

AUSLANDER-REITEN THEORY VIA NAKAYAMA DUALITY IN ABELIAN CATEGORIES

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Dedicated to the memory of Idun Reiten

ABSTRACT. Using the Nakayama duality induced by a Nakayama functor, we provide a novel and concise account of the existence of Auslander-Reiten dualities and almost split sequences in abelian categories with enough projective objects or enough injective objects. As an example, we establish the existence of almost split sequences ending with finitely presented modules and those starting with finitely copresented modules in the category of all modules over a small endo-local Hom-reflexive category. Specializing to algebras given by (not necessarily finite) quivers with relations, we further investigate when the categories of finitely presented modules, finitely copresented modules and finite dimensional modules have almost split sequences on either or both sides.

INTRODUCTION

The Auslander-Reiten theory of irreducible maps and almost split sequences, introduced by Auslander and Reiten in the 1970's; see [8, 9], is a powerful tool for studying categories of various kinds: most prominently, categories of modules over algebras and their derived categories; see [10, 19, 21, 38], and also, categories of Cohen-Macaulay modules over commutative algebras; see [32, 39], categories of sheaves on varieties and schemes; see [6, 25], derived categories of differential graded algebras arising from topological spaces; see [24], and most recently, extriangulated categories; see [23, 34].

In this paper, we are mainly concerned with the existence of almost split sequences in an R -category \mathfrak{A} , where R is a commutative ring. Write D for the dual functor $\text{Hom}_R(-, I_R)$, where I_R is the minimal injective cogenerator for R -modules. Typically, one derives an almost split sequence $0 \longrightarrow \tau M \longrightarrow E \longrightarrow M \longrightarrow 0$ from an Auslander-Reiten duality $\text{Ext}_{\mathfrak{A}}^1(X, \tau M) \cong D\mathfrak{A}(M, X)$ or $\text{Ext}_{\mathfrak{A}}^1(M, X) \cong D\mathfrak{A}(X, \tau M)$ for all $X \in \mathfrak{A}$, or from their weak forms as described in [35, (2.3)], where τ is the AR-translation; see [4, 10, 19, 31]. The classical AR-translation for module categories is the dual of the transpose; see [4, (I.3)], [7, (7.4)], [8, (2.2)] and [37, (1.6.1)]. For lack of a transpose functor, we shall take a different approach.

Let \mathcal{P} be a strictly full additive subcategory of projective objects in \mathfrak{A} . As defined in [34, (5.4)], a Nakayama functor $\nu : \mathcal{P} \rightarrow \mathfrak{A}$ induces a *Nakayama duality* $D\mathfrak{A}(P, X) \cong \mathfrak{A}(X, \nu P)$ for $P \in \mathcal{P}$ and $X \in \mathfrak{A}$. It has been shown that a Nakayama duality induces Auslander-Reiten dualities in the derived category of \mathfrak{A} ; see [34, (5.7)]. Inspired by our recent work on the Auslander-Reiten dualities for graded modules; see [33, (3.4)], we show that it also induces, in an elegant way, Auslander-Reiten dualities in \mathfrak{A} . Indeed, defining the *AR-translate* $\tau_s M$ of an object M with

2010 *Mathematics Subject Classification.* 16E35, 16G20, 16G70, 16W50.

Key words and phrases. Algebras given by quivers with relations; modules over categories; Auslander-Reiten duality; almost split sequences; Nakayama functor; Krull-Schmidt categories.

a projective presentation δ over \mathcal{P} ; see (2.3.1), we show straightforwardly that $\text{Ext}_{\mathfrak{A}}^1(X, \tau_{\delta}M) \cong D\mathfrak{A}(M, X)$ for $X \in \mathfrak{A}$ if \mathfrak{A} has enough projective objects, and $D\text{Ext}_{\mathfrak{A}}^1(M, X) \cong \mathfrak{A}(X, \tau_{\delta}M)$ for $X \in \mathfrak{A}$ if \mathfrak{A} has enough injective objects; see (2.3.2) and (2.3.3). Observe that the second isomorphism induces only a monomorphism $\text{Ext}_{\mathfrak{A}}^1(M, X) \rightarrow D\mathfrak{A}(X, \tau_{\delta}M)$, not an isomorphism as stated above.

In order to derive almost split sequences from Auslander-Reiten dualities or their weak forms, one requires a key property that the AR-translation preserves strong indecomposability. For this purpose, we assume that \mathcal{P} is Hom-reflexive and Krull-Schmidt; see (1.2). In this case, we obtain a *Nakayama equivalence* $\nu : \mathcal{P} \rightarrow \nu\mathcal{P}$; see [34, (5.6)] and an AR-translation uniquely defined (up to isomorphism) for objects finitely presented over \mathcal{P} ; see (2.4.3). These lead directly to the desired property; see (2.4.5), without involving a transpose functor as the classical approach; see [7, (7.4)] and [10, (IV.1)]. This enables us to apply the result in [35, (2.3)] to establish the existence of almost split sequences ending with objects finitely presented over \mathcal{P} if \mathfrak{A} has enough projective objects, and almost split sequences starting with objects finitely copresented over $\nu\mathcal{P}$ if \mathfrak{A} has enough injective objects; see (3.6.3).

Our results will be applicable to many concrete abelian categories, including potentially categories of sheaves on topological spaces. In Section 3, for example, we shall consider the category $\text{Mod}\mathcal{C}$ of all left modules over a small endo-local Hom-reflexive R -category \mathcal{C} . By the Yoneda Lemma, the full subcategory $\text{proj}\mathcal{C}$ of finitely generated projective modules in $\text{Mod}\mathcal{C}$ is Hom-reflexive and Krull-Schmidt. By constructing a Nakayama functor $\nu : \text{proj}\mathcal{C} \rightarrow \text{Mod}\mathcal{C}$; see (3.4.1), we obtain an existence theorem for almost split sequences in $\text{Mod}\mathcal{C}$ ending with finitely presented modules and for those starting with finitely copresented modules; see (2.5.1). Note that the first part of this existence theorem was stated without proof or reference by Auslander in [4]. However, the most relevant result that we can find in the existent literature is the existence of the Auslander-Reiten duality, and consequently, the existence of almost split sequences, in the category of finitely presented \mathcal{C} -modules when \mathcal{C} is a dualizing R -variety; see [7, (7.4)]. It is not clear how this approach works for $\text{Mod}\mathcal{C}$, whether \mathcal{C} is a dualizing R -variety or not.

In section 4, we shall specialize to an algebra Λ given by a quiver with relations. In case Λ is locally semiperfect; see (4.4.1), which can be viewed as a small endo-local Hom-reflexive category, we obtain immediately a general existence theorem for almost split sequences in $\text{Mod}\Lambda$; see (4.7.1). So we will focus on the subcategories $\text{mod}^+\Lambda$, $\text{mod}^-\Lambda$ and $\text{mod}^b\Lambda$ of finitely presented modules, finitely copresented modules and finite dimensional modules in $\text{Mod}\Lambda$, respectively. In case Λ is locally semiprimary given by locally finite quivers; see (4.5.1), the almost split sequences in $\text{mod}^+\Lambda$ and $\text{mod}^-\Lambda$ are almost split in $\text{Mod}\Lambda$; see (4.7.4) and (4.7.6). And we find conditions for these subcategories to have almost split sequences on either or both sides; see (4.8.1), (4.8.3) and (4.8.5). In particular, we obtain examples in which $\text{mod}^b\Lambda$ has almost split sequences, but it has neither enough projective objects nor enough injective objects; see (4.8.9). In the hereditary case, our results strengthen the corresponding ones in [13]. We conclude by emphasizing the broad significance of the representation theory of algebras defined by infinite quivers with or without relations: it allows us to apply Bongartz and Gabriel's covering theory to study representations of finite dimensional algebras; see [11, 12, 14, 18]; and it establishes deep connections to other domains such as coalgebras and comodules; see [15, 16], non-commutative algebraic geometry; see [38] and cluster theory; see [22, 36].

1. PRELIMINARIES

The objective of this section is to fix some terminology and notation, which will be used throughout this paper, and collect some preliminary results.

1.1. REFLEXIVE MODULES. Throughout this paper, R stands for a commutative ring with an identity. We denote by $\text{Mod}R$ the category of R -modules and by I_R a minimal injective cogenerator for $\text{Mod}R$. A significant role in our investigation will be played by the *dual functor* $D = \text{Hom}_R(-, I_R) : \text{Mod}R \rightarrow \text{Mod}R$ will play. Given an R -module M , there exists a canonical R -linear monomorphism $\sigma_M : M \rightarrow D^2M$, sending m to the evaluating at m function. We shall say that M is *reflexive* if σ_M is bijective. The following statement is known; see [34, (1.3)].

Reflex

1.1.1. LEMMA. *Let R be a commutative ring. Then the category $\text{mod}R$ of reflexive R -modules is abelian, and the dual functor restricts to a duality $D : \text{mod}R \rightarrow \text{mod}R$.*

sp_rmod

1.1.2. REMARK. (1) The category $\text{mod}R$ contains all R -modules of finite length.

(2) In case R is complete noetherian local, $\text{mod}R$ contains all noetherian R -modules and all artinian R -modules.

KS_section

1.2. KRULL-SCHMIDT CATEGORIES. An R -category is a category in which the morphism sets are R -modules and the composition of morphisms is R -bilinear. We shall compose morphisms in any R -category from right to left. A full subcategory of an R -category is called *strictly full* if it is closed under isomorphisms. All functors between R -categories are assumed to be R -linear.

Let \mathcal{A} be an R -category. Given objects $X, Y \in \mathcal{A}$, we write $\mathcal{A}(X, Y)$ for the R -module of morphisms $f : X \rightarrow Y$. A non-zero object X in \mathcal{A} is called *strongly indecomposable* if $\mathcal{A}(X, X)$ is local; and *Krull-Schmidt* if it is a finite direct sum of strongly indecomposable objects. We shall say that \mathcal{A} is *Krull-Schmidt* if every non-zero object is Krull-Schmidt, and *Hom-reflexive* (respectively, *Hom-noetherian*, *Hom-finite*) if the R -modules $\mathcal{A}(X, Y)$ are reflexive (respectively, noetherian, of finite length), for all objects X, Y in \mathcal{A} . Note that a Hom-finite R -category is Hom-reflexive and Hom-noetherian. Moreover, if R is complete noetherian local, then a Hom-noetherian R -category is Hom-reflexive.

Ku-sum

1.2.1. PROPOSITION. *Let \mathcal{A} be a strictly full additive subcategory of an abelian R -category \mathfrak{A} . If \mathcal{A} is Krull-Schmidt, then it is closed under direct summands in \mathfrak{A} . The converse holds if \mathcal{A} is either Hom-finite or Hom-noetherian with R being complete local noetherian.*

Proof. Firstly, assume that \mathcal{A} is Krull-Schmidt. Let $X \in \mathcal{A}$ with a nonzero direct summand Y in \mathfrak{A} . Then, $X = X_1 \oplus \cdots \oplus X_n$, where $X_i \in \mathcal{A}$ with $\mathcal{A}(X_i, X_i)$ being local. Since $\mathfrak{A}(X_i, X_i) = \mathcal{A}(X_i, X_i)$, we see that Y is a direct sum of some of the X_i ; see [35, (1.1)]. In particular, $Y \in \mathcal{A}$.

Conversely, assume that \mathcal{A} is closed under direct summands in \mathfrak{A} . Then, the idempotent endomorphisms in \mathcal{A} split. Assume further that \mathcal{A} is either Hom-finite or Hom-noetherian with R being complete noetherian local. Then, for any $X \in \mathcal{A}$, the R -algebra $\mathcal{A}(X, X)$ is either artianian; or noetherian with R being complete noetherian local. In either case, $\mathcal{A}(X, X)$ is semi-perfect; see [30, (21.35)]. Thus, \mathcal{A} is Krull-Schmidt; see [35, (1.1)]. The proof of the proposition is completed.

1.3. MINIMAL MORPHISMS. Let \mathcal{A} be an R -category. An epimorphism $f : X \rightarrow Y$ in \mathcal{A} is called *superfluous* if any morphism $g : M \rightarrow X$ in \mathcal{A} such that fg is

an epimorphism is an epimorphism. Dually, a monomorphism $f : X \rightarrow Y$ is called *essential* if any morphism $h : Y \rightarrow N$ such that hf is a monomorphism is a monomorphism. More importantly, a morphism $f : X \rightarrow Y$ in \mathcal{A} is called *left minimal* if every morphism $g : Y \rightarrow Y$ such that $gf = f$ is an automorphism; and *right minimal* if every morphism $h : X \rightarrow X$ such that $fh = f$ is an automorphism. For later reference, we quote the following statement from [29, (1.4)].

min_decom

1.3.1. **PROPOSITION.** *Let \mathcal{A} be an additive R -category.*

- (1) *Every nonzero morphism $f : X \rightarrow Y$ in \mathcal{A} , where Y is Krull-Schmidt, can be decomposed as $f = (f_1, 0)^T : X \rightarrow Y_1 \oplus Y_2$, where $f_1 : X \rightarrow Y_1$ is left minimal.*
- (2) *Every nonzero morphism $g : X \rightarrow Y$ in \mathcal{A} , where X is Krull-Schmidt, can be decomposed as $g = (g_1, 0) : X_1 \oplus X_2 \rightarrow Y$, where $g_1 : X_1 \rightarrow Y$ is right minimal.*

1.4. **ALMOST SPLIT SEQUENCE.** Let \mathcal{A} be an exact R -category, that is, an extension-closed subcategory of an abelian R -category \mathfrak{A} . An object X in \mathcal{A} is called *ext-projective* if every short exact sequence $0 \rightarrow Z \rightarrow Y \rightarrow X \rightarrow 0$ in \mathcal{A} splits; and *ext-injective* if every short exact sequence $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$ in \mathcal{A} splits. If \mathcal{A} is abelian, then the projective objects in \mathcal{A} coincide with the Ext-projective objects, and the injective objects coincide with the Ext-injective objects.

Let $f : X \rightarrow Y$ be a morphism in \mathcal{A} . One says that f is *irreducible* if f is neither a section nor a retraction, and given any factorization $f = gh$, either h is a section or g is a retraction. Moreover, f is called *left almost split* if f is not a section and every non-section morphism $g : X \rightarrow Z$ factors through f ; and *minimal left almost split* if it is left minimal and left almost split. Dually, one defines *right almost split morphisms* and *minimal right almost split morphisms* in \mathcal{A} . Furthermore, an *almost split sequence* in \mathcal{A} is a short exact sequence

$$0 \rightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \rightarrow 0,$$

where f is minimal left almost split and g is minimal right almost. In such an almost split sequence, X is the *starting term* and Z is the *ending term*.

We shall say that \mathcal{A} has almost split sequence *on the right* if every strongly indecomposable and not ext-projective object is the ending term of an almost split sequence, and \mathcal{A} has almost split sequences *on the left* if every strongly indecomposable and not ext-injective object is the starting term of an almost split sequence. Finally, we shall say that \mathcal{A} has almost split sequences if it has almost split sequences on the right and on the left.

1.5. **STABLE CATEGORIES.** Let \mathcal{A} be an exact R -category. A morphism $f : X \rightarrow Y$ in \mathcal{A} is called *projectively trivial* if it factors through every proper epimorphism $g : M \rightarrow Y$ in \mathcal{A} , and *injectively trivial* if it factors through every proper monomorphism $h : X \rightarrow N$ in \mathcal{A} . In case \mathcal{A} is abelian with enough projective objects, a morphism in \mathcal{A} is projectively trivial if and only if it factors through a projective object. And in case \mathcal{A} is abelian with enough injective objects, a morphism in \mathcal{A} is injectively trivial if and only if it factors through an injective object.

Write \mathcal{P} and \mathcal{I} for the ideals in \mathcal{A} generated by the projectively trivial morphisms and the injectively trivial morphisms, respectively. The *projectively stable category* and the *injectively stable category* of \mathcal{A} are the quotient categories $\underline{\mathcal{A}} = \mathcal{A}/\mathcal{P}$ and $\bar{\mathcal{A}} = \mathcal{A}/\mathcal{I}$, respectively; see [19, (9.2)], [31, (2.1)], [35, Section 2].

2. AUSLANDER-REITEN THEORY IN ABELIAN CATEGORIES

The main objective of this section is to provide a novel and concise account of the Auslander-Reiten theory in general abelian categories. First, we derive an Auslander-Reiten duality from the Nakayama duality induced by a Nakayama functor. In case the Nakayama functor is defined on a “small” subcategory of projective objects, we show that the AR-translation is uniquely defined (up to isomorphism) on finitely presented objects and preserves the strong indecomposability. This enables us to establish the existence of almost split sequences ending with finitely presented objects and those starting with finitely copresented objects.

2.1. PROJECTIVE PRESENTATIONS AND INJECTIVE COPRESENTATIONS. Throughout this section, \mathfrak{A} stands for an abelian R -category. Let \mathcal{P} and \mathcal{I} be strictly full additive subcategories of projective objects and injective objects in \mathfrak{A} , respectively. Consider an object X in \mathfrak{A} . A *projective cover* of X over \mathcal{P} is a right minimal epimorphism $f : P \rightarrow X$ in \mathfrak{A} with $P \in \mathcal{P}$; and an *injective envelope* of X over \mathcal{I} is a left minimal monomorphism $g : X \rightarrow I$ in \mathfrak{A} with $I \in \mathcal{I}$. Moreover, a *projective presentation* of X over \mathcal{P} is an exact sequence $P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} X \rightarrow 0$ in \mathfrak{A} with $P_1, P_0 \in \mathcal{P}$, which is *minimal* if d_1, d_0 are right minimal. Dually, a *injective copresentation* of X over \mathcal{I} is an exact sequence $0 \rightarrow X \xrightarrow{d^0} I^0 \xrightarrow{d^1} I^1$ in \mathfrak{A} with $I^0, I^1 \in \mathcal{I}$, which is *minimal* if d^0, d^1 are left minimal. Clearly, projective cover, injective envelope, minimal projective presentation and minimal injective copresentation are unique up to isomorphism if they exist. We denote by $\mathfrak{A}^+(\mathcal{P})$ and $\mathfrak{A}^-(\mathcal{I})$ the full subcategories of objects in \mathfrak{A} with a projective presentation over \mathcal{P} and with an injective copresentation over \mathcal{I} , respectively.

subcat_pres

2.1.1. PROPOSITION. *Let \mathfrak{A} be an abelian R -category, and let \mathcal{P} be a strictly full additive subcategory of projective objects in \mathfrak{A} . Then, $\mathfrak{A}^+(\mathcal{P})$ is closed under cokernels and extension-closed in \mathfrak{A} . Moreover,*

- (1) *if \mathcal{P} is Hom-reflexive (respectively, Hom-noetherian, Hom-finite), then $\mathfrak{A}^+(\mathcal{P})$ is Hom-reflexive (respectively, Hom-noetherian, Hom-finite);*
- (2) *if \mathcal{P} is Krull-Schmidt, then $\mathfrak{A}^+(\mathcal{P})$ is closed under direct summands in \mathfrak{A} , and every object in $\mathfrak{A}^+(\mathcal{P})$ admits a minimal projective presentation over \mathcal{P} .*

Proof. It follows from Proposition 2.1 in [2] that $\mathfrak{A}^+(\mathcal{P})$ is closed under cokernels and extension-closed in \mathfrak{A} .

(1) Let X, X' be objects in $\mathfrak{A}^+(\mathcal{P})$. By definition, there exist projective presentations $P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} X \rightarrow 0$ and $P'_1 \xrightarrow{d'_1} P'_0 \xrightarrow{d'_0} X' \rightarrow 0$ over \mathcal{P} . Set $\mathcal{M}(X, X') := \{(f_1, f_0) \in \mathfrak{A}(P_1, P'_1) \oplus \mathfrak{A}(P_1, P'_1) \mid f_0 d_1 = d'_1 f_1\}$. It is evident that we have an R -linear epimorphism $\varphi : \mathcal{M}(X, X') \rightarrow \mathfrak{A}(X, X')$, given by $(f_1, f_0) \mapsto f$, where f is the unique morphism making the diagram

$$\begin{array}{ccccccc} P_1 & \xrightarrow{d_1} & P_0 & \xrightarrow{d_0} & X & \rightarrow & 0 \\ \downarrow f_1 & & \downarrow f_0 & & \downarrow f & & \\ P'_1 & \xrightarrow{d'_1} & P'_0 & \xrightarrow{d'_0} & X' & \rightarrow & 0 \end{array}$$

commute. Suppose that \mathcal{P} is Hom-reflexive. Then $\mathfrak{A}(P_1, P'_1)$ and $\mathfrak{A}(P_1, P'_1)$ are reflexive R -modules. Since $\text{mod}R$ is abelian; see (1.1.1), $\mathcal{M}(X, X')$ is reflexive, and so is $\mathfrak{A}(X, X')$. Similarly, if \mathcal{P} is Hom-noetherian or Hom-finite, then $\mathfrak{A}^+(\mathcal{P})$ is Hom-noetherian or Hom-finite, respectively.

(2) Assume that \mathcal{P} is Krull-Schmidt. Then, \mathcal{P} is closed under direct summands in \mathfrak{A} . Let $X \in \mathfrak{A}^+(\mathcal{P})$ with a projective presentation $P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} X \rightarrow 0$ over \mathcal{P} . By Proposition 1.3.1, $d_0 = (d'_0, 0) : P_0 = P'_0 \oplus N_0 \rightarrow X$, where d'_0 is a right minimal epimorphism. Thus, $d'_0 : P'_0 \rightarrow X$ is a projective cover with $P'_0 \in \mathcal{P}$. Moreover, $\text{Im}(d_1) = \text{Ker}(d_0) = \text{Ker}(d'_0) \oplus N_0$. Thus, we have a decomposition $d_1 = (f, 0)^T : P_1 \rightarrow P'_0 \oplus N_0$, where $f : P_1 \rightarrow P'_0$ is such that $\text{Im}(f) = \text{Ker}(d'_0)$. By Proposition 1.3.1 again, $f = (d_1, 0) : P'_1 \oplus N_1 \rightarrow P'_0$, where $d'_1 : P'_1 \rightarrow P'_0$ is right minimal. It is easy to check that $\text{Im}(d'_1) = \text{Im}(f)$. This yields a minimal projective presentation $P'_1 \xrightarrow{d'_1} P'_0 \xrightarrow{d'_0} X \rightarrow 0$ over \mathcal{P} .

Suppose now that $X = X_1 \oplus X_2$ with canonical projections $p_i : X \rightarrow X_i$. In particular, we obtain epimorphisms $p_i d'_0 : P'_0 \rightarrow X_i$, for $i = 1, 2$. As seen above, there exist short exact sequences $0 \rightarrow K_i \xrightarrow{f_i} L_i \xrightarrow{g_i} X_i \rightarrow 0$, where $L_i \in \mathcal{P}$ and g_i is a projective cover, for $i = 1, 2$. By the uniqueness of projective cover, we see easily that $K_1 \oplus K_2 \cong \text{Ker}(d'_0) = \text{Im}(d'_1)$. Therefore, we have epimorphisms $h_i : P'_1 \rightarrow K_i$, for $i = 1, 2$. This yields projective presentations $P'_1 \xrightarrow{f_i h_i} L_i \xrightarrow{g_i} X_i \rightarrow 0$ over \mathcal{P} , for $i = 1, 2$. So, $X_1, X_2 \in \mathfrak{A}^+(\mathcal{P})$. The proof of the proposition is completed.

Dually, we have the following statement.

subcat_copres

2.1.2. PROPOSITION. Let \mathfrak{A} be an abelian R -category, and let \mathcal{I} be a strictly full additive subcategory of injective objects in \mathfrak{A} . Then, $\mathfrak{A}^+(\mathcal{I})$ is closed under kernels and extension-closed in \mathfrak{A} . Moreover,

- (1) if \mathcal{I} is Hom-reflexive (respectively, Hom-noetherian, Hom-finite), then $\mathfrak{A}^-(\mathcal{I})$ is Hom-reflexive (respectively, Hom-noetherian, Hom-finite);
- (2) if \mathcal{I} is Krull-Schmidt, then $\mathfrak{A}^-(\mathcal{I})$ is closed under direct summands in \mathfrak{A} , and every object in $\mathfrak{A}^-(\mathcal{I})$ admits a minimal injective copresentation over \mathcal{I} .

2.2. NAKAYAMA FUNCTOR. The key ingredient in our approach to the Auslander-Reiten duality is a Nakayama functor defined as follows; see [34, (5.4)].

N_functor

2.2.1. DEFINITION. Let \mathfrak{A} be an abelian R -category, and let \mathcal{P} be a strictly full additive subcategory of projective objects in \mathfrak{A} . A functor $\nu : \mathcal{P} \rightarrow \mathfrak{A}$ is called a *Nakayama functor* provided, for all $P \in \mathcal{P}$ and $X \in \mathfrak{A}$, that there exists an R -linear isomorphism $\phi_{P,X} : \mathfrak{A}(X, \nu P) \rightarrow D\mathfrak{A}(P, X)$, which is natural in P and X .

NF_im

2.2.2. REMARK. Since the dual functor D is exact, the image \mathcal{I}_ν of a Nakayama functor $\nu : \mathcal{P} \rightarrow \mathfrak{A}$ is a strictly full additive subcategory of injective objects in \mathfrak{A} .

2.3. AUSLANDER-REITEN DUALITY. In order to have an Auslander-Reiten duality, we need to introduce the following notion, which depends on not only the objects in $\mathfrak{A}^+(\mathcal{P})$ but also their projective presentations over \mathcal{P} .

pseu_AR_trasl

2.3.1. DEFINITION. Let \mathfrak{A} be an abelian R -category equipped with a Nakayama functor $\nu : \mathcal{P} \rightarrow \mathfrak{A}$. Given an object M in $\mathfrak{A}^+(\mathcal{P})$ with a projective presentation $\delta : P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \rightarrow 0$ over \mathcal{P} , we put $\tau_\delta M = \text{Ker}(\nu d_1)$, called the *Auslander-Reiten translate* of M associated with δ .

The following statement is essential for the Auslander-Reiten duality.

Naka-fcon

2.3.2. PROPOSITION. Let \mathfrak{A} be an abelian R -category equipped with a Nakayama functor $\nu : \mathcal{P} \rightarrow \mathfrak{A}$. Consider a projective presentation $\delta : P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \rightarrow 0$

over \mathcal{P} and a short exact sequence $0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$ in \mathfrak{A} . Then, we have an R -linear exact sequence

$$\begin{aligned} 0 \longrightarrow \mathfrak{A}(Z, \tau_\delta M) &\xrightarrow{g^*} \mathfrak{A}(Y, \tau_\delta M) \xrightarrow{f^*} \mathfrak{A}(X, \tau_\delta M) \xrightarrow{\eta} \\ &D\mathfrak{A}(M, Z) \xrightarrow{Dg_*} D\mathfrak{A}(M, Y) \xrightarrow{Df_*} D\mathfrak{A}(M, X) \longrightarrow 0. \end{aligned}$$

Proof. Applying the Nakayama functor ν to δ , we obtain an injective copresentation $0 \longrightarrow \tau_\delta M \longrightarrow \nu P_1 \xrightarrow{\nu d_1} \nu P_0$ over \mathcal{I}_ν . Given an object N in \mathfrak{A} , applying $\mathfrak{A}(N, -)$ to this injective copresentation yields an exact sequence

$$(*) \quad 0 \longrightarrow \mathfrak{A}(N, \tau_\delta M) \longrightarrow \mathfrak{A}(N, \nu P_1) \xrightarrow{(\nu d_1)_*} \mathfrak{A}(N, \nu P_0).$$

On the other hand, applying $D\mathfrak{A}(-, N)$ to δ and using the Nakayama duality, we obtain a commutative diagram

$$\begin{array}{ccccccc} \mathfrak{A}(N, \nu P_1) & \xrightarrow{(\nu d_1)_*} & \mathfrak{A}(N, \nu P_0) & & & & \\ \cong \downarrow & & \cong \downarrow & & & & \\ D\mathfrak{A}(P_1, N) & \xrightarrow{Dd_1^*} & D\mathfrak{A}(P_0, N) & \xrightarrow{Dd_0^*} & D\mathfrak{A}(M, N) & \longrightarrow 0, & \end{array}$$

where the bottom row is exact. Combining this with $(*)$ yields an exact sequence

$$0 \longrightarrow \mathfrak{A}(N, \tau_\delta M) \longrightarrow \mathfrak{A}(N, \nu P_1) \xrightarrow{(\nu d_1)_*} \mathfrak{A}(N, \nu P_0) \longrightarrow D\mathfrak{A}(M, N) \longrightarrow 0.$$

Therefore, we obtain a commutative diagram with exact rows and columns

$$\begin{array}{ccccc} & 0 & 0 & 0 & \\ & \downarrow & \downarrow & \downarrow & \\ 0 \longrightarrow \mathfrak{A}(Z, \tau_\delta M) & \xrightarrow{g^*} & \mathfrak{A}(Y, \tau_\delta M) & \xrightarrow{f^*} & \mathfrak{A}(X, \tau_\delta M) \\ & \downarrow & \downarrow & \downarrow & \\ 0 \longrightarrow \mathfrak{A}(Z, \nu P_1) & \longrightarrow & \mathfrak{A}(Y, \nu P_1) & \longrightarrow & \mathfrak{A}(X, \nu P_1) \longrightarrow 0 \\ & \downarrow & \downarrow & \downarrow & \\ 0 \longrightarrow \mathfrak{A}(Z, \nu P_0) & \longrightarrow & \mathfrak{A}(Y, \nu P_0) & \longrightarrow & \mathfrak{A}(X, \nu P_0) \longrightarrow 0 \\ & \downarrow & \downarrow & \downarrow & \\ D\mathfrak{A}(M, Z) & \xrightarrow{Dg_*} & D\mathfrak{A}(M, Y) & \xrightarrow{Df_*} & D\mathfrak{A}(M, X) \longrightarrow 0, \\ & \downarrow & \downarrow & \downarrow & \\ & 0 & 0 & 0 & \end{array}$$

where the two middle rows are exact because νP_1 and νP_0 are injective. Applying the Snake Lemma, we obtain the desired exact sequence stated in the proposition. The proof of the proposition is completed.

We are ready to establish the Auslander-Reiten duality; compare [28, (1.1), (1.2)]. Recall that $\underline{\mathfrak{A}} = \mathfrak{A}/\mathcal{P}$, the projectively stable category; and $\overline{\mathfrak{A}} = \mathfrak{A}/\mathcal{I}$, the injectively stable category, of \mathfrak{A} .

AR-formu

2.3.3. THEOREM. *Let \mathfrak{A} be an abelian R -category equipped with a Nakayama functor $\nu : \mathcal{P} \rightarrow \mathfrak{A}$. Consider an object M in $\mathfrak{A}^+(\mathcal{P})$ with a projective presentation $\delta : P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \longrightarrow 0$ over \mathcal{P} .*

- (1) If \mathfrak{A} has enough projective objects, then there exists an R -linear isomorphism $D\underline{\mathfrak{A}}(M, X) \cong \text{Ext}_{\mathfrak{A}}^1(X, \tau_{\delta}M)$ for $X \in \mathfrak{A}$, which is natural in X .
- (2) If \mathfrak{A} has enough injective objects, then there exists an R -linear isomorphism $\overline{\mathfrak{A}}(X, \tau_{\delta}M) \cong D\text{Ext}_{\mathfrak{A}}^1(M, X)$ for $X \in \mathfrak{A}$, which is natural in X .

Proof. (1) Suppose that \mathfrak{A} has enough projective objects. Fix an object $X \in \mathfrak{A}$. We have a short exact sequence $0 \longrightarrow L \xrightarrow{q} P \xrightarrow{p} X \longrightarrow 0$, where P is a projective object in \mathfrak{A} . It is easy to see that there exists an R -linear exact sequence

$$0 \longrightarrow D\underline{\mathfrak{A}}(M, X) \longrightarrow D\underline{\mathfrak{A}}(M, X) \xrightarrow{Dp_*} D\underline{\mathfrak{A}}(M, P).$$

Thus, we have an R -linear isomorphism $D\underline{\mathfrak{A}}(M, X) \cong \text{Ker}(Dp_*)$, which is clearly natural in X . On the other hand, applying $\mathfrak{A}(-, \tau_{\delta}M)$ to the above short exact sequence yields an R -linear exact sequence

$$0 \longrightarrow \mathfrak{A}(X, \tau_{\delta}M) \xrightarrow{p^*} \mathfrak{A}(P, \tau_{\delta}M) \xrightarrow{q^*} \mathfrak{A}(L, \tau_{\delta}M) \longrightarrow \text{Ext}_{\mathfrak{A}}^1(X, \tau_{\delta}M) \longrightarrow 0.$$

Thus, we obtain an R -linear isomorphism $\text{Coker}(q^*) \cong \text{Ext}_{\mathfrak{A}}^1(X, \tau_{\delta}M)$, which is also natural in X . Further, by Proposition 2.3.2, there exists an exact sequence

$$\mathfrak{A}(P, \tau_{\delta}M) \xrightarrow{q^*} \mathfrak{A}(L, \tau_{\delta}M) \xrightarrow{\eta} D\underline{\mathfrak{A}}(M, X) \xrightarrow{Dp_*} D\underline{\mathfrak{A}}(M, P)$$

This yields an R -linear isomorphism $\text{Ker}(Dp_*) = \text{Im}(\eta) \cong \text{Coker}(q^*)$, which is natural in X . Combining the above isomorphisms, we obtain an R -linear isomorphism $D\underline{\mathfrak{A}}(M, X) \cong \text{Ext}_{\mathfrak{A}}^1(X, \tau_{\delta}M)$, which is natural in X .

(2) Suppose that \mathfrak{A} has enough injective objects. Fix an object $X \in \mathfrak{A}$. We have a short exact sequence $0 \longrightarrow X \xrightarrow{q} I \xrightarrow{p} L \longrightarrow 0$, where I is an injective object in \mathfrak{A} . Then, we have an exact sequence

$$\mathfrak{A}(I, \tau_{\delta}M) \xrightarrow{q^*} \mathfrak{A}(X, \tau_{\delta}M) \longrightarrow \overline{\mathfrak{A}}(X, \tau_{\delta}M) \longrightarrow 0.$$

So, we have an R -linear isomorphism $\overline{\mathfrak{A}}(X, \tau_{\delta}M) \cong \text{Coker}(q^*)$, which is natural in X . On the other hand, applying $\mathfrak{A}(M, -)$ yields an R -linear exact sequence

$$0 \longrightarrow \mathfrak{A}(M, X) \xrightarrow{q_*} \mathfrak{A}(M, I) \xrightarrow{p^*} \mathfrak{A}(M, L) \longrightarrow \text{Ext}_{\mathfrak{A}}^1(M, X) \longrightarrow 0.$$

This gives rise to an R -linear isomorphism $D\text{Ext}_{\mathfrak{A}}^1(M, X) \cong \text{Ker}(Dp_*)$, which is natural in X . Further, by Proposition 2.3.2, we have an R -linear exact sequence

$$\mathfrak{A}(I, \tau_{\delta}M) \xrightarrow{q^*} \mathfrak{A}(X, \tau_{\delta}M) \xrightarrow{\eta} D\underline{\mathfrak{A}}(M, L) \xrightarrow{Dp_*} D\underline{\mathfrak{A}}(M, I).$$

This gives rise to an R -linear isomorphism $\text{Ker}(Dp_*) = \text{Im}(\eta) \cong \text{Coker}(q^*)$, which is natural in X . Combining the above R -linear isomorphisms, we obtain an R -linear isomorphism $\overline{\mathfrak{A}}(X, \tau_{\delta}M) \cong D\text{Ext}_{\mathfrak{A}}^1(M, X)$, which is natural in X . The proof of the theorem is completed.

2.4. AUSLANDER-REITEN TRANSLATIONS. Let $\nu : \mathcal{P} \rightarrow \mathfrak{A}$ be a Nakayama functor. Given a strongly indecomposable object M with a projective presentation δ over \mathcal{P} , in order to derive an almost split sequence from the Auslander-Reiten duality stated in Theorem 2.3.3(1), we need to ensure that $\tau_{\delta}M$ is strongly indecomposable. For this purpose, we assume that \mathcal{P} is Hom-reflexive and Krull-Schmidt. In this setting, we have the following result; see [34, (5.6)].

fu-fai

2.4.1. PROPOSITION. *Let \mathfrak{A} be an abelian R -category equipped with a Nakayama functor $\nu : \mathcal{P} \rightarrow \mathfrak{A}$, where \mathcal{P} is Hom-reflexive and Krull-Schmidt. Then ν restricts to an equivalence $\nu : \mathcal{P} \rightarrow \mathcal{I}_{\nu}$. In particular, \mathcal{I}_{ν} is Hom-reflexive and Krull-Schmidt.*

2.4.2. REMARK. We call the equivalence $\nu : \mathcal{P} \rightarrow \mathcal{I}_\nu$ stated in Proposition 2.4.1 the *Nakayama equivalence*, and we fix a quasi-inverse $\nu^- : \mathcal{I}_\nu \rightarrow \mathcal{P}$ for it.

By Propositions 2.1.1 and 2.1.2, the exact categories $\mathfrak{A}^+(\mathcal{P})$ and $\mathfrak{A}^-(\mathcal{I}_\nu)$ are Hom-reflexive and closed under direct summands in \mathfrak{A} . Moreover, every object in $\mathfrak{A}^+(\mathcal{P})$ admits a minimal projective presentation over \mathcal{P} , and every object in $\mathfrak{A}^-(\mathcal{I}_\nu)$ admits a minimal injective copresentation over \mathcal{I}_ν . This allows us to define, unique up to isomorphism, the *AR-translations* τ and τ^- as follows.

AR-trsl

2.4.3. DEFINITION. Let \mathfrak{A} be an abelian R -category equipped with a Nakayama functor $\nu : \mathcal{P} \rightarrow \mathfrak{A}$, where \mathcal{P} is Hom-reflexive and Krull-Schmidt.

- (1) Given $M \in \mathfrak{A}^+(\mathcal{P})$ of which $P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \rightarrow 0$ is a minimal projective presentation over \mathcal{P} , we define $\tau M = \text{Ker}(\nu d_1)$.
- (2) Given $N \in \mathfrak{A}^-(\mathcal{I}_\nu)$ of which $0 \rightarrow N \xrightarrow{d^0} I_0 \xrightarrow{d^1} I_1$ is a minimal injective copresentation over \mathcal{I}_ν , we define $\tau^- N = \text{Coker}(\nu^- d^1)$.

In order to show that τ and τ^- preserve the strong indecomposability, we slightly generalize a well-known fact stated in [4, (II.4.3)].

LR-minimal

2.4.4. LEMMA. Let \mathfrak{A} be an abelian R -category, and let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be an exact sequence in \mathfrak{A} .

- (1) If $\mathfrak{A}(Z, Z)$ is local and g is a nonsplit epimorphism, then f is left minimal.
- (2) If $\mathfrak{A}(X, X)$ is local and f is a nonsplit monomorphism, then g is right minimal.

Proof. We shall only prove Statement (1). Suppose that $\mathfrak{A}(Z, Z)$ is local and g is a nonsplit epimorphism. Then, there exists a nonsplit short exact sequence $0 \rightarrow M \xrightarrow{j} Y \xrightarrow{g} Z \rightarrow 0$ in \mathfrak{A} . By Lemma 4.3 in [4, Chapter II], j is left minimal. And since $M = \text{Im}(f)$, there exists an epimorphism $v : X \rightarrow M$ such that $f = jv$. Assume that $hf = f$ for some $h : Y \rightarrow Y$. Then $hjv = jv$, and hence, $hj = j$. Therefore, h is an automorphism. The proof of the lemma is completed.

The classical approach for showing that the Auslander-Reiten translation preserves indecomposability goes through the morphism category of projective modules and the stable categories; see [10, (IV.1)]. Here we provide a direct proof.

tau1

2.4.5. PROPOSITION. Let \mathfrak{A} be an abelian R -category equipped with a Nakayama functor $\nu : \mathcal{P} \rightarrow \mathfrak{A}$, where \mathcal{P} is Hom-reflexive and Krull-Schmidt. Consider a strongly indecomposable object M in \mathfrak{A} .

- (1) If $M \in \mathfrak{A}^+(\mathcal{P})$ is not projective, then $\tau M \in \mathfrak{A}^-(\mathcal{I}_\nu)$ is strongly indecomposable and not injective such that $\tau^-(\tau M) \cong M$.
- (2) If $M \in \mathfrak{A}^-(\mathcal{I}_\nu)$ is not injective, then $\tau^- M \in \mathfrak{A}^+(\mathcal{P})$ is strongly indecomposable and not projective such that $\tau(\tau^- M) \cong M$.

Proof. We shall only prove Statement (1). Assume that $M \in \mathfrak{A}^+(\mathcal{P})$ is nonprojective with a minimal projective presentation $P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \rightarrow 0$ over \mathcal{P} . By definition, τM admits an injective copresentation $0 \rightarrow \tau M \xrightarrow{j} \nu P_1 \xrightarrow{\nu d_1} \nu P_0$ over \mathcal{I}_ν , which we claim is minimal. Indeed, d_1 is right minimal and d_0 is not a retraction. By Lemma 2.4.4, d_1 is also left minimal. And by Proposition 2.4.1, νd_1 is left and right minimal. Since νP_1 is Krull-Schmidt, it follows from Proposition 1.3.1 that $j = (l, 0)^T : \tau M \rightarrow L \oplus U = \nu P_1$, where $l : \tau M \rightarrow L$ is left minimal. Then, $\text{Ker}(\nu d_1) = \text{Im}(j) = \text{Im}(l)$. Hence, $\nu d_1 = (f, g) : \nu P_1 = L \oplus U \rightarrow \nu P_0$, where $g : U \rightarrow \nu P_0$ is a monomorphism. Since U is injective, g is a section. And

since $\nu : \mathcal{P} \rightarrow \mathcal{I}_\nu$ is an equivalence, we can write $d_1 = (p, q) : P_1 = V \oplus W \rightarrow P_0$ such that $g = \nu(q : W \rightarrow P_0)$. Then, q is a section such that $d_0q = 0$. Since $d_0 : P_0 \rightarrow M$ is right minimal, we have $W = 0$, and consequently, $U = 0$. That is, j is left minimal. Thus, τM admits a minimal injective copresentation over \mathcal{I}_ν as claimed above. Applying ν^- to it, we see that $\tau^-(\tau M) \cong M$.

Suppose that τM is injective. Then, j is a section. Being left minimal, j is an isomorphism. Then, $\nu d_1 = 0$. Since νd_1 is left minimal, $\nu P_0 = 0$, and hence, $P_0 = 0$, absurd. It remains to show that $\mathfrak{A}(\tau M, \tau M)$ is local. Given any $f \in \mathfrak{A}(\tau M, \tau M)$, we have a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & \tau M & \xrightarrow{j} & \nu P_1 & \xrightarrow{\nu d_1} & \nu P_0 \\ & & \downarrow f & & \downarrow f_1 & & \downarrow f_0 \\ 0 & \longrightarrow & \tau M & \xrightarrow{j} & \nu P_1 & \xrightarrow{\nu d_1} & \nu P_0. \end{array}$$

Again since ν is an equivalence, we have a commutative diagram with exact rows

$$\begin{array}{ccccccc} P_1 & \xrightarrow{d_1} & P_0 & \xrightarrow{d_0} & M & \longrightarrow & 0 \\ \downarrow g_1 & & \downarrow g_0 & & \downarrow g & & \\ P_1 & \xrightarrow{d_1} & P_0 & \xrightarrow{d_0} & M & \longrightarrow & 0 \end{array}$$

such that $\nu g_0 = f_0$ and $\nu g_1 = f_1$. Assume that g is invertible. Since d_0 is superfluous; see [27, (3.4)] and P_0 is Krull-Schmidt, g_0 is an isomorphism. Similarly, g_1 is also an isomorphism. Thus, f_0 and f_1 are isomorphisms. So f is invertible. Suppose that g is not invertible. Since $\mathfrak{A}(M, M)$ is local, $1_M - g$ is invertible. Observe that we obtain commutative diagrams from the above commutative diagrams by replacing (f, f_1, f_0) and (g_1, g_0, g) by $(1 - f, 1 - f_1, 1 - f_0)$ and $(1 - g_1, 1 - g_0, 1 - g)$, respectively. Using the same argument, we deduce that $1_{\tau M} - f$ is invertible. So, $\mathfrak{A}(\tau M, \tau M)$ is local. The proof of the proposition is completed.

We conclude this subsection with some sufficient conditions for $\mathfrak{A}^+(\mathcal{P})$ and $\mathfrak{A}^-(\mathcal{I}_\nu)$ to be Krull-Schmidt.

2.4.6. LEMMA. *Let \mathfrak{A} be an abelian R -category equipped with a Nakayama functor $\nu : \mathcal{P} \rightarrow \mathfrak{A}$, where \mathcal{P} is Krull-Schmidt. If \mathcal{P} is either Hom-finite or Hom-noetherian with R being complete noetherian local, then $\mathfrak{A}^+(\mathcal{P})$ and $\mathfrak{A}^-(\mathcal{I}_\nu)$ are Krull-Schmidt.*

Proof. Let \mathcal{P} be Hom-finite. By Proposition 2.4.1, \mathcal{I}_ν is Hom-finite and Krull-Schmidt. By Propositions 2.1.1 and 2.1.2, $\mathfrak{A}^+(\mathcal{P})$ and $\mathfrak{A}^-(\mathcal{I}_\nu)$ are Hom-finite and closed under direct summands in \mathfrak{A} . By Proposition 1.2.1, they are Krull-Schmidt. In case R is complete noetherian local and \mathcal{P} is Hom-noetherian, we can prove the result by the same argument. The proof of the lemma is completed.

2.5. THE EXISTENCE THEOREM. We are ready to apply the result in [35, (2.3)] to derive almost split sequences from the Auslander-Reiten dualities in Theorem 2.3.3.

2.5.1. THEOREM. *Let \mathfrak{A} be an abelian R -category equipped with a Nakayama functor $\nu : \mathcal{P} \rightarrow \mathfrak{A}$, where \mathcal{P} is Hom-reflexive and Krull-Schmidt.*

(1) *If \mathfrak{A} has enough projective objects, then it has an almost split sequence*

$$0 \longrightarrow \tau M \longrightarrow E \longrightarrow M \longrightarrow 0,$$

for every strongly indecomposable nonprojective object M in $\mathfrak{A}^+(\mathcal{P})$.

(2) *If \mathfrak{A} has enough injective objects, then it has an almost split sequence*

$$0 \longrightarrow N \longrightarrow E \longrightarrow \tau^-N \longrightarrow 0,$$

for every strongly indecomposable noninjective object N in $\mathfrak{A}^-(\mathcal{I}_\nu)$.

fpres_cat_KS

AR-seq

Poof. (1) Assume that \mathfrak{A} has enough projective objects. Consider a strongly indecomposable nonprojective $M \in \mathfrak{A}^+(\mathcal{P})$. By Proposition 2.4.5(1), τM is strongly indecomposable, and by Theorem 2.3.3(1), there exists a functorial isomorphism $\Psi_M : \text{Ext}_{\mathfrak{A}}^1(-, \tau M) \rightarrow D\mathfrak{A}(M, -)$. In particular, $\text{Ext}_{\mathfrak{A}}^1(M, \tau M) \cong D\mathfrak{A}(M, M)$ as right $\mathfrak{A}(M, M)$ -modules. Since $\mathfrak{A}(M, M)$ is local, we see that $D\mathfrak{A}(M, M)$ has a nonzero $\mathfrak{A}(M, M)$ -socle, and so does $\text{Ext}_{\mathfrak{A}}^1(M, \tau M)$. By Theorem 2.3 in [35], we obtain a desired almost split sequence as stated in Statement (1).

(2) Assume that \mathfrak{A} has enough injective objects. Consider a strongly indecomposable noninjective object $N \in \mathfrak{A}^-(\mathcal{I}_\nu)$. By Proposition 2.4.5(2), τ^-N is strongly indecomposable, and by Theorem 2.3.3(2), there exists a functorial isomorphism $\Phi_N : D^2\text{Ext}_{\mathfrak{A}}^1(\tau^-N, -) \rightarrow D\bar{\mathfrak{A}}(-, N)$. This induces a functorial monomorphism $\Psi_N : \text{Ext}_{\mathfrak{A}}^1(\tau^-N, -) \rightarrow D\bar{\mathfrak{A}}(-, N)$. Moreover, we have a left $\mathfrak{A}(N, N)$ -linear isomorphism $\Phi_{N,N} : D^2\text{Ext}_{\mathfrak{A}}^1(\tau^-N, N) \rightarrow D\bar{\mathfrak{A}}(N, N)$. By Propositions 2.4.1 and 2.1.2, $\mathfrak{A}(N, N)$ is R -reflexive, and by Lemma 1.1.1, both $D\bar{\mathfrak{A}}(N, N)$ and $D^2\text{Ext}_{\mathfrak{A}}^1(\tau^-N, N)$ are R -reflexive. Therefore, $\text{Ext}_{\mathfrak{A}}^1(\tau^-N, N) \cong D^2\text{Ext}_{\mathfrak{A}}^1(\tau^-N, N) \cong D\bar{\mathfrak{A}}(N, N)$ as left $\mathfrak{A}(N, N)$ -modules. Since $\mathfrak{A}(N, N)$ is local, $D\bar{\mathfrak{A}}(N, N)$ has a nonzero $\mathfrak{A}(N, N)$ -socle, and so does $\text{Ext}_{\mathfrak{A}}^1(\tau^-N, N)$. By Theorem 2.3 in [35], we obtain a desired almost split sequence as stated in Statement (2). The proof of the theorem is completed.

As an easy example, we obtain Auslander's results stated in [4, (II.6.3), (II.6.6)].

2.5.2. THEOREM (AUSLANDER). *Let A be a noetherian R -algebra with an identity with R complete noetherian local, and let M be an indecomposable module in $\text{Mod}A$.*

- (1) *If M is noetherian and not projective, then there exists an almost split sequence $0 \longrightarrow \tau M \longrightarrow E \longrightarrow M \longrightarrow 0$ in $\text{Mod}A$, where τM is artinian.*
- (2) *If M is artinian and not injective, then there exists an almost split sequence $0 \longrightarrow M \longrightarrow E \longrightarrow \tau^-M \longrightarrow 0$ in $\text{Mod}A$, where τ^-M is noetherian.*

Proof. The category $\text{Mod}A$ of left A -modules has enough projective and injective modules. And the full subcategories of noetherian modules and artinian modules in $\text{Mod}A$ are Krull-Schmidt; see [4, (I.5.1), (I.5.2)]. In particular, the full subcategory $\text{proj}A$ of finitely generated projective modules in $\text{Mod}A$ is Krull-Schmidt and Hom-noetherian; see [4, (I.4.2)]. By Lemma 5.5 in [34], there exists a Nakayama functor $\nu = D\text{Hom}_A(-, A) : \text{proj}A \rightarrow \text{Mod}A$. Further, a module in $\text{Mod}A$ is noetherian if and only if it has a projective presentation over $\text{proj}A$; and artinian if and only if it has an injective copresentation over $\nu(\text{proj}A)$; see [4, (I.5.2)]. Now, the statement follows from Theorem 2.5.1. The proof of the corollary is completed.

2.5.3. REMARK. Let Σ be a ring with an identity. Auslander obtained an existence theorem for almost split sequences in $\text{Mod}\Sigma$ ending with finitely presented modules; see [4, (II.5.1)]. To the best of our knowledge, there exists no such statement for almost split sequences in $\text{Mod}\Sigma$ starting with finitely copresented modules.

3. APPLICATIONS TO FUNCTOR CATEGORIES

An R -category is called *small* if its objects form a set and *endo-local* if all endomorphism algebras are local. The objective of this section is to apply our previous results to establish the existence of almost split sequences for finitely presented modules and for finitely copresented modules in the category of all modules over a small endo-local Hom-reflexive R -category. The key ingredients for this objective include a Nakayama functor and the sufficiency of projective and injective objects. The latter fact was stated in [41], but its proof seems to have never been published.

3.1. MODULES OVER A CATEGORY. Throughout this section, \mathcal{C} stands for a small R -category. A *left \mathcal{C} -module* is a covariant functor $M : \mathcal{C} \rightarrow \text{Mod}R$. Given left \mathcal{C} -modules M, N , a *\mathcal{C} -linear morphism* $f : M \rightarrow N$ consists of a family of R -linear maps $f_x : M(x) \rightarrow N(x)$ with $x \in \mathcal{C}$ such that $N(u) \circ f_x = f_y \circ M(u)$, for all morphisms $u : x \rightarrow y$ in \mathcal{C} . Write $\text{Hom}_{\mathcal{C}}(M, N)$ for the R -module of \mathcal{C} -linear morphisms $f : M \rightarrow N$. The category $\text{Mod}\mathcal{C}$ of all left \mathcal{C} -modules is an abelian R -category with arbitrary products and coproducts; see [4, Section 2]. A module $M \in \text{Mod}\mathcal{C}$ is called *locally R -reflexive* if $M(x)$ is an R -reflexive for all $x \in \mathcal{C}$. Since the category $\text{mod}R$ of reflexive R -modules is abelian, the full subcategory $\text{mod}\mathcal{C}$ of locally R -reflexive modules in $\text{Mod}\mathcal{C}$ is abelian.

We denote by \mathcal{C}° the opposite category of \mathcal{C} : the objects are those of \mathcal{C} and the morphisms $u^\circ : y \rightarrow x$ are induced by the morphisms $u : x \rightarrow y$ in \mathcal{C} . Given a module M in $\text{Mod}\mathcal{C}^\circ$, we define a module $\mathfrak{D}M$ in $\text{Mod}\mathcal{C}$ by $(\mathfrak{D}M)(x) = D(M(x))$ and $(\mathfrak{D}M)(u) = D(M(u^\circ))$ for all objects x and morphisms u in \mathcal{C} . This yields an exact contravariant functor $\mathfrak{D} : \text{Mod}\mathcal{C}^\circ \rightarrow \text{Mod}\mathcal{C}$.

LRMod

3.1.1. PROPOSITION. *Let \mathcal{C} be a small R -category. The exact contravariant functor $\mathfrak{D} : \text{Mod}\mathcal{C}^\circ \rightarrow \text{Mod}\mathcal{C}$ restricts to a duality $\mathfrak{D} : \text{mod}\mathcal{C}^\circ \rightarrow \text{mod}\mathcal{C}$.*

Proof. Let $M \in \text{Mod}\mathcal{C}^\circ$. By Lemma 1.1.1, there exist canonical R -linear isomorphisms $\sigma_{M,x} : M(x) \rightarrow D^2(M(x))$ with $x \in \mathcal{C}$, which form a \mathcal{C} -linear isomorphism, natural in M . The proof of the proposition is completed.

3.2. PROJECTIVE MODULES. For each object x in \mathcal{C} , we obtain a left \mathcal{C} -module $P_x = \mathcal{C}(x, -) : \mathcal{C} \rightarrow \text{Mod}R$. And a morphism $u : y \rightarrow x$ in \mathcal{C} induces a \mathcal{C} -linear morphism $\mathcal{C}(u, -) : P_x \rightarrow P_y$ with $\mathcal{C}(u, -)_z = \mathcal{C}(u, z) : \mathcal{C}(x, z) \rightarrow \mathcal{C}(y, z)$ for $z \in \mathcal{C}$. We denote by 1_x the identity morphism associated with an object x in \mathcal{C} .

Mor_proj_mod

3.2.1. PROPOSITION. *Let \mathcal{C} be a small R -category. Given $M \in \text{Mod}\mathcal{C}$ and $x \in \mathcal{C}$, we have an R -linear isomorphism, which is natural in M , as follows:*

$$\Phi_{P_x, M} : \text{Hom}_{\mathcal{C}}(P_x, M) \rightarrow M(x) : f \mapsto f_x(1_x).$$

Proof. Fix $M \in \text{Mod}\mathcal{C}$ and $x \in \mathcal{C}$. By the Yoneda Lemma, $\Phi_{P_x, M}$ is an R -linear isomorphism, which is natural in M and x . If $\varphi : P_x \rightarrow P_y$ is a \mathcal{C} -linear morphism, by the Yoneda Lemma again, $\varphi = \mathcal{C}(u, -)$ for some $u : y \rightarrow x$ in \mathcal{C} . Thus $\Phi_{P_x, M}$ is natural in P_x because it is natural in x . The proof of the proposition is completed.

proj_Cmod

3.2.2. REMARK. By Proposition 3.2.1, P_x is projective in $\text{Mod}\mathcal{C}$, for all $x \in \mathcal{C}$.

The following result is stated with a sketched proof in [41]. We include a short proof for the reader's convenience.

Proj_C_mcat

3.2.3. PROPOSITION. *Let \mathcal{C} be a small R -category. Then $\text{Mod}\mathcal{C}$ has enough projective objects.*

Proof. Fix $M \in \text{Mod}\mathcal{C}$. Given $x \in \mathcal{C}$ with $m \in M(x)$, we write $x_m = x$. Then, $P = \coprod_{x \in \mathcal{C}; m \in M(x)} \mathcal{C}(x_m, -)$ is projective in $\text{Mod}\mathcal{C}$; see (3.2.2). For $y \in \mathcal{C}$, we have an R -linear map $\psi_y : P(y) = \coprod_{x \in \mathcal{C}; m \in M(x)} \mathcal{C}(x_m, y) \rightarrow M(y)$, defined by $\psi_y(u_{x,m}) = M(u_{x,m})(m)$ for $u_{x,m} \in \mathcal{C}(x_m, y)$ such, for any $v : y \rightarrow z$ in \mathcal{C} , that

$$\begin{array}{ccc} \coprod_{x \in \mathcal{C}; m \in M(x)} \mathcal{C}(x_m, y) & \xrightarrow{\psi_y} & M(y) \\ \downarrow \coprod_{x \in \mathcal{C}; m \in M(x)} \mathcal{C}(x_m, v) & & \downarrow M(v) \\ \coprod_{x \in \mathcal{C}; m \in M(x)} \mathcal{C}(x_m, z) & \xrightarrow{\psi_z} & M(z) \end{array}$$

commutes. Given $m \in M(y)$, we have $\psi_y(1_y) = M(1_y)(m) = m$. Thus, the R -linear epimorphisms ψ_y with $y \in \mathcal{C}$ form a \mathcal{C} -linear epimorphism $\psi : P \rightarrow M$. The proof of the proposition is completed.

3.3. INJECTIVE MODULES. Given $x \in \mathcal{C}$, we have a module $I_x = \mathfrak{D}(P_x^\circ)$ in $\text{Mod}\mathcal{C}$, where $P_x^\circ = \mathcal{C}^\circ(x, -) = \mathcal{C}(-, x) \in \text{Mod}\mathcal{C}^\circ$. More explicitly, $I_x(y) = D\mathcal{C}(y, x)$ and $I_x(u) = D\mathcal{C}(u, x) : D\mathcal{C}(y, x) \rightarrow D\mathcal{C}(z, x)$, for objects y and morphisms $u : y \rightarrow z$ in \mathcal{C} . A morphism $v : x \rightarrow y$ in \mathcal{C} induces a \mathcal{C} -linear morphism $D\mathcal{C}(-, v) : I_y \rightarrow I_x$. And for $x \in \mathcal{C}$, we have an R -linear function $\theta_x : \mathcal{C}(x, x) \rightarrow I_R$ given by $f \mapsto f(1_x)$.

Mor-mod-inj

3.3.1. PROPOSITION. *Let \mathcal{C} be a small R -category. Given $M \in \text{Mod}\mathcal{C}$ and $x \in \mathcal{C}$, we have an R -linear isomorphism, which is natural in M , as follows:*

$$\Psi_{M, I_x} : \text{Hom}_{\mathcal{C}}(M, I_x) \rightarrow D(M(x)) : f \mapsto \theta_x \circ f_x.$$

Proof. Fix $M \in \text{Mod}\mathcal{C}$ and $x \in \mathcal{C}$. Clearly, we have an R -linear map Ψ_{M, I_x} as stated in the proposition. Assume that $f \in \text{Hom}_{\mathcal{C}}(M, I_x)$ such that $\Psi_{M, I_x}(f) = 0$. Given $y \in \mathcal{C}$ and $u \in \mathcal{C}(y, x)$, since $D\mathcal{C}(u, x) \circ f_y = f_x \circ M(u)$, we have

$$\begin{aligned} f_y(m)(u) &= f_y(m)(\mathcal{C}(u, x)(1_x)) \\ &= \theta_x(D\mathcal{C}(u, x)(f_y(m))) \\ &= \theta_x(f_x(M(u)(m))) \\ &= \Psi_{M, I_x}(f)(M(u)(m)) \\ &= 0, \end{aligned}$$

for all $m \in M(y)$ and $u \in \mathcal{C}(y, x)$. Thus, $f_y(m) = 0$, for all $m \in M(y)$. That is, $f_y = 0$ for all $y \in \mathcal{C}$, and hence, $f = 0$. So, Ψ_{M, I_x} is a monomorphism. Conversely, given any R -linear function $g : M(x) \rightarrow I_R$, we shall construct a \mathcal{C} -linear morphism $f : M \rightarrow I_x$. Indeed, for any $y \in \mathcal{C}$, we have an R -linear map

$$f_y : M(y) \rightarrow I_x(y) = D\mathcal{C}(y, x) : m \mapsto f_y(m)$$

such that $f_y(m)(u) = g(M(u)(m))$, for $u \in \mathcal{C}(y, x)$. Given $v \in \mathcal{C}(y, z)$, we claim that $D\mathcal{C}(v, x) \circ f_y = f_z \circ M(v)$. Indeed, for $m \in M(y)$ and $u \in \mathcal{C}(z, x)$, we have

$$\begin{aligned} (D\mathcal{C}(v, x))(f_y(m))(u) &= (f_y(m) \circ \mathcal{C}(v, x))(u) \\ &= f_y(m)(uv) \\ &= g(M(uv)(m)) \\ &= f_z(M(v)(m))(u). \end{aligned}$$

This establishes our claim. Hence, the R -linear maps $f_y : M(y) \rightarrow I_x(y)$ with $y \in \mathcal{C}$ form a \mathcal{C} -linear morphism $f : M \rightarrow I_x$. Moreover, for any $m \in M(x)$, we have

$$\Psi_{M, I_x}(f)(m) = \psi_x(f_x(m)) = f_x(m)(1_x) = (g \circ M(1_x))(m) = g(m).$$

That is, $\Psi_{M, I_x}(f) = g$. So, Ψ_{M, I_x} is surjective. Finally, a routine verification shows that Ψ_{M, I_x} is natural in M . The proof of the proposition is completed.

inj_Cmod

3.3.2. REMARK. Since the dual functor D is exact, we deduce from Proposition 3.3.1 that I_x is an injective object in $\text{Mod}\mathcal{C}$, for all $x \in \mathcal{C}$.

The following result was stated without any proof in [41]. We include a short proof for the reader's convenience.

Inj_C_mcat

3.3.3. PROPOSITION. *Let \mathcal{C} be a small R -category. Then, $\text{Mod}\mathcal{C}$ has enough injective objects.*

Proof. Fix $M \in \text{Mod}\mathcal{C}$. For $x \in \mathcal{C}$ and $\varphi \in D(M(x))$, we write $x_\varphi = x$. Then, $I = \prod_{x \in \mathcal{C}; \varphi \in D(M(x))} D\mathcal{C}(-, x_\varphi)$ is an injective object in $\text{Mod}\mathcal{C}$; see (3.3.2). For any $y \in \mathcal{C}$, we have an R -linear map

$\psi_y : M(y) \rightarrow I(y) = \prod_{x \in \mathcal{C}; \varphi \in D(M(x))} D\mathcal{C}(y, x_\varphi) : m \mapsto (f_{x, \varphi, m})_{x \in \mathcal{C}; \varphi \in D(M(x))}$, where $f_{x, \varphi, m} \in D\mathcal{C}(y, x_\varphi)$ such that $f_{x, \varphi, m}(u) = \varphi(M(u)(m))$, for $u \in \mathcal{C}(y, x_\varphi)$. Given a morphism $v : y \rightarrow z$ in \mathcal{C} , it is not hard to verify that the diagram

$$\begin{array}{ccc} M(y) & \xrightarrow{\psi_y} & \prod_{x \in \mathcal{C}; \varphi \in D(M(x))} \mathcal{C}(y, x_\varphi) \\ M(v) \downarrow & & \downarrow \prod_{x \in \mathcal{C}; \varphi \in D(M(x))} D\mathcal{C}(x_\varphi, v) \\ M(z) & \xrightarrow{\psi_z} & \prod_{x \in \mathcal{C}; \varphi \in D(M(x))} D\mathcal{C}(z, x_\varphi) \end{array}$$

commutes. Assume that $\psi_y(m) = 0$ for some $m \in M(y)$ with $y \in Q_0$. Then, for any $\varphi \in D(M(y))$, we have $0 = f_{y, \varphi, m}(1_y) = \varphi(M(1_y)(m)) = \varphi(m)$. Since I_R is a cogenerator for $\text{Mod}R$, we have $m = 0$. So, ψ_y is a monomorphism. Therefore, the R -linear maps ψ_y with $y \in \mathcal{C}$ form a \mathcal{C} -linear monomorphism $\psi : M \rightarrow I$. The proof of the proposition is completed.

3.4. THE NAKAYAMA FUNCTOR. Let \mathcal{C} be a small R -category. We shall construct a Nakayama functor for $\text{Mod}\mathcal{C}$. Write $\text{proj}\mathcal{C}$ and $\text{inj}\mathcal{C}$ for the strictly full additive subcategories of $\text{Mod}\mathcal{C}$ generated by the projective modules P_x with $x \in \mathcal{C}$; see (3.2.2), and by the injective modules I_x with $x \in \mathcal{C}$; see (3.3.2), respectively.

NK_functor

3.4.1. THEOREM. *Let \mathcal{C} be a small R -category. Then, there exists a Nakayama functor $\nu : \text{proj}\mathcal{C} \rightarrow \text{Mod}\mathcal{C}$, sending P_x to I_x for all objects $x \in \mathcal{C}$.*

Proof. We begin with the full subcategory \mathcal{P} of $\text{Mod}\mathcal{C}$ generated by the modules P_x with $x \in \mathcal{C}$. By the Yoneda Lemma, $\text{Hom}_{\mathcal{C}}(P_x, P_y) = \{\mathcal{C}(u, -) \mid u \in \mathcal{C}(y, x)\}$. Thus, setting $\nu P_x = I_x$ and $\nu \mathcal{C}(u, -) = D\mathcal{C}(-, u)$ for objects x and morphisms u in \mathcal{C} , we obtain an R -linear functor $\nu : \mathcal{P} \rightarrow \text{Mod}\mathcal{C}$.

Fix $M \in \text{Mod}\mathcal{C}$ and $x \in \mathcal{C}$. By Proposition 3.2.1, we have an R -linear isomorphism $\Phi_{P_x, M} : \text{Hom}_{\mathcal{C}}(P_x, M) \rightarrow M(x)$, which is natural in P_x and M . This induces an R -linear isomorphism $D\Phi_{P_x, M} : D(M(x)) \rightarrow D\text{Hom}_{\mathcal{C}}(P_x, M)$, which is natural in P_x and M . And by Proposition 3.3.1, we have an R -linear isomorphism

$$\Psi_{M, \nu P_x} : \text{Hom}_{\mathcal{C}}(M, \nu P_x) \rightarrow D(M(x)) : f \mapsto \theta_x \circ f_x,$$

which is natural in M . We claim that $\Psi_{M, \nu P_x}$ is natural in P_x , or equivalently, $\Psi_{M, \nu P_y} \circ \text{Hom}_{\mathcal{C}}(M, \nu \mathcal{C}(u, -)) = D(M(y)) \circ \Psi_{M, \nu P_x}$, for any $u : y \rightarrow x$ in \mathcal{C} . Indeed, given $f \in \text{Hom}_{\mathcal{C}}(M, \nu P_x)$ and $m \in M(y)$, we have

$$\begin{aligned} (\Psi_{M, \nu P_y} \circ \text{Hom}_{\mathcal{C}}(M, \nu \mathcal{C}(u, -))(f))(m) &= (\theta_y \circ (D\mathcal{C}(-, u) \circ f)_y)(m) \\ &= \theta_y(D\mathcal{C}(y, u)(f_y(m))) \\ &= (f_y(m) \circ \mathcal{C}(y, u))(1_y) \\ &= f_y(m)(u). \end{aligned}$$

And since f is \mathcal{C} -linear, we have $D\mathcal{C}(u, x) \circ f_y = f_x \circ M(u)$. This yields

$$\begin{aligned} (D(M(u)) \circ \Psi_{M, \nu P_x}(f))(m) &= (\theta_x \circ D\mathcal{C}(u, x) \circ f_y)(m) \\ &= \theta_x(D\mathcal{C}(u, x)(f_y(m))) \\ &= (f_y(m) \circ \mathcal{C}(u, x))(1_x) \\ &= f_y(m)(u). \end{aligned}$$

This establishes our claim. Composing $\Psi_{M, \nu P_x}$ and $D\Phi_{P_x, M}$ yields an R -linear isomorphism $\Theta_{M, P_x} : \text{Hom}_{\mathcal{C}}(M, \nu P_x) \rightarrow D\text{Hom}_{\mathcal{C}}(P_x, M)$, which is natural in P_x and M . Now, since every object in $\text{proj}\mathcal{C}$ is isomorphic to a finite direct sum of the P_x with $x \in \mathcal{C}$, we can extend, in a canonical way, the functor $\nu : \mathcal{P} \rightarrow \text{Mod}\mathcal{C}$ to a functor $\nu : \text{proj}\mathcal{C} \rightarrow \text{Mod}\mathcal{C}$. Moreover, the isomorphism Θ_{M, P_x} can also be

extended to an R -linear isomorphism $\Theta_{M,P} : \text{Hom}_{\mathcal{C}}(M, \nu P) \rightarrow D\text{Hom}_{\mathcal{C}}(P, M)$, for $P \in \text{proj}\mathcal{C}$ and $M \in \text{Mod}\mathcal{C}$, and it is natural in P and M . This shows that $\nu : \text{proj}\mathcal{C} \rightarrow \text{Mod}\mathcal{C}$ is a Nakayama functor. The proof of the theorem is completed.

3.5. FINITELY PRESENTED AND FINITELY COPRESENTED MODULES. Let \mathcal{C} be a small R -category. Fix a module M in $\text{Mod}\mathcal{C}$. We shall say that M is *finitely generated* if there exists an epimorphism $f : P \rightarrow M$ with $P \in \text{proj}\mathcal{C}$; and *finitely presented* if it admits a projective presentation over $\text{proj}\mathcal{C}$, or equivalently, there exists an epimorphism $f : P \rightarrow M$ with P in $\text{proj}\mathcal{C}$ and $\text{Ker}(f)$ finitely generated. Dually, M is called *finitely cogenerated* if there exists a monomorphism $g : M \rightarrow I$ with $I \in \text{inj}\mathcal{C}$; and *finitely copresented* if it admits an injective copresentation over $\text{inj}\mathcal{C}$, or equivalently, there exists a monomorphism $g : M \rightarrow I$ with I in $\text{inj}\mathcal{C}$ and $\text{Coker}(g)$ finitely cogenerated. We denote by $\text{mod}^+\mathcal{C}$ and $\text{mod}^-\mathcal{C}$ the full subcategories of finitely presented modules and finitely copresented modules in $\text{Mod}\mathcal{C}$, respectively. By Propositions 2.1.1 and 2.1.2, $\text{mod}^+\mathcal{C}$ and $\text{mod}^-\mathcal{C}$ are exact R -categories. It is interesting to know when they are abelian categories.

fpres_AB

3.5.1. PROPOSITION. *Let \mathcal{C} be a small R -category. Then*

- (1) *$\text{mod}^+\mathcal{C}$ is abelian if and only if the finitely generated submodules of the modules in $\text{proj}\mathcal{C}$ are finitely presented;*
- (2) *$\text{mod}^-\mathcal{C}$ is abelian if and only if the finitely cogenerated quotient-modules of the modules in $\text{inj}\mathcal{C}$ are finitely copresented.*

Proof. We shall only prove Statement (1). Assume that $\text{mod}^+\mathcal{C}$ is abelian. Let M be a finitely generated submodule of a module P_0 in $\text{proj}\mathcal{C}$. Then, we have an epimorphism $f : P_1 \rightarrow M$ with $P_1 \in \text{proj}\mathcal{C}$, and hence, a morphism $jf : P_1 \rightarrow P_0$ in $\text{mod}^+\mathcal{C}$, where $j : M \rightarrow P_0$ is the inclusion map. Note that f and jf have the same kernel K , which lies in $\text{mod}^+\mathcal{C}$. Thus, M is finitely presented.

Assume that the finitely generated submodules of modules in $\text{proj}\mathcal{C}$ are finitely presented. Consider a morphism $f : P_1 \rightarrow P_0$ in $\text{proj}\mathcal{C}$ and its kernel $j : K \rightarrow P_1$ in $\text{Mod}\mathcal{C}$. By the assumption, there exists an epimorphism $f' : P'_1 \rightarrow \text{Im}(f)$ having a kernel $q : K' \rightarrow P'_1$, where $P'_1 \in \text{proj}\mathcal{C}$ and K' is finitely generated. By Schanule's Lemma, $K' \oplus P_1 \cong K \oplus P'_1$. Thus, K is finitely generated with an epimorphism $g : P_2 \rightarrow K$, where $P_2 \in \text{proj}\mathcal{C}$. So, $P_2 \xrightarrow{jg} P_1 \xrightarrow{f} P_0$ is an exact sequence in $\text{proj}\mathcal{C}$. Thus, $\text{mod}^+\mathcal{C}$ is closed under kernels; see [2, (2.1)]. Being also closed under cokernels; see (2.1.1), $\text{mod}^+\mathcal{C}$ is abelian. The proof of the proposition is completed.

3.6. THE EXISTENCE THEREOM. In this subsection, we need to assume that \mathcal{C} is endo-local and Hom-reflexive. In this setting, we have the following statement.

C-proj

3.6.1. PROPOSITION. *Let \mathcal{C} be a small endo-local Hom-reflexive R -category. Then*

- (1) *$\text{proj}\mathcal{C}$ and $\text{inj}\mathcal{C}$ are Hom-reflexive Krull-Schmidt subcategories of $\text{mod}\mathcal{C}$;*
- (2) *the functor $\mathfrak{D} : \text{Mod}\mathcal{C} \rightarrow \text{Mod}\mathcal{C}$ restricts to a duality $\mathfrak{D} : \text{mod}^+\mathcal{C}^\circ \rightarrow \text{mod}^-\mathcal{C}$.*

Proof. Let $x, y \in \mathcal{C}$. Since $P_x(y) = \mathcal{C}(x, y)$, $\text{proj}\mathcal{C} \subseteq \text{mod}\mathcal{C}$. By Proposition 3.2.1, there exists an R -linear isomorphism $\varphi_{x,y} : \text{Hom}_{\mathcal{C}}(P_x, P_y) \rightarrow \mathcal{C}(y, x)$, natural in x and y . So, $\text{proj}\mathcal{C}$ is Hom-reflexive. Since $\varphi_{x,x} : \text{End}_{\mathcal{C}}(P_x) \rightarrow \mathcal{C}(x, x)$ is an algebra anti-isomorphism, $\text{proj}\mathcal{C}$ is Krull-Schmidt. Since \mathcal{C}° is Hom-reflexive and Krull-Schmidt, $\text{proj}\mathcal{C}^\circ$ is Hom-reflexive, Krull-Schmidt and contained in $\text{mod}\mathcal{C}^\circ$. Then, the duality $\mathfrak{D} : \text{mod}\mathcal{C}^\circ \rightarrow \text{mod}\mathcal{C}$ restricts a duality $\mathfrak{D} : \text{proj}\mathcal{C}^\circ \rightarrow \text{inj}\mathcal{C}$; see (3.1.1). Therefore, $\text{inj}\mathcal{C}$ is Hom-reflexive, Krull-Schmidt and contained in $\text{mod}\mathcal{C}$.

Since $\text{mod}\mathcal{C}^\circ$ and $\text{mod}\mathcal{C}$ are abelian, $\text{mod}^+\mathcal{C}^\circ \subseteq \text{mod}\mathcal{C}^\circ$ and $\text{mod}^-\mathcal{C} \subseteq \text{mod}\mathcal{C}$. For any $M \in \text{mod}^+\mathcal{C}^\circ$ with a projective presentation $P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$ over $\text{proj}\mathcal{C}^\circ$, we have an injective copresentation $0 \rightarrow \mathfrak{D}M \rightarrow \mathfrak{D}P_0 \rightarrow \mathfrak{D}P_1$ over $\text{inj}\mathcal{C}$. Hence, $\mathfrak{D}M \in \text{mod}^-\mathcal{C}$. So, the duality $\mathfrak{D} : \text{mod}\mathcal{C}^\circ \rightarrow \text{mod}\mathcal{C}$ restricts to a duality $\mathfrak{D} : \text{mod}^+\mathcal{C}^\circ \rightarrow \text{mod}^-\mathcal{C}$. The proof of the proposition is completed.

3.6.2. REMARK. Let \mathcal{C} be a small R -category, where R is a commutative artinian ring. By Proposition 3.6.1(2), we see that \mathcal{C} is a dualizing R -variety as defined in [7, Section 2] if and only if \mathcal{C} is endo-local and Hom-finite with $\text{mod}^+\mathcal{C} = \text{mod}^-\mathcal{C}$.

We are ready to obtain the main result of this section.

C_mod_ass 3.6.3. THEOREM. *Let \mathcal{C} be a small endo-local Hom-reflexive R -category. Consider a strongly indecomposable module M in $\text{Mod}\mathcal{C}$.*

- (1) *If $M \in \text{mod}^+\mathcal{C}$ is not projective, then there exists an almost split sequence $0 \rightarrow \tau M \rightarrow E \rightarrow M \rightarrow 0$ in $\text{Mod}\mathcal{C}$, where $\tau M \in \text{mod}^-\mathcal{C}$.*
- (2) *If $M \in \text{mod}^-\mathcal{C}$ is not injective, then there exists an almost split sequence $0 \rightarrow M \rightarrow E \rightarrow \tau^-M \rightarrow 0$ in $\text{Mod}\mathcal{C}$ where $\tau^-M \in \text{mod}^+\mathcal{C}$.*

Proof. By Theorem 3.4.1, we have a Nakayama functor $\nu : \text{proj}\mathcal{C} \rightarrow \text{Mod}\mathcal{C}$, where $\text{proj}\mathcal{C}$ is Hom-reflexive and Krull-Schmidt; see (3.6.1). Since $\text{Mod}\mathcal{C}$ has enough projective and injective objects; see (3.2.3) and (3.3.3), the result follows immediately from Theorem 2.5.1. The proof of the theorem is completed.

ass_cln 3.6.4. REMARK. Let \mathcal{C} be a small endo-local Hom-noetherian R -category with R being complete noetherian local.

- (1) In Theorem 3.6.3, it suffices to assume that M is indecomposable; see (2.4.6).
- (2) Theorem 3.6.3(1) is stated as Theorem 6 in [5]. However, we cannot find any proof in the existing literature, and we do not see how to establish an Auslander-Reiten duality for $\text{Mod}\mathcal{C}$ by the classical approach as used in [7, (7.4)] or [4, (I.3.4)].

4. APPLICATION TO REPRESENTATIONS OF QUIVERS WITH RELATIONS

The objective of this section is mainly to apply our previous results to study the existence of almost split sequences over algebras defined by quivers with relations. In the local semiperfect case; see (4.4.1), a general existence theorem in the category of all modules is a special case for small endo-local Hom-reflexive categories; see (4.7.1). So, we shall focus on the subcategories of finitely presented modules, finitely copresented modules and finite dimensional modules. For locally semiprimary algebras given by locally finite quivers; see (4.5.1), we show that the almost split sequences in the first two subcategories are almost split in the category of all modules; see (4.7.4) and (4.7.6). More importantly, we shall study when these subcategories have almost split sequences; see (4.8.1), (4.8.3) and (4.8.5).

Throughout this section, k denotes a field. Note that the dual functor for $\text{Mod}k$ is $D = \text{Hom}_k(-, k)$, and a module in $\text{Mod}k$ is reflexive if and only if it is finite dimensional. In particular, a k -category is Hom-reflexive if and only if it is Hom-finite. In this section, a k -algebra has enough idempotents but not necessarily an identity, in which an ideal is always two-sided.

4.1. QUIVERS. Let $Q = (Q_0, Q_1)$ be a quiver, where Q_0 is the set of vertices and Q_1 is the set of arrows. Given $\alpha : x \rightarrow y$ in Q_1 , we call x the *starting point* and y the *ending point*, and write $s(\alpha) = x$ and $e(\alpha) = y$. For each $x \in Q_0$, one associates

a *trivial path* ε_x with $s(\varepsilon_x) = e(\varepsilon_x) = x$. A *path* of length $n (> 0)$ is a sequence $\rho = \alpha_n \cdots \alpha_1$, where $\alpha_i \in Q_1$ such that $s(\alpha_{i+1}) = e(\alpha_i)$, for $1 \leq i < n$. Two paths are called *comparable* if one is the subpath of the other, and a set of paths is called *comparable* if every two paths in the set are comparable. For $x \in Q_0$ and $n \geq 0$, we denote by $Q_n(x, -)$ and $Q_n(-, x)$ the sets of paths of length n starting with x , and those ending with x , respectively. We say that Q is *left locally finite* if $Q_1(x, -)$ is finite for all $x \in Q_0$; *right locally finite* if $Q_1(-, x)$ is finite for all $x \in Q_0$, and *locally finite* if Q is left and right locally finite. Further, Q is called *interval-finite* if the set of paths from x to y is finite, for all $x, y \in Q_0$; and *strongly locally finite* if Q is locally finite and interval-finite.

The *opposite* quiver Q° of Q is defined in such a way that $(Q^\circ)_0 = Q_0$ and $(Q^\circ)_1 = \{\alpha^\circ : y \rightarrow x \mid \alpha : x \rightarrow y \in Q_1\}$. A non-trivial path $\rho = \alpha_n \cdots \alpha_1$ in Q , where $\alpha_i \in Q_1$, induces a non-trivial path $\rho^\circ = \alpha_1^\circ \cdots \alpha_n^\circ$ in Q° . However, we shall identify the trivial path in Q° at a vertex x with that in Q .

4.2. ALGEBRAS GIVEN BY QUIVERS WITH RELATIONS. Let Q be a quiver. Write kQ for the path algebra of Q over k . An ideal \mathfrak{I} in kQ is a *relation ideal* if $\mathfrak{I} \subseteq (kQ^+)^2$, where kQ^+ is the ideal in kQ generated by the arrows in Q . In this case, (Q, \mathfrak{I}) is called a *bound quiver*, and a path ρ in Q is called *nonzero in Λ* , or simply *nonzero*, if $\rho \notin \mathfrak{I}$. Given $\rho = \sum \lambda_i \rho_i \in kQ$, where $\lambda_i \in k$ and ρ_i is a path in Q , we write $\rho^\circ = \sum \lambda_i \rho_i^\circ \in kQ^\circ$. In this way, an ideal \mathfrak{I} in kQ is a relation ideal if and only if $\mathfrak{I}^\circ = \{\rho^\circ \mid \rho \in \mathfrak{I}\}$ is a relation ideal in kQ° .

Consider now a k -algebra $\Lambda = kQ/\mathfrak{I}$, where Q is a quiver and \mathfrak{I} is a relation ideal in kQ . Write $\bar{\gamma} = \gamma + \mathfrak{I} \in \Lambda$ for $\gamma \in kQ$. Then Λ has a complete orthogonal set of idempotents $\{e_x \mid x \in Q_0\}$, where $e_x = \bar{\varepsilon}_x$. Set $J_\Lambda = \{\bar{\rho} \mid \rho \in kQ^+\} := J$. Note that $kQ^\circ/\mathfrak{I}^\circ$ is the opposite algebra of Λ . Write $\bar{\rho}^\circ = \rho^\circ + \mathfrak{I}^\circ$ for $\rho \in kQ$ but $e_x = \varepsilon_x + \mathfrak{I}^\circ$ for $x \in Q_0$. In particular, $J_{\Lambda^\circ} = \{\bar{\rho}^\circ \mid \rho \in kQ^+\} = J^\circ$.

A left Λ -module M is called *unitary* if $M = \sum_{x \in Q_0} e_x M$. We denote by $\text{Mod}\Lambda$ the category of all unitary left Λ -modules. Observe that the k -algebra Λ can also be regarded as a small k -category, whose objects are the vertices of Q and whose morphism spaces are $\Lambda(x, y) = e_y \Lambda e_x$ with $x, y \in Q_0$. In this viewpoint, a k -linear representation of (Q, \mathfrak{I}) is a covariant functor from Λ to $\text{Mod}k$. It is well known that $\text{Mod}\Lambda$ is equivalent to the category $\text{Rep}(Q, \mathfrak{I})$ of all k -linear representations of (Q, \mathfrak{I}) , in such a way that a module M corresponds to a representation \tilde{M} defined by $\tilde{M}(x) = e_x M$ and $\tilde{M}(u) : \tilde{M}(x) \rightarrow \tilde{M}(y)$ being the left multiplication by u , where $x, y \in Q_0$ and $u \in \Lambda(x, y)$. Identifying M with \tilde{M} , we shall freely use the terminology, the notation and the results stated in Section 4 for Λ -modules. In particular, for each $x \in Q_0$, we have a projective module $P_x = \Lambda e_x = \Lambda(x, -)$; see (3.2.2) and a simple module $S_x = \Lambda e_x/J e_x$ in $\text{Mod}\Lambda$. And we have a projective module $P_x^\circ = \Lambda^\circ e_x = \Lambda(-, x)$ and a simple module $S_x^\circ = \Lambda^\circ e_x/J^\circ e_x$ in $\text{Mod}\Lambda^\circ$.

For a module M in $\text{Mod}\Lambda^\circ$, we have a module $\mathfrak{D}M = \oplus_{x \in Q_0} D(e_x M)$ in $\text{Mod}\Lambda$ such, for all $\varphi \in D(e_x M)$ and $u \in e_y \Lambda e_x$, that $u\varphi \in D(e_y M)$, which is defined by $(u\varphi)(m) = \varphi(u^\circ m)$, for $m \in e_y M$. For a morphism $f : M \rightarrow N$ in $\text{Mod}\Lambda^\circ$, we have a morphism $\mathfrak{D}f : \mathfrak{D}N \rightarrow \mathfrak{D}M$ in $\text{Mod}\Lambda$ such, for $\varphi \in D(e_x N)$ with $x \in Q_0$, that $(\mathfrak{D}f)(\varphi) = \varphi \circ f_x$, where $f_x : e_x M \rightarrow e_x N$ is the restriction of f . This yields a contravariant functor $\mathfrak{D} : \text{Mod}\Lambda^\circ \rightarrow \text{Mod}\Lambda$. For each $x \in Q_0$, we see that $D(S_x^\circ) \cong S_x$ and $I_x = \mathfrak{D}(P_x^\circ) = \mathfrak{D}(-, x)$, an injective module in $\text{Mod}\Lambda$; see (3.3.2).

A module M in $\text{Mod}\Lambda$ is called *locally finite dimensional* if $\dim_k(e_x M) < \infty$ for all $x \in Q_0$. We denote by $\text{mod}\Lambda$ and $\text{mod}^b\Lambda$ the full subcategories of locally finite

dimensional modules and finite dimensional modules in $\text{Mod}\Lambda$, respectively. As a special case of Proposition 3.1.1, we obtain the following statement.

duality_lfdm

4.2.1. PROPOSITION. *Let $\Lambda = kQ/\mathfrak{I}$, where Q is a quiver and \mathfrak{I} is an ideal in kQ . Then the contravariant functor $\mathfrak{D} : \text{Mod}\Lambda^\circ \rightarrow \text{Mod}\Lambda$ restricts to dualities $\mathfrak{D} : \text{mod}\Lambda^\circ \rightarrow \text{mod}\Lambda$ and $\mathfrak{D} : \text{mod}^b\Lambda^\circ \rightarrow \text{mod}^b\Lambda$.*

4.3. LOCALLY NOETHERIAN ALGEBRAS. Let $\Lambda = kQ/\mathfrak{I}$, where Q is a quiver and \mathfrak{I} is an ideal in kQ . Recall that Λ is *locally left noetherian* if Λe_x is noetherian for all $x \in Q_0$, *locally right noetherian* if $e_x\Lambda$ is noetherian for all $x \in Q_0$, and *locally noetherian* if Λ is locally left and right noetherian. Moreover, Λ is *locally left bounded* if Λe_x is finite dimensional for all $x \in Q_0$, *locally right bounded* if $e_x\Lambda$ is finite dimensional for all $x \in Q_0$, and *locally bounded* if Λ is locally left and right bounded. Locally (left, right) bounded algebras are clearly locally (left, right) noetherian, and so are the special multiserial algebras; see [33, (1.3)]. To provide more classes of locally noetherian algebras, we introduce the following notion.

mserial

4.3.1. DEFINITION. Let $\Lambda = kQ/\mathfrak{I}$, where Q is a quiver and \mathfrak{I} is a relation ideal in kQ . We shall say that Λ is

- (1) *locally left eventually multiserial* if Q is left locally finite, and for each $x \in Q_0$, there exists an integer $n_x (\geq 0)$ such, for any $\rho \in Q_{n_x}(x, -)$, that the set of nonzero paths in Q starting with ρ is comparable;
- (2) *locally right eventually multiserial* if Q is right locally finite, and for any $x \in Q_0$, there exists an integer $n_x (\geq 0)$ such, for any $\rho \in Q_{n_x}(-, x)$, that the set of nonzero paths in Q ending with ρ is comparable;
- (3) *locally eventually multiserial* if Λ is locally left and right eventually multiserial.

Note that special multiserial algebras are locally left and right eventually multiserial, while locally bounded algebras are not necessarily so.

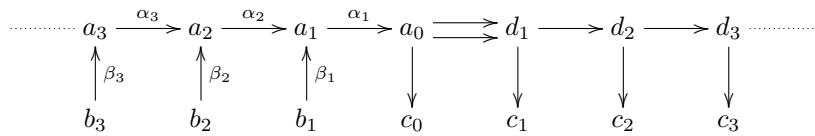
mul-noe

4.3.2. PROPOSITION. *Let $\Lambda = kQ/\mathfrak{I}$, where Q is a quiver and \mathfrak{I} is a relation ideal in kQ . If Λ is locally left or right eventually multiserial, then it is locally left or right noetherian, respectively.*

Proof. We shall consider only the case where Λ is locally left eventually multiserial. Let x be an vertex in Q with an integer n_x as stated in Definition 4.3.1(1). Fix $\rho \in Q_{n_x}(x, -)$. Then $\Lambda \bar{\rho} = \sum_{\eta \in \Omega(\rho)} k\bar{\eta}$, where $\Omega(\rho)$ denotes the set of nonzero paths in Q starting with ρ . If $\Omega(\rho)$ is finite, then $\Lambda \bar{\rho}$ is noetherian. Otherwise, since $\Omega(\rho)$ is comparable, there exist arrows $\alpha_i : x_i \rightarrow x_{i+1}$ in Q with $i \geq 1$ and $x_1 = x$, such that $\Omega(\rho) = \{\rho, \alpha_1\rho, \dots, \alpha_i \cdots \alpha_1\rho, \dots\}$. Set $u_0 = \bar{\rho}$ and $u_i = \bar{\alpha}_i \cdots \bar{\alpha}_1 \bar{\rho}$ for $i \geq 1$. Given $0 \neq u \in \Lambda \bar{\rho}$, there exists a minimal $d(u) \geq 0$ such that $u = \sum_{i=0}^{d(u)} \lambda_i u_i$, where $\lambda_i \in k$ with $\lambda_{d(u)} \neq 0$. If $u, v \in \Lambda \bar{\rho}$ are nonzero, then $v = qu + w$, where $q \in \Lambda$ and $w \in \Lambda \bar{\rho}$ with $w = 0$ or $d(w) < d(u)$. Thus, every nonzero submodule of $\Lambda \bar{\rho}$ is generated by some $u \in \Lambda \bar{\rho}$. Thus, $\Lambda \bar{\rho}$ is noetherian. Since Q is left locally finite, $J^{n_x} e_x = \sum_{\rho \in Q_{n_x}(x, -)} \Lambda \bar{\rho}$ with $Q_{n_x}(x, -)$ finite is noetherian, and $\Lambda e_x / J^{n_x} e_x$ is finite dimensional. So, Λe_x is noetherian. The proof of the proposition is completed.

trun_mser_ex

4.3.3. EXAMPLE. Consider $\Lambda = kQ/\mathfrak{I}$, where Q is the quiver



and $\mathfrak{I} = \langle \alpha_i \beta_i \mid i \geq 1 \rangle$. Then Λ is locally right eventually multiserial. By Proposition 4.3.2, it is locally right noetherian. On the other hand, Λ is neither locally left eventually multiserial nor locally left noetherian.

4.4. LOCALLY SEMIPERFECT ALGEBRAS. A ring Σ with an identity is semiperfect if and only if it has a complete orthogonal set of idempotents $\{e_1, \dots, e_n\}$ such that $e_i \Sigma e_i$ is local; see [1, (27.6)]. This inspires the following definition; see [20, (1.4.3)].

Ispa

4.4.1. DEFINITION. Let $\Lambda = kQ/\mathfrak{I}$, where Q is a quiver and \mathfrak{I} is a relation ideal in kQ . One says that Λ is *locally semiperfect* provided, for all $x, y \in Q_0$, that

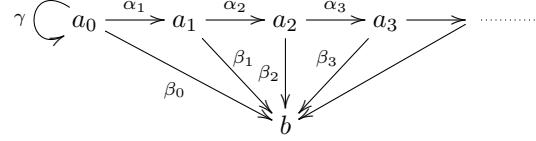
- (1) $e_x \Lambda e_x$ is a local k -algebra;
- (2) $e_y \Lambda e_x$ is a finite dimensional k -vector space.

4.4.2. REMARK. Let $\Lambda = kQ/\mathfrak{I}$ be a locally semiperfect algebra. Regarded as a k -category, Λ is small endo-local and Hom-finite.

spa_ex

4.4.3. EXAMPLE. (1) Let Λ be the k -algebra defined by a single loop α at a vertex with relation $\alpha^2 = \alpha^3$. Then Λ is locally bounded but not locally semiperfect.

- (2) Consider the k -algebra $\Lambda = kQ/\mathfrak{I}$, where Q is the quiver



and $\mathfrak{I} = \langle \gamma^2, \beta_1 \alpha_1 - \beta_i \alpha_i \cdots \alpha_2 \alpha_1 \mid i \geq 2 \rangle$. Then Λ is locally semiperfect with Q being left locally finite but not right locally finite.

The following result collects some basic properties of locally semiperfect algebras.

lfd_rad

4.4.4. PROPOSITION. Let $\Lambda = kQ/\mathfrak{I}$ be locally semiperfect, where Q is a quiver and \mathfrak{I} is a relation ideal in kQ .

- (1) If $x \in Q_0$, then Je_x is the unique maximal submodule of P_x .
- (2) If $x \in Q_0$, then $\text{soc } I_x$ is isomorphic to S_x and essential in I_x .
- (3) The non-isomorphic simple modules in $\text{Mod } \Lambda$ are the S_x with $x \in Q_0$.
- (4) If $M \in \text{Mod } \Lambda$, then $\text{soc } M = \{m \in M \mid Jm = 0\}$.

Proof. (1) Let $x \in Q_0$. Clearly, Je_x is a maximal submodule of P_x . For the uniqueness, it suffices to show that $e_x - \bar{\rho}$ is invertible for $\rho \in \varepsilon_x(kQ^+)_x$. Assume that this is not the case. Since $e_x \Lambda e_x$ is local, $\bar{\rho}$ is invertible for some $\rho \in \varepsilon_x(kQ^+)_x$. Then, $e_x = \bar{\rho}\bar{\eta}$ for some $\eta \in \varepsilon_x(kQ)_x$. So $e_x - \rho\eta \in \mathfrak{I}$, contrary to $\mathfrak{I} \subseteq (kQ^+)^2$.

(2) Let $x \in Q_0$. Since Λ° is locally semiprimary, by Statement (1), the canonical projection $p : \Lambda^\circ e_x \rightarrow S_x^\circ$ is a projective cover. So, p is superfluous; see [27, (3.4)]. Since $\Lambda^\circ e_x \in \text{mod } \Lambda^\circ$, by Proposition 4.2.1, $\mathfrak{D}p : S_x \rightarrow I_x$ is an essential monomorphism. Thus, $\text{Im}(\mathfrak{D}p)$ is an essential submodule of I_x . Since $\text{Im}(\mathfrak{D}p) \cong S_x$, we see that $\text{soc } I_x = \text{Im}(\mathfrak{D}p)$.

(3) Let S be a simple module in $\text{Mod } \Lambda$. Then, there exists $0 \neq m \in e_x S$ for some $x \in Q_0$. If $(Je_x)m = S$, then $m = \bar{\rho}m$ for some $\bar{\rho} \in e_x Je_x$. Since $e_x - \bar{\rho}$ is invertible, we have $m = 0$, absurd. Thus, $(Je_x)m = 0$, and hence, $S \cong \Lambda e_x / Je_x$.

(4) Given $m \in \text{soc } M$, by Statement (3), $Jm = 0$. Suppose that $Jm = 0$ for some $0 \neq m \in M$. Then, $m = e_{x_1}m + \cdots + e_{x_r}m$, where $e_{x_i}m \neq 0$ and $x_i \in Q$ are pairwise distinct. Since $Je_{x_i}(e_{x_i}m) = (Je_{x_i})m \subseteq Jm = 0$, we deduce that $\Lambda(e_{x_i}m_i) \cong S_{x_i}$, for $i = 1, \dots, r$. Thus, $m \in \text{soc } M$. The proof of the proposition is completed.

4.4.5. **REMARK.** By Proposition 4.4.4(1), the condition (1) in Definition 4.4.1 can be replaced by the condition that every oriented cycle in Q is nilpotent in Λ .

fd_soc

4.4.6. **LEMMA.** *Let $\Lambda = kQ/\mathfrak{I}$ be locally semiperfect, where Q is a quiver and \mathfrak{I} is a relation ideal in kQ . Then a module M in $\text{Mod}\Lambda$ is finitely cogenerated if and only if $\text{soc}M$ is finite dimensional and essential in M .*

Proof. Assume that M is finitely cogenerated with an injective envelope $g : M \rightarrow I$, where $I \in \text{inj}\Lambda$. By Proposition 4.4.4(2), $\text{soc}I \in \text{mod}^b\Lambda$ is essential in I . Since $g(\text{soc}M) \subseteq \text{soc}I$, we have $\text{soc}M \in \text{mod}^b\Lambda$ and $g(\text{soc}M) = \text{soc}I$. Thus, $\text{soc}M$ is essential in M . Conversely, assume that $\text{soc}M \in \text{mod}^b\Lambda$ is essential in M . Then, $\text{soc}M = km_1 \oplus \dots \oplus km_s$, where $0 \neq m_i \in e_{x_i}(\text{soc}M)$ with $x_i \in Q_0$. By Proposition 4.4.4(4), $km_i \cong S_{x_i}$, and in view of Proposition 4.4.4(2), we have a monomorphism $h_i : km_i \rightarrow I_{x_i}$ with $\text{Im}(h_i) = \text{soc}I_{x_i}$, for $i = 1, \dots, s$. This yields a monomorphism $h : \text{soc}M \rightarrow I = \bigoplus_{i=1}^s I_{x_i}$ with $h(\text{soc}M) = \text{soc}I$. Thus, h extends to a monomorphism $g : M \rightarrow I$. The proof of the lemma is completed.

4.5. **LOCALLY SEMIPRIMARY ALGEBRAS.** For our later study on almost split sequences in $\text{mod}^+\Lambda$ and $\text{mod}^-\Lambda$, we need to impose some stronger conditions on Λ . Recall that a ring Σ with an identity is called *semiprimary* if $\text{rad}\Sigma$ is nilpotent and $\Sigma/\text{rad}\Sigma$ is semisimple; see [30, (4.15)]. This motivates the following definition.

lspri_def

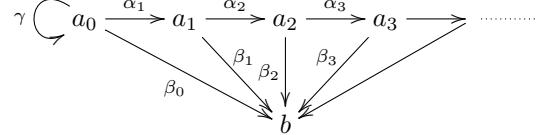
4.5.1. **DEFINITION.** Let $\Lambda = kQ/\mathfrak{I}$, where Q is a quiver and \mathfrak{I} is a relation ideal in kQ . We say that Λ is *locally semiprimary* provided, for any $x, y \in Q_0$, that Q contains at most finitely many nonzero paths from x to y .

lsprm_lsp

4.5.2. **REMARK.** Clearly, locally semiprimary algebras are locally semiperfect.

lspri_ex

4.5.3. **EXAMPLE.** (1) If Q is an interval-finite quiver, then kQ is locally semiprimary.
(2) Consider $\Lambda = kQ/\mathfrak{I}$, where Q is the quiver



and $\mathfrak{I} = \langle \gamma^2, \beta_n \alpha_n \cdots \alpha_i \mid i \geq 1; n \geq 2i \rangle$. Then Λ is locally semiprimary with Q not locally finite. In contrast, the algebra in Example 4.4.3(2) defined by the same quiver is not locally semiprimary.

We shall need the following statement for describing almost split sequences in the subcategories $\text{mod}^+\Lambda$ and $\text{mod}^-\Lambda$ of $\text{Mod}\Lambda$.

fdmod

4.5.4. **LEMMA.** *Let $\Lambda = kQ/\mathfrak{I}$ be locally semiprimary, where Q is a quiver and \mathfrak{I} is a relation ideal in kQ . Then a module in $\text{Mod}\Lambda$ is finite dimensional if and only if it is finitely generated and finitely cogenerated.*

Proof. Let $M \in \text{mod}^b\Lambda$. In particular, M is finitely generated and $\text{soc}M \in \text{mod}^b\Lambda$. And there exist $x_1, \dots, x_r \in Q_0$ such that $e_x M = 0$, for $x \in Q_0 \setminus \{x_1, \dots, x_r\}$. Let $0 \neq m \in e_{x_i} M$ for some $1 \leq i \leq r$. Suppose that $(\text{soc}M) \cap (Am) = 0$. By Proposition 4.4.4(4), there exist $\alpha_n : a_n \rightarrow a_{n+1}$ in Q_1 with $a_1 = x_i$, such that $0 \neq \bar{\alpha}_n \cdots \bar{\alpha}_1 m \in e_{a_n} M$, for all $n \geq 1$. In particular, $a_n \in \{x_1, \dots, x_r\}$ and $\alpha_n \cdots \alpha_1$ is a nonzero path in Q from x_i to a_n , for all $n \geq 1$. So, there exists some $1 \leq j \leq r$ such that Q has infinitely many nonzero paths from x_i to x_j , a contradiction. Thus, $\text{soc}M$ is essential in M , and hence, M is finitely cogenerated; see (4.4.6).

Assume now that M is finitely cogenerated, and finitely generated with an epimorphism $f : P_{x_1} \oplus \cdots \oplus P_{x_r} \rightarrow M$, where $x_i \in Q_0$. Then $M = \sum_{i=1}^r \Lambda f(e_{x_i})$, where $f(e_{x_i}) \in e_{x_i} M$, and by Lemma 4.4.6, $\text{soc } M = e_{y_1}(\text{soc } M) \oplus \cdots \oplus e_{y_s}(\text{soc } M)$, where $y_i \in Q_0$. For any $x \in Q_0$, $\dim(e_x M) \leq \sum_{i=1}^r \dim(e_x \Lambda e_{x_i}) < \infty$. Suppose that $e_x M \neq 0$ for some $x \in Q_0$. Then, Q contains a path ρ from some x_i to x such that $\bar{\rho}f(e_{x_i}) \neq 0$. Since $\text{soc } M$ is essential in M ; see (4.4.6), Q contains a path η from x to some y_j such that $\bar{\eta}(\bar{\rho}f(e_{x_i})) \neq 0$. So, $\eta\rho$ is a nonzero path in Q from x_i to y_j passing through x . By Definition 4.5.1, $e_x M \neq 0$ for only finitely many $x \in Q_0$. Thus, $M \in \text{mod}^b \Lambda$. The proof of the lemma is completed.

fg_cfg_nfd

4.5.5. REMARK. Lemma 4.5.4 does not hold for locally semiperfect algebras. Consider the locally semiperfect algebra $\Lambda = kQ/\mathfrak{I}$, where Q is the quiver

$$\begin{array}{ccccccccccc} a_0 & \xrightarrow{\alpha_0} & a_1 & \xrightarrow{\alpha_1} & a_2 & \xrightarrow{\alpha_2} & a_3 & \xrightarrow{\alpha_3} & a_4 & \cdots & \cdots \\ & & \downarrow \gamma_1 & & \downarrow \gamma_2 & & \downarrow \gamma_3 & & \downarrow \gamma_4 & & \cdots \\ b_0 & \xleftarrow{\beta_0} & b_1 & \xleftarrow{\beta_1} & b_2 & \xleftarrow{\beta_2} & b_3 & \xleftarrow{\beta_3} & b_4 & \cdots & \cdots \end{array}$$

and \mathfrak{I} is the ideal generated by $\beta_0\gamma_1\alpha_0 - \beta_0\beta_1 \cdots \beta_i\gamma_{i+1}\alpha_i \cdots \alpha_1\alpha_0$ with $i \geq 1$, and $\beta_i\gamma_{i+1}\alpha_i - \beta_i\beta_{i+1} \cdots \beta_j\gamma_{j+1}\alpha_j \cdots \alpha_{i+1}\alpha_i$ with $j > i \geq 1$. Note that $P_{a_0} \cong I_{b_0}$. Thus, P_{a_0} is finitely generated, finitely cogenerated and infinite dimensional.

The following result is handy for constructing examples of locally semiprimary algebras, which are also locally left or right noetherian.

4.5.6. LEMMA. *Let $\Lambda = kQ/\mathfrak{I}$, where Q is a quiver and \mathfrak{I} is a relation ideal in kQ . If Λ is locally left or right eventually multiserial, then Λ is locally semiprimary if and only if all oriented cycles in Q are nilpotent in Λ .*

Proof. We consider only the case where Λ is locally left eventually multiserial. The necessity is evident. Assume that all oriented cycles in Q are nilpotent in Λ . Suppose that Q has infinitely many nonzero paths from x to y , for some $x, y \in Q_0$. Since Q is left locally finite, it contains an infinite path

$$\zeta : x = y_0 \xrightarrow{\alpha_1} y_1 \xrightarrow{\alpha_2} y_2 \xrightarrow{\alpha_3} \cdots \xrightarrow{\alpha_i} y_i \xrightarrow{\alpha_{i+1}} \cdots$$

such that Q has a path $\rho_i : y_i \rightsquigarrow y$ such that $\rho_i\alpha_i \cdots \alpha_1$ is a nonzero path, for all $i \geq 1$. Let n_x be an integer as in Definition 4.3.1(1). Then, for any $i \geq n_x$, the infinite subpath ζ_i of ζ starting with y_i starts with ρ_i . So, we may assume that $y_{n_x} = y$. It is not difficult to see that ζ_{n_x} starts with $\rho_{n_x}^s$ for infinitely many $s > 0$, contrary to the assumption. The proof of the lemma is completed.

4.6. FINITE PRESENTED AND FINITELY COPRESENTED MODULES. We begin with the following statement, which is a special case of Proposition 3.6.1.

lfd_fpres

4.6.1. PROPOSITION. *Let $\Lambda = kQ/\mathfrak{I}$ be locally semiperfect, where Q is a quiver and \mathfrak{I} is a relation ideal in kQ . Then*

- (1) $\text{mod}^+ \Lambda$ and $\text{mod}^- \Lambda$ are Hom-finite Krull-Schmidt subcategories of $\text{mod} \Lambda$;
- (2) The duality $\mathfrak{D} : \text{mod} \Lambda^\circ \rightarrow \text{mod} \Lambda$ restricts to a duality $\mathfrak{D} : \text{mod}^+ \Lambda^\circ \rightarrow \text{mod}^- \Lambda$.

We shall need the property that finite dimensional modules are all finitely presented or all finitely copresented.

fd_fp

4.6.2. LEMMA. *Let $\Lambda = kQ/\mathfrak{I}$ be locally semiperfect, where Q is a quiver and \mathfrak{I} is a relation ideal in kQ . Then $\text{mod}^b \Lambda \subseteq \text{mod}^+ \Lambda$ if and only if Q is left locally finite; and $\text{mod}^b \Lambda \subseteq \text{mod}^- \Lambda$ if and only if Q is right locally finite.*

Proof. Since mod^+A is extension-closed in $\text{Mod}A$, by Proposition 4.4.4(3), we see that $\text{mod}^bA \subseteq \text{mod}^+A$ if and only if $S_x \in \text{mod}^+A$ for all $x \in Q_0$. And for each $x \in Q_0$, there exists a short exact sequence $0 \longrightarrow Je_x \longrightarrow P_x \longrightarrow S_x \longrightarrow 0$ in $\text{Mod}A$. Since $\{\bar{\alpha} \mid \alpha \in Q_1(x, -)\}$ is a minimal generating set for Je_x , by Schanul's Lemma, $S_x \in \text{mod}^+A$ if and only if $\{\bar{\alpha} \mid \alpha \in Q_1(x, -)\}$ is finite, that is, $Q_1(x, -)$ is finite. The first part of the lemma holds. Next, by Propositions 4.2.1 and 4.6.1(2), $\text{mod}^bA \subseteq \text{mod}^-A$ if and only if $\text{mod}^bA^\circ \subseteq \text{mod}^+A^\circ$, or equivalently, A° is left locally finite, that is, A is right locally finite. The proof of the lemma is completed.

We shall also need the following easy statement.

11b_lrb 4.6.3. **LEMMA.** *Let $A = kQ/R$ be semiperfect, where Q is a quiver and \mathfrak{I} is a relation ideal in kQ . Then, $\text{mod}^+A = \text{mod}^bA$ if and only if A is locally left bounded; and $\text{mod}^-A = \text{mod}^bA$ if and only if A is locally right bounded.*

Proof. We shall only prove the sufficiency of the first part of the lemma. Assume that A is locally left bounded. Then, $\text{proj}A \subseteq \text{mod}^bA$, and hence, $\text{mod}^+A \subseteq \text{mod}^bA$. Moreover, Q is clearly left locally finite, and hence, $\text{mod}^bA \subseteq \text{mod}^+A$; see (4.6.2). The proof of the lemma is completed.

Clearly, the projective modules in mod^+A are ext-projective, and the injective modules in mod^-A are ext-injective. The converses are not true in general.

ext_proj_ab 4.6.4. **LEMMA.** *Let $A = kQ/R$, where Q is a quiver and \mathfrak{I} is a relation ideal in kQ .*

- (1) *If mod^+A is abelian, then its ext-projective objects are the modules in $\text{proj}A$.*
- (2) *If mod^-A is abelian, then its ext-injective objects are the modules in $\text{inj}A$.*

Proof. We shall only prove Statement (1). Suppose that mod^+A be abelian. Consider a nonprojective module M in mod^+A . Then there exists a nonsplit short exact sequence $0 \longrightarrow L \longrightarrow P \longrightarrow M \longrightarrow 0$ in $\text{Mod}A$ where $P \in \text{proj}A$ and L is finitely generated. By Proposition 3.5.1, $L \in \text{mod}^+A$, and hence, M is not ext-projective in mod^+A . The proof of the lemma is completed.

We provide some classes of algebras A for which mod^+A or mod^-A is abelian.

fpres_ab 4.6.5. **PROPOSITION.** *Let $A = kQ/\mathfrak{I}$ be locally semiperfect, where Q is a quiver and \mathfrak{I} is a relation ideal in kQ .*

- (1) *If $\mathfrak{I} = 0$, then mod^+A and mod^-A are abelian.*
- (2) *If A is locally left (resp. right) noetherian, then mod^+A (resp. mod^-A) is abelian.*

Proof. (1) Assume that $\mathfrak{I} = 0$. Then A is hereditary; see [19, (8.2)]. Thus, $\text{proj}A$ is closed under finitely generated submodules and $\text{inj}A$ is closed under finitely cogenerated quotient-modules. By Proposition 3.5.1, mod^+A and mod^-A are abelian.

(2) Suppose that A is locally left noetherian. Then the modules in $\text{proj}A$ are noetherian. Thus, finitely generated modules are finitely presented. By Proposition 3.5.1, mod^+A is abelian. Assume that A is locally right noetherian. Then A° is locally left noetherian, and hence, mod^+A° is abelian. By Proposition 4.6.1(2), mod^-A is abelian. The proof of the proposition is completed.

Ab_subcat_ex 4.6.6. **EXAMPLE.** Consider $A = kQ/\mathfrak{I}$, where Q is the quiver

$$\begin{array}{ccccccccccc} \cdots & a_3 & \xrightarrow{\alpha_3} & a_2 & \xrightarrow{\alpha_2} & a_1 & \xrightarrow{\alpha_1} & a_0 & \xrightarrow{\alpha} & a_{-1} & \longrightarrow & a_{-2} & \longrightarrow & a_{-3} & \cdots \\ & \uparrow \beta_3 & & \uparrow \beta_2 & & \uparrow \beta_1 & & & & \downarrow c_1 & & \downarrow c_1 & & \downarrow c_2 & \\ & b_3 & & b_2 & & b_1 & & & & & & & & & & \end{array}$$

and $\mathfrak{I} = \langle \alpha\alpha_1, \beta\alpha_1, \alpha_i\beta_i \mid i \geq 1 \rangle$. Then, Λ is locally right eventually multiserial. By Propositions 4.3.2 and 4.6.5, $\text{mod}^-\Lambda$ is abelian. Although Λ is neither locally left noetherian nor hereditary, we claim that $\text{mod}^+\Lambda$ is abelian.

Indeed let $\Lambda' = kQ'$, where Q' is the full subquiver of Q generated by the a_i and c_j with $i \leq 0$ and $j \geq 1$. Then $\text{proj}\Lambda' \subseteq \text{proj}\Lambda$. Let M be a finitely generated submodule of some module P in $\text{proj}\Lambda$. Write P' for the maximal direct summand of P lying in $\text{proj}\Lambda'$. Note that $M' = \sum_{a_0 \neq x \in Q'_0} e_x M$ is a finitely generated submodule of $P' \cap M$ such that $M/M' \in \text{mod}^b\Lambda \subseteq \text{mod}^+\Lambda$; see (4.6.2). Since Λ' is hereditary, $M' \in \text{proj}\Lambda' \subseteq \text{mod}^+\Lambda$. Therefore, $M \in \text{mod}^+\Lambda$. Now it follows from Proposition 3.5.1(1) that $\text{mod}^+\Lambda$ is abelian.

4.7. ALMOST SPLIT SEQUENCES. We start with the locally semiperfect case. Since $\text{mod}^+\Lambda$ and $\text{mod}^-\Lambda$ are Krull-Schmidt; see (4.6.1), as a special case of Theorem 3.6.3, we immediately obtain a general existence theorem for $\text{Mod}\Lambda$ as follows.

ass_lfd

4.7.1. THEOREM. *Let $\Lambda = kQ/\mathfrak{I}$ be a locally semiperfect algebra, where Q is a quiver and \mathfrak{I} is a relation ideal in kQ . Consider an indecomposable module $M \in \text{Mod}\Lambda$.*

- (1) *If $M \in \text{mod}^+\Lambda$ is not projective, then there exists an almost split sequence $0 \longrightarrow \tau M \longrightarrow N \longrightarrow M \longrightarrow 0$ in $\text{Mod}\Lambda$, where $\tau M \in \text{mod}^-\Lambda$.*
- (2) *If $M \in \text{mod}^-\Lambda$ is not injective, then there exists an almost split sequence $0 \longrightarrow M \longrightarrow N \longrightarrow \tau^-M \longrightarrow 0$ in $\text{Mod}\Lambda$, where $\tau^-M \in \text{mod}^+\Lambda$.*

4.7.2. REMARK. (1) In case Q is strongly locally finite and $\mathfrak{I} = 0$, Theorem 4.7.1 strengthens the result of Theorem 2.8 in [13].

(2) In case Λ is locally left or right bounded, Theorem 4.7.1 strengthens the result of Theorem 3.4.1 in [20]. Note that the latter is obtained using the classical approach, and contains an error claiming that τM and τ^-M are finite dimensional.

Next, we shall concentrate on almost split sequences in $\text{mod}^+\Lambda$ and $\text{mod}^-\Lambda$ in the locally semiprimary case. This will be based on the following description of irreducible morphisms in $\text{mod}^+\Lambda$ and $\text{mod}^-\Lambda$, which covers the result in [13, (3.1)].

ker_irr_epi

4.7.3. LEMMA. *Let $\Lambda = kQ/\mathfrak{I}$ be locally semiprimary, where Q is a quiver and \mathfrak{I} is a relation ideal in kQ .*

- (1) *If Q is left locally finite, then every irreducible epimorphism in $\text{mod}^+\Lambda$ has a finite dimensional kernel.*
- (2) *If Q is right locally finite, then every irreducible monomorphism in $\text{mod}^-\Lambda$ has a finite dimensional cokernel.*

Proof. We shall only prove Statement (1). Assume that Q is left locally finite. By Lemma 4.6.2, $\text{mod}^b\Lambda \subseteq \text{mod}^+\Lambda$. Consider an irreducible epimorphism $f : M \rightarrow N$ in $\text{mod}^+\Lambda$. We may assume that N is indecomposable. In view of Theorem 4.7.1(1), we can construct a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & L & \xrightarrow{g} & M & \xrightarrow{f} & N \longrightarrow 0 \\ & & \downarrow u & & \downarrow v & & \parallel \\ 0 & \longrightarrow & X & \xrightarrow{h} & Y & \xrightarrow{p} & N \longrightarrow 0, \end{array}$$

where the bottom row is an almost split sequence in $\text{Mod}\Lambda$ with $X \in \text{mod}^-\Lambda$. Writing $X' = \text{Im}(u)$ and $Y' = \text{Im}(v)$, we obtain factorizations $u = ju'$ and $v = qv'$, where $u' : L \rightarrow X'$ and $v' : M \rightarrow Y'$ are epimorphisms; $j : X' \rightarrow X$ and $q : Y' \rightarrow Y$ are inclusion maps. This yields a commutative diagram with exact rows

$$\begin{array}{ccccccc}
 0 & \longrightarrow & L & \xrightarrow{g} & M & \xrightarrow{f} & N \longrightarrow 0 \\
 & & \downarrow u' & & \downarrow v' & & \parallel \\
 0 & \longrightarrow & X' & \xrightarrow{h'} & Y' & \xrightarrow{pq} & N \longrightarrow 0.
 \end{array}$$

Since L is finitely generated, so is X' . And since X is finitely cogenerated, so is X' ; see (4.4.6). Thus, $X' \in \text{mod}^b\Lambda \subseteq \text{mod}^+\Lambda$; see (4.5.4), and hence, $Y' \in \text{mod}^+\Lambda$; see (2.1.1). Since p is not a retraction, neither is $pq : Y' \rightarrow N$. Thus, v' is a section. Then, v' is an isomorphism, and so is u' . So, L is finite dimensional. The proof of the lemma is completed.

We are now ready to describe the almost split sequences in $\text{mod}^+\Lambda$ which, in the hereditary case, strengthens the result stated in [13, (3.6)].

ass_fpres

4.7.4. THEOREM. *Let $\Lambda = kQ/\mathfrak{I}$ be a locally semiprimary algebra, where Q is a locally finite quiver and \mathfrak{I} is a relation ideal in kQ . Then the almost split sequences in $\text{mod}^+\Lambda$ are those in $\text{Mod}\Lambda$ with a finite dimensional starting term. Moreover,*

- (1) *an indecomposable nonprojective module M in $\text{mod}^+\Lambda$ is the ending term of an almost split sequence in $\text{mod}^+\Lambda$ if and only if τM is finite dimensional;*
- (2) *an indecomposable noninjective module N in $\text{mod}^+\Lambda$ is the starting term of an almost split sequence in $\text{mod}^+\Lambda$ if and only if N is finite dimensional.*

Proof. Since Q is locally finite, $\text{mod}^b\Lambda = \text{mod}^+\Lambda \cap \text{mod}^-\Lambda$; see (4.5.4) and (4.6.2). Consider an almost split sequence $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ in $\text{Mod}\Lambda$, where X is finite dimensional. By Theorem 4.7.1(2), $Z \cong \tau^-X \in \text{mod}^+\Lambda$. Since $\text{mod}^+\Lambda$ is extension-closed in $\text{Mod}\Lambda$, this is an almost split sequence in $\text{mod}^+\Lambda$. Conversely, let $(*) 0 \longrightarrow M \longrightarrow N \longrightarrow L \longrightarrow 0$ be an almost split sequence in $\text{mod}^+\Lambda$. By Lemma 4.7.3(1), $M \in \text{mod}^b\Lambda$, and by Theorem 4.7.1(2), there exists an almost split sequence $0 \longrightarrow M \longrightarrow E \longrightarrow \tau^-M \longrightarrow 0$ in $\text{Mod}\Lambda$, where $\tau^-M \in \text{mod}^+\Lambda$. Since $M \in \text{mod}^+\Lambda$, this is an almost split sequence in $\text{mod}^+\Lambda$, and hence, it is isomorphic to $(*)$. This establishes the first part of the theorem. Combining this part with Theorem 4.7.1, we deduce easily Statements (1) and (2). The proof of the theorem is completed.

4.7.5. REMARK. In view of Theorem 4.7.4(2), we see that the finite dimensional ext-injective objects in $\text{mod}^+\Lambda$ are the finite dimensional modules in $\text{inj}\Lambda$.

Dually, we can describe the almost split sequences in $\text{mod}^-\Lambda$ as follows.

ass_fcp

4.7.6. THEOREM. *Let $\Lambda = kQ/\mathfrak{I}$ be a locally semiprimary algebra, where Q is a locally finite quiver and \mathfrak{I} is a relation ideal in kQ . Then the almost split sequences in $\text{mod}^-\Lambda$ are those in $\text{Mod}\Lambda$ with a finite dimensional ending term. Moreover,*

- (1) *an indecomposable noninjective module N in $\text{mod}^-\Lambda$ is the starting term of an almost split sequence in $\text{mod}^-\Lambda$ if and only if τ^-N is finite dimensional;*
- (2) *an indecomposable nonprojective module M in $\text{mod}^-\Lambda$ is the ending term of an almost split sequence in $\text{mod}^-\Lambda$ if and only if M is finite dimensional.*

4.7.7. REMARK. In view of 4.7.6(2), we see that the finite dimensional ext-projective objects in $\text{mod}^-\Lambda$ are the finite dimensional modules in $\text{proj}\Lambda$.

4.8. SUBCATEGORIES HAVING ALMOST SPLIT SEQUENCES. The objective of this subsection is to study when $\text{mod}^+\Lambda$, $\text{mod}^-\Lambda$ and $\text{mod}^b\Lambda$ have almost split sequences on one or both sides in case Λ is locally semiprimary given by a locally finite quiver.

ass_fp_cat

4.8.1. THEOREM. Let $\Lambda = kQ/R$ be a locally semiprimary algebra, where Q is a locally finite quiver and \mathfrak{I} is a relation ideal in kQ .

- (1) If Λ is locally left bounded, then $\text{mod}^+\Lambda$ has almost split sequences on the left. And the converse holds in case $\text{mod}^+\Lambda$ is abelian.
- (2) If the indecomposable noninjective modules in $\text{mod}^-\Lambda$ are finite dimensional (in particular, if Λ is locally right bounded), then $\text{mod}^+\Lambda$ has almost split sequences on the right. And the converse holds in case $\text{mod}^+\Lambda$ is abelian.

Proof. (1) Suppose that Λ is locally left bounded. Then, $\text{mod}^+\Lambda = \text{mod}^b\Lambda$ by Lemma 4.6.3. If $M \in \text{mod}^+\Lambda$ is indecomposable and not ext-injective, then it is not injective, and by Theorem 4.7.4(2), M is the starting term of an almost split sequence in $\text{mod}^+\Lambda$. Thus, $\text{mod}^+\Lambda$ has almost split sequences on the left.

Suppose now that Λ is not locally left bounded and $\text{mod}^+\Lambda$ is abelian. Then P_x is infinite dimensional for some $x \in Q_0$. Since Q is locally finite, by Proposition 3.5.1(1), $\text{rad}P_x \in \text{mod}^+\Lambda$. Let N be an infinite dimensional indecomposable direct summand of $\text{rad}P_x$. Then, we obtain a nonsplit short exact sequence $0 \longrightarrow N \longrightarrow P_x \longrightarrow P_x/N \longrightarrow 0$ in $\text{mod}^+\Lambda$. In particular, N is not ext-injective in $\text{mod}^+\Lambda$. By Theorem 4.7.4(2), N is not the starting term of any almost split sequence in $\text{mod}^+\Lambda$. Thus, $\text{mod}^+\Lambda$ does not have almost split sequences on the left.

(2) Suppose that all indecomposable noninjective modules in $\text{mod}^-\Lambda$ are finite dimensional. If $M \in \text{mod}^+\Lambda$ is indecomposable and not ext-projective, then M is not projective, and by Theorem 4.7.1(1), there exists an almost split sequence $0 \longrightarrow \tau M \longrightarrow N \longrightarrow M \longrightarrow 0$ in $\text{Mod}\Lambda$, where $\tau M \in \text{mod}^-\Lambda$. Since $\tau M \in \text{mod}^b\Lambda$ by the assumption, this is an almost split sequence in $\text{mod}^+\Lambda$. Therefore, $\text{mod}^+\Lambda$ has almost split sequences on the right.

Assume now that $\text{mod}^+\Lambda$ is abelian and $\text{mod}^-\Lambda$ contains an infinite dimensional indecomposable noninjective module M . By Theorem 4.7.1(2), there exists an almost split sequence $0 \longrightarrow M \longrightarrow N \longrightarrow L \longrightarrow 0$ in $\text{Mod}\Lambda$, where $L \in \text{mod}^+\Lambda$. Since $\tau L \cong M$, by Theorem 4.7.4(1), $\text{mod}^+\Lambda$ has no almost split sequence ending with L . Since L is not ext-projective in $\text{mod}^+\Lambda$ by Lemma 4.6.4(1), $\text{mod}^+\Lambda$ does not have almost split sequences on the right. The proof of the theorem is completed.

4.8.2. REMARK. In case $\mathfrak{I} = 0$, we see from Lemma 4.6.5(1) that Theorem 4.8.1 covers the result in [13, (3.7)]; compare also [33, (3.6.1)].

Dually, we have the following statement.

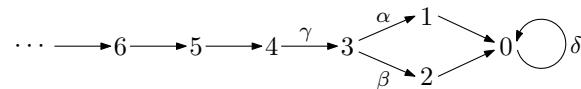
ass_fcp_cat

4.8.3. THEOREM. Let $\Lambda = kQ/R$ be a locally semiprimary algebra, where Q is a locally finite quiver and \mathfrak{I} is a relation ideal in kQ .

- (1) If Λ is locally right bounded, then $\text{mod}^-\Lambda$ has almost split sequences on the right. And the converse holds if $\text{mod}^-\Lambda$ is abelian.
- (2) If the indecomposable nonprojective modules in $\text{mod}^+\Lambda$ are finite dimensional (in particular, if Λ is locally left bounded), then $\text{mod}^-\Lambda$ has almost split sequences on the left. And the converse holds if $\text{mod}^-\Lambda$ is abelian.

ex_ass_fp_cat

4.8.4. EXAMPLE. (1) Consider $\Lambda = kQ/\mathfrak{I}$, where Q is the quiver



and $\mathfrak{I} = \langle \alpha\gamma, \beta\gamma, \delta^3 \rangle$. Then, Λ is locally eventually multiserial. By Lemmas 4.3.2 and 4.6.5, $\text{mod}^-\Lambda$ and $\text{mod}^+\Lambda$ are abelian. Since Λ is locally left bounded but

not locally right bounded, by Theorem 4.8.3, $\text{mod}^-\Lambda$ has almost split sequences on the left but not on the right. And by Theorem 4.8.1(1), $\text{mod}^+\Lambda$ has almost split sequences on the left. We claim that it also has almost split sequences on the right.

Indeed let $\Lambda' = kQ'$ and $\Lambda'' = (kQ'')/\langle \delta^3 \rangle$, where Q' and Q'' are the full subquivers of Q generated by the vertices $i \geq 3$ and by the vertices j with $0 \leq j \leq 3$, respectively. Then, every module M in $\text{Mod}\Lambda$ is decomposed as $M = M' \oplus M''$, where $M' \in \text{Mod}\Lambda'$ and $M'' \in \text{Mod}\Lambda''$. Thus, if M is an infinite dimensional indecomposable module in $\text{mod}^-\Lambda$, then $M \in \text{mod}^-\Lambda'$, and hence, $M \cong I_i$ for some $i \geq 3$. By Theorem 4.8.1(2), $\text{mod}^+\Lambda$ has almost split sequences on the right.

(2) Consider $\Lambda = kQ/\mathfrak{J}$, where Q is the quiver

$$\begin{array}{ccccccccccc} \dots & a_3 & \xrightarrow{\alpha_3} & a_2 & \xrightarrow{\alpha_2} & a_1 & \xrightarrow{\alpha_1} & a_0 & \xrightarrow{\alpha} & a_{-1} & \longrightarrow & a_{-2} & \longrightarrow & a_{-3} & \dots \\ & \uparrow \beta_3 & & \uparrow \beta_2 & & \uparrow \beta_1 & & & \downarrow & & & \downarrow & & \downarrow & \\ & b_3 & & b_2 & & b_1 & & & c_1 & & c_1 & & c_2 & & \end{array}$$

and $\mathfrak{J} = \langle \alpha\alpha_1, \beta\alpha_1, \alpha_i\beta_i \mid i \geq 1 \rangle$. We claim that neither $\text{mod}^+\Lambda$ nor $\text{mod}^-\Lambda$ has almost split sequences on either side. As seen in Example 4.6.6, $\text{mod}^+\Lambda$ and $\text{mod}^-\Lambda$ are abelian. Since Λ is not left locally bounded, by Theorem 4.8.1(1), $\text{mod}^+\Lambda$ does not have almost split sequences on the left. On the other hand, let M be the submodule of I_{a_1} such that $I_{a_1}/M \cong S_{b_1}$. Then M is an infinite dimensional indecomposable noninjective module in $\text{mod}^-\Lambda$. By Theorem 4.8.1(2), $\text{mod}^+\Lambda$ does not have almost split sequences on the right. Similarly, we see from Theorem 4.8.3 that $\text{mod}^-\Lambda$ does not have almost split sequences on either side.

To conclude the paper, we study when $\text{mod}^b\Lambda$ has almost split sequences.

ass_fdm_cat

4.8.5. THEOREM. *Let $\Lambda = kQ/R$ be a locally semiprimary algebra, where Q is a locally finite quiver and \mathfrak{J} is a relation ideal in kQ .*

- (1) *If the indecomposable noninjective modules in $\text{mod}^-\Lambda$ are finite dimensional, then $\text{mod}^b\Lambda$ has almost split sequences on the right.*
- (2) *If the indecomposable nonprojective modules in $\text{mod}^+\Lambda$ are finite dimensional, then $\text{mod}^b\Lambda$ has almost split sequences on the left.*
- (3) *In each of the above two situations, the almost split sequences in $\text{mod}^b\Lambda$ are almost split sequences in $\text{Mod}\Lambda$.*

Proof. We shall only consider the case where all indecomposable noninjective modules in $\text{mod}^-\Lambda$ are finite dimensional. Let $M \in \text{mod}^b\Lambda$ be indecomposable and not ext-projective. By Lemma 4.6.2, $M \in \text{mod}^+\Lambda$ is not projective, and by Theorem 4.7.1, $\text{Mod}\Lambda$ has an almost split sequence $0 \longrightarrow \tau M \longrightarrow N \longrightarrow M \longrightarrow 0$ with $\tau M \in \text{mod}^-\Lambda$. Since $\tau M \in \text{mod}^b\Lambda$, this is an almost split sequence in $\text{mod}^b\Lambda$. Thus, Statements (1) and (3) hold in this case. The proof of the theorem is completed.

rlb-ass

4.8.6. REMARK. In case Λ is a finite dimensional algebra, Theorem 4.8.5(3) is known to some specialists.

By Lemma 4.6.3 and Theorem 4.8.5, we have immediately the following result.

ass_fdm_cor

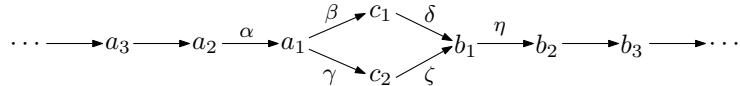
4.8.7. COROLLARY. *Let $\Lambda = kQ/R$ be a locally semiprimary algebra, where Q is a locally finite quiver and \mathfrak{J} is a relation ideal in kQ . If Λ is locally (left, right) bounded, then $\text{mod}^b\Lambda$ has almost split sequences (on the left, on the right).*

4.8.8. REMARK. Let $\Lambda = kQ/\mathfrak{J}$ be finite dimensional, where Q is a finite quiver and \mathfrak{J} is a relation ideal in kQ containing $(kQ^+)^n$ for some $n \geq 2$. Let $\tilde{\Lambda} = k\tilde{Q}/\tilde{\mathfrak{J}}$ such

that there exists a quiver-covering $\phi : \tilde{Q} \rightarrow Q$; see [11, (4.1)] and $\tilde{\mathfrak{I}}$ is the pre-image of \mathfrak{I} under the k -linearly induced map $\phi' : k\tilde{Q} \rightarrow kQ$. Then, ϕ' induces a covering $\psi : \tilde{\Lambda} \rightarrow \Lambda$ as defined in [14, (3.1)]. It is easy to check that $\tilde{\Lambda}$ is locally semiprimary and locally bounded. By Corollary 4.8.7, $\text{mod}^b\tilde{\Lambda}$ has almost split sequences, and by Theorem 4.8.5, the almost split sequences in $\text{mod}^b\tilde{\Lambda}$ are almost split in $\text{Mod}\tilde{\Lambda}$.

ex_ass_fdim_cat

4.8.9. EXAMPLE. Consider $\Lambda = kQ/\mathfrak{I}$, where Q is the quiver



and $\mathfrak{I} = \langle \beta\alpha, \gamma\alpha, \eta\delta, \zeta\eta \rangle$. Clearly, Λ is neither locally left bounded nor locally right bounded. As seen in Example 4.8.4(1), the infinite dimensional indecomposable modules in $\text{mod}^-\Lambda$ are the injective modules I_{a_i} with $i \geq 1$, and the infinite dimensional indecomposable modules in $\text{mod}^+\Lambda$ are the projective modules P_{b_j} with $j \geq 1$. By Theorem 4.8.5, $\text{mod}^b\Lambda$ has almost split sequences. Observe that $\text{mod}^b\Lambda$ has neither enough projective objects nor enough injective objects.

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