Q$, and there exists some integer s with $y^\star = (y, s) \in \mathcal{C}_0$. Let $z \in \Gamma_0$, and write $n_z = d(x^*, z)$. Choose a walk $\tilde{w}$ in Γ from x^* to z, which is necessarily of degree n_z. Then $\phi(\tilde{w})$ is a walk in Q from y to $\phi(z)$ of degree n_z. By Lemma 7.2, $Q^{\mathbb{Z}}$ contains a walk from $y^\star = (y, s)$ to $(\phi(z), s + n_z)$. In particular, $(\phi(z), s + n_z) \in \mathcal{C}$. Define $\psi_0(z) = (\phi(z), s + n_z)$. If $(b, i) \in \mathcal{C}_0$ with $b \in Q_0$ then, by Lemma 7.2, Q has a walk w from y to b of degree $i - s$. Since ϕ is a quiver-covering, Γ contains a walk $\tilde{u}$ from x^* to some vertex b^* such that $\phi(\tilde{u}) = w$. Observing that $\partial(\tilde{u}) = \partial(w) = i - s$, we see that $\psi_0(b^*) = (b, i)$. This yields a surjection $\psi_0 : \Gamma_0 \to \mathcal{C}_0$. Next, let $\alpha : z \to z_1$ be an arrow in Γ. Then $n_{z_1} = n_z + 1$. Since $\phi(\alpha) : \phi(z) \to \phi(z_1)$ is an arrow in Q, we obtain an arrow $(\phi(\alpha), n_z) : (\phi(z), n_z) \to (\phi(z_1), n_{z_1})$ in $\mathcal{C}$. Define $\psi_1(\alpha) = (\phi(\alpha), n_z)$. This gives rise to a quiver-morphism $\psi : \Gamma \to \mathcal{C}$ such that $q_c \circ \psi = \phi$. Since ϕ, q_c are both quiver-coverings, so is ψ. The proof of the theorem is completed.

Remark. By Lemma 7.2(1), the connected components of $Q^{\mathbb{Z}}$ are pairwise isomorphic. If $\mathcal{C}$ is such a component, due to the property stated in Theorem 7.5(5), we call $q_c : \mathcal{C} \to Q$ a *minimal gradable covering* of Q.

Let k be a field. Recall that the *path category* kQ of Q over k is a skeletal k-linear category in which the objects are the vertices in Q; and a morphism space $(kQ)(x, y)$, with $x, y \in Q_0$, has $Q(x, y)$ as a k-basis. We shall be interested in the following locally bounded k-linear category

$$A = kQ/(kQ^+)^2$$

with $\mathrm{rad}^2(A) = 0$, where kQ^+ is the ideal in kQ generated by the arrows. Sometimes, it will be convenient to regard A as a k-algebra with a complete set of pairwise orthogonal primitive idempotents $\{e_a = \bar{\varepsilon}_a \mid a \in Q_0\}$, where $\bar{u} = u + (kQ^+)^2 \in A$, for $u \in kQ$. Accordingly, a left module over the category A will be identified with a left module over the algebra A which is unitary with respect to $\{e_a \mid a \in Q_0\}$. In particular, for each $a \in Q_0$, we have an indecomposable projective left A-module $P[a] = Ae_a$; and for each arrow $\alpha : a \to b$ in Q, we have an A-linear morphism $P[\alpha] : P[b] \to P[a]$, the right multiplication by $\bar{\alpha}$; and for a trivial path ε_a, we write $P[\varepsilon_a] = 1_{P[a]}$. All tensor products are over the base field k.

7.6. Lemma. *Let $A = kQ/(kQ^+)^2$, where Q is a connected locally finite quiver. If $a, b \in Q_0$ and U, V are k-spaces, then an A-linear morphism $\phi : P[b] \otimes U \to P[a] \otimes V$ can be uniquely written as*

$$\phi = \textstyle\sum_{\gamma \in Q_{\le 1}(a,\, b)} P[\gamma] \otimes f_\gamma, \ \textit{where}\ f_\gamma \in \mathrm{Hom}_k(U, V).$$

Moreover, ϕ is radical if and only if $\phi = \sum_{\alpha \in Q_1(a,b)} P[\alpha] \otimes f_\alpha,\ f_\alpha \in \mathrm{Hom}_k(U, V)$.

Proof. Let U, V be k-spaces. Suppose that $\phi : P[b] \otimes U \to P[a] \otimes V$, with $a, b \in Q_0$, is an A-linear morphism. Observe that $\phi(e_b \otimes U) \subset e_b A e_a \otimes V$. Since $\mathrm{rad}^2(A) = 0$, we have $e_b A e_a \otimes V = \oplus_{\gamma \in Q_{\le 1}(a,\, b)} k\bar{\gamma} \otimes V$. Thus, for each $u \in U$, the element

$\phi(e_b \otimes u)$ is uniquely written as

$$(*) \qquad \phi(e_b \otimes u) = \textstyle\sum_{\gamma \in Q_{\le 1}(a,\, b)} \bar{\gamma} \otimes v_\gamma,\ v_\gamma \in V.$$

This yields, for each $\gamma \in Q_{\le 1}(a,\, b)$, a k-linear map $f_\gamma : U \to V : u \mapsto v_\gamma$. Being A-linear, $\phi = \sum_{\gamma \in Q_1(a,\, b)} P[\gamma] \otimes f_\gamma$, and this expression is unique by the uniqueness of the v_γ in $(*)$.

If $\phi = \sum_{\alpha \in Q_1(a,\, b)} P[\alpha] \otimes f_\alpha$, then ϕ is clearly radical. Otherwise, $b = a$ and $f_{\varepsilon_b}(u) \neq 0$ for some $u \in U$. Thus $\phi(e_b \otimes u) = e_b \otimes f_{\varepsilon_b}(u)$, which is not in the radical of $P[a] \otimes V$, that is, ϕ is not radical. The proof of the lemma is completed.

The following result is essential for our investigation.

7.7. Proposition. *Let $A = kQ/(kQ^+)^2$, where Q is a connected gradable locally finite quiver. Let $P^\bullet$ be an indecomposable complex in $RC^-(\operatorname{Proj} A)$. If $a \in Q_0$, then there exists an integer s such that, for every integer i, we have*

$$P^i = \oplus_{x \in Q^{(a,s-i)}} P[x] \otimes V_x^i, \quad \textit{where } V_x^i \in \operatorname{Mod} k.$$

Proof. If $P^n \neq 0$, then we may assume that $P^n = \oplus_{x \in \Omega(n)} P[x] \otimes V_x^n$, where $\Omega(n) \subseteq Q_0$ and the V_x^n are non-zero k-spaces; and if $d_P^n \neq 0$, then we write

$$d_P^n = (d_P^n(y,x))_{(y,x) \in \Omega(n+1) \times \Omega(n)},$$

where $d_P^n(y,x) : P[x] \otimes V_x^n \to P[y] \otimes V_y^{n+1}$ is a radical A-linear morphism which, by Lemma 7.6, can be written as

$$d_P^n(y,x) = \textstyle\sum_{\alpha \in Q_1(y,x)} P[\alpha] \otimes f_\alpha^n, \quad \text{where } f_\alpha^n \in \operatorname{Hom}_k(V_y^n, V_x^{n+1}).$$

Fix $a \in Q_0$. Let $x \in \Omega(n)$ and $y \in \Omega(m)$ be distinct vertices, where n, m are integers. Then $x \in Q^{(a,s-n)}$ and $y \in Q^{(a,t-m)}$, where $s = n + d(a,x)$ and $t = m + d(a,y)$. In particular, $d(x,y) = (t-m)-(s-n)$. Now, $P[x] \otimes V_x^n$ is a non-zero direct summand of P^n, while $P[y] \otimes V_y^m$ is a non-zero direct summand of P^m. Since $P^\bullet$ is indecomposable, there exist integers $n = n_0, n_1, \ldots, n_r = m$ with $n_{i+1} = n_i \pm 1$; and vertices $x = y_0, y_1, \cdots, y_r = y$ with $y_i \in \Omega(n_i)$ such that $d_P^{n_i}(y_{i+1}, y_i) \neq 0$ in case $n_{i+1} = n_i + 1$ or $d_P^{n_{i+1}}(y_i, y_{i+1}) \neq 0$ in case $n_{i+1} = n_i - 1$. Using the above description of the maps d_P^n, we obtain a walk $w = \alpha_r^{n_{r-1}-n_r} \cdots \alpha_2^{n_1-n_2} \alpha_1^{n_0-n_1}$ in Q from x to y. This yields

$$(t-s) + (n-m) = d(x,y) = \partial(w) = \textstyle\sum_{i=0}^{r-1} (n_{i+1} - n_i) = n - m,$$

and consequently, $t = s$. Therefore, $\Omega(i) \subseteq Q^{(a,s-i)}$ for every i such that $P^i \neq 0$. Setting $V_z^j = 0$ in case $P^j = 0$ or $z \notin \Omega(j)$, we obtain $P^i = \oplus_{x \in Q^{(a,s-i)}} P[x] \otimes V_x^i$, for every i. The proof of the proposition is completed.

For the rest of the paper, we fix a connected component $\tilde{Q}$ of $Q^{\mathbb{Z}}$, and put

$$\tilde{A} = k\tilde{Q}/(k\tilde{Q}^+)^2,$$

a connected locally bounded k-linear category with $\operatorname{rad}^2(\tilde{A}) = 0$. Let G be the translation group of $\tilde{Q}$. By linearity, the G-action on $\tilde{Q}$ induces a G-action on $\tilde{A}$. Moreover, the minimal gradable covering $\pi : \tilde{Q} \to Q$ induces a k-linear functor $\tilde{A} \to A$ which, for the simplicity of notation, will be denoted by π again.

7.8. LEMMA. *Let $A = kQ/(kQ^+)^2$ and $\tilde{A} = k\tilde{Q}/(k\tilde{Q}^+)^2$, where Q is a connected locally finite quiver and $\tilde{Q}$ is a connected component of $Q^{\mathbb{Z}}$. The minimal gradable covering $\pi : \tilde{Q} \to Q$ induces a G-invariant Galois G-covering $\pi : \tilde{A} \to A$, where G is the translation group of $\tilde{Q}$.*

Proof. First of all, it is easy to see that the G-action on $\tilde{A}$ is admissible and the functor $\pi : \tilde{A} \to A$ is G-invariant and satisfies the conditions (1), (2) and (3) stated in Definition 2.8. It suffices to show, for $x^*, y^* \in \tilde{Q}_0$, that

$$\pi_{x^*,y^*} : \oplus_{g\in G}\tilde{A}(x^*, g\cdot y^*) \to A(\pi(x^*), \pi(y^*)) : (u_g)_{g\in G} \to \sum\nolimits_{g\in G} \pi(u_g)$$

is an isomorphism. Indeed, write $\pi(x^*) = x$ and $\pi(y^*) = y$. By definition, $A(x, y)$ has a k-basis $\mathcal{B} = \{\bar{\eta} \mid \alpha \in Q_{\le 1}(x, y)\}$, and $\tilde{A}(x^*, g \cdot y^*)$ with $g \in G$ has a k-basis $\mathcal{B}_g = \{\bar{\xi} \mid \xi \in \tilde{Q}_{\le 1}(x^*, g \cdot y^*)\}$. Since $\pi : \tilde{Q} \to Q$ is a Galois G-covering, it induces a bijection from $\cup_{g\in G}\tilde{Q}_{\le 1}(x^*, g \cdot y^*)$ onto $Q_{\le 1}(x, y)$. As a consequence, π_{x^*,y^*} is a k-linear isomorphism. The proof of the lemma is completed.

In the sequel, the G-invariant Galois G-covering $\pi : \tilde{A} \to A$ stated in Lemma 7.8 will be called a *minimal gradable covering* of A. Recall that the push-down functor $\pi_\lambda : \mathrm{Mod}\tilde{A} \to \mathrm{Mod}A$ induces a functor $\pi^C_\lambda : C(\mathrm{Mod}\tilde{A}) \to C(\mathrm{Mod}A)$.

7.9. LEMMA. *Let $A = kQ/(kQ^+)^2$ and $\tilde{A} = k\tilde{Q}/(k\tilde{Q}^+)^2$, where Q is a connected locally finite quiver and $\tilde{Q}$ is a connected component of $Q^{\mathbb{Z}}$. Let $\pi : \tilde{A} \to A$ be the minimal gradable covering. If $P^\bullet \in RC^{-,b}(\mathrm{Proj}A)$, then $P^\bullet \cong \pi^C_\lambda(L^\bullet)$, where $L^\bullet \in RC^{-,b}(\mathrm{Proj}\tilde{A})$, which lies in $RC^{-,b}(\mathrm{proj}\tilde{A})$ in case $P^\bullet \in RC^{-,b}(\mathrm{proj}A)$.*

Proof. By Theorem 7.5(3), we may assume that Q is of grading period $r > 0$. Let Σ be the translation group of $Q^{\mathbb{Z}}$, which is generated by the translation σ of $Q^{\mathbb{Z}}$. By Theorem 7.5, we have a canonical Galois Σ-covering $q : Q^{\mathbb{Z}} \to Q$. Consider the locally bounded k-linear category $A^{\mathbb{Z}} = kQ^{\mathbb{Z}}/I^{\mathbb{Z}}$, where $I^{\mathbb{Z}}$ is the square of the ideal in $kQ^{\mathbb{Z}}$ generated by the arrows. In view of Lemma 7.8, we see that $q : Q^{\mathbb{Z}} \to Q$ induces a Σ-invariant Galois Σ-covering $q : A^{\mathbb{Z}} \to A$. The push-down functor $q_\lambda : \mathrm{Mod}A^{\mathbb{Z}} \to \mathrm{Mod}A$ induces an exact functor $q^C_\lambda : C(\mathrm{Mod}A^{\mathbb{Z}}) \to C(\mathrm{Mod}A)$.

Let $P^\bullet \in RC^{-,b}(\mathrm{Proj}A)$. For each $i \in \mathbb{Z}$, write $P^i = \oplus_{x\in Q_0}P[x] \otimes V^i_x$, where V^i_x is a k-space; and $d^i_P = (d^i_P(y, x))_{(y,x)\in Q_0\times Q_0}$, where $d^i_P(y, x)$ is a radical A-linear map from $P[x] \otimes V^i_x$ to $P[y] \otimes V^{i+1}_y$. By Lemma 7.6,

$$d^i_P(y, x) = \sum\nolimits_{\alpha\in Q_1(y,x)}P[\alpha] \otimes f^i_\alpha, \text{ where } f^i_\alpha \in \mathrm{Hom}_k(V^i_x, V^{i+1}_y).$$

Define a complex $X^\bullet \in RC^-(\mathrm{Proj}A^{\mathbb{Z}})$ by setting $X^i = \oplus_{x\in Q_0} P[(x, -i)] \otimes V^i_x$ and $d^i_X = (d^i_X(y, x))_{(y,x)\in Q_0\times Q_0} : X^i \to X^{i+1}$, where

$$d^i_X(y, x) = \sum\nolimits_{\alpha\in Q_1(y,x)}P[(\alpha, -i-1)] \otimes f^i_\alpha : P[(x, -i)] \otimes V^i_x \to P[(y, -i-1)] \otimes V^{i+1}_y.$$

By Lemma 6.3(2), $q^C_\lambda(X^\bullet) = P^\bullet$.

On the other hand, it follows from Lemma 7.4 that the connected components of $Q^{\mathbb{Z}}$ are $\mathcal{C}_j = \sigma^j(\tilde{Q})$, $j = 0, \cdots, r-1$. Thus, we may write $X^\bullet = \oplus^{r-1}_{j=0}X^\bullet_j$, where $X^\bullet_j \in RC^{-,b}(\mathrm{Proj}A^{\mathbb{Z}})$ is supported by $\mathcal{C}_j$. Clearly, there exist complexes $L^\bullet_j \in RC^-(\mathrm{Proj}\tilde{A})$ such that $X^\bullet_j = \sigma^j \cdot L^\bullet_j$, and hence, $q^C_\lambda(X^\bullet_j) \cong q^C_\lambda(L^\bullet_j)$, for $j = 0, \ldots, r-1$. Thus, $L^\bullet = \oplus^{r-1}_{j=0} L^\bullet_j \in RC^-(\mathrm{Proj}\tilde{A})$ is such that

$$q^C_\lambda(L^\bullet) \cong \oplus^{r-1}_{j=0} q^C_\lambda(L^\bullet_j) \cong \oplus^{r-1}_{j=0} q^C_\lambda(X^\bullet_j) \cong q^C_\lambda(X^\bullet) \cong P^\bullet.$$

Since $\pi : \tilde{Q} \to Q$ is the restriction of q, we see that $\pi_\lambda^C : C(\mathrm{Mod}\tilde{A}) \to C(\mathrm{Mod}A)$ is the restriction of q_λ^C. This yields $\pi_\lambda^C(L^\cdot) = q_\lambda^C(L^\cdot) \cong P^\cdot$. Moreover, since π_λ^C is exact and faithful, $L^\cdot \in RC^{-,b}(\mathrm{Proj}\,\tilde{A})$.

Finally, if $P^\cdot \in RC^{-,b}(\mathrm{proj}\,A)$, then the V_x^i are finite dimensional for all $x \in Q_0$ and $i \in \mathbb{Z}$. Hence, by our construction, $X^\cdot \in RC^{-}(\mathrm{proj}\,A^{\mathbb{Z}})$, and as a consequence, $L^\cdot \in RC^{-,b}(\mathrm{proj}\,\tilde{A})$. The proof of the lemma is completed.

We are now ready to have the main result of this section.

7.10. Theorem. *Let $A = kQ/(kQ^+)^2$ and $\tilde{A} = k\tilde{Q}/(k\tilde{Q}^+)^2$, where Q is a connected locally finite quiver and $\tilde{Q}$ is a connected component of $Q^{\mathbb{Z}}$. Let $\pi : \tilde{A} \to A$ be the minimal gradable covering, and G the translation group of $\tilde{Q}$.*

(1) *The push-down functor $\pi_\lambda^D : D^b(\mathrm{mod}^b\tilde{A}) \to D^b(\mathrm{mod}^b A)$ is a Galois G-covering.*

(2) *The push-down functor $\pi_\lambda^D : D^b(\mathrm{Mod}^b\tilde{A}) \to D^b(\mathrm{Mod}^b A)$ is a Galois G-covering.*

Proof. By Proposition 7.8, the functor $\pi : \tilde{A} \to A$ is a G-invariant Galois G-covering. By Theorem 7.5(3), we may assume that Q is of grading period $r > 0$. Then, the translation ρ of $\tilde{Q}$ is of infinite order. In particular, G is torsion-free. The G-actions on $D^b(\mathrm{Mod}^b\tilde{A})$ and $D^b(\mathrm{mod}^b\tilde{A})$ are locally bounded by Lemma 6.6, and free by Lemma 2.2. By Theorem 6.7, both $\pi_\lambda^D : D^b(\mathrm{mod}^b\tilde{A}) \to D^b(\mathrm{mod}^b A)$ and $\pi_\lambda^D : D^b(\mathrm{Mod}^b\tilde{A}) \to D^b(\mathrm{Mod}^b A)$ are G-precoverings.

(1) By Theorem 6.7(2), $\pi_\lambda^D : D^b(\mathrm{mod}^b\tilde{A}) \to D^b(\mathrm{mod}^b A)$ satisfies the conditions (2) and (3) stated in Definition 2.8. Let $X^\cdot \in D^b(\mathrm{mod}^b A)$ be indecomposable. By Lemma 6.8, $X^\cdot \cong P^\cdot$ in $D(\mathrm{mod}^b A)$, where $P^\cdot$ is an indecomposable object in $RC^{-,b}(\mathrm{proj}\,\tilde{A})$. By Lemma 7.9, $P^\cdot \cong \pi_\lambda^C(L^\cdot)$ for some $L^\cdot \in RC^{-,b}(\mathrm{proj}\,\tilde{A})$, and then, $P^\cdot \cong \pi_\lambda^D(L^\cdot)$ in $D(\mathrm{mod}^b\tilde{A})$. It is well known that there exists some $Y^\cdot \in C^b(\mathrm{mod}^b\tilde{A})$ such that $L^\cdot \cong Y^\cdot$ in $D(\mathrm{mod}^b\tilde{A})$. This yields the following isomorphisms

$$\pi_\lambda^D(Y^\cdot) \cong \pi_\lambda^D(L^\cdot) \cong P^\cdot \cong X^\cdot$$

in $D(\mathrm{mod}^b A)$. Since $\pi_\lambda^D(Y^\cdot)$ and $X^\cdot$ are bounded, we obtain $\pi_\lambda^D(Y^\cdot) \cong X^\cdot$ in $D^b(\mathrm{mod}^b A)$; see, for example, [10, (6.15)]. That is, $\pi_\lambda^D : D^b(\mathrm{mod}^b\tilde{A}) \to D^b(\mathrm{mod}^b A)$ is almost dense, and hence, a Galois G-covering.

(2) As argued above, the functor $\pi_\lambda^D : D^b(\mathrm{Mod}^b\tilde{A}) \to D^b(\mathrm{Mod}^b A)$ is almost dense. We claim that the G-action on $D^b(\mathrm{Mod}^b\tilde{A})$ is directed. Indeed, let $X^\cdot, Y^\cdot$ be indecomposable objects in $D^b(\mathrm{Mod}^b\tilde{A})$. By Lemma 6.8, there exist indecomposable complexes $P^\cdot, L^\cdot \in RC^{-}(\mathrm{Proj}\,\tilde{A})$ such that $X^\cdot \cong P^\cdot$ and $Y^\cdot \cong L^\cdot$ in $D(\mathrm{Mod}^b\tilde{A})$.

Fix $a^* = (a, i_0) \in \tilde{Q}_0$, and write $\tilde{Q}^{(i)} = \tilde{Q}^{(a^*,i)}$ for $i \in \mathbb{Z}$. By Proposition 7.7, there exist integers s, t such that

$$P^i = \oplus_{x^* \in \tilde{Q}^{(s-i)}} P[x^*] \otimes U^i_{x^*}; \quad L^i = \oplus_{y^* \in \tilde{Q}^{(t-i)}} P[y^*] \otimes V^i_{y^*},$$

where $U^i_{x^*}, V^i_{y^*}$ are k-spaces, for any $i \in \mathbb{Z}$.

Suppose that $D^b(\mathrm{Mod}^b\tilde{A})(Y^\cdot, X^\cdot) \neq 0$ and $D^b(\mathrm{Mod}^b\tilde{A})(X^\cdot, Y^\cdot) \neq 0$. As a consequence, $D(\mathrm{Mod}^b\tilde{A})(L^\cdot, P^\cdot) \neq 0$ and $D(\mathrm{Mod}^b\tilde{A})(P^\cdot, L^\cdot) \neq 0$. In view of Lemma 1.9(1), we deduce that $C(\mathrm{Mod}^b\tilde{A})(L^\cdot, P^\cdot) \neq 0$ and $C(\mathrm{Mod}^b\tilde{A})(P^\cdot, L^\cdot) \neq 0$. Let $u^\cdot : P^\cdot \to L^\cdot$ and $v^\cdot : L^\cdot \to P^\cdot$ be non-zero morphisms in $C(\mathrm{Mod}^b\tilde{A})$. In particular, $f^m \neq 0$ for some m. Then there exist $x^* \in \tilde{Q}^{(s-m)}$ and $y^* \in \tilde{Q}^{(t-m)}$ such that $\mathrm{Hom}_{\tilde{A}}(P[x^*], P[y^*]) \neq 0$. By Lemma 7.6, $\tilde{Q}_{\leq 1}(x^*, y^*) \neq \emptyset$. Therefore,

$(t-m)-(s-m)=d(x^*,y^*)\geq 0$, that is, $t\geq s$. Similarly, since $v^n\neq 0$ for some n, we may deduce that $s\geq t$. That is, $s=t$.

Let $g=\rho^j\in G$ with $j\in\mathbb{Z}$. Then $d(z^*,g\cdot z^*)=jr$ for any $z^*\in\tilde{Q}$. Hence, $g\cdot\tilde{Q}^{(i)}=\tilde{Q}^{(i+jr)}$, for any $i\in\mathbb{Z}$. This implies that $g\cdot L^\bullet\in RC^-(\mathrm{Proj}\tilde{A})$ such that, for each $i\in\mathbb{Z}$, we have

$$(g\cdot L)^i=g\cdot L^i=\oplus_{y^*\in Q^{(s-i)}}P[g\cdot y^*]\otimes V^i_{y^*}=\oplus_{z^*\in Q^{((s+jr)-i)}}P[z^*]\otimes V^i_{g^{-1}\cdot z^*}.$$

Suppose that $D^b(\mathrm{Mod}^b\tilde{A})(g\cdot Y^\bullet,X^\bullet)\neq 0$ and $D^b(\mathrm{Mod}^b\tilde{A})(X^\bullet,g\cdot Y^\bullet)\neq 0$. Since $g\cdot Y^\bullet\cong g\cdot L^\bullet$ in $D(\mathrm{Mod}^b\tilde{A})$, as shown above, we deduce that $s+jr=s$, that is, $j=0$. This establishes our claim.

Since idempotents in $D^b(\mathrm{Mod}^b\tilde{A})$ split, the indecomposable objects in $D^b(\mathrm{Mod}^b\tilde{A})$ are properly indecomposable. By Lemma 2.10, $\pi^D_\lambda: D^b(\mathrm{Mod}^b\tilde{A})\to D^b(\mathrm{Mod}^b A)$ satisfies the conditions (2) and (3) stated in Definition 2.8, and hence, is a Galois G-covering. The proof of the theorem is completed.

As an immediate consequence of Theorems 7.10 and 4.7, we obtain the following interesting result.

7.11. Corollary. *Let $A=kQ/(kQ^+)^2$ and $\tilde{A}=k\tilde{Q}/(k\tilde{Q}^+)^2$, where Q is a connected locally finite quiver and $\tilde{Q}$ is a connected component of $Q^{\mathbb{Z}}$. If G is the translation group of $\tilde{Q}$, then the minimal gradable covering $\pi:\tilde{A}\to A$ induces a Galois G-covering $F_\pi:\mathit{\Gamma}_{D^b(\mathrm{mod}^b\tilde{A})}\to\mathit{\Gamma}_{D^b(\mathrm{mod}^b A)}$.*

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